THE DECOMPOSABILITY OF ADDITIVE HEREDITARY PROPERTIES OF GRAPHS

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Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphisms. If $P_1, \ldots, P_n$ are properties of graphs, then a $(P_1, \ldots, P_n)$-decomposition of a graph $G$ is a partition $E_1, \ldots, E_n$ of $E(G)$ such that $G[E_i]$ is in $P_i$, for $i = 1, \ldots, n$. We define $P_1 \oplus \cdots \oplus P_n$ as the property $\{G \in I : G$ has a $(P_1, \ldots, P_n)$-decomposition $\}$. A property $P$ is said to be decomposable if there exist non-trivial hereditary properties $P_1$ and $P_2$ such that $P = P_1 \oplus P_2$. We study the decomposability of the well-known properties of graphs $I_k$, $O_k$, $W_k$, $T_k$, $S_k$, $D_k$ and $O^p$.

Keywords: property of graphs, additive, hereditary, decomposable property of graphs.

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1 Introduction

Following [2] we denote the class of all finite simple graphs by $I$. A property of graphs is a non-empty isomorphism-closed subclass of $I$. The fact that $H$ is a subgraph of $G$ is denoted by $H \subseteq G$ and the disjoint union of two graphs $G$ and $H$ is denoted by $G \cup H$. A property $P$ is called hereditary if $G \in P$ and $H \subseteq G$ implies $H \in P$; $P$ is called additive if $G \cup H \in P$ whenever $G \in P$ and $H \in P$. Throughout this paper the term property (of graphs) is used to refer to an additive hereditary property (of graphs).
Example. Some well-known properties are given in the list below.
\[\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\},\]
\[\mathcal{I}_k = \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\},\]
\[\mathcal{O}_k = \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k+1 \text{ vertices}\},\]
\[\mathcal{W}_k = \{G \in \mathcal{I} : \text{the length of any path in } G \text{ is at most } k\},\]
\[\mathcal{T}_k = \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \text{ or } K_{\left\lfloor \frac{k+3}{2}\right\rfloor, \left\lceil \frac{k+3}{2}\right\rceil}\},\]
\[\mathcal{S}_k = \{G \in \mathcal{I} : \text{the maximum degree of } G \text{ is at most } k\},\]
\[\mathcal{D}_k = \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate, i.e. every subgraph of } G \text{ has a vertex of degree at most } k\} .\]

We remark that \(\mathcal{D}_1 = \mathcal{T}_1\) is the class of all forests so that
\[SF = \mathcal{D}_1 \cap \mathcal{W}_2 = \{G \in \mathcal{I} : \text{every component of } G \text{ is a star}\} .\]

The properties \(\mathcal{I}\) and \(\mathcal{O}\) are defined to be the trivial properties and an edgeless graph is called a trivial graph. We use the phrase \(G \text{ has property } P\) to denote the fact that \(G \in P\).

2 Decomposability

Let \(P_1, \ldots, P_n\) be properties of graphs. A \((P_1, \ldots, P_n)\)-decomposition of a graph \(G\) is a partition \(E_1, \ldots, E_n\) of \(E(G)\) such that \(G[E_i]\), the subgraph of \(G\) induced by \(E_i\), has property \(P_i\), for \(i = 1, \ldots, n\). We denote by \(P_1 \oplus \cdots \oplus P_n\) the property \(\{G \in \mathcal{I} : G \text{ has a } (P_1, \ldots, P_n)\text{-decomposition}\}\). If \(G \in P_1 \oplus \cdots \oplus P_n\) we also write \(G = G_1 \oplus \cdots \oplus G_n\) where \(G_i = G[E_i]\) for \(i = 1, \ldots, n\). If \(P_1 = \cdots = P_n = P\), then the property \(P_1 \oplus \cdots \oplus P_n\) is also denoted by \(nP\). Note that \(O\) is the identity element for \(\oplus\) in the sense that \(P \oplus O = P\) for every property \(P\). It is easy to see that if \(P_i\) is additive and hereditary for every \(i\) then \(P_1 \oplus \cdots \oplus P_n\) is also additive and hereditary.

If \(L\) is a set of hereditary properties and \(P \in L\) then \(P\) is said to be decomposable in \(L\) if there exist non-trivial hereditary properties \(P_1\) and \(P_2\) in \(L\) such that \(P = P_1 \oplus P_2\); otherwise \(P\) is said to be indecomposable in \(L\). Throughout this paper we use for \(L\) the lattice \(L^a\) of all additive hereditary properties of graphs — see [2] for more details on this lattice.

The property \(P \circ Q\) is the vertex-analogue of \(P \oplus Q\). For the sake of completeness we give the necessary definitions: For given properties \(P_1, \ldots, P_n\), a vertex \((P_1, \ldots, P_n)\)-partition of a graph \(G\) is a partition \(V_1, \ldots, V_n\) of \(V(G)\) such that for each \(i = 1, \ldots, n\) the induced subgraph \(G[V_i]\) has property \(P_i\).
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(In this context it is convenient to regard the empty set \( \emptyset \) as a set inducing a subgraph with every property \( \mathcal{P} \).) The product \( \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n \) of the properties \( \mathcal{P}_1, \ldots, \mathcal{P}_n \) is now defined as the set of all graphs having a vertex \( (\mathcal{P}_1, \ldots, \mathcal{P}_n) \)-partition; each \( \mathcal{P}_i \) is called a factor of this product. If \( \mathcal{P}_1 = \cdots = \mathcal{P}_n = \mathcal{P} \), then we write \( \mathcal{P}^n = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n \). As an example we note that \( \mathcal{O}^k \) denotes the class of all \( k \)-colourable graphs.

These operations come together in the following distributive law which is proved in [3].

**Lemma 21.** Let \( \mathcal{P} \), \( \mathcal{Q}_1 \) and \( \mathcal{Q}_2 \) be additive and hereditary properties of graphs. Then \( \mathcal{P} \oplus (\mathcal{Q}_1 \circ \mathcal{Q}_2) = (\mathcal{P} \oplus \mathcal{Q}_1) \circ (\mathcal{P} \oplus \mathcal{Q}_2) \).

A property \( \mathcal{R} \) is reducible if there are properties \( \mathcal{P} \) and \( \mathcal{Q} \) such that \( \mathcal{R} = \mathcal{P} \circ \mathcal{Q} \); otherwise it is irreducible. Reducible properties play an important role in the lattice \( L_\subseteq \) because of following Unique Factorization Theorem of Mihók, Semanišin and Vasky which appears in [5].

**Theorem 22.** Every reducible property \( \mathcal{P} \neq \mathcal{I} \) is uniquely factorizable into irreducible factors (up to the order of the factors).

In the light of this result it seems imperative to study the decomposability of properties of graphs. It is our aim to study the decomposability of the well-known properties of graphs \( \mathcal{I}_k \), \( \mathcal{O}_k \), \( \mathcal{W}_k \), \( \mathcal{T}_k \), \( \mathcal{S}_k \), \( \mathcal{D}_k \) and \( \mathcal{O}^p \).

### 3 The Indecomposability of \( \mathcal{I}_k \), \( \mathcal{O}_k \), \( \mathcal{W}_k \), and \( \mathcal{T}_k \)

The indecomposability of \( \mathcal{I}_k \) is easy to show using the following well-known result of Nešetřil and Rödl — see [6]. In it we use the notation \( \omega(G) \) to denote the clique number of a graph \( G \).

**Theorem 31.** For every graph \( G \) there is a graph \( H \) with \( \omega(G) = \omega(H) \) such that for every partition \( E_1, E_2 \) of \( E(H) \) there is an induced subgraph \( G' \) of \( H \) with \( G' \) isomorphic to \( G \) and \( E(G') \subseteq E_1 \) or \( E(G') \subseteq E_2 \).

**Theorem 32.** For every positive integer \( k \) the property \( \mathcal{I}_k \) is indecomposable in \( L_\subseteq \).

**Proof.** Suppose to the contrary that \( \mathcal{I}_k = \mathcal{P} \oplus \mathcal{Q} \) with both \( \mathcal{P} \) and \( \mathcal{Q} \) non-trivial. Applying Theorem 31 to \( G = K_{k+2} - e \in \mathcal{I}_k \) it follows that there is a graph \( H \in \mathcal{I}_k \) so that \( H \) has a \( (\mathcal{P}, \mathcal{Q}) \)-decomposition and by considering
any \((P, Q)\)-decomposition of \(H\) we see that \(K_{k+2} - e \in P\) or \(K_{k+2} - e \in Q\). But then \(K_{k+2} \in P \oplus Q = I_k\), a contradiction. 

The following lemma will prove to be useful to show the indecomposability of some properties.

**Lemma 33.** If \(P\) is decomposable in \(L^a_{\subseteq}\) then \(2S_1 \subseteq P\).

**Proof.** If \(P = Q \oplus R\) with \(Q\) and \(R\) non-trivial then \(S_1 \subseteq Q\) and \(S_1 \subseteq R\) since \(S_1\) is the smallest non-trivial property in \(L^a_{\subseteq}\). It follows that \(2S_1 \subseteq Q \oplus R\).

**Corollary 34.** For every positive integer \(k\) the properties \(O_k\) and \(W_k\) are indecomposable in \(L^a_{\subseteq}\).

**Proof.** The path \(P_{k+2}\) of order \(k+2\) is in \(2S_1\) but not in \(O_k\), hence \(O_k\) is indecomposable by Lemma 33. Since \(P_{k+2}\) is not in \(W_k\) either, \(W_k\) is also indecomposable.

**Theorem 35.** If \(SF \subseteq P\) and \(P \neq \emptyset\) is closed under taking subdivisions of graphs, then \(P\) is indecomposable in \(L^a_{\subseteq}\).

**Proof.** Suppose that \(P = Q \oplus R\) with \(Q\) and \(R\) non-trivial. Let \(G\) be any graph in \(P\) and let \(G'\) be the graph obtained from \(G\) by subdividing every edge of \(G\) twice. Clearly, \(G' \in SF \oplus S_1\). Since \(G' \notin P\) it follows that \(SF \notin Q\) and \(SF \notin R\). Therefore there exist positive integers \(a\) and \(b\) such that \(K_1, a \notin Q\) and \(K_1, b \notin R\). But then \(K_{1,a+b-1} \notin Q \oplus R\), contradicting the assumption that \(SF \subseteq P\).

**Corollary 36.** For every positive integer \(k\) the property \(T_k\) is indecomposable in \(L^a_{\subseteq}\).

## 4 The Decomposability of \(S_k\)

**Theorem 41.** Let \(p\) and \(q\) be any positive integers. Then

1. \(S_p \oplus S_q \subseteq S_{p+q}\).
2. \(S_p \subseteq (p+1)S_1\).
3. \(S_{p+q} \subseteq S_{p+1} \oplus S_q\).
4. \(S_{2p} = pS_2\).
Proof. (1) If \( G \not\in S_{p+q} \) then \( G \) has a vertex of degree at least \( p + q + 1 \), hence \( K_{1,p+q+1} \subseteq G \). But since \( K_{1,p+q+1} \not\subseteq S_p \oplus S_q \), it then follows that \( G \not\in S_p \oplus S_q \).

(2) This is Vizing’s fundamental result on edge colourings of graphs.

(3) From (2) and (1) it follows that \( S_{p+q} \subseteq (p + q + 1)S_1 \subseteq S_{p+1} \oplus S_q \).

(4) The inclusion \( pS_2 \subseteq S_{2p} \) follows from (1). For the other inclusion, suppose that \( G \in S_{2p} \). Then \( G \) is a subgraph of a 2\( p \)-regular graph \( H \). Since \( H \) is regular of even degree, \( H \) is 2-factorable by Petersen’s Theorem. This 2-factorization of \( H \) clearly is a \( (pS_2) \)-decomposition of \( H \) and therefore induces a \( (pS_2) \)-decomposition of \( G \). Therefore \( G \in pS_2 \).

We are now ready to determine exactly when equality holds in part (1) of Theorem 41.

Corollary 42. Let \( p \) and \( q \) be any even positive integers. Then \( S_p \oplus S_q = S_{p+q} \).

Proof. An easy calculation using part (4) of Theorem 41.

Theorem 43. Let \( p \) and \( q \) be positive integers with \( q \) odd. Then \( S_{p+q} \not\subseteq S_p \oplus S_q \).

Proof. Suppose first that \( p \) is odd too. Since \( K_{p+q+1} \in S_{p+q} \) it is sufficient to show that \( K_{p+q+1} \not\subseteq S_p \oplus S_q \): Suppose, to the contrary, that \( E_1, E_2 \) is an \( (S_p, S_q) \)-decomposition of \( E(K_{p+q+1}) \). Then \( E_1 \) has at most \( \frac{1}{2}[p(p+q+1)-1] \) edges, since any graph of maximum degree \( p \) on \( n \) vertices has at most \( \frac{1}{2}pn \) edges and if \( n \) is odd this inequality is strict. Similarly, \( E_2 \) has at most \( \frac{1}{2}[q(p+q+1)-1] \) edges. But \( K_{p+q+1} \) has \( \frac{1}{2}(p+q)(p+q+1) > \frac{1}{2}[p(p+q+1)-1] + \frac{1}{2}[q(p+q+1)-1] \) edges, a contradiction.

Next let \( p = 2 \). Let \( F = K_{q+1,q+1} \) with (vertex) partite sets \( V = \{v_1, \ldots, v_{q+1}\} \) and \( W = \{w_1, \ldots, w_{q+1}\} \). Let \( H \) be the graph obtained from \( F \) by adding vertices \( x \) and \( y \) and edges \( \{xv_i : i = 1, \ldots, q+1\}, \{w_iw_{i+1} : i = 1, 3, \ldots, q\} \) and \( xy \). Suppose now that \( E_1, E_2 \) is any \( (S_2, S_q) \)-decomposition of \( E(H) \). Since every vertex other than \( y \) has degree \( q + 2 \) in \( H \), every such vertex must have degree 2 in \( H[E_1] \). Therefore \( xy \not\in E_1 \), otherwise \( H[E_1] \) would have an odd number of odd vertices. Now let \( G \) be the graph obtained from \( q + 1 \) copies of \( H \) by identifying the vertices of degree one. Then \( G \in S_{2+q} \), but, by the above argument, \( G \not\in S_2 \oplus S_q \). Hence \( S_{2+q} \not\subseteq S_2 \oplus S_q \) for every odd positive integer \( q \).
Finally, let $p = 2k$ with $k \geq 2$ and suppose that $S_{2k+q} \subseteq S_{2k} \oplus S_q$. Then $S_{2+(2(k-1)+q)} = S_{2k+q} \subseteq S_{2k} \oplus S_q = S_2 \oplus (S_{2(k-1)} \oplus S_q) \subseteq S_2 \oplus S_{2(k-1)+q}$, contradicting the statement proven in the previous paragraph.

**Lemma 44.** If $S_k = P \oplus Q$ with $P \in L^a_\subseteq$ and $Q \in L^a_\subseteq$ both non-trivial, then $S_k = S_p \oplus S_q$ for some positive integers $p$ and $q$ with $p + q = k$.

**Proof.** If $G \in P$ has a vertex $v$ of degree $a$ and $H \in Q$ has a vertex $w$ of degree $b$ then there is a graph in $P \oplus Q$ with a vertex of degree $a + b$, for example the graph obtained from $G \cup H$ by identifying $v$ and $w$. Therefore there exist positive integers $p$ and $q$ such that $P \subseteq S_p$, $Q \subseteq S_q$ and $p + q \leq k$. If $p + q < k$ then $K_{1,k} \in S_k$ but $K_{1,k} \notin P \oplus Q$; therefore $p + q = k$. Now $S_k = P \oplus Q \subseteq S_p \oplus S_q \subseteq S_k$, hence $S_k = S_p \oplus S_q$.

We are now ready to prove the main result of this section.

**Theorem 45.** $S_k$ is decomposable in $L^a_\subseteq$ if and only if $k = 2n$ for some $n \geq 2$.

**Proof.** Suppose $S_k$ is decomposable and $k$ is odd or $k = 2$. Then, by Lemma 44, $S_k = S_p \oplus S_q$, where $p + q = k$. In either case $p$ or $q$ is odd and Theorem 43 is contradicted.

The other direction follows from part (4) of Theorem 41.

## 5 The Decomposability of $D_k$

**Theorem 51.** $D_1$ is indecomposable in $L^a_\subseteq$.

**Proof.** The cycle $C_4$ is in $2S_1$ but not in $D_1$, hence $D_1$ is indecomposable in $L^a_\subseteq$ by Lemma 33.

The indecomposability of $D_2$ is also known — see [4]. We state their result as

**Theorem 52.** $D_2$ is indecomposable.

In order to prove that $D_k$ is indecomposable for each $k \geq 3$, we first discuss some decomposable bounds for $D_k$ which are useful. The first bound is due to Borowiecki and Hałuszcak — see [1].

**Theorem 53.** For all positive integers $a$ and $b$ the inclusion $D_{a+b} \subseteq D_a \oplus D_b$ holds.
In order to show that \( D_a \oplus D_b \) is not an upper bound for \( D_{a+b+1} \), we need the following lemma.

**Lemma 54.** For all positive integers \( a, b \) and \( n \) there is a positive integer \( t \) such that for every \((D_a, D_b)\)-decomposition \( E_1, E_2 \) of \( K_{a+b,t} \), the inclusions \( K_{a,n} \subseteq K_{a+b,t}[E_1] \) and \( K_{b,n} \subseteq K_{a+b,t}[E_2] \) hold.

**Proof.** In order to simplify the arguments we assume throughout that \( n > a + b \). For \( a = b = 1 \) we can take \( t = 2n + 2 \). Assume now inductively that the statement is true for \( a, b, n \) and \( t \) and consider any \((D_a, D_{b+1})\)-partition \( E_1, E_2 \) of \( E(G) \) where \( G = K_{a+b+1,2t} \). Let \( V \) be the (vertex) partite set of order \( a + b + 1 \) of \( G \), \( U = V(G) - V \) and let \( v \) be any vertex of \( V \). We consider two cases:

1. At least \( t \) edges incident with \( v \) are in \( E_2 \): In this case we apply the induction hypothesis to the subgraph of \( G \) induced by \( (V - v) \cup \{u \in U : vu \in E_2\} \) to obtain \( K_{a,n} \subseteq G[E_1] \) and \( K_{b,n} \subseteq G[E_2] \). Together with the \( t \) edges in \( E_2 \) incident with \( v \) we then also have \( K_{b+1,n} \subseteq G[E_2] \). (Note that the \( b \) independent vertices of the \( K_{b,n} \) in \( G[E_2] \) is necessarily a subset of \( V \) since \( n > b \).)

2. At least \( t \) edges incident with \( v \) are in \( E_1 \): By the same arguments as in case (1) we now obtain that \( K_{a+1,n} \subseteq G[E_1] \). Since \( n > a \) we have a contradiction, hence this case is impossible. \( \square \)

**Theorem 55.** Let \( a \) and \( b \) be positive integers. Then \( D_{a+b+1} \not\subseteq D_a \oplus D_b \).

**Proof.** Clearly \( K_{a+b+1,t} \in D_{a+b+1} \) for every positive integer \( t \). Further, by Lemma 54, every \((D_a, D_{b+1})\)-decomposition \( E_1, E_2 \) of \( K_{a+b+1,t} \) has \( K_{t+1,b+1} \subseteq K_{a+b+1,t}[E_2] \) for \( t \) large enough. Since a \((D_a, D_b)\)-decomposition is also a \((D_a, D_{b+1})\)-decomposition, \( K_{a+b+1,t} \not\subseteq D_a \oplus D_b \). \( \square \)

If \( b = 1 \) we can demonstrate the sharpness of the inclusion in Theorem 53 in a stronger sense.

**Theorem 56.** Let \( a \) be a positive integer. If \( D_1 \not\subseteq P \) then \( D_{a+1} \not\subseteq D_a \oplus P \).

**Proof.** If \( D_1 \not\subseteq P \), then there is a tree \( T \) with \( T \in D_1 \) and \( T \not\subseteq P \); suppose \( T \) is of size \( m \). We construct a sequence \( G_0, G_1, \ldots, G_m \) of graphs in \( D_{a+1} \) as follows: \( G_0 = K_1 \) and \( G_{i+1} \) is obtained from \( a \) copies of \( G_i \) by adding, for every set \( V \) consisting of one vertex from each copy of \( G_i \), \( a \) copies of \( T \) together with all edges between \( V \) and these copies of \( T \).
We now prove by induction on \(i\) that for every \((\mathcal{D}_a, \mathcal{P})\)-decomposition \(E_1, E_2\) of \(G_i\) the graph \(G_i[E_2]\) contains every tree of size \(i\). It then follows that \(G_m \not\in \mathcal{D}_a \oplus \mathcal{P}\).

For \(i = 0\) this is trivial. Suppose therefore the statement is true for \(i - 1\) and consider any \((\mathcal{D}_a, \mathcal{P})\)-decomposition \(E_1, E_2\) of \(E(G_i)\) and any tree \(T'\) of size \(i\). Let \(v\) be a leaf of \(T'\) and let \(u\) be the neighbour of \(v\). Every copy of \(G_{i-1}\) contains a copy of \(T' - v\) that is in \(E_2\). Let \(U = \{u_1, u_2, \ldots, u_a\}\) be the set of vertices corresponding to \(v\) in each copy of \(G_{i-1}\). Now consider the subgraph of \(G_i\) isomorphic to \(aK_1 + aT\) consisting of \(U\) together with the corresponding \(a\) copies of \(T\) added in the construction of \(G_i\). Since \(T \not\in \mathcal{P}\), every copy of \(T\) has an edge in \(E_1\). The end-vertices of these \(a\) edges together with \(U\) induce a subgraph \(H\) isomorphic to \(aK_2 + aK_1 \not\in \mathcal{D}_a\). Therefore some edge of \(H\) incident with an element of \(U\) must be in \(E_2\), hence \(T' \subseteq G_i[E_2]\).

**Corollary 57.** If \(\mathcal{P} \subset \mathcal{D}_1\) then \(\mathcal{D}_{a+1} \not\subseteq \mathcal{D}_a \oplus \mathcal{P}\).

The next bound is also due to Borowiecki and Hałuszczak — see [1].

**Theorem 58.** Let \(n_1, \ldots, n_k\) be positive integers. Then \(\mathcal{D}_{n_1} \oplus \cdots \oplus \mathcal{D}_{n_k} \subseteq \mathcal{D}_2 \sum_{i=1}^k n_i - 1\).

The sharpness of the inclusion of Theorem 58 is demonstrated in Theorem 510.

**Lemma 59.** Suppose \(\mathcal{P}\) and \(\mathcal{Q}\) are properties, \(G \in \mathcal{P}\) and \(H \in \mathcal{Q}\) are graphs and \(a, b, c, d, k\) and \(n\) are positive integers such that the following holds:

1. \(G\) and \(H\) both have order \(n\),
2. \(G\) has \(k\) independent vertices of degree \(a\) and the other \(n - k\) vertices have degree \(c\),
3. \(H\) has \(n - k\) independent vertices of degree \(b\) and the other \(k\) vertices have degree \(d\).

Then \(\mathcal{P} \oplus \mathcal{Q} \not\subseteq \mathcal{D}_m\) where \(m = \min\{a + d, b + c\} - 1\).

**Proof.** Let \(G_i\) be a copy of \(G\) and \(H_i\) be a copy of \(H\) and suppose \(U_i\) is the set of \(k\) vertices of degree \(a\) in \(G_i\), \(V_i\) is the set of \(k\) vertices of degree \(d\) in \(H_i\), \(W_i = V(G_i) - U_i\) and \(X_i = V(H_i) - V_i\), \(i = 1, 2\). Let \(F\) be the graph obtained from these four graphs by identifying the vertices of \(U_1\) one
by one with the vertices of $V_2$, those of $W_2$ with $X_2$, those of $V_1$ with $U_2$ and those of $X_1$ with $W_1$. Then $F \in \mathcal{P} \oplus \mathcal{Q}$ and, since $U_1$, $U_2$, $X_1$ and $X_2$ are independent, every vertex in $F$ has degree $a + d$ or $b + c$ so that $F \notin \mathcal{D}_m$. 

If $G$ is a graph with vertex set $\{v_1, \ldots, v_n\}$ then we denote by $G[k_1, \ldots, k_n]$ the graph with vertex set $\cup_{i=1}^n (v_i, j) : j = 1, \ldots, k_i$ and edge set $\{(v_i, j) : v_iv_m \in E(G)\}$. If $k_1 = \cdots = k_n = k$ we write $G[k]$ for $G[k_1, \ldots, k_n].$

**Theorem 510.** For all positive integers $a$ and $b$ we have that $\mathcal{D}_a \oplus \mathcal{D}_b \not\subseteq \mathcal{D}_{2a+2b-2}$.

**Proof.** Let $G_p$ have vertex set $\{v_0, v_1, \ldots, v_{p+1}, u_1, u_2, \ldots, u_p\}$ and edge set $\{v_iv_{i+1} : i = 0, 1, \ldots, p\} \cup \{u_iu_i : i = 1, 2, \ldots, p\}$. Let $G_p[k, m] = G_p[k, k, \ldots, k, m, k, m, \ldots, m]$ (with $p + 2k$'s and $p$'s). Note that $G_p[k, m]$ has $pk$ vertices of degree $2k + m$ and $2k + mp$ independent vertices of degree $k$. Now take $x = 2(a^2 + b^2 - a) - (a + b - 2) > 0$, $y = 2(a^2 + b^2 - b) - (a + b - 2) > 0$ and let $G = G_x[a, b - 1] \cup (a + b - 2)G_1[a, b - 1] \in \mathcal{D}_a$ and $H = G_y[b, a - 1] \cup (a + b - 2)G_1[b, a - 1] \in \mathcal{D}_b$.

$G$ has $a(x + a + b - 2) = 2a(a^2 + b^2 - a)$ vertices of degree $2a + b - 1$ and $2a(1+a+b-2)+(b-1)(x+a+b-2) = 2b(a^2+b^2-b)$ independent vertices of degree $a$. By symmetry, $H$ has $2b(a^2+b^2-b)$ vertices of degree $2b+a-1$ and $2a(a^2+b^2-a)$ independent vertices of degree $b$. Therefore $G$ and $H$ satisfy the conditions of Lemma 59, with $c = 2a + b - 1$, $d = 2b + a - 1$, $k = 2b(a^2+b^2-b)$ and $n = 2(a^2+b^2)(a+b-1)$, hence $\mathcal{D}_a \oplus \mathcal{D}_b \not\subseteq \mathcal{D}_m$ where $m = 2a + 2b - 2$. 

**Theorem 511.** For every positive integer $k$ the property $\mathcal{D}_k$ is indecomposable in $\mathbb{L}^0_{\ll}$.

**Proof.** Suppose that $\mathcal{D}_k = \mathcal{P} \oplus \mathcal{Q}$ with $\mathcal{P}$ and $\mathcal{Q}$ non-trivial. Note that if $G \in \mathcal{P}$ and $H \in \mathcal{Q}$ then $G \times H \in \mathcal{P} \oplus \mathcal{Q}$ and $\delta(G \times H) = \delta(G) + \delta(H)$. Therefore there exist positive integers $a$ and $b$ with $a + b = k$ such that $\delta(G) \leq a$ and $\delta(H) \leq b$ for all $G \in \mathcal{P}$ and $H \in \mathcal{Q}$. It follows that $\mathcal{P} \subseteq \mathcal{D}_a$ and $\mathcal{Q} \subseteq \mathcal{D}_b$.

Let $P_n[l,m]$ be the graph $P_n[l,m,m,\ldots,m]$ where we take $V(P_n)$ to be $\{v_1, v_2, \ldots, v_n\}$ and $E(P_n)$ to be $\{v_iv_{i+1} : i = 1, 2, \ldots, n - 1\}$. We now show that $P_n[a + b, a] \notin \mathcal{P}$ and $P_n[a + b, b] \in \mathcal{Q}$ for every $n$. We do this by finding a graph $G_n \in \mathcal{D}_k$ such that for every $(\mathcal{D}_a, \mathcal{D}_b)$-partition $E_1, E_2$ of $E(G_n)$, $P_n[a + b, a] \subseteq G_n[E_1]$ and $P_n[a + b, b] \subseteq G_n[E_2]$. For $n = 2$ we can take $G_2 = K_{a+b,t}$ with $t$ large enough, by Lemma 54. Assume therefore that
\[ G_n \text{ has been found and consider a } (P_a, D_b)\text{-partition } E_1, E_2 \text{ of } E(G_{n+1}), \]
where \( G_{n+1} \) is obtained from \( G_n \) by adding, for every subset of \( a+b \) vertices of \( V(G_n) \), \( t \) vertices together with all edges connecting the new vertices with these \( a+b \) vertices. We know that \( P_n[a+b, a] \subseteq G_n[E_1] \). By applying Lemma 54 to the appropriate set of \( a+b \) independent vertices and the \( t \) new vertices adjacent to them it now follows that \( P_{n+1}[a+b, a] \subseteq E_1 \). Similarly, \( P_{n+1}[a+b, b] \subseteq E_2 \).

In particular, \( P_4[a] \in P \) and \( P_4[b] \in Q \). Then \( G = bP_4[a] \) and \( H = aP_4[b] \)
satisfy the conditions of Lemma 59 with \( c = 2a \), \( d = 2b \), \( n = 4ab \) and \( k = 2ab \). Therefore \( P + Q \not\subseteq D_m \) where \( m = \min\{a+2b, b+2a\} - 1 \geq a+b \), a contradiction.

\section{The Decomposability of \( P^k \)}

\textbf{Corollary 61.} If \( P \neq O \) is a property of graphs and \( k \geq 2 \) then \( P^k \) is decomposable.

\textbf{Proof.} It follows easily from Lemma 21 that \( P^k = P \oplus O^k \).

In the rest of this section we consider the decomposability of the property \( O^k \). First we define two parameters for a property \( P \): Let \( \chi(P) \) denote the least \( k \) such that \( P \subseteq O^k \) and let \( \chi^*(P) \) denote the largest \( k \) such that \( O^k \subseteq P \). The next lemma is useful in the proof of the main result of this section.

\textbf{Lemma 62.} Let \( P \) and \( Q \) be properties of graphs. Then \( \chi^*(P \oplus Q) \leq \chi^*(P) \chi(Q) \leq \chi(P \oplus Q) \).

\textbf{Proof.} Note that the result holds if \( \chi(Q) \) is infinite. We may therefore suppose that \( a = \chi^*(P) \) and \( b = \chi(Q) \) are finite. For the first inequality, let \( t \) be such that \( K_{a+1}[t] \not\in P \). In order to show that \( \chi^*(P \oplus Q) \leq \chi^*(P) \chi(Q) \), it is sufficient to show that \( G = K_{ab+1}[bt] \not\in P \oplus Q \). Suppose, to the contrary, that \( E_1, E_2 \) is a \((P, Q)\)-decomposition of \( E(G) \). Let \( U_1, \ldots, U_{ab+1} \) be the partition sets of \( G \), let \( V_1, \ldots, V_b \) be a \( b \)-colouring of \( G[E_2] \) and set \( V_{i,j} = V_i \cap U_j \). For every \( j \) there must be an \( i \) such that \( |V_{i,j}| \geq t \) since \( |U_j| = bt \) and \( U_j \) is a union of only \( b \) of the \( V_{i,j} \)'s. Therefore there are at least \( ab + 1 \) \( V_{i,j} \)'s with \( |V_{i,j}| \geq t \). On the other hand, for every \( i \) there are at most \( a \) \( j \)'s such that \( |V_{i,j}| \geq t \) since every edge of \( G[V_i] \) is in \( E_1 \) and \( K_{a+1}[t] \not\in P \). Therefore there are at most \( ab \) \( V_{i,j} \)'s with \( |V_{i,j}| \geq t \), a contradiction.
For the second inequality, suppose $G \in Q$ has chromatic number $b$. Then the graph $H = G + G + \cdots + G$ (a $G$’s) has chromatic number $ab$ and is in $\mathcal{P} \oplus \mathcal{Q}$.

We are now ready to discuss the decomposability of $\mathcal{O}^p$.

**Theorem 63.** $\mathcal{O}^p$ is indecomposable in $\mathcal{L}_a^\mathcal{L}$ if and only if $p$ is prime.

**Proof.** If $p$ is not prime, say $p = ab$, then it is easy to see from Lemma 21 that $\mathcal{O}^p = \mathcal{O}^a \oplus \mathcal{O}^b$.

Conversely, suppose that $\mathcal{O}^p = \mathcal{P} \oplus \mathcal{Q}$ with $\mathcal{P}$ and $\mathcal{Q}$ non-trivial properties of graphs. Then $\chi(\mathcal{P}), \chi(\mathcal{Q}), p \geq 2$ and, since $\chi(\mathcal{O}^p) = \chi^*(\mathcal{O}^p) = p$, it follows from Lemma 62 that $p = \chi^*(\mathcal{P}) = \chi(\mathcal{Q}) = \chi(\mathcal{P}) \chi^*(\mathcal{Q})$. If $\chi^*(\mathcal{P}) > 1$ or $\chi^*(\mathcal{Q}) > 1$ we are done. Suppose therefore that $\chi^*(\mathcal{P}) = \chi^*(\mathcal{Q}) = 1$. Then $\chi(\mathcal{P}) = \chi(\mathcal{Q}) = p$, hence there exist graphs $F \in \mathcal{P}$ and $H \in \mathcal{Q}$ which both have chromatic number $p$. Since $F + K_1 \not\in \mathcal{O}^p$ and $F + K_1 \in \mathcal{P} \oplus \mathcal{S}\mathcal{F}$ it follows that $\mathcal{S}\mathcal{F} \not\subseteq \mathcal{Q}$. Similarly, $\mathcal{S}\mathcal{F} \not\subseteq \mathcal{P}$. But then $\mathcal{S}\mathcal{F} \not\subseteq \mathcal{P} \oplus \mathcal{Q} = \mathcal{O}^p$, a contradiction, since $p \geq 2$.

**References**


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