DIAGONAL FLIPS IN PSEUDO-TRIANGULATIONS
ON CLOSED SURFACES WITHOUT LOOPS

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Abstract. A pseudo-triangulation on a closed surface without loops is a graph embedded on the surface so that each face is triangular and may have multiple edges, but no loops. We shall establish a theory of diagonal flips in those pseudo-triangulations. Our theory will work in parallel to that for simple triangulations basically, but it will present more concrete theorems than the latter.

Introduction

A triangulation on a closed surface is a simple graph embedded on the surface so that each face is triangular and that any two faces share at most one edge. A diagonal flip of an edge ac in such a triangulation is to replace the diagonal ac with bd in the quadrilateral abcd consisting of the two faces sharing ac. We do not perform a diagonal flip if it results in a nonsimple graph.

After Negami [13] proved the following theorem, many studies have appeared to establish a theory on diagonal flips in triangulations; [2], [4], [5], [9], [10], [14] and so on.

THEOREM 1. For any closed surface \( F^2 \), there exists a natural number \( N = N(F^2) \) such that two triangulations \( G_1 \) and \( G_2 \) on \( F^2 \) can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips if \( |V(G_1)| = |V(G_2)| \geq N \).

Let \( N(F^2) \) denote its minimum value which makes the theorem valid. For example, the results given by Wanger [18], Dewdney [3], Negami and Watanabe [11] imply that \( N(S^2) = 4 \), \( N(T^2) = 7 \), \( N(P^2) = 6 \) and \( N(K^2) = 8 \) for the sphere \( S^2 \), the projective plane \( P^2 \), the torus \( T^2 \) and the Klein bottle \( K^2 \) in order. These values coincide with the minimum number of vertices of triangulations on these surfaces, but it does not hold in general. It is so difficult to determine the precise

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value of $N(F^2)$ for a given closed surface $F^2$. Also Negami [15] has already shown that

$$N(F^2) \leq 19 V_{irr}(F^2) - 18 \chi(F^2)$$

where $\chi(F^2)$ denotes the Euler characteristic of $F^2$. However, this bound includes an unknown quantity $V_{irr}(F^2)$, which is the maximum order of irreducible triangulations of $F^2$. We have $V_{irr}(S^2) = 4$, $V_{irr}(P^2) = 7$, $V_{irr}(T^2) = 10$ and $V_{irr}(K^2) = 11$ (see [17], [1], [6] and [7], for irreducible triangulations of these surface in order) but it has been known only $|V(F^2)| \leq 171(2 - \chi(F^2)) - 72$ for other surfaces [8], which implies the above upper bound for $N(F^2)$ is of linear order with respect to the genus of $F^2$.

One of points in the difficulty is that we have to keep the simpleness of graphs during flipping edges in triangulations. What happens if we neglect the simpleness of graphs? For example, Negami [15] has already given an answer to this question, which we shall present as Theorem 10 in Section 3, and has shown the previous upper bound for $N(F^2)$, as an application of his answer. We shall show another answer in this paper, establishing a theory which is more concrete than that for simple triangulations.

A pseudo-triangulation on a closed surface $F^2$ is a triangular embedding of a graph on $F^2$ which may have loops and multiple edges, according to Negami’s definition in [15]. However, we shall exclude the loops and show the following theorem in the same style as Theorem 1:

**Theorem 2.** Given a closed surface $F^2$, there exists a natural number $n(F^2)$ such that two pseudo-triangulations $G_1$ and $G_2$ on $F^2$ without loops can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips through those pseudo-triangulations if $|V(G_1)| = |V(G_2)| \geq n(F^2)$.

Let $n(F^2)$ denote its minimum value hereafter, as well as $N(F^2)$. We shall give the following upper bound for $n(F^2)$, which does not include any unknown quantity.

**Theorem 3.** If a closed surface $F^2$ is one of the sphere, the projective plane, the torus and the Klein bottle, then $n(F^2) = 3$. Otherwise, we have:

$$4 \leq n(F^2) \leq 18 - 5\chi(F^2)$$

For convenience, we say that two pseudo-triangulations without loops are equivalent under diagonal flips if they can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips through those pseudo-triangulations without loops, and often call a pseudo-triangulation without loops simply a pseudo-triangulation hereafter, omitting “without loops”.

In the next section, we shall define the notions of minimal, pseudo-minimal and frozen pseudo-triangulations to carry out the same arguments as for simple triangulations developed in [13] and [16]. Distinguishing these notions is important in the theory for simple triangulations, but they are the same for pseudo-triangulations without loops, which enables us to establish the above concrete bound for $n(F^2)$.

1. **Minimal pseudo-triangulations**

A pseudo-triangulation on a closed surface without loops is said to be minimal if it has the fewest vertices among those. Since it has no loop, the three corners of each face consist of three distinct vertices. Thus, it is clear that any minimal pseudo-triangulation without loops has at least three vertices and also it is easy to construct pseudo-triangulations with precisely three vertices under the following conditions.

**Lemma 4.** Let $G$ be a minimal pseudo-triangulation on a closed surface $F^2$ with $V$ vertices, $E$ edges and $F$ faces and without loops. Then we have

$$V = 3, \quad E = 9 - 3\chi(F^2), \quad F = 6 - 2\chi(F^2)$$

and $G$ is an $F$-regular graph such that all faces are incident to each vertex. Thus, $G$ can be obtained from a wheel $W_{2n}$ by identifying the vertices and edges along its rim of length $2n$ suitably.

**Proof.** It is easy to show that

$$E = 3(V - \chi(F^2)), \quad F = 2(V - \chi(F^2))$$

for a pseudo-triangulation $G$ on a closed surface $F^2$ in general, using Euler's formula. Since we can construct a minimal pseudo-triangulation with precisely three vertices actually, we obtain the three equalities in the lemma, assigning 3 to $V$ in the above.

Let $V(G) = \{u, v, w\}$. Then each face of $G$ has to have these three vertices $u$, $v$ and $w$ at its corners. This implies all of $F$ faces are incident to $v$ (and also to $u$ and $w$) and they form a wheel with $v$ at its center. The rim of this wheel $W_{2n}$ is a closed walk of length $F = 2n$ representing the link of $v$, denoted by $\text{lk}(v)$, and includes only $u$ and $w$. So we need to identify the vertices which come from the same vertex, $u$ or $w$, to obtain the actual form of $G$. $\blacksquare$

**Lemma 5.** A minimal pseudo-triangulation without loops is unique for each of the sphere, the projective plane, the torus and the Klein bottle.
Proof. By Lemma 4, it is clear that the only minimal pseudo-triangulation of the sphere is $K_3$, the cycle of length 3, which has two faces. Also, the unique minimal pseudo-triangulation of the projective plane can be obtained from the wheel $W_4$ by identifying each pair of antipodal points on its boundary.

Those of the torus and the Klein bottle can be obtained from $W_6$ by suitable identification along its boundary. For the torus, the identification is clear; each parallel pair of edges should be identified. To represent it we give each edge a label so that two edges which should be identified have the same label. In this case, we have $xyzxyz$. Since the vertices has been labeled with $u$ and $w$, the labeling on edges determines the identification uniquely.

On the other hand, we need a slight argument on the identification of $W_6$ for the Klein bottle. To obtain a nonorientable surface, we have to identify at least one pair of edges so that the surface includes a Möbius band. To do this, the identification should be represent with labeling "$x \cdot x \cdot x \cdot x \cdot x \cdot x$" or its cyclic shift, where each "$\cdot$" stands for one label. It is not difficult to determine the unknown labels and it will be $xyzxyz$ uniquely up to symmetry. Otherwise, the resulting pseudo-triangulation would have more than three vertices. $lacksquare$

**Lemma 6.** Any closed surface $F^2$ with $\chi(F^2) < 0$ admits two or more minimal pseudo-triangulations without loops.

**Proof.** First, consider minimal pseudo-triangulations on the orientable closed surface of genus $g \geq 2$. By Lemma 4, they can be constructed from $W_F$ with $F = 4g + 2$ by identifying vertices and edges on its rim. For example, the two identification with labeling

$$x_1x_2 \cdots x_Fx_1x_2 \cdots x_F; \quad x_1x_2 \cdots x_{F-2}x_Fx_1x_{F-1}x_Fx_2 \cdots x_{F-2}$$

yield two pseudo-triangulations with three vertices. They are not homeomorphic to each other since their duals are not isomorphic as abstract 3-regular graphs.

Similarly, we can give two identifications on the boundary of $W_F$ with $F = 2q + 2$ for the nonorientable closed surface of genus $q \geq 3$:

$$x_1x_2 \cdots x_Fx_1x_F \cdots x_2; \quad x_1x_2 \cdots x_{F-1}x_Fx_1x_Fx_2 \cdots x_{F-1}$$

They also yield non-homeomorphic pseudo-triangulations with three vertices whose duals are not isomorphic. $lacksquare$

Here, we shall show an easy way to construct a series of minimal pseudo-triangulations inductively. Let $G_1$ and $G_2$ be pseudo-triangulations on two disjoint closed surfaces $F^2_1$ and $F^2_2$, respectively. Choose one face of $G_1$ and of $G_2$, say $A_1$ and $A_2$. Paste $F^2_1$ and $F^2_2$ along $A_1$ and $A_2$, and remove the open 2-cell
$A_1 = A_2$. Then we obtain a pseudo-triangulation on the connected sum $F_1^2 \# F_2^2$ of the two surfaces $F_1^2$ and $F_2^2$. The resulting pseudo-triangulation also is called a connected sum of $G_1$ and $G_2$ and is denoted by $G_1 \# G_2$. If each of $G_1$ and $G_2$ has precisely three vertices, then $G_1 \# G_2$ also has precisely three vertices. By Lemma 4, $G_1 \# G_2$ is a minimal pseudo-triangulation of $F_1^2 \# F_2^2$.

For example, a series of minimal pseudo-triangulations of the orientable closed surfaces of genus 2, 3, 4, . . . can be constructed from many copies of the unique minimal pseudo-triangulation of the torus by joining them repeatedly in the above way. Each of their duals has a nontrivial 3-edge-cut, that is, a set of three edges whose removal disconnects it into nontrivial components. Thus, we cannot construct the first type given in the proof of Lemma 6 in this way since its dual does not have such a 3-edge-cut.

2. Pseudo-minimal pseudo-triangulations

Let $G$ be a pseudo-triangulation on a closed surface $F^2$ without loops and $ac$ an edge in $G$ with two faces $abc$ and $adc$ incident to it. The contraction of $ac$ is to shrink $ac$ to a point and to replace the resulting two digonal faces with edges $ab = cb$ and $ad = cd$, respectively. We perform the contraction of an edge only when it results in another pseudo-triangulation on $F^2$ without loops, denoted by $G/ac$, and call such an edge a contractible edge.

A pseudo-triangulation is said to be contractible if it has a contractible edge and to be irreducible otherwise. For example, any minimal pseudo-triangulation is irreducible since an edge contraction decreases the number of vertices. A pseudo-triangulation is said to be pseudo-minimal if it cannot be transformed into any contractible pseudo-triangulation by diagonal flips. Any pseudo-triangulation equivalent to a pseudo-minimal one is pseudo-minimal.

Let $G$ be a pseudo-triangulation on a closed surface $F^2$ without loops and let $\delta(G)$ denote the minimum degree of $G$. In general, we have $\delta(G) \geq 2$; otherwise, we could find a loop around a vertex of degree 1. Suppose that $\delta(G) = 2$ and let $v$ be a vertex of degree 2 in $G$. Then $v$ has two distinct neighbors $u$ and $w$ and there are multiple edges between $u$ and $w$ which bound a digonal region including the path $uwv$ of length 2. Replace this digonal part with a single edge $uw$ to obtain another pseudo-triangulation without loops. We call this deformation the elimination of a vertex $v$ of degree 2. Note that each of the two edges incident to a vertex $v$ of degree 2 is contractible and its contraction realizes the elimination of $v$.

Lemma 7. A pseudo-triangulations on a closed surface without loops, except $K_3$, is pseudo-minimal if and only if it is equivalent to no pseudo-triangulation with a vertex of degree 2 under diagonal flips.
Proof. The necessity is clear since a pseudo-triangulation is contractible if it has a vertex of degree 2. To prove the sufficiency, it suffices to show that a contractible pseudo-triangulation is equivalent to one with a vertex of degree 2 under diagonal flips.

Let \( v \) be a vertex and \( u_1, \ldots, u_n \) its neighbors lying on \( \text{lk}(v) \) around \( v \) in this cyclic order. Suppose that \( vu_n \) is a contractible edge in \( G \). Since \( G/vu_n \) has no loops, each of \( u_1, \ldots, u_{n-1} \) is distinct from \( u_n \). Thus, we can flip \( vu_1 \) to \( u_nu_2 \), \( vu_2 \) to \( u_nu_3, \ldots, vu_{n-2} \) to \( u_nu_{n-1} \). The vertex \( v \) will have degree 2 finally.

The next lemma follows from the above immediately:

**Lemma 8.** Any pseudo-triangulation on a closed surface without loops can be transformed into a pseudo-minimal one by a sequence of diagonal flips and elimination of vertices of degree 2.

Negami [13] has defined the pseudo-minimal triangulations in a similar style, related to contraction of edges. They also play an important role to determine the value of \( N(F^2) \). However, they are just theoretical objects and we know nothing about their concrete forms. (We can find several examples of pseudo-minimal triangulations in [16]). On the other hand, we can give a good characterization of the pseudo-minimal pseudo-triangulations, as follows, which suggests how to construct them.

Recall that we must not flip an edge in a pseudo-triangulation without loops if it yields a loop. A pseudo-triangulation is said to be frozen if any diagonal flip is not applicable to it. That is, any frozen pseudo-triangulation is not equivalent to any other pseudo-triangulation under diagonal flips.

**Lemma 9.** For a pseudo-triangulation \( G \) on a closed surface without loops, the following four are equivalent to one another:

(i) \( G \) is frozen.
(ii) \( G \) is pseudo-minimal.
(iii) \( G \) is minimal.
(iv) \( G \) has precisely three vertices.

Proof. By Lemma 4, the equivalence between (iii) and (iv) is obvious. So we shall show the equivalence among (i), (ii) and (iv) below.

(i) implies (ii): Suppose that there is a vertex \( v \) of degree 2. Then it has two distinct neighbors \( u \) and \( w \) and they are joined by multiple edges. Each of the multiple edges between \( u \) and \( w \) is flippable in \( G \). Thus, any frozen pseudo-triangulation has minimum degree at least 3. Since it is not equivalent to any other pseudo-triangulation, it is pseudo-minimal by Lemma 7.
(ii) implies (iv): Let $G$ be a pseudo-minimal pseudo-triangulation. We may suppose that $\delta(G)$ is the smallest among those pseudo-triangulations that are equivalent to $G$ under diagonal flips. Let $v$ be a vertex of $G$ with $\deg v = \delta(G) \geq 3$. By the minimality of $\delta(G)$, each edge incident to $v$ is not flippable; otherwise, flipping it would decrease $\deg v$ by one. This implies that $\deg v$ is an even number $\geq 4$ and that two distinct vertices $u$ and $w$ lie alternately along $\text{lk}(v)$. Each face incident to $v$ consists of the three vertices $\{u, v, w\}$.

Consider the link $\text{lk}(w)$ of $w (= w_1)$ and suppose that there is a fourth vertex $x$ on $\text{lk}(w)$, different from $u$, $v$ and $w$. Then we can find a segment $xu_1vu_2$ along $\text{lk}(w)$. To distinguish two $u$'s in the segment, we don't it by $xu_1vu_2$ and let $u_1w_1u_2w_2\cdots$ be the walk along $\text{lk}(v)$ starting at $u_1$. Flip $w_1u_1$ to $vx$, $vw_1$ to $w_2x$ and $vu_2$ to $w_2x$. This sequence of diagonal flips decreases $\deg v$ finally by one, contrary to the minimality of $\delta(G)$. Therefore, $\text{lk}(w)$ consists of only $v$ and $u$, and $\text{lk}(u)$ also consists of only $v$ and $w$, similarly. This implies $\{u, v, w\}$ induces a connected component of $G$. Since $G$ is connected, $G$ has only these three vertices $u$, $v$ and $w$.

(iv) implies (i): If $V(G) = \{u, v, w\}$, then flippings any edge, say $uv$, yields a loop at $w$. Thus, no diagonal flip is applicable to $G$. $\blacksquare$

Any pseudo-minimal pseudo-triangulation is irreducible. However, we can make those irreducible pseudo-triangulations that are not pseudo-minimal, for each closed surface $F^2$ except the sphere and the projective plane, as follows.

Prepare the wheel $W_{4g}$ which subdivides a $4g$-gonal disk, for the orientable closed surface of genus $g \geq 1$ and identify the boundary of the disk to obtain the surface so that all of the $4g$ vertices of $W_{4g}$ except its center $v$ become a single vertex, say $u$. The resulting graph has two vertices $u$ and $2g$ loops, which come from edges on the rim of $W_{4g}$, and the $4g$ spokes form multiple edges between $v$ and $u$. Subdivide each loop into a pair of multiple edges with its middle point as a vertex and join the new vertex to the center $v$ with an edge.

Now we obtain a pseudo-triangulation without loops which has precisely $2g+2$ vertices, and hence it is not minimal or equivalently not pseudo-minimal by Lemmas 4 and 9. Each of its edges lies on a cycle of length 2 and hence it is irreducible. Similarly, we can construct those with $q + 2$ vertices from $W_{2q}$ for the nonorientable closed surface of genus $q \geq 2$. It is not difficult to see that the irreducible pseudo-triangulations of the sphere and of the projective plane are the unique minimal ones given in Lemma 5.

3. Proof of theorems

Negami [15] has shown the following theorem for pseudo-triangulations possibly with loops. In such pseudo-triangulations, there is no restriction to flip
edges. His proof of this theorem suggests an algorithm to transform $G_1$ into $G_2$, which is greedy in a sense, and gives an upper bound for the length of a sequence of diagonal flips from $G_1$ to $G_2$. The quantity $cr_{\nabla}(G_1, G_2)$ is called the crossing number of $G_1$ and $G_2$ under vertex coincidence and is the minimum number of crossing points in $G_1 \cup G_2$ when we embed $G_1$ and $G_2$ together on the same surface $F^2$ with $V(G_1) = V(G_2)$.

**THEOREM 10.** Let $G_1$ and $G_2$ be two labeled pseudo-triangulations on a closed surface $F^2$ with the same number of vertices. Then they can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips of length at most $cr_{\nabla}(G_1, G_2)$.

As an application of this theorem, we shall prove Theorems 2 and 3 for pseudo-triangulations without loops, as follows.

Let $G$ be a pseudo-triangulation on a closed surface $F^2$ without loops and $uv$ an edge in $G$. Replace $uv$ with a pair of multiple edges between $u$ and $v$ bounding a digonal region which includes a path $uv$ of length 2. Then we obtain another pseudo-triangulation $G'$ with a new vertex $x$ of degree 2. We call this local deformation of $G$ into $G'$ the insertion of a vertex $x$ of degree 2 along an edge $uv$ and denote $G'$ by $G + \Theta_1$. Furthermore, let $G + \Theta_m$ denote a pseudo-triangulation without loops obtained from $G$ by inserting $m$ vertices of degree 2 along edges in order. The insertion of a vertex of degree 2 is the inverse operation of the elimination of a vertex of degree 2.

Let $uvw$ be a face of a pseudo-triangulation $G$ and insert a vertex $x$ of degree 2 along an edge $uv$ with multiple edges $e_1$ and $e_2$ so that $e_1$ lies in the face $uvw$. Flip $e_1$ to $xw$ and $xu$ to $vw$. The resulting pseudo-triangulation can be regarded as the one obtained from $G$ by inserting $x$ along $uv$. Repeating this deformation, we can move a vertex of degree 2 freely to anywhere. This fact implies that any two pseudo-triangulations with the same notation $G + \Theta_1$ are equivalent to each other under diagonal flips and hence it is the same for $G + \Theta_m$ with any natural number $m$.

The following theorem will give an essence of our proof of Theorems 2 and 3:

**THEOREM 11.** Let $G_1$ and $G_2$ be two pseudo-triangulations on a closed surface $F^2$ without loops which have the same number of vertices. Then $G_1 + \Theta_m$ can be transformed into $G_2 + \Theta_m$, up to homeomorphism, by a sequence of diagonal flips through pseudo-triangulations without loops if $m \geq 5(|V(G)| - \chi(F^2))$.

**Proof.** By Theorem 10, $G_1$ can be transformed into $G_2$ by a sequence of diagonal flips, but this sequence $T_0, T_1, \ldots, T_n$ might include pseudo-triangulations with many loops although $G_1 = T_0$ and $G_2 = T_n$ have no loops. We shall translate this sequence into that from $G_1 + \Theta_m$ to $G_2 + \Theta_m$, as follows.
Consider the barycentric subdivision $G'_1$ of $G_1$. That is, $G'_1$ can be obtained from $G_1$ by putting a new vertex at the middle point of each edge and adding the barycenter of each face as a vertex adjacent to all of six vertices along its boundary. The number of vertices added to $G_1$, say $m_0$, is equal to $|E(G)| + |F(G)| = 5(|V(G)| - \chi(F^2))$. Flipping edges in faces of $G_1$, we can make the additional vertices have degree 2. This implies that $G'_1$ is equivalent to $G_1 + \Theta_{m_0}$ under diagonal flips.

$\blacksquare$

Proof of Theorems 2 and 3. Let $G_1$ be a pseudo-triangulation on $F^2$ without loops. If $G_1$ is not pseudo-minimal, then $G_1$ can be transformed into a pseudo-minimal one, say $Q_1$, by a sequence of diagonal flips and elimination of vertices of degree 2, by Lemma 8, and hence $G_1$ is equivalent to $Q_1 + \Theta_m$ under diagonal flips, where $m = |V(G_1)| - |V(Q_1)|$.

Similarly, let $G_2$ be a pseudo-triangulation on $F^2$ with the same number of vertices as $G_1$ and let $Q_2$ be the pseudo-minimal one such that $G_2$ is equivalent to $Q_2 + \Theta_m$ under diagonal flips. By Lemma 9, both $Q_1$ and $Q_2$ has precisely three
vertices. By [Theorem 11], if \( m \geq 5(3 - \chi(F^2)) = 15 - 5\chi(F^2) \), then \( Q_1 + \Theta_m \) and \( Q_2 + \Theta_m \) are equivalent under diagonal flips. This implies that \( G_1 \) is equivalent to \( G_2 \) via \( Q_1 + \Theta_m \) and \( Q_2 + \Theta_m \) under diagonal flips if \( |V(G_1)| = |V(G_2)| \geq 18 - 5\chi(F^2) \). That is, \( n(F^2) \leq 18 - 5\chi(F^2) \).

By Lemma 5, if \( F^2 \) is one of the sphere, the projective plane, the torus and the Klein bottle, then \( Q_1 \) and \( Q_2 \) are identical. Thus, there is no restriction on the number of vertices to transform two pseudo-triangulations into each other and hence \( n(F^2) = 3 \). Otherwise, there are two or more frozen pseudo-triangulations on \( F^2 \) which have precisely three vertices and no two of which are equivalent under diagonal flips. Thus, we have \( n(F^2) \geq 4 \).

It is not difficult to see that there are precisely two minimal pseudo-triangulations of the orientable closed surface \( S_2 \) of genus 2, up to homeomorphism. They are the ones obtained in the proof of Lemma 6, denoted by \( T_1 \) and \( T_2 \) here. We have already observed that \( T_1 + \Theta_1 \) and \( T_2 + \Theta_1 \) are equivalent under diagonal flips, which implies that \( n(S_2) = 4 \). We conjecture that \( n(F^2) = 4 \) for any closed surface \( F^2 \) with \( \chi(F^2) < 0 \), orientable or nonorientable.

References


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