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Some Aspects of Free Fall Simulation

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Abstract. This paper deals with motion of a falling body in a quiescent fluid. The objective of first part is to give some mathematical analysis of free fall and to introduce the mathematical model of this problem. We show how to solve problems with domain which is apriori unknown and changes in time. We also present equation of motion with rotation and its solution and at the end some numerical results.

Introduction

The ability to predict the motion of a solid body falling in a fluid has a far reaching impact in areas ranging from meteorology, sedimentology, aerospace engineering and biology. Extensive experiments have been carried out in order to classify and quantify the distinct types of free fall motions which may appear at different regimes. For example the trajectory behavior of falling disks can be reduced to four distinct types of motions. At lower Reynolds numbers, a body dropped at any initial orientation settles down toward a steady fall with horizontal orientation. At higher Reynolds numbers the motion appears to be periodic-oscillating. Depending on the dimensionless moment of inertia the motion may then branch from side-to-side oscillations (flutter) to end-over-end rotation (tumble). Chaotic motions are found for moderately large Reynolds numbers and dimensionless moments of inertia.

Let us suppose that a rigid body is released from rest in an otherwise quiscent liquid, under the action of the force of gravity. We assume, that the liquid fills the whole space. After a certain time interval, body will eventually execute a motion where its angular velocity and the velocity of its center of mass will be constant. We shall call this motion terminal state. Regarding this simple phenomenon, several interesting mathematical questions can be formulated. For example, is the set of terminal states always non-empty, no matter what the shape and density of body, and the property of the liquid is? How many terminal states do exist for given body and liquid, and which are those that can be attained or which are the stable ones?

Mathematical formulation

Let us consider a rigid body $\mathcal{B} \subset I\!\!R^N$ $(N = 2, 3)$ moving through an incompressible liquid $\mathcal{L}$ that fills the whole space. We indicate by $\boldsymbol{U} = \boldsymbol{U}(x,t)$ the velocity field associated with the motion of $\mathcal{B}$ with respect to an inertial frame $\mathcal{I}$. Thus, denoting by C the center of mass of $\mathcal{B}$ and O the origin of $\mathcal{I}$, we have

$$\boldsymbol{U}(x,t) = \mathcal{U}_C(t) + \Omega(t) \times (x - x_C), \tag{1}$$

where $\mathcal{U}_C(t) = \frac{\partial x_C}{\partial t}$, and Ω is the angular velocity of $\mathcal{B}$. The Eulerian velocity and pressure fields associated to the motion of $\mathcal{L}$ in $\mathcal{I}$, are denoted by $\boldsymbol{u} = \boldsymbol{u}(x,t)$ and p=p(x,t). The region occupied by $\mathcal{B}$ at time t is described by $\mathcal{B}(t)$.

The equations of conservation of linear momentum and mass of $\mathcal{L}$, with respect to $\mathcal{I}$, are then given by

$$\rho \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{dt}} = \operatorname{div} \mathcal{T}(\boldsymbol{u}, \mathrm{p}) + \rho\, \boldsymbol{g}, \tag{2}$$

$$\operatorname{div} \boldsymbol{u} = 0, \qquad (x,t) \in \cup_{t>0}[I\!\!R^N \backslash \mathcal{B}(t)] \times \{t\},$$

where ρ is the constant density of $\mathcal{L}$, $\rho\boldsymbol{g}$ is the force of gravity which is assumed to be the only external force, and $\mathcal{T}$ is the Cauchy stress tensor. We assume a Navier-Stokes liquid model for the liquid, where the Cauchy stress tensor is given by

$$\mathcal{T}(\boldsymbol{u}, \mathrm{p}) = -\mathrm{p}\mathbb{I} + 2\mu\mathbb{D}, \tag{3}$$

where μ is the dynamic viscosity, $\mathbb{I}$ is the unit matrix and $\mathbb{D}$ is the deformation velocity tensor

$$\mathbb{D} = \mathbb{D}(\boldsymbol{u}) = (d_{ij})_{i,j=1}^{N}, \quad d_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{4}$$

Thus we can rewrite equations (2) in the following form

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} - \mu\Delta\boldsymbol{u} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\mathrm{p} = \rho\boldsymbol{g} \tag{5}$$

$$\operatorname{div}\boldsymbol{u} = 0, \qquad (x,t) \in \cup_{t>0}[\mathbb{R}^N \backslash \mathcal{B}(t)] \times \{t\},$$

We assume that the liquid is at rest at infinity, so that we impose

$$\lim_{|x|\to\infty} \boldsymbol{u}(x,t) = 0. \tag{6}$$

The initial and boundary conditions are given by

$$\boldsymbol{u}(x,0) = 0 \;\; \text{for } \mathrm{x} \in \mathrm{R}^{\mathrm{N}} \backslash \mathcal{B}(\mathrm{t}) \tag{7}$$

and

$$\boldsymbol{u}(x,t) = \mathcal{U}_C(t) + \Omega(t) \times (x - x_C(t)) \;\; \text{for } \mathrm{x} \in \partial\mathcal{B}(\mathrm{t}), \text{respectively.}$$

From the balance of the linear and angular momentum we get

$$m\frac{\mathrm{d}\mathcal{U}_C}{\mathrm{d}t} = m\boldsymbol{g} - \int_{\partial\mathcal{B}(t)} \mathcal{T}(\boldsymbol{u}, \mathrm{p}) \cdot \mathbf{N}, \tag{8}$$

$$\frac{\mathrm{d}(\mathbf{J}\cdot\Omega)}{\mathrm{dt}} = -\int_{\partial\mathcal{B}(t)} (x - x_C) \times [\mathcal{T}(\boldsymbol{u}, \mathrm{p}) \cdot \mathbf{N}],$$

where m is the mass of the body, $\mathbf{N}$ is the unit normal to $\partial\mathcal{B}(t)$ oriented toward the body and $\mathbf{J}$ is the inertia tensor with respect to the mass center C. As initial condition, we take

$$\mathcal{U}_C(0) = 0, \quad \Omega(0) = 0. \tag{9}$$

This formulation (5)-(9) has a disadvantage that the region occupied by the liquid $\mathcal{L}$ is time dependent and apriori unknown, because we do not know how will the body move, under constant gravity.

So we reformulate the problem in a frame $\mathcal{S}$ attached to $\mathcal{B}$, where the region occupied by the fluid remains the same at all times and the force of gravity changes.

We take the origin of coordinates of $\mathcal{S}$ coinciding with C and assume $\mathcal{I} \equiv \mathcal{S}$ at time $t = 0$. Thus if y denotes the position vector of a point P in $\mathcal{S}$ and x the position vector of the same point in $\mathcal{I}$, we have

$$x = Q(t)\cdot y + x_C(t), \quad Q(0) = \mathbb{I}, \quad x_C(0) = 0 \tag{10}$$

with orthogonal linear transformation Q i.e. $Q(t)\cdot Q^T(t) = Q^T(t)\cdot Q(t) = \mathbb{I}$.

Now we can reformulate the original system of equations with transformed fields,

$$\begin{aligned}
\boldsymbol{v}(y,t) &:= Q^T(t)\cdot \boldsymbol{u}(Q(t)\cdot y + x_C(t),t), & p(y,t) &:= \mathrm{p}(Q(t)\cdot y + x_C(t),t)\\
\boldsymbol{V}(y,t) &:= Q^T(t)\cdot(\mathcal{U}_C(t)+\Omega(t)\times(Q(t)\cdot y)), & \mathbf{G}(t) &:= Q^T(t)\cdot \boldsymbol{g}\\
\boldsymbol{V}_C(t) &:= Q^T(t)\cdot \mathcal{U}_C(t), & \omega(t) &:= Q^T(t)\cdot\Omega(t)\\
T(\boldsymbol{v},p) &:= Q^T\cdot \mathcal{T}(Q\cdot\boldsymbol{v},p)\cdot Q, & (y,t) &\in [I\!R^N\backslash\mathcal{B}(0)]\times(0,\infty)
\end{aligned}$$

We obtain transformed system of equations

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} - \mu\Delta\boldsymbol{v} + \rho\,((\boldsymbol{v}-\boldsymbol{V})\cdot\nabla)\boldsymbol{v} + \rho\,\omega\times\boldsymbol{v} - \nabla p = \rho\,\mathbf{G}(t) \tag{11}$$
$$\operatorname{div}\boldsymbol{v} = 0, \qquad (y,t)\in[I\!R^N\backslash\mathcal{B}(0)]\times(0,\infty),$$

The aditional term $\omega\times\boldsymbol{v}$ corresponds to the Corriolis force induced by the frame transformation (10). We shall deal with this term later.

The initial and boundary conditions are given by

$$\boldsymbol{v}(y,0)=0, \qquad \lim_{|y|\to\infty}\boldsymbol{v}(y,t)=0 \text{ for } y\in I\!R^N\backslash\mathcal{B}(0), \tag{12}$$
$$\boldsymbol{v}(y,t) = \boldsymbol{V}_C(t)+\omega(t)\times y \qquad \text{for } y\in\partial\mathcal{B}(0).$$

The system of equations (8) describing the motion of the body is transformed to

$$\begin{aligned}
m\frac{\mathrm{d}\boldsymbol{V}_C}{\mathrm{d}t} + m\,(\omega\times\boldsymbol{V}_C) &= m\,\mathbf{G}(t) - \int_{\partial\mathcal{B}} T(\boldsymbol{v},p)\times\boldsymbol{n},\\
I\frac{\mathrm{d}\omega}{\mathrm{d}t} + \omega\times(\mathbf{I}\cdot\omega) &= -\int_{\partial\mathcal{B}} y\times[T(\boldsymbol{v},p)\cdot\boldsymbol{n}],\\
\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t} &= \mathbf{G}\times\omega,
\end{aligned}\tag{13}$$

where

$$\boldsymbol{n} := Q^T\cdot\mathbf{N}, \qquad \mathbf{I} := Q^T\cdot\mathbf{J}\cdot Q, \qquad \partial\mathcal{B} := \partial\mathcal{B}(0).$$

The initial condition (9) is transformed to

$$\boldsymbol{V}_C(0)=0, \qquad \omega(0)=0. \tag{14}$$

Let us recall, that in the body frame $\mathcal{S}$ the direction of the gravitational force $\mathbf{G}$ depends on the time t and becomes therefore an unknown to be solved. The initial value for $\mathbf{G}$ is clearly

$$\mathbf{G}(0) = \boldsymbol{g}. \tag{15}$$

The unsteady free fall problem can be now formulated as follows:

Given ρ, μ, ρ_B (density of the body) and $\boldsymbol{g}$, find $\boldsymbol{v}$, p, $\boldsymbol{V}_C$, $\omega$and$\mathbf{G}$ such that the equations (11)-(15) hold.

More about the model you can find in [*Galdi*, 2002].

Numerical solution of the free fall problem

Now we know how the free fall problem is formulated. To find its solution, we introduce one possible method [*Böhnisch et al.*, 2004]. We divide the time interval in which we observe the free fall into subintervals (time levels). On each time level all functions depend only on spatial coordinates x. Let us assume that we know solution at time t_{n-1}. It means that we have $\boldsymbol{v}^{n-1}$, p^{n-1}, $\boldsymbol{V}_C^{n-1}$, ω^{n-1}, $\mathbf{G}^{n-1}$. In the first step we solve equations (11) with known $\boldsymbol{V}_C^{n-1}$, ω^{n-1}, $\mathbf{G}^{n-1}$ and unknown $\boldsymbol{v}^n$, p^n. In the second step we solve equations (13) with known $\boldsymbol{v}^n$, p^n and unknown $\boldsymbol{V}_C^n$, ω^n, $\mathbf{G}^n$. These equations are ODE and we need some ODE solver. The time derivative in equations (11) is discretized with the Crank-Nicolson method. Other possibility is using a fractional-step-scheme.

Navier-Stokes equations with rotation

To solve the nonstationary free fall problem we must very often solve equations (11) i.e. the Navier-Stokes equations with rotation. But the term $\omega \times \boldsymbol{v}$ deteriorates the stability of the standard Galerkin finite element method when the viscosity is small.

Let us consider the stationary Navier-Stokes equation. Because now we are dealing with stabilization, which is damage by Corriolis force and nonstationary flow is another problem. So we need to solve these equations:

$$-\nu \Delta \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} + \omega \times \boldsymbol{v} + \nabla p = \boldsymbol{f}, \tag{16}$$

$$\operatorname{div} \boldsymbol{v} = 0.$$

The classical Galerkin finite element aproximation gives us the following formulation of our problem: find $\boldsymbol{v} \in V$ and $p \in Q$ such that

$$a(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{u}) - b(p, \boldsymbol{u}) + c(\boldsymbol{v}, \boldsymbol{u}) = l(\boldsymbol{u}) \qquad \forall \boldsymbol{u} \in V, \tag{17}$$

$$b(q, \boldsymbol{v}) = 0 \qquad \forall q \in Q,$$

where $V = (H_0^1(\Omega))^N,\ Q = \{q \in L^2(\Omega); \int_\Omega q\, dx = 0\}$ and

$$\begin{aligned}
a(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}) &= \nu \int_\Omega \sum_{i,j=1}^N \frac{\partial v_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}\, dx + \int_\Omega \sum_{i,j=1}^N v_j \frac{\partial w_i}{\partial x_j} u_i\, dx, \\
b(q, \boldsymbol{u}) &= \int_\Omega q \operatorname{div} \boldsymbol{u}\, dx, \\
c(\boldsymbol{v}, \boldsymbol{u}) &= \int_\Omega (\omega \times \boldsymbol{v}) \cdot \boldsymbol{u}\, dx, \\
l(\boldsymbol{u}) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}\, dx
\end{aligned}$$

The classical Galerkin finite element aproximation you can find in [Š*edivý*, 2004].
The idea of the stabilization is to add a new term to the classical Galerkin discretization a new term. We used two methods. The first one Divergence of the residual stabilization (DRS) [*Codina et al.*, 1997]. If we solve the problem with the finite element method, we look for $\boldsymbol{v}_h \in V_h \approx V$ and $p_h \in Q_h \approx Q$, which satisfy

$$a(\boldsymbol{v}_h, \boldsymbol{v}_h, \boldsymbol{u}_h) - b(p_h, \boldsymbol{u}_h) + c(\boldsymbol{v}_h, \boldsymbol{u}_h) = l(\boldsymbol{u}_h) \qquad \forall \boldsymbol{u}_h \in V_h, \tag{18}$$

$$b(q_h, \boldsymbol{v}_h) = 0 \qquad \forall q_h \in Q_h. \tag{19}$$

In DRS we solve following equations

$$\begin{aligned}
&a(\boldsymbol{v}_h, \boldsymbol{v}_h, \boldsymbol{u}_h) - b(p_h, \boldsymbol{u}_h) + c(\boldsymbol{v}_h, \boldsymbol{u}_h) \\
&+ \sum_{K \in \mathcal{T}_h} \int_K \tau [\nabla(\omega \times \boldsymbol{u}_h)][\Delta p_h + \nabla \cdot (\omega \times \boldsymbol{v}_h) - \nabla \cdot \boldsymbol{f}]\, dx = l(\boldsymbol{u}_h) \qquad \forall \boldsymbol{u}_h \in V_h, \\
&b(q_h, \boldsymbol{v}_h) \\
&+ \sum_{K \in \mathcal{T}_h} \int_K \tau \Delta q_h [\Delta p_h + \nabla \cdot (\omega \times \boldsymbol{v}_h) - \nabla \cdot \boldsymbol{f}]\, dx = 0 \qquad \forall q_h \in Q_h,
\end{aligned}$$

where K are elements of triangulation $\mathcal{T}_h$ and $\tau = |\omega| \frac{h_K^2}{\nu}$, h_K is diameter of element K. The second possible method is the Galerkin least-squares fomulation, where we look for $\boldsymbol{v}_h \in V_h \approx V$

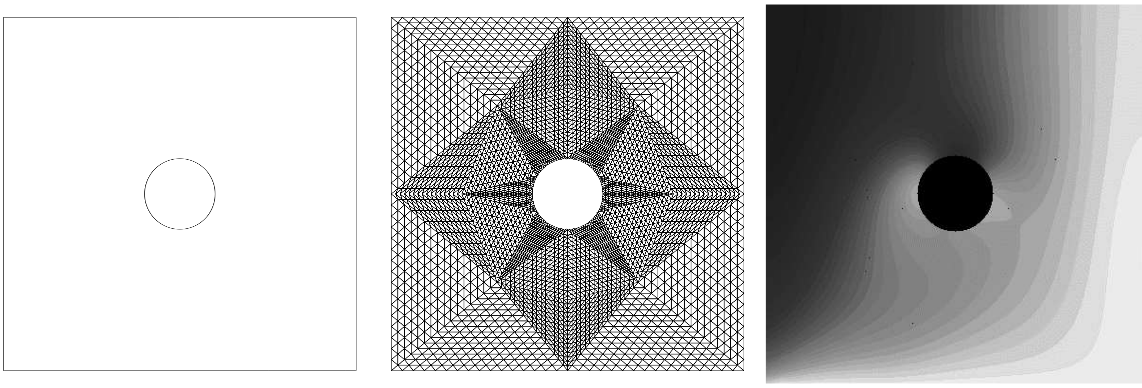

Figure 1. computational domain, its triangulation and pressure

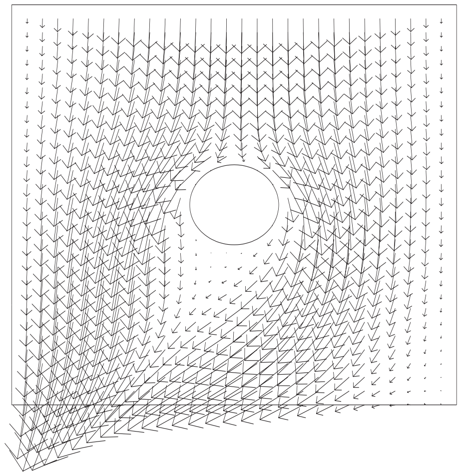

Figure 2. velocity

and $p_h \in Q_h \approx Q$, which satisfy

$$a(\boldsymbol{v}_h, \boldsymbol{v}_h, \boldsymbol{u}_h) - b(p_h, \boldsymbol{u}_h) + c(\boldsymbol{v}_h, \boldsymbol{u}_h)$$
$$+ \sum_{K \in \mathcal{T}_h} \int_K [\tau_1(-\nu \Delta \boldsymbol{u}_h + \omega \times \boldsymbol{u}_h) \cdot (-\nu \Delta \boldsymbol{v}_h + \nabla p_h \omega \times \boldsymbol{v}_h - \boldsymbol{f}) + \tau_2 (\nabla \cdot \boldsymbol{u}_h)(\nabla \cdot \boldsymbol{v}_h)]\, dx$$
$$= l(\boldsymbol{u}_h) \qquad \forall \boldsymbol{u}_h \in V_h,$$
$$b(q_h, \boldsymbol{v}_h)$$
$$+ \sum_{K \in \mathcal{T}_h} \int_K \tau_1 \nabla q_h \cdot (-\nu \Delta \boldsymbol{v}_h + \nabla p_h + \omega \times \boldsymbol{v}_h - \boldsymbol{f}) = 0 \qquad \forall q_h \in Q_h.$$

Results

We solved the following Navier-Stokes equations with rotation:

$$-\nu \Delta \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} + \boldsymbol{e}_3 \times \boldsymbol{v} + \nabla p = \boldsymbol{f}, \qquad \operatorname{div} \boldsymbol{v} = 0,$$

where $\rho = 1,\ \nu = 0.01,\ f_1 = 0,\ f_2 = 0$ and $\boldsymbol{e}_3$ is unit vector. Computational domain is a square $[-1,1] \times [-1,1]$ with the solid ball inside having the radius 0.2 and the centre [0,0]. Input boundary part $\partial\Omega$ is the boundary between the points [-1,1] and [1,1]. Output boundary part is situated between [-1,-1] and [1,-1]. The remaining part of boundary is fixed wall. We have used $P2 - P1$ elements (Taylor-Hood).

Boundary condition on input: $v_1 = 0,\ v_2 = 0.5x_1^2 - 0.5$.
Boundary condition on output: $\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} = 0$.
Boundary condition on the fixed wall and ball: $\boldsymbol{v} = 0$.
And we used DRS.
In the Figure 1. you can see the domain, triangulation, which we used and isolines of the pressure. The velocity of the fluid is shown in the Figure 2.

Discussion

We have introduced the basic free fall simulation problems. We have mentioned the difficulties with unknown domain and presented a solution of this problem. We have presented a method how to solve the free fall problem and we have described difficulties arrising when the Navier-Stokes equations are solved numerically.

Conclusion

We develop a computer program which is able to solve the free fall problem. This program together with the numerical experiments is the main result of this work. We can solve instationary Navier-Stokes equations with Corriolis force and we presented these numerical results. Now we plan simulate the free fall of a rigid body. Our aim is to numerically confirm some theoretical results and to simulate the free fall in nonnewtonian fluid.

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