

Realization of Voevodsky's motives

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Introduction

The theory of motives has always had two faces. One is the geometric face where a universal cohomology theory for varieties is cooked up from geometric objects like cycles. The other one is the linear algebra face where restricting conditions are put on objects of linear algebra like vector spaces with an operation of the Galois group. The ideal theorem would be an equivalence of these two approaches.

For pure motives, Grothendieck proposed a geometric construction. The linear algebra side is covered by l -adic cohomology for all primes l together with singular cohomology equipped with its Hodge structure. The relation between the two sides is made by the Tate or the Hodge conjecture which tell us that geometry and linear algebra should be very close to each other. However, we neither know whether Grothendieck's construction has the required properties nor what the image of the category of motives on the linear algebra side is. I.e., we cannot tell whether a Galois module is motivic just from checking linear algebra conditions (there are conjectures though).

For mixed motives, Voevodsky's work goes a long way in constructing the geometric side of the story. The linear algebra side is given by Deligne's absolute Hodge motives, independently considered by Jannsen under the name of mixed realizations. By Beilinson's conjectures the interplay between geometry and linear algebra should be measured by special values of L -functions of motives. However, we are far from proving the ideal theorem.

The main aim of the present article is to provide the expected functor between the two sides. More precisely, we construct a realization functor from Voevodsky's triangulated category of geometrical motives (which should be thought of as the derived category of mixed motives) to the "derived category" of mixed realizations which we constructed in [Hu1]. Indeed, most of the present article is a follow-up of loc. cit. where the realization functor was constructed on the category of simplicial varieties. As a direct corollary we also obtain realization functors to continuous l -adic cohomology and to absolute Hodge cohomology. Their existence is not a surprise (cf. [Vo2]) but was not in the literature yet.

We want to mention that Levine has a triangulated category of motives ([Le3]). Over a field of characteristic zero it is equivalent to Voevodsky's. He also constructs realization functors in his setting starting from a different

set of axioms.

We can show (2.3.6) that the motivic objects in the category of mixed realizations obtained from Voevodsky's category are contained in the category of motivic objects considered before. We do not get new motives on the linear algebra side.

We can also prove that the Chern classes from higher algebraic K -theory to mixed realizations factor through Voevodsky's category. The key to this fact is the computation of the motive of BGL in Voevodsky's category.

Besides these formal insights, the mixed realization functor should be very useful wherever an attempt is made to prove Beilinson's conjectures on L -values. They require the construction of elements in motivic cohomology. In the known cases, Adams eigenspaces of K -theory were used as the definition of motivic cohomology. The formal properties of motivic cohomology in the sense of Voevodsky are lot better , e.g. localization sequences involving singular varieties. Its main advantage is that it allows to do computations in two variables. Motivic cohomology in the sense of K -theory always fixes the second variable as a Tate motive. We hope that explicit applications will follow.

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1 Voevodsky's triangulated category of motives

We start with a quick review of Voevodsky's category. Then we give some extra notions and properties needed in the present text.

1.1 Review

We repeat the definition of the triangulated category of motives as given by Voevodsky in [Vo1]. For more details and properties we refer to his paper. He has developed an integral theory. We mostly need the \mathbb{Q} -rational version.

Let k be a fixed ground field of characteristic zero. Let Var be the category of varieties over k , i.e. separated schemes of finite type over k . Let Sm be the full subcategory of smooth varieties. Let $A = \mathbb{Q}$ or \mathbb{Z} be the coefficient ring.

Definition 1.1.1 ([Vo1] 2.1). *Let X be a smooth variety and Y a general variety. A prime correspondence from X to Y is an integral closed subscheme W of $X \times Y$ which is finite over X and surjective over a connected component of X . Let $c(X, Y)_A$ be the free A -module generated by prime correspondences. Its elements are called finite correspondences. Let SmCor be the category with objects smooth varieties and morphisms given by finite correspondences.*

We want to recall the composition of correspondences. Assume $\Gamma_1 \subset X \times Y$ and $\Gamma_2 \subset Y \times Z$ are prime correspondences. $\Gamma_1 \times Z$ and $X \times \Gamma_2$ are cycles in $X \times Y \times Z$. All irreducible components C_i (with reduced structure) of their intersection are finite over X . In particular they all have the right dimension. We have

$$(\Gamma_1 \times Z).(X \times \Gamma_2) = \sum n_i C_i$$

where the intersection multiplicity n_i of C_i is the usual one in the Chow group, e.g.[Fu] 20.4. Let $\pi : X \times Y \times Z \rightarrow X \times Z$ be the natural projection. It induces a map $\pi_* : c(X, Y \times Z) \rightarrow c(X, Z)$ which takes a primitive correspondence C to the closure of its image $\pi(C)$ times the degree of the covering $C \rightarrow \pi(C)$. By definition

$$\Gamma_2 \circ \Gamma_1 = \pi_*((\Gamma_1 \times Z).(X \times \Gamma_2)) .$$

There is a functor

$$[\cdot] : \text{Sm} \rightarrow \text{SmCor} .$$

It maps a morphism to its graph.

Note that SmCor is additive. Hence we can consider complexes of objects in SmCor as well as homotopies between maps of complexes.

Definition 1.1.2 ([Vo1] 2.1.1). *The triangulated category of effective geometrical motives $DM_{gm}^{eff}(k, A)$ is the localization of the homotopy category $K^b(\text{SmCor})$ with respect to the smallest thick subcategory containing the following:*

1. For any smooth scheme X the complex

$$[X \times \mathbb{A}^1] \rightarrow [X] .$$

2. For any smooth scheme X and any Zariski-covering $X = U \cup V$ the complex

$$[U \cap V] \xrightarrow{j'_U + j'_V} [U] \oplus [V] \xrightarrow{j_U - j_V} [X]$$

where j'_U, j'_V, j_U and j_V are the obvious inclusions.

The fibre product of varieties induces a tensor product structure of $DM_{gm}^{eff}(k, A)$.

Remark: If the distinction is not important or the setting clear, we will drop k and A from the notation.

For any smooth variety X , we have the complex $[X]$ concentrated in zero. We also have the complex $[X]^\sim = [X] \rightarrow [\text{Spec } k]$ sitting in degrees 0 and 1. In this normalization, there is an exact triangle

$$[X]^\sim \rightarrow [X] \rightarrow [\text{Spec } k] \rightarrow [X]^\sim[1] .$$

Definition 1.1.3 ([Vo1] after 2.1.3). *Let*

$$A(0) = [\text{Spec } k] \in DM_{gm}^{eff} .$$

It is the unit object for the tensor structure on DM_{gm}^{eff} . Let

$$A(1) = [\mathbb{P}^1]^\sim[-2] \in DM_{gm}^{eff}$$

be the Tate motive. For $k \geq 1$ let

$$A(k) = A(1)^{\otimes k} .$$

If M is a motive, we put $M(k) = M \otimes A(k)$.

$M \mapsto M(1)$ is a triangulated functor on DM_{gm}^{eff} .

Definition 1.1.4 ([Vo1] end of 2.1). *The category of geometrical motives $DM_{gm}(k, A)$ is obtained from $DM_{gm}^{eff}(k, A)$ by formally inverting the Tate motive. Explicitly, objects of DM_{gm} are pairs (M, n) with $M \in DM_{gm}^{eff}$ and $n \in \mathbb{Z}$. Morphisms are given by*

$$\mathrm{Hom}_{DM_{gm}}((B, n), (C, m)) = \lim_{k \gg 0} \mathrm{Hom}_{DM_{gm}^{eff}}(B(n+k), C(n+k)) .$$

We write $A(n) = (A(0), n)$ for $n \in \mathbb{Z}$. By construction $A(n) = (A(n), 0)$ for $n \geq 1$.

This definition is very much in the spirit of Grothendieck's definition of pure motives. There is a second category which is a lot more useful in computations.

Definition 1.1.5 ([Vo1] 3.1.1, 3.1.8, after 3.1.10). *A presheaf with transfers on Sm is an additive contravariant functor from SmCor to the category of abelian groups. It is called a Nisnevich sheaf with transfers if the corresponding presheaf on Sm is a sheaf in the Nisnevich topology (see [Fr] Ch.2). The category is denoted $Sh_{\mathrm{Nis}}(\mathrm{SmCor})$. A presheaf F on Sm is called homotopy invariant if the natural map $F(X) \rightarrow F(X \times \mathbb{A}^1)$ is an isomorphism for all smooth varieties X .*

The category of motivic complexes $DM_-(k, A)$ is the full subcategory of the derived category $D^-(Sh_{\mathrm{Nis}}(\mathrm{SmCor}))$ whose objects have homotopy invariant cohomology sheaves.

DM_- has a natural t -structure. Its heart is the abelian category of homotopy invariant Nisnevich sheaves with transfers.

Proposition 1.1.6 ([Vo1] 3.1.2, 3.1.11, 3.1.13). *There is a functor*

$$L : \mathrm{SmCor} \rightarrow Sh_{\mathrm{Nis}}(\mathrm{SmCor})$$

given by $L(X)(U) = c(U, X)_A$. It also defines a functor on Var . The natural inclusion of DM_- in $D^-(Sh_{\mathrm{Nis}}(\mathrm{SmCor}))$ has a left adjoint. For any Nisnevich sheaf with transfers F , it is given by the complex $C_(F)$*

$$C_{-n}(F)(X) = F(X \times \Delta^n)$$

where Δ^ is the standard cosimplicial object. The composition $C_* \circ L$ induces a functor*

$$M : DM_{gm}^{eff} \rightarrow DM_- .$$

It is fully faithful and hence identifies DM_{gm}^{eff} with a full subcategory of DM_- .

Remark: It is rather formal to show that we get a functor

$$DM_{gm}^{eff} \rightarrow DM_- .$$

However, it is very hard to show that $A(1)$ is quasi-invertible in DM_{gm}^{eff} ([Vo1] 4.2.4) and hence that DM_{gm}^{eff} is a full subcategory of DM_{gm} . $A(1)$ is *not* invertible in DM_- .

Remark: Note that $M : \text{SmCor} \rightarrow DM_-$ is induced by a functor to the category of complexes $C^{\leq 0}(S_{\text{Nis}}(\text{SmCor}))$.

Definition 1.1.7. *The pseudo abelian hull of the image of the composite functor*

$$M : \text{SmCor} \rightarrow DM_{gm} \rightarrow DM_-$$

is called the category of effective (generalized) Chow motives. The category of (generalized) Chow motives is obtained from it by formally inverting $A(1)$.

The reason for this terminology will become clear after 1.1.12 below. DM_- is pseudo abelian because the derived category of an abelian category with enough injectives (as $Sh_{\text{Nis}}(\text{SmCor})$) is by [Le2] Thm A.5.3. Hence the pseudo abelian hull of the definition is still a subcategory of DM_- . In particular, $A(1)[2]$ is a Chow motive because it is the cokernel of the projector induced by $\mathbb{P}^1 \rightarrow \text{Spec } k \xrightarrow{0} \mathbb{P}^1$. By definition

$$M(\mathbb{P}^1) = A \oplus A(1)[2] .$$

The splitting is independent of the choice of k -rational point of \mathbb{P}^1 by homotopy invariance.

Remark: There is some confusion with the older literature. In the category of Grothendieck motives we would decompose $h(\mathbb{P}^1) = \mathbb{Q} \oplus \mathbb{Q}(-1)$ and call $h^2(\mathbb{P}^1) = \mathbb{Q}(-1)$ the *Lefschetz motive*. Its dual then is the Tate motive. Voevodsky's Tate motive really *is* the Lefschetz motive in old terminology. The difference in signs is only a matter of choice in the definition. It makes sense because Voevodsky's functor M is covariant whereas Grothendieck's functor is contravariant.

Lemma 1.1.8. *The functor $M : \text{SmCor} \rightarrow D\mathcal{M}_-$ extends to complexes in $C^-(\text{SmCor})$. Let $f : X_* \rightarrow Y_*$ be a morphism of complexes such that all $f_n : X_n \rightarrow Y_n$ induce isomorphisms $M(f_n)$. Then $M(f)$ is an isomorphism.*

Proof. Recall that the functor M is induced by the functor C_*L which takes values in the category of bounded above complexes of Nisnevich sheaves with transfers. More precisely, they are bounded above by 0. It extends by taking double complexes. There is no problem because in each component only finitely many direct summands contribute. For the second assertion, the assumption implies that all $H^p(C_*L(f_q))$ are isomorphisms. A spectral sequence argument in the surrounding category of Nisnevich sheaves with transfer shows that $M(f)$ induces an isomorphism on all $H^n(C_*L(f))$. \square

Definition 1.1.9. *The full subcategory $DM_{gm}^{eff}(k, A)$ of $D\mathcal{M}_-(k, A)$ generated by the image of $C^-(\text{SmCor})$ which is triangulated, pseudo abelian and closed under tensor products is called the category of complexes of effective generalized Chow motives.*

Note that DM_{gm}^{eff} is a full subcategory of DM_{gm}^{eff} .

The Tate motive is also used in order to define motivic cohomology.

Definition 1.1.10 (Voevodsky). *Let X be a variety. Then*

$$H_{\mathcal{M}}^i(X, A(j)) = \text{Hom}_{D\mathcal{M}_-(k, A)}(M(X), A(j)[i])$$

is the motivic cohomology of X .

Morphisms of motives in $D\mathcal{M}_-$ are “known”. At least we relate them to other theories.

Proposition 1.1.11. *Let X and Y be smooth and proper varieties pure of dimension d and d' .*

$$\text{Hom}_{D\mathcal{M}_-}(M(Y), M(X)) = \text{CH}^d(X \times Y) \otimes A$$

where the right hand side denotes cycles of codimension d up to rational equivalence.

Proof. For a proper variety, $M^c(X) = M(X)$ (by definition see [Vo1] 4.1). By [Vo1] 4.2.3 (cf. [Fr] Prop. 4.9 where the sign is correct), 4.2.2 3 and 4.2.5)

$$\begin{aligned} \text{Hom}_{D\mathcal{M}_-}(Y, X) &= A_{0,0}(Y, X) = A_{-d,0}(Y \times X, \text{Spec } k) \\ &= \text{Hom}_{D\mathcal{M}_-}(M(Y \times X), A(d)[2d]) = \text{CH}^d(X \times Y)_A . \end{aligned}$$

\square

Corollary 1.1.12 (Voevodsky). *The full subcategory of DM_- with objects direct summands of motives $M(X)$ with smooth and proper X is equivalent to the category of effective Chow motives.*

Proof. Clear by definition of the category of Chow motives, e.g. [Sch] 1.2. \square

Corollary 1.1.13. *Let k be a number field. Then $\mathrm{Hom}_{DM_-(k, \mathbb{Q})}(\mathbb{Q}, \mathbb{Q}(j)[i])$ vanishes in the cases*

1. $i < 0$,
2. $i > 1$,
3. $i = 0, j \neq 0$,
4. $i = 1, j \leq 0$.

If $k = \mathbb{Q}$ then $\mathrm{Hom}_{DM_-}(\mathbb{Q}, \mathbb{Q}(j)[1])$ is zero or one-dimensional.

Proof. By [Vo1] 2.2, morphisms of Tate motives are given by higher Chow groups which are known to be isomorphic to the graded pieces of the γ -filtration on higher K -groups. In the case of a number field the ranks are known by Borel's result [Bo]. \square

Remark: This theorem will be needed in the computations in the sequel. Note, however, that we do not need to understand the precise isomorphisms.

1.2 Mixed Tate motives

We now study a particularly simple subcategory of DM_{gm} where the morphisms are understood. The computations in chapter 3 are carried out in this subcategory. The use of this category for question of this type was suggested to me by Kahn. The main computational tool, the weight filtration was introduced by Levine in the setting of his triangulated category ([Le1]). Let $k = \mathbb{Q}$ and $A = \mathbb{Q}$.

Definition 1.2.1 (Kahn, Levine). *The triangulated category of mixed Tate motives DMT is the full triangulated category of DM_{gm} generated by $\mathbb{Q}(i)$ for $i \in \mathbb{Z}$. By $DMT_{\geq N}, DMT_{\leq N}, DMT_{[N, M]}$ we denote the full triangulated subcategories generated by $\mathbb{Q}(i)$ with $i \geq N, i \leq N, N \leq i \leq M$ respectively.*

The category DMT is closed under tensor products.

Lemma 1.2.2. 1. For all $N \in \mathbb{Z}$, $M_1 \in DM\mathcal{T}_{\geq N}$ and $M_2 \in DM\mathcal{T}_{\leq N-1}$, we have $\text{Hom}_{DM\mathcal{T}}(M_1, M_2) = 0$.

2. The categories $DM\mathcal{T}_{[N,N]}$ are isomorphic to the category of finite dimensional graded \mathbb{Q} -vector spaces.

Proof. For $M_n = \mathbb{Q}(i_n)[j_n]$, $n = 1, 2$, this is a consequence of 1.1.13. If the assertion is true for two vertices of an exact triangle (in the first or second argument), then it is true for the third. This proves 1. For 2. we show by induction that $M \in DM\mathcal{T}_{[0,0]}$ is direct sum of $\mathbb{Q}(0)[j]$'s. Consider a triangle

$$M \rightarrow \bigoplus \mathbb{Q}(0)^{e_j}[j] \xrightarrow{\phi} \bigoplus \mathbb{Q}(0)^{f_j}[j] .$$

Let $\phi_j = \phi |_{\mathbb{Q}(0)^{e_j}[j]}$. Again by 1.1.13 it maps to $\mathbb{Q}(0)^{f_j}[j]$. Such morphisms are given matrices with rational numbers as entries. Composition of matrices is composition of morphisms. This means precisely that for fixed j the category of powers of $\mathbb{Q}(0)[j]$ is isomorphic to the category of finite dimensional \mathbb{Q} -vector spaces. We decompose $\mathbb{Q}(0)^{e_j}[j] = K_j[j] \oplus I_j[j]$, $\mathbb{Q}(0)^{f_j}[j] = I_j[j] \oplus L_j$ such that ϕ_j vanishes on K_j and is an isomorphism on I_j . Then M also stands in the exact triangle

$$M \rightarrow \bigoplus K_j[j] \xrightarrow{0} \bigoplus L_j[j]$$

hence $M \cong \bigoplus K_j[j] \oplus \bigoplus L_j[j-1]$. As we have already seen, the morphisms of such direct sums are the same as the morphisms of graded \mathbb{Q} -vector spaces. \square

Remark: We will construct later (proposition 2.1.7) a faithful fibre functor from $DM\mathcal{T}_{[N,N]}$ to the category of graded \mathbb{Q} -vector spaces given by singular cohomology.

The first part of this lemma is enough to deduce in a formal way the existence of an extra structure on $DM\mathcal{T}$.

Proposition 1.2.3 (Levine). For all $M \in DM\mathcal{T}$ and integers N , the functor

$$\text{Hom}_{DM\mathcal{T}}(\cdot, M) : DM\mathcal{T}_{\geq N}^{\circ} \rightarrow \underline{ab}$$

is representable by an object $W_{\geq N}M$ together with a morphism

$$W_{\geq N}M \rightarrow M .$$

$W_{\geq N}$ is an exact functor $DM\mathcal{T} \rightarrow DM\mathcal{T}_{\geq N}$. Dually, the functor

$$\mathrm{Hom}_{DM\mathcal{T}(M, \cdot)} : DM\mathcal{T}_{< N} \rightarrow \underline{ab}$$

is representable by

$$M \rightarrow W_{< N}M .$$

This defines an exact functor to $DM\mathcal{T}_{< N}$. For each object M there is an exact triangle

$$W_{\geq N}M \rightarrow M \rightarrow W_{< N}M) .$$

Let $\mathrm{Gr}_N M = W_{< N+1}W_{\geq N}M$ be the graded pieces.

Proof. We put $N = 0$ in order to simplify notations. As always, it is enough to construct some object $W_{\geq 0}M \rightarrow M$ with the universal property. It is necessarily unique up to unique isomorphism. We start with $M = \mathbb{Q}(i)[j]$. We put

$$W_{\geq 0}M = \begin{cases} M & \text{if } i \geq 0, \\ 0 & \text{else.} \end{cases}$$

It satisfies the universal property by the lemma.

Now assume $W_{\geq 0}$ is constructed on M_1 and M_2 . We want to define it on M_3 which sits in the triangle

$$M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_1[1] .$$

By the axioms of a triangulated category there is an object W_0M_3 sitting in the triangle

$$W_{\geq 0}M_1 \rightarrow W_{\geq 0}M_2 \rightarrow W_{\geq 0}M_3 \rightarrow W_{\geq 0}M_1[1]$$

and a morphism $W_{\geq 0}M_3 \rightarrow M_3$ such that the obvious morphism of triangles is defined. Now we have to check the universal property. Let K be an object with weights less or equal to 0. There are long exact sequences

$$\begin{array}{ccccccccc} \mathrm{Hom}(K, M_1) & \longrightarrow & \mathrm{Hom}(K, M_2) & \longrightarrow & \mathrm{Hom}(K, M_3) & \longrightarrow & \mathrm{Hom}(K[-1], M_1) & \longrightarrow & \mathrm{Hom}(K[-1], M_2) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathrm{Hom}(K, W_{\geq 0}M_1) & \longrightarrow & \mathrm{Hom}(K, W_{\geq 0}M_2) & \longrightarrow & \mathrm{Hom}(K, W_{\geq 0}M_3) & \longrightarrow & \mathrm{Hom}(K[-1], W_{\geq 0}M_1) & \longrightarrow & \mathrm{Hom}(K[-2], W_{\geq 0}M_2) \end{array}$$

The outer arrows are isomorphisms by assumption. By the five lemma the middle arrow is also an isomorphism. The same proof works for the dual assertion. The universal property allows to construct $W_{\geq N}$ and $W_{< N}$ on morphisms. For $M = \mathbb{Q}(i)[j]$ the exact triangle clearly exists. By construction and lemma 1.2.4 below we get exact triangles as claimed. \square

Remark: Usually it is not a good idea to define a functor by choosing the third vertex of a triangle. In our case it is well-defined because we also have the universal property working for us.

The above amounts to the construction of the weight cofiltration on Tate motives. Application of contravariant realization functors maps it to the weight filtration on the realization side. Objects of $DM\mathcal{T}_{[N,N]}$ are (triangulated) pure Tate motives of weight $\pm 2N$ depending on conventions. The category of pure Tate motives of fixed weight is the derived category of a semi-simple abelian category. Obviously there is a “weight spectral sequence” with initial terms the graded pieces and converging to M .

Lemma 1.2.4. *Let D be a triangulated category. Consider the diagram of exact triangles in D*

$$\begin{array}{ccccc}
 A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 \\
 \downarrow & & \downarrow & & \\
 B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 \\
 \downarrow & & \downarrow & & \\
 C_1 & & C_2 & &
 \end{array}$$

Then there is an object C_3 and morphisms such that the last line and row are also triangles.

Proof. [BBD] 1.1.11. □

1.3 More Technical background

Let again k be a field of characteristic zero and $A = \mathbb{Z}$ or \mathbb{Q} .

Definition 1.3.1. *Let K be a complex in SmCor . The stupid filtration $\sigma_n K$ is defined by*

$$(\sigma_n K)^i = \begin{cases} K^i & \text{for } i \geq n, \\ 0 & \text{for } i < n. \end{cases}$$

$\sigma_n K$ is a subcomplex of K . Its graded pieces are the K^i .

Let L be an object in DM_- . The canonical truncation $\tau_{\geq n} L$ is the truncation functor on DM_- with respect to the t -structure with heart the homotopy invariant sheaves with transfers. It is a quotient complex of L .

$\tau_{\geq n}$ is a functor on DM_- whereas σ_n does not pass to DM_{gm} . Note that

$$\tau_{\geq n}M(K) = \tau_{\geq n}M(\sigma_{n-1}K) .$$

Proposition 1.3.2. *Let $K \rightarrow K'$ be a morphism in $C^-(\text{SmCor})$. Assume that it induces isomorphisms*

$$\sigma_n K \rightarrow \sigma_n K'$$

in DM_{gm} for all n . Then $M(K) \rightarrow M(K')$ is an isomorphism in DM_- .

Proof. We consider the corresponding objects in the derived category of Nisnevich sheaves with transfers. We have to check that

$$H^i(M(K)) \rightarrow H^i(M(K'))$$

is an isomorphism for all i where H^i denotes the cohomological functor with respect to the natural t -structure. Only finitely many K^k contribute to $H^i(M(K))$, namely those with $k \geq i - 1$. In other words,

$$H^i(M(K)) = H^i(M(\sigma_{i-1}K)) .$$

Hence the assumption on the stupid filtration is enough to prove the quasi-isomorphism. \square

Lemma 1.3.3. *For $K \in C^-(\text{SmCor})$ we have in DM_-*

$$M(K) = \varinjlim M(\sigma_n K) .$$

Proof. Clearly, the equality holds in the abelian category $C^-(Sh_{\text{Nis}}(\text{SmCor}))$. We have to prove that it passes to $D^-(Sh_{\text{Nis}}(\text{SmCor}))$. Assume we are given a direct system of morphisms in the derived category

$$f_n : M(\sigma_n K) \rightarrow L .$$

They are represented by morphisms of complexes

$$M(\sigma_n K) \xrightarrow{f_n^!} L_n \xleftarrow{g_n} L$$

where g_n is a quasi-isomorphism. The L_n can be chosen such that the $f_n^!$ and the g_n give a direct system up to homotopy. Let L_∞ the direct limit of the

L_n . As direct limits are exact in $Sh_{\text{Nis}}(\text{SmCor})$ it is also quasi-isomorphic to L . Let

$$f_n'' : M(\sigma_n K) \rightarrow L_n \rightarrow L_\infty .$$

They form a direct system up to homotopy. Let s_i be the stupid filtration of complexes of Nisnevich sheaves, i.e.

$$(s_i C)^p = \begin{cases} C^p & \text{for } p \geq i \\ 0 & \text{else.} \end{cases}$$

Note that $s_i M(\sigma_n K) = s_i M(K)$ for $i \geq n$. Hence f_n'' induces a direct system of maps up to homotopy on $s_i M(K)$. We can define their limit f on $M(K)$ by descending induction. Assume f is defined on $s_n M(K)$. f and f_{n-1}'' differ by a homotopy $h^* : M(K)^* \rightarrow L_\infty^{*-1}$. We modify f by $h^n \circ d$ on $M(K)^n$ and extend it to $M(K)^{n-1}$ using $f_{n-1}''|_{M(K)^{n-1}}$. This gives a new morphism of complexes $s_{n-1} M(K) \rightarrow L_\infty$ which is homotopic to f_{n-1}'' . Clearly the morphism f is homotopic to f_n'' on the subcomplex $M(\sigma_n K)$ as well. \square

This lemma allows to reduce all questions on $DM_{gm^-}^{eff}$ (see definition 1.1.9) to questions in DM_{gm} .

Proposition 1.3.4. *Let $M(K)$ and $M(L)$ be in $DM_{gm^-}^{eff}$. Then*

$$\text{Hom}_{DM_{gm^-}^{eff}}(M(K), M(L)) = \varprojlim \text{Hom}_{DM_{gm^-}^{eff}}(M(\sigma_n K), M(L)) .$$

Moreover, for fixed K and n there is $N \in \mathbb{N}$ such that

$$\text{Hom}_{DM_{gm^-}^{eff}}(M(\sigma_n K), M(L)) = \text{Hom}_{DM_{gm}}(M(\sigma_n K), M(\sigma_N L)) .$$

Proof. The first equality is nothing but the lemma. In order to show the existence of N we can assume that K is of the form $M(X)[p]$ for some p and some smooth variety X . Then by [Fr] 3.3

$$\text{Hom}_{DM_{gm^-}^{eff}}(M(K), M(L)) = H_{Zar}^p(X, M(L)) .$$

X is finite dimensional hence Zariski cohomology has finite cohomological dimension. Hence for N small enough

$$H_{Zar}^p(X, M(L)) = H_{Zar}^p(X, \tau_{\geq N} M(L)) = H_{Zar}^p(X, \tau_{\geq N} M(\sigma_{N-1} L)) .$$

\square

Lemma 1.3.5. *Let X be a variety such that $M(X)$ is a mixed Tate motive. Then $\tau_{>0} \text{Gr}_n M(X) = 0$ for all n .*

Let X_ be in $C^b(\text{SmCor})$ such that all X_i are mixed Tate motives. Then*

$$\tau_{>0} W_{\geq n} M(X_*) = \tau_{>0} W_{\geq n} \sigma_0 M(X_*) .$$

Proof. By assumption $\text{Gr}_n M(X) = \bigoplus_{j_i} \mathbb{Q}(n)[j_i]$. It follows from Hodge theory, [De1] Thm 8.2.4, that $0 \leq j_i \leq n$. Now we use the fact that $\tau_{>n} \mathbb{Q}(n) = 0$. ([Ka] 2. property C or from $\tau_{\geq 1} M(\mathbb{G}_m^n) = 0$). For the second assertion it suffices to show that

$$\tau_{\geq 0} W_{\geq n} (M(X_*)) = 0$$

for complexes concentrated in negative degrees. On the level of Gr_n this follows from the first part. By induction on the weight filtration it follows for all $W_{\geq n}$. \square

Proposition 1.3.6. *Let X_* and Y_* be objects of $C^-(\text{SmCor})$ such that all X_i and Y_i are mixed Tate motives. Let*

$$f : X_* \rightarrow Y_*$$

be a morphism in DM_- . For all integers N and Tate weights n (see 1.2.3) we assume that kernel and cokernel of the morphism

$$\text{Gr}_n \sigma_{-N} X_* \rightarrow \text{Gr}_n \sigma_{-N} Y_*$$

is direct sum of $\mathbb{Q}(n)[j]$ with $j \geq N$. Then f is an isomorphism.

Proof. Recall that $\tau_{\geq -N+1} \mathbb{Q}(n)[j] = 0$ for $j \geq N + n$. By assumption the map

$$\text{Gr}_n \sigma_{-N} X_* \rightarrow \text{Gr}_n \sigma_{-N} Y_*$$

fails to be an isomorphism by direct sums of $\mathbb{Q}(n)[j]$ with $j \geq N$. Hence application of $\tau_{\geq -N+1}$ yields that

$$(*) \quad \tau_{\geq -N+1} \text{Gr}_n \sigma_{-N-n} X_* \rightarrow \tau_{\geq -N+1} \text{Gr}_n \sigma_{-N-n} Y_*$$

is an isomorphism for all n and N . We want to show from (*) that $\tau_{\geq -N+1}(f)$ is an isomorphism for all N . Only finitely many weights occur in $\sigma_{-N} X_*$ and $\sigma_{-N} Y_*$. Let n_0 be the maximal one. The weight filtration has at most n_0 non-trivial steps because all X_i and Y_i are effective. By assumption (*),

$\tau_{\geq -N-n_0+1} \text{Gr}_n \sigma_{-N-2n_0}(f)$ is an isomorphism for all weights up to n_0 . By induction on the exact triangles for the weight filtration (see 1.2.3) and the five lemma in each step we deduce that

$$\tau_{\geq -N+1} W_{\geq n_0} \sigma_{-N-2n_0}(f)$$

is an isomorphism. By lemma 1.3.5 this implies that

$$\tau_{\geq -N+1} W_{\geq n_0} \sigma_{-N}(f) = \tau_{\geq -N+1} \sigma_{-N}(f) = \tau_{\geq -N+1}(f)$$

is an isomorphism as claimed. \square

2 The realization functor

We construct triangulated realization functors from $D\mathcal{M}_{gm}(k, A)$ to various triangulated categories. On the level of cohomology objects this corresponds e.g. to the l -adic or Hodge realization. In order to stress the logic we first give an axiomatic construction. Then we recall the definition of the “derived” category of mixed realizations. In a last step we apply the construction to this case. The main result is 2.3.3 and its corollaries.

2.1 Axiomatic construction

Again k is a base field of characteristic zero. In this section we assume throughout that \mathcal{A} is an abelian category with enough injectives in which arbitrary direct sums exist and are exact. We start with a contravariant functor

$$\tilde{R} : \text{Sm} \longrightarrow C^{\geq 0}(\mathcal{A}) .$$

The aim of this section is to deduce from it (under the assumption of certain axioms) a triangulated functor

$$R : D\mathcal{M}_{gm} \longrightarrow D^+(\mathcal{A}) .$$

Such a functor is called a *realization functor*. All realization functors induce by functoriality *regulator maps*, i.e. transformations of functors from Var to \underline{ab}

$$H_{\mathcal{M}}^i(\Gamma, \mathbb{Z}(n)) \longrightarrow H_R^i(\Gamma, n) := \text{Hom}_{D(\mathcal{A})}(R(\mathbb{Z}(n)), R(\Gamma)) .$$

For technical reason we are also interested in extensions of R to $D\mathcal{M}_{gm}^{eff}$.

Example: In the case $k = \mathbb{C}$, let $\underline{\mathbb{Q}}$ be the category of \mathbb{Q} -vector spaces and \tilde{R}_{sing} the singular cochain complex, i.e., for a smooth variety X , let

$$\tilde{R}_{sing}(X) = \mathbb{Q}[\text{Cont}(\Delta^*, X(\mathbb{C}))]^\vee$$

where $X(\mathbb{C})$ is considered as a complex manifold, Δ^n is the standard topological n -simplex, Cont denotes continuous maps of topological spaces and \cdot^\vee is the \mathbb{Q} -dual. By definition

$$H^i(\tilde{R}_{sing}(X)) = H_{sing}^i(X(\mathbb{C}), \mathbb{Q}) .$$

Clearly, $\tilde{R}_{sing}(X)$ is a functor to $C^{\geq 0}(\underline{\mathbb{Q}})$.

Definition 2.1.1. *A functor*

$$\tilde{R} : \text{Sm} \longrightarrow C^{\geq 0}(\mathcal{A})$$

or

$$\tilde{R} : \text{Var} \longrightarrow C^{\geq 0}(\mathcal{A})$$

has descent for open covers respectively for proper covers, if application of \tilde{R} to the covering map for the nerve of a Cech-covering respectively for a proper hypercovering yields a quasi-isomorphism . It satisfies the homotopy property if for all varieties X the morphism

$$\tilde{R}(X) \rightarrow \tilde{R}(X \times \mathbb{A}^1)$$

is a quasi-isomorphism.

The first realization result is rather obvious:

Proposition 2.1.2. • *Assume we have a functor*

$$\tilde{R} : \text{SmCor} \rightarrow C^{\geq 0}(\mathcal{A})$$

which has descent for open covers and satisfies the homotopy property. Then it extends to an exact functor

$$R : DM_{gm-}^{eff} \rightarrow D^+(\mathcal{A}) .$$

- *Assume in addition that \mathcal{A} is a tensor category and \tilde{R} is compatible with tensor products. Then R is a triangulated tensor functor.*

- Finally, if the functor $\cdot \otimes R(\mathbb{Q}(1))$ on $D^+(\mathcal{A})$ is an equivalence of categories, then R extends to a functor

$$R : DM_{gm} \rightarrow D^+(\mathcal{A}) .$$

Proof. Applying \tilde{R} to objects of $C^b(\text{SmCor})$ we get double complexes. We take the associate simple complex in $C^+(\mathcal{A})$. This involves a universal choice of signs which we fix once and for all. \tilde{R} is a functor

$$K^b(\text{SmCor}) \rightarrow K^+(\mathcal{A}) \rightarrow D^+(\mathcal{A}) .$$

(See [Hu1] 2.2.3 for details on signs.) All other statements are immediate. We obtain

$$R : DM_{gm} \rightarrow D^+(\mathcal{A}) .$$

Finally we extend it to DM_{gm}^{eff} . On objects the same definition as on bounded complexes works. On morphisms we use the description of lemma 1.3.4 to reduce to the bounded case. \square

Remark: It is tempting to weaken the assumption of the proposition to functors with values $D(\mathcal{A})$ or even triangulated categories in general. However, the above proof does not work in that generality. In fact, all constructions of Chern classes or cycles classes on higher Chow groups - a question very similar to the above - assume the existence of functorial complexes rather than objects in a derived category.

Example: (Etale cohomology) Fix a prime l and an integer n . We choose an injective resolution \mathcal{I} of \mathbb{Z}/l^n on the étale site on the category of smooth schemes over our base field k . Let $M(X)_{et} \otimes \mathbb{Z}/l^n$ be the étale sheafification of the complex of Nisnevich sheaves modulo l^n . We put

$$\tilde{R}\Gamma(X, \mathbb{Z}/l^n) = \text{Hom}_{Sh_{et}}^*(M(X)_{et} \otimes \mathbb{Z}/l^n, \mathcal{I}) .$$

Clearly this is a contravariant functor

$$\text{SmCor} \rightarrow C(\underline{ab}) .$$

By [Vol] Prop. 3.2.3 the cohomology of this complex computes the étale version of motivic cohomology or equivalently étale cohomology of X . Let $\tilde{R}(X, \mathbb{Z}/l^n)$ be the sheafification of $R\Gamma(X_{k'}, \mathbb{Z}/l^n)$ for finite field extensions of k in the étale topology. Clearly $\tilde{R}(X, \mathbb{Z}/l^n)$ computes $Rp_*(\mathbb{Z}/l^n)_X$ if

$p : X \rightarrow \text{Spec } k$ is the structural morphism. From this we get the exact realization functor

$$R : DM_{gm} \rightarrow D^b((\text{Spec } k)_{et}, \mathbb{Z}/l^n)$$

to the derived category of l^n -torsion Galois modules. Formulated like this, many properties of étale cohomology, e.g. localization sequences for smooth pairs, are a *consequence* of the existence of the functor. By functoriality, we get regulator maps

$$H_{\mathcal{M}}^i(X, \mathbb{Z}(j)) \rightarrow H_{et}^i(X, \mu_{l^n}^{\otimes j}).$$

Example: (Continuous étale cohomology) Let $R(X, \mathbb{Z}_l)$ be the projective system of complexes $R(X, \mathbb{Z}/l^n)$ for varying n . It allows to define an exact realization functor

$$R : DM_{gm} \rightarrow D^b((\text{Spec } k)_{et}, \mathbb{Z}_l)$$

into Ekedahl's triangulated category of \mathbb{Z}_l -sheaves ([Ek]). By functoriality, we get regulator maps

$$H_{\mathcal{M}}^i(X, \mathbb{Z}(j)) \rightarrow H_{cont}^i(X, \mathbb{Z}_l(j))$$

into Jannsen's continuous étale cohomology.

The rest of this section will be concerned with giving criteria for the existence of functors on SmCor.

Proposition 2.1.3. *Let $\tilde{R} : \text{Var} \rightarrow C^{\geq 0}(\mathcal{A})$ be a contravariant functor which is also covariant for all finite and surjective maps between irreducible varieties.*

- We assume that for a finite surjective map $f : X' \rightarrow X$ the composition

$$\tilde{R}(X) \xrightarrow{f^*} \tilde{R}(X') \xrightarrow{f_*} \tilde{R}(X)$$

equals multiplication with the degree of f .

- We also assume that f_* is compatible with direct products in the following sense: for f as before and all Y the diagram commutes:

$$\begin{array}{ccc} \tilde{R}(Y \times X') & \longleftarrow & \tilde{R}(X') \\ (\text{id}_Y \times f)_* \left\{ \begin{array}{c} \downarrow \\ \downarrow \end{array} \right. & & \left\{ \begin{array}{c} \downarrow \\ \downarrow \end{array} \right. f_* \\ \tilde{R}(Y \times X) & \longleftarrow & \tilde{R}(X) \end{array}$$

- Let $g : Y \rightarrow X$ be an arbitrary map and f a closed immersion. We put $Y' = \coprod C_i$ the disjoint union of the irreducible components of $X' \times_X Y$ with reduced structure. For all i let n_i be the ramification index of $X' \rightarrow X$ at C_i (see [Fu] Ex. 4.3.7). We assume that

$$\begin{array}{ccc} \tilde{R}(Y') & \xleftarrow{g'^*} & \tilde{R}(X') \\ \Sigma n_i f_{C_i}^* \downarrow & & \downarrow f^* \\ \tilde{R}(Y) & \xleftarrow{g^*} & \tilde{R}(X) \end{array}$$

also commutes.

Then \tilde{R} extends to a functors on SmCor .

Remark: If \tilde{R} has descent for open covers and satisfies the homotopy property, then it extends at least to DM_{gm}^{eff} . Hence we have potentially two definitions on singular varieties - the original one and the one from DM_{gm} . We do not know if they agree in general.

Proof. Let X and Y be smooth connected varieties. Let Γ in $X \times Y$ be an irreducible subvariety which is finite and surjective over X . Consider the diagram

$$X \leftarrow \Gamma \rightarrow Y .$$

It induces a map

$$R(\Gamma) : \tilde{R}(Y) \rightarrow \tilde{R}(\Gamma) \rightarrow \tilde{R}(X)$$

by composition of contravariant and covariant functoriality. If Γ is the graph of a morphism f , then $\tilde{R}(f) = R(\Gamma)$.

We claim that R is a functor. Let $\Gamma_1 \subset X \times Y$ and $\Gamma_2 \subset Y \times Z$ be prime correspondences. Then

$$(\Gamma_1 \times Z).(X \times \Gamma_2) = \sum n_i C_i$$

is a cycle in $X \times Y \times Z$. The C_i are the irreducible components of $\Gamma_1 \times_Y \Gamma_2$. Let pr be the projection map to $X \times Z$. The prime correspondence C_i is finite and dominant over its image $pr(C_i)$. Let d_i be the degree of this covering. The composition of correspondences is given by the cycle

$$\Gamma_2 \circ \Gamma_1 = \sum n_i d_i pr(C_i) .$$

The morphism

$$R(\Gamma_2 \circ \Gamma_1) : R(Z) \rightarrow R(\Gamma_2 \circ \Gamma_1) \rightarrow R(X)$$

can equivalently be computed via $R(\Gamma_1.\Gamma_2)$. Note that the multiplicities work out as they have to.

The intersection multiplicity n_i is equal to the ramification index of $X \times \Gamma_2 \rightarrow X \times Y$ at C_i (see [Fu] 4.3.7) by loc. cit. 7.1.15 and because Z is flat over k . By assumption

$$\begin{array}{ccccc} \tilde{R}(\coprod C_i) & \longleftarrow & \tilde{R}(X \times \Gamma_2) & \longleftarrow & \tilde{R}(\Gamma_2) \\ \tilde{R}(\cdot) \downarrow & & \downarrow & & \downarrow \tilde{R}''(\cdot) \\ \tilde{R}(\Gamma_1) & \longleftarrow & \tilde{R}(X \times Y) & \longleftarrow & \tilde{R}(Y) \end{array}$$

commutes. This implies

$$R(\Gamma_2 \circ \Gamma_1) = \tilde{R}(\Gamma_1) \circ \tilde{R}(\Gamma_2) .$$

□

By Deligne's method we can often extend a functor from Sm to Var . This will allow us to weaken our assumptions further.

Convention: Let \mathcal{V}_2 be a category and \mathcal{V}_1 a subcategory. Typically, \mathcal{V}_1 will be Sm and \mathcal{V}_2 will be Var or SmCor . Given a functor $\tilde{R} : \mathcal{V}_1 \rightarrow C(\mathcal{A})$ we say that it extends to \mathcal{V}_2 if there is a functor $R : \mathcal{V}_2 \rightarrow C(\mathcal{A})$ and a natural transformation $\eta : \tilde{R} \rightarrow R$ of functors on \mathcal{V}_1 such that all $\eta(\Gamma)$ are *quasi-isomorphisms*.

Proposition 2.1.4. *Let $\tilde{R} : \text{Sm} \rightarrow C^{\geq 0}(\mathcal{A})$ be a functor with descent for proper covers. Then it extends to a functor $\tilde{R}' : \text{Var} \rightarrow C^{\geq 0}(\mathcal{A})$. \tilde{R}' has descent for proper covers on Var as well.*

Proof. Let X be a variety. By [Del] 6.2, there exists a proper hypercovering of X such that all components are smooth. We put

$$\tilde{R}'(X) = \varinjlim \text{Tot}(\tilde{R}(X.))$$

where $X.$ runs through the inverse system of all proper hypercoverings of X by smooth varieties. The construction works because direct limits exist in \mathcal{A} . Descent for proper covers is a consequence of the same property for the original functor. □

Example: The functor \tilde{R}_{sing} has descent for proper covers (e.g.[Hu1] 6.2.2). By abuse of notation we call the extension also \tilde{R}_{sing} .

Lemma 2.1.5. *Let \mathcal{A} be \mathbb{Q} -linear. Let G be a finite group operating on an object $K \in \mathcal{A}$. Let K^G be the subobject of G -invariant elements, i.e., the intersection of $\ker(g - \text{id})$ for all $g \in G$. Then the functor \cdot^G is exact and there is a canonical section $K \rightarrow K^G$ of the inclusion.*

Proof. The section is given by

$$\sigma = \frac{1}{\#G} \sum_{g \in G} g \cdot$$

This is where we need invertibility of $\#G$. Using the section it is easy to show that short exact sequences remain exact. \square

Theorem 2.1.6. *Let \mathcal{A} be a \mathbb{Q} -linear abelian category with enough injectives in which arbitrary direct sums exist. Let*

$$\tilde{R} : \text{Sm} \longrightarrow C^{\geq 0}(\mathcal{A})$$

be a functor. We assume:

- \tilde{R} has descent for open and proper covers and satisfies the homotopy property (cf. 2.1.1)

Let \tilde{R} also denote the extension to Var (cf. 2.1.4).

- *For any surjective and finite morphism $f : X \rightarrow Y$ between normal varieties the group $G = \text{Aut}_Y(X)$ operates on $\tilde{R}(X)$ by contravariant functoriality. If the covering is generically Galois with Galois group G , then we assume that*

$$\tilde{R}(Y) \rightarrow \tilde{R}(X) \xrightarrow{\sigma} \tilde{R}(X)^G$$

is a quasi-isomorphism.

Then \tilde{R} extends to a functor

$$\tilde{R}'' : \text{SmCor} \rightarrow C^{\geq 0}(\mathcal{A})$$

together with a natural transformation of functors on Sm

$$\eta : \tilde{R} \rightarrow \tilde{R}''$$

such the $\eta(\cdot)$ are quasi-isomorphisms. \tilde{R}'' induces a functor

$$R : DM_{gm}^{eff} \rightarrow D^+(\mathcal{A}) .$$

Finally assume that \mathcal{A} is a tensor category and \tilde{R} is compatible with tensor products. Then R is a triangulated tensor functor. Assume

- The functor $\cdot \otimes R(\mathbb{Q}(1))$ on $D^+(\mathcal{A})$ is an equivalence of categories.

Then R passes to a functor

$$R : DM_{gm} \rightarrow D^+(\mathcal{A}) .$$

This can be seen as a prototype theorem. The same methods also works in other settings. An important such case is the realization functor $R_{\mathcal{MR}} : \text{Sm} \rightarrow C_{\mathcal{MR}}$, see theorem 2.3.1.

Remark: Levine also proves a realization theorem in the setting of his triangulated category ([Le3] Part I Ch. V). It has a different flavour from the above. His axioms seem to amount to a contravariant functor which is covariant for proper maps between smooth schemes plus the existence of cycle classes.

Before actually proving the theorem, we want to consider our example:

Proposition 2.1.7. \tilde{R}_{sing} satisfies all conditions of the theorem. It extends to a functor

$$R_{sing} : DM_{gm} \rightarrow D^+(\underline{\mathbb{Q}}) .$$

Proof. We have to check several properties of singular cohomology of complex analytic spaces. Proper descent was mentioned above. Descent for open covers is nothing but the Mayer-Vietoris sequence. The only non-trivial statement is the isomorphism

$$H_{sing}^i(Y, \mathbb{Q}) \rightarrow H_{sing}^i(X, \mathbb{Q})^G$$

for finite covering maps $f : X \rightarrow Y$ (between normal spaces) which are generically Galois with Galois group G . Note that G operates on X by functoriality of the normalization of X in $K(Y)$. By the Leray spectral sequence it is enough to show the isomorphism for the direct image of the

constant sheaf \mathbb{Q} (there are no higher direct images for a finite morphism). Its stalks are given by [GraRem] Ch. 2 §3.3

$$(f_*\mathbb{Q})_y = \prod_{x \in f^{-1}(y)} \mathbb{Q} .$$

The covering group $\text{Aut}_Y(X)$ permutes these factors. On the generic fibre it operates transitively by assumption. G operates transitively on all fibres, e.g. [Ma] Theorem 9.3. Hence $(f_*\mathbb{Q})_y^G$ is at most one-dimensional. As the image of \mathbb{Q} in $f_*f^*\mathbb{Q}$ is invariant under G it is at least one-dimensional. \square

Corollary 2.1.8. *Let $k = \mathbb{Q}$. Then R_{sing} identifies the category $DM\mathcal{T}_{[N,N]}$ of Tate motives with fixed weight with the category of finite dimensional graded \mathbb{Q} -vector spaces.*

Proof. Clear because $R_{\text{sing}}(\mathbb{Q}(N)) = \mathbb{Q}$ and morphisms are the same. \square

In the rest of this section, we are going to prove our theorem in several steps. The main step is the following remark:

Lemma 2.1.9. *Let X be a normal variety and \tilde{R} a functor on Var as in the theorem. Let $\tilde{R}'(X) = \varinjlim \tilde{R}(X')^{\text{Aut}(X'/X)}$ where the X' run through the category of normal X -schemes which are finite surjective over X with generically Galois covering map. Then \tilde{R}' is a contravariant functor on normal varieties and also covariant for finite surjective maps between them. The morphism $\tilde{R}(X) \rightarrow \tilde{R}'(X)$ is a quasi-isomorphism.*

Proof. As transition maps in the system we use only finite surjective X -morphisms which are generically Galois. The system is well-defined. By assumption $\tilde{R}(X) \rightarrow \tilde{R}(X')^{\text{Aut}(X'/X)}$ is a quasi-isomorphism. By exactness of direct limits in \mathcal{A} this remains true in the direct limit. We first check contravariant functoriality. Let $X \rightarrow Y$ be an arbitrary morphism of normal varieties and Y' a covering of Y in our direct category. Then the normalization X' of $Y' \times_Y X$ is also finite and surjective over X . All irreducible components are generically Galois by the lemma below. We thus get a morphism of direct systems. For covariant functoriality, consider finite surjective morphism $\pi : X \rightarrow Y$ and a covering X' of X in the direct system. There is another such covering of X' which is also generically Galois over Y . Let π_* be the projection

$$\tilde{R}(X') \rightarrow \tilde{R}(X')^{\text{Aut}(X'/Y)} .$$

It induces a map of direct systems and hence

$$\pi_* : \tilde{R}'(X) \rightarrow \tilde{R}'(Y) .$$

□

Note that in this normalization the composition

$$\tilde{R}'(Y) \xrightarrow{\pi^*} \tilde{R}'(X) \xrightarrow{\pi_*} \tilde{R}'(Y)$$

is the identity.

Lemma 2.1.10. *Let Y, Y' be normal irreducible varieties in characteristic zero and let $\pi : Y' \rightarrow Y$ be finite surjective and generically Galois with covering group G . Let*

$$f : \Gamma \rightarrow Y$$

be a morphism. Then G operates transitively on the fibres of

$$\Gamma \times_Y Y' \rightarrow \Gamma .$$

Let C be the reduction of an irreducible component of $\Gamma \times_Y Y'$. Then $C \rightarrow \Gamma$ is surjective and generically Galois. The Galois group is the quotient of the stabilizer of C in G by the fixgroup of C .

Proof. First we assume that $\Gamma = \text{Spec } L$ is a (not necessarily closed) point of Y . We can assume that $Y = \text{Spec } A, Y' = \text{Spec } B$ are affine. Now the assertion is easy to check directly using the fact that $B^G = A$ and that everything is \mathbb{Q} -linear. More generally let $\Gamma = \text{Spec } L$ where L is a field. The reduction of $\Gamma \times_Y Y' = \Gamma \times_{f(\Gamma)} \pi^{-1}(f(\Gamma))$ is given by $\Gamma \times_{f(\Gamma)} \pi^{-1}(f(\Gamma))^{\text{red}}$. Hence the assertion follows from the special case.

Clearly the map $\Gamma \times_Y Y' \rightarrow \Gamma$ is surjective. G stabilizes the subvariety of components which dominate Γ . Moreover the operation of G is transitive on all fibres, hence the closure of the fibre over the generic point is everything. □

Proof. (of Theorem) We only have to extend \tilde{R}'' of the lemma to the category of smooth correspondences. The method is the same as in the proof of 2.1.3. Let X and Y be smooth connected varieties. Let Γ in $X \times Y$ be an irreducible subvariety which is finite and surjective over X . Let $\tilde{\Gamma}$ be its normalization. Consider the diagram

$$X \leftarrow \tilde{\Gamma} \rightarrow Y .$$

It induces a map

$$\tilde{R}''(Y) \rightarrow \tilde{R}''(\tilde{\Gamma}) \rightarrow \tilde{R}''(X)$$

by composition of contravariant and covariant functoriality. Let the morphism $R(\Gamma)$ be this composition times the degree of Γ over Y . If Γ is the graph of a morphism f , then $\tilde{R}''(f) = R(\Gamma)$ in $D^+(\mathcal{A})$.

We claim that R is a functor. Let $\Gamma_1 \subset X \times Y$ and $\Gamma_2 \subset Y \times Z$ be prime correspondences. Then

$$(\Gamma_1 \times Z).(X \times \Gamma_2) = \sum n_i C_i$$

is a cycle in $X \times Y \times Z$. The C_i are the irreducible components of $\Gamma_1 \times_Y \Gamma_2$. Let pr be the projection map to $X \times Z$. The prime correspondence C_i is finite and dominant over its image $pr(C_i)$. Let d_i be the degree of this covering. The composition of correspondences is given by the cycle

$$\Gamma_2 \circ \Gamma_1 = \sum n_i d_i pr(C_i).$$

Let $\widetilde{\Gamma_1 \cdot \Gamma_2} = \sum n_i \tilde{C}_i$ be the normalization of $\Gamma_1 \cdot \Gamma_2$. The morphism

$$R(\Gamma_2 \circ \Gamma_1) : R(Z) \rightarrow R(\widetilde{\Gamma_2 \circ \Gamma_1}) \rightarrow R(X)$$

can equivalently be computed via $R(\widetilde{\Gamma_1 \cdot \Gamma_2})$. Note that the multiplicities work out as they have to.

The intersection multiplicities n_i are equal to the ramification index of $X \times \Gamma_2 \rightarrow X \times Y$ at C_i (see [Fu] 4.3.7) by loc. cit. 7.1.15 and because Z is flat over k . Let Γ'_2 be normal variety, finite and surjective over Γ_2 which is Galois over Y . We put

$$\Gamma_1.(X \times \Gamma'_2) = \sum n'_{ij} C'_{ij}$$

where the C'_{ij} are the irreducible components of $\Gamma_1 \times_Y \Gamma'_2$ covering C_i and the n'_{ij} are the ramification indices of $X \times \Gamma'_2 \rightarrow X \times Y$. By the last lemma the covering of Γ_1 is generically Galois, hence all degrees $d(C'_{ij}/\Gamma_1) = d'$ agree. As Y is smooth all ramification indices are equal to

$$e' = \frac{d(\Gamma'_2/Y)}{d' \cdot \#\{C'_{ij}\}}$$

by the degree formula loc. cit. 4.3.7. By the same degree formula, we can replace $R(\widetilde{\Gamma_1 \cdot \Gamma_2})$ by $R(\Gamma_1 \cdot X \times \Gamma'_2)$ in the computation of our morphism.

Finally we have to show that the diagram

$$\begin{array}{ccc}
\tilde{R}''(\coprod \tilde{C}'_{ij}) & \longleftarrow & \tilde{R}''(\tilde{\Gamma}'_2) \\
e' d' \tilde{R}''(\cdot) \downarrow & & \downarrow d(\Gamma'_2/Y) \tilde{R}''(\cdot) \\
\tilde{R}''(\tilde{\Gamma}'_1) & \longleftarrow & \tilde{R}''(Y)
\end{array}$$

commutes. For this we have to go back to the definition of \tilde{R}'' . We use the original contravariant functoriality and then project to the invariants under G . We have already seen that the multiplicities fit. The rest of the theorem follows from 2.1.2. \square

2.2 Review of mixed realizations

We review the basic notions of [Hu1], i.e., define the category of mixed realizations and the surrounding triangulated category. Everything in this section is pure linear algebra. Let k be a field of characteristic zero which can be embedded into \mathbb{C} . Let S be the set of embeddings.

We first recall the definition of \mathcal{MR} . It is a slight modification of Jannsen's in [Ja]. It is equivalent to the notion of absolute Hodge motive which was independently given by Deligne [De2].

Definition 2.2.1 ([Hu1] 11.1.1). *An object A in the category of mixed realizations \mathcal{MR} is given by the following data:*

- a bifiltered k -vector space A_{DR} ;
- for each prime l a filtered \mathbb{Q}_l -vector space A_l with a continuous operation of G_k ;
- for each prime l and each $\sigma \in S$ a filtered \mathbb{Q}_l -vector space $A_{\sigma,l}$;
- for each $\sigma \in S$ a filtered \mathbb{Q} -vector space A_σ ;
- for each $\sigma \in S$ a filtered \mathbb{C} -vector space $A_{\sigma,\mathbb{C}}$;
- for each $\sigma \in S$ a filtered isomorphism

$$I_{\text{DR},\sigma} : A_{\text{DR}} \otimes_\sigma \mathbb{C} \longrightarrow A_{\sigma,\mathbb{C}} ;$$

- for each $\sigma \in S$ a filtered isomorphism

$$I_{\sigma,\mathbb{C}} : A_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow A_{\sigma,\mathbb{C}} ;$$

- for each $\sigma \in S$ and each prime l a filtered isomorphism

$$I_{\bar{\sigma},l} : A_{\sigma} \otimes_{\mathbb{Q}} \mathbb{Q}_l \longrightarrow A_{\sigma,l} ;$$

- for each prime l and each $\sigma \in S$ a filtered isomorphism

$$I_{l,\sigma} : A_l \otimes_{\mathbb{Q}} \mathbb{Q}_l \longrightarrow A_{\sigma,l} .$$

Additionally we require that the tuples $(A_{\sigma}, A_{\text{DR}}, A_{\sigma, \mathbb{C}}, I_{\text{DR}, \sigma}, I_{\sigma, \mathbb{C}})$ give Hodge structures ([Hu1] 8.1.1) and that the A_l are constructible Galois modules equipped with the filtrations by weights ([Hu1] 9.1.4).

Morphisms of mixed realizations are morphisms of this data compatible with the comparison isomorphisms.

\mathcal{MR} is an abelian category because morphisms are automatically strictly compatible with all filtrations. Kernels and cokernels are computed componentwise. Recall that a morphism between filtered objects is called strict if coimage and image are isomorphic as filtered objects.

Now we need to recall the definition of the category $D_{\mathcal{MR}}$, cf. [Hu1] 11.1.3. It should be thought of as the derived category of \mathcal{MR} .

Definition 2.2.2. Let C^+ be the category with objects given by a tuple of complexes in the additive categories in the definition of \mathcal{MR} plus filtered quasi-isomorphisms between them. Let $C_{\mathcal{MR}}$ be the subcategory of complexes with strict differentials whose cohomology objects are in \mathcal{MR} . Let $D_{\mathcal{MR}}$ be the localization of the homotopy category of $C_{\mathcal{MR}}$ (see [Hu1] 4.1.5) with respect to the class of quasi-isomorphisms (see [Hu1] 4.1.7).

Lemma 2.2.3. Morphisms of objects in $C_{\mathcal{MR}}$ induce strict morphisms on cohomology. In particular, the category is abelian. $D_{\mathcal{MR}}$ is a triangulated category with t -structure whose heart is \mathcal{MR} .

Proof. The first assertion holds because morphisms in \mathcal{MR} are automatically strict. The second is [Hu1] 11.1.4. \square

We need two lemmas which were implicit in [Hu1] but not stated.

Lemma 2.2.4. Let K_i for $i \in I$ be a direct system of complexes in $C_{\mathcal{MR}}$. Assume that all direct limits $\varinjlim H^k(K_i)$ exist in \mathcal{MR} . Then $\varinjlim K_i$ exists in $C_{\mathcal{MR}}$.

Proof. Direct limits exist in C^+ . The direct limit functor is exact, hence strictness of differentials is preserved and cohomology commutes with the functor. The second condition on objects in $C_{\mathcal{MR}}$ holds by assumption. \square

A direct system where all transition maps are quasi-isomorphisms is a special case of a direct system to which the lemma applies.

Lemma 2.2.5. *Let K^* be a bounded below complex of objects in $C_{\mathcal{MR}}$ which are concentrated in positive degrees. Then the total complex $\text{Tot } K^*$ is in $C_{\mathcal{MR}}$.*

Proof. We can take the total complex in C^+ as usual. Strictness of differentials of the total complex is [Hu1] 3.1.8. Clearly the cohomology of the total complex is obtained from the cohomology of the K^i via a spectral sequence. The boundedness conditions ensure that only finitely many cohomology objects of the K^i contribute to one cohomology object of the total complex. In particular all vector spaces involved in the definition are indeed finite dimensional. We have to check that the filtration on the Galois modules is the filtration by weights, i.e. that Gr_W^i is pure of weight i . We pass to graded pieces in the spectral sequence. This is possible because the differentials are strict. Note that purity of weight i is stable under subquotients and extensions. For the Hodge condition we also pass to the weight graded piece. We now have to check the condition of a pure Hodge structure. Again the condition is stable under extensions. \square

Proposition 2.2.6. *The category $D_{\mathcal{MR}}$ is pseudo-abelian.*

Proof. This is not a special case of Levine's result in [Le2] A.5. However, his proof can be modified so that it works in our case. Let C be an object of $D_{\mathcal{MR}}$ and $p : C \rightarrow C$ an idempotent, i.e. $p^2 = p$. p is represented by a morphism of complexes $p^{(1)} : C \rightarrow C^{(1)}$ where $C^{(1)}$ is quasi-isomorphic to C via $c^{(1)}$. Choose $\tilde{C}^{(1)}$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{p^{(1)}} & C^{(1)} \\ c^{(1)} \downarrow & & \downarrow \tilde{c}^{(1)} \\ C^{(1)} & \xrightarrow{\tilde{p}^{(1)}} & \tilde{C}^{(1)} \end{array}$$

commutes. The equation $p^2 = p$ implies that there is a quasi-isomorphism $\tilde{C}^{(1)} \rightarrow C^{(2)}$ such that $\tilde{p}^{(1)} \circ p^{(1)}$ and $\tilde{c}^{(1)} \circ p^{(1)}$ become homotopic. In fact, there is a whole chain of morphisms of complexes

$$p^{(n)} : C^{(n-1)} \rightarrow C^{(n)}$$

where all $C^{(n)}$ are quasi-isomorphic to C (fix the quasi-isomorphisms $c^{(n)}$ once and for all), $p^{(n)}$ represents p and $p^{(n+1)} \circ p^{(n)}$ and $\tilde{c}^{(n)} \circ p^{(n)}$ are

homotopic. By 2.2.4 the limit over $C^{(n)}$ with transition maps the quasi-isomorphisms $c^{(n)}$ exists in $C_{\mathcal{MR}}$. Now p is represented by the induced morphism of complexes

$$\varinjlim p^{(n)} : \varinjlim C^{(n)} \rightarrow \varinjlim C^{(n)}$$

Replace C by the limit. By this procedure we have succeeded in representing p by an endomorphism of a complex such that the identity $p^2 = p$ holds up to homotopy of complexes. From now we can argue precisely as Levine in loc. cit. Theorem A.5.3. There is one little change, however, because the maps $f(p)$ and $f(\text{id})$ (notation of loc.cit.) are not homotopy equivalences but only quasi-isomorphisms. This suffices for the argument. \square

2.3 The mixed realization functor

We proceed by constructing a realization functor from Voevodsky's geometrical motives to mixed realizations.

One of the main results of [Hu1] was the following:

Theorem 2.3.1 (loc. cit. 11.2). *There is a contravariant functor*

$$\tilde{R}_{\mathcal{MR}} : \text{Sm} \rightarrow C_{\mathcal{MR}}$$

whose cohomology objects compute the mixed realizations of a smooth variety. Composed with the natural projections to the category of Galois modules or to the category of Hodge complexes it computes the l -adic realizations respectively the Hodge realization of a variety.

$\tilde{R}_{\mathcal{MR}}$ has descent for proper hypercovers, hence the functor extends as in 2.1.4 (cf. loc. cit. 11.2.2) to all varieties.

Definition 2.3.2 (loc. cit. 11.3.1). *Let*

$$H_{\mathcal{MR}}^i(X, n) = \text{Hom}_{D_{\mathcal{MR}}}(\mathbb{Q}(-n), R_{\mathcal{MR}}(X)[i])$$

be the absolute mixed realization cohomology.

As shown in loc. cit. part III this is part of a Bloch-Ogus cohomology theory. By functoriality, there is a map to absolute Hodge cohomology and to continuous l -adic cohomology. Note, however, that giving an element in absolute realization cohomology is stronger than giving elements in these standard cohomologies. This is parallel to the fact that giving a mixed realization is stronger than giving a mixed Hodge structure and various Galois-modules - we also fix comparison isomorphisms.

We immediately get:

Theorem 2.3.3. *Let k be a field which is embeddable into \mathbb{C} and $A = \mathbb{Z}, \mathbb{Q}$. $\tilde{R}_{\mathcal{M}\mathcal{R}}$ extends to contravariant functors*

$$\begin{aligned} R_{\mathcal{M}\mathcal{R}} : D\mathcal{M}_{gm}(k, A) &\rightarrow D_{\mathcal{M}\mathcal{R}} , \\ R_{\mathcal{M}\mathcal{R}} : D\mathcal{M}_{gm}^{eff}(k, A) &\rightarrow D_{\mathcal{M}\mathcal{R}} . \end{aligned}$$

It maps the Tate motive $A(n)$ to $\mathbb{Q}(-n)$ (cf. the remark after 1.1.7). In particular, it induces a transformation of functors

$$H_{\mathcal{M}}^i(X, A(n)) \longrightarrow H_{\mathcal{M}\mathcal{R}}^i(X, n)$$

which is compatible with all structures (products, localization sequences etc.).

Proof. We repeat the proof of section 2.1. $C_{\mathcal{M}\mathcal{R}}$ itself is not a category of complexes over an abelian category and certainly arbitrary direct limits do not exist. However, the constructions of theorem 2.1.6 go through by lemma 2.2.4 and 2.2.5. We only have to check that $\tilde{R}_{\mathcal{M}\mathcal{R}}$ satisfies the conditions of theorem 2.1.6. All of them can be checked in the singular component. Hence they hold by proposition 2.1.7. The realization of the Tate motive can be computed in $D_{\mathcal{M}\mathcal{R}}$, e.g. as the decomposition of \mathbb{P}^1 in the category of Chow motives ([Hu1] 20.2.1). The transformation of functors is nothing but functoriality. \square

Remark: The same theorem (with the same proof) also holds for the more refined functor $R_{\mathcal{M}\mathcal{R}^P}$ with values in the category $D_{\mathcal{M}\mathcal{R}^P}$ which takes into account the polarizability of the graded pieces with respect to the weight filtration (see [Hu1] Ch. 21.)

By functoriality (or using the same arguments again), the theorem also implies the existence of other realization functors.

Corollary 2.3.4. *Let D_l be the “derived category” of constructible \mathbb{Q}_l -sheaves on $\text{Spec}(k)$ in [Ek] (or in the number field case the refined version in [Hu2]). Then there is a realization functor*

$$D\mathcal{M}_{gm}(k, A) \longrightarrow D_l .$$

It induces a transformation of functors

$$H_{\mathcal{M}}^i(X, A(n)) \longrightarrow H_{cont}^i(X, \mathbb{Q}_l(n))$$

where H_{cont}^i is continuous étale cohomology respectively the horizontal version of [Hu2].

Corollary 2.3.5. *Let $D_{\mathcal{H}}$ be the category of Hodge complexes as in [Be1] 3.2 or [Hu1] 8.1.5. Then there is a realization functor*

$$DM_{gm}(\mathbb{C}, A) \longrightarrow D_{\mathcal{H}}$$

It induces a transformation of functors

$$H_{\mathcal{M}}^i(X, A(n)) \longrightarrow H_{\mathcal{H}}^i(X, \mathbb{Q}_l(n))$$

where $H_{\mathcal{H}}^i$ is absolute Hodge cohomology as introduced by Beilinson in [Be1].

Beilinson's category of Hodge complexes differs from Deligne's by décalage of the weight filtration. Note also that absolute Hodge cohomology agrees with Deligne cohomology in the good range of indices, see [Be1] 5.7.

Other regulators which we get from this are to De Rham cohomology, singular cohomology again, and geometric étale cohomology. This is certainly not a surprise. The existence of such functors is already stated in [Vo2].

Recall ([Hu1] 22.1.3) that an object of \mathcal{MR} is called *motivic* if it is subquotient of an object $H^i(R_{\mathcal{MR}}(X_*))$ where X_* is a complex of varieties with morphisms formal \mathbb{Q} -linear combinations of morphisms of varieties.

Theorem 2.3.6. *Let X_* be an object of DM_{gm}^{eff} . Then $H^i(R_{\mathcal{MR}}(X_*))$ is motivic.*

Proof. As $H^i(R_{\mathcal{MR}}(X_*))$ only depends on $\sigma_{-i-1}X_*$ we can as well assume $X_* \in DM_{gm}^{eff}$. First consider the special case of a complex of length one, i.e. $X_* = [X_0 \xrightarrow{f} X_1]$. The morphism f in SmCor is finite linear combination $f = \sum \alpha_i f_i$ with $\alpha_i \in \mathbb{Q}$ and f_i a primitive finite correspondence. Recall that it is finite over a connected component of X_0 . Let $Y_i = \text{supp}(f_i)$ and \tilde{Y}_i a normal finite cover of it. It is finite surjective over a connected component of X_0 . We assume that this cover is generically Galois. Let X'_0 be the union of those connected components of X_0 which are not covered by any Y_i . Let

$$\tilde{X}_0 = X'_0 \amalg \coprod \tilde{Y}_i .$$

Let f'_i be the projection map $\tilde{Y}_i \rightarrow X_1$ and $\tilde{f} = \sum \alpha_i f'_i$. Now we put

$$\tilde{X}_* = [\tilde{X}_0 \xrightarrow{\tilde{f}} X_1] .$$

By construction of the realization functor for correspondences the diagram

$$\begin{array}{ccc} \tilde{R}_{\mathcal{M}\mathcal{R}}(\tilde{X}_0) & \xleftarrow{\tilde{R}_{\mathcal{M}\mathcal{R}}(f)} & \tilde{R}_{\mathcal{M}\mathcal{R}}(X_1) \\ \uparrow \left\{ \right. & & \uparrow \left\{ = \right. \\ \tilde{R}_{\mathcal{M}\mathcal{R}}(X_0) & \xleftarrow{\tilde{R}_{\mathcal{M}\mathcal{R}}(\tilde{f})} & \tilde{R}_{\mathcal{M}\mathcal{R}}(X_1) \end{array}$$

commutes and the left vertical map has a compatible splitting. Hence $\tilde{R}_{\mathcal{M}\mathcal{R}}(X_*)$ is a direct summand of $R_{\mathcal{M}\mathcal{R}}(\tilde{X}_*)$. Clearly cohomology of $R_{\mathcal{M}\mathcal{R}}(X_*)$ is a direct summand of the cohomology of $R_{\mathcal{M}\mathcal{R}}(\tilde{X}_*)$.

Now we have to extend this to longer complexes. We do this inductively. Assume X_* is a complex in degrees $-n$ to k with X_i a general variety for $i < 0$ and X_i a smooth variety for $i \geq 0$. We assume that the boundaries in negative degrees are linear combinations of morphisms of varieties and the boundaries in positive degrees are finite correspondences. It can be considered as an object in DM_- because the functor M on SmCor is also defined on Var . Apply the previous construction to $X_0 \rightarrow X_1$. This yields a finite covering \tilde{X}_0 of X_0 . We have to construct \tilde{X}_k for $k < 0$. Let

$$\{f_i : X_p \rightarrow X_0\}$$

be the set of morphisms which occur as compositions of the morphisms of varieties making up the complex X_* . Let

$$\tilde{X}_p = \coprod f_i^* \tilde{X}_0$$

where

$$f_i^* \tilde{X}_0 = (X_p \times_{X_0, f_i} \tilde{X}_0)^{\text{red}} .$$

By lemma 2.1.10 its components are surjective over X_p . The covering group operates transitively on all fibres.

We have to define boundary morphisms in \tilde{X}_p such that

$$\tilde{X}_* \rightarrow X_*$$

is a morphism of complexes for $* \leq 0$. For simplicity, assume that X_p is connected, let $g : X_p \rightarrow X_{p+1}$ one of the morphisms occurring in the boundary. For each of the morphisms $f_i : X_{p+1} \rightarrow X_0$, we lift g to $g(f_i) : (gf_i)^* \tilde{X}_0 \rightarrow f_i^* \tilde{X}_0$. The coefficient of $g(f_i)$ is taken as the coefficient of g . It is easy to see that this yields indeed a complex. The crucial diagram in degrees 0 and 1 is

treated as in the in the first case. At least after application of $R_{\mathcal{MR}}(X_*)$ it commutes and has a splitting. $R_{\mathcal{MR}}(X_*)$ is a direct summand of $R_{\mathcal{MR}}(\tilde{X}_*)$ because the morphism between them is surjective and finite. The splitting is constructed by projecting to the invariants under the covering group. The components of \tilde{X}_* are not normal so this is not an application of our axioms for the existence of a realization functor. But still the same proof works as in the normal case. \square

Remark: It is easy to see that the category of motivic objects in \mathcal{MR} remains unchanged when we restrict to \mathbb{Z} -linear combination of morphisms rather than \mathbb{Q} -linear ones. Replacing a singular variety by a smooth proper hypercovering, it is easy to see that we can assume X_* to be a complex of smooth varieties in the definition of motivic objects. However, it is not clear at all whether it might suffice to assume that X is a smooth variety rather than a complex.

3 The motive of BGL

The main result of this chapter is corollary 3.2.5 where we completely determine the motive of the classifying space of GL. We work over the base \mathbb{Q} . By base change the results follows over any base field of characteristic zero. As before let DM_- be Voevodsky's category of motivic complexes with rational coefficients. In the next chapter the result will be used in order to give a very easy construction of Chern classes in motivic cohomology and check their relation with Chern classes in mixed realization cohomology.

3.1 Set-up

Let G be a connected algebraic group (we only need $G = \mathrm{GL}_n$ and $G = \mathbb{G}_m^n$.) We define the simplicial variety

$$\begin{array}{ccccccc}
 & & & & & & \leftarrow \\
 & & & & & & \leftarrow \\
 EG = G & \leftarrow & G \times G & \leftarrow & G \times G \times G & \leftarrow & \leftarrow \\
 & \leftarrow & & \leftarrow & & \leftarrow & \\
 & & & & & & \leftarrow
 \end{array}$$

where the face morphisms are induced by the various projections and the degeneracy maps by the section e . Note that EG is contractible. EG is a homogeneous space under G with the diagonal action. Put

$$BG = EG/G .$$

We identify $B_i G = G^i$. This corresponds to the classical construction of the classifying space as quotient of the universal cover. EG and BG are obviously functorial.

We need to understand the motive of \mathbb{G}_m . Let $e : \mathbb{Z}(0) \rightarrow \mathbb{G}_m$ the unit section. The multiplication map is denoted μ .

Lemma 3.1.1. *The section e induces a decomposition of \mathbb{G}_m into*

$$M(\mathbb{G}_m) \cong \mathbb{Z}(0) \oplus M(\mathbb{G}_m)^\sim$$

in $DM_{gm}(\mathbb{Q}, \mathbb{Z})$. $M(\mathbb{G}_m)^\sim$ is isomorphic to $\mathbb{Z}(1)[1]$ via residue at 0. For the multiplication map we have

$$\mathrm{Gr}_w(\mu) = \begin{cases} \mathrm{id} & w = 0, \\ \Delta & w = 1, \\ 0 & w \neq 0, 1. \end{cases}$$

Proof. Recall that by definition

$$M(\mathbb{A}^1) = \mathbb{Z}(0) .$$

Now consider the localization triangle for the smooth pair $(\mathbb{G}_m, \mathbb{A}^1)$:

$$M(\mathbb{G}_m) \longrightarrow M(\mathbb{A}^1) \longrightarrow M(0)(1)[2] ,$$

i.e.

$$\mathbb{Z}(0) \oplus M(\mathbb{G}_m)^\sim \longrightarrow \mathbb{Z}(0) \longrightarrow \mathbb{Z}(1)[2] .$$

The last map is an element of $H^{-2}(\mathbb{Z}(1)(\mathbb{Q})) = 0$. Hence $M(\mathbb{G}_m)^\sim$ is isomorphic to $\mathbb{Z}(1)[1]$. The decomposition of the multiplication map uses the commutative diagram for the properties of a left and right unit. \square

In particular \mathbb{G}_m is a mixed Tate motive. The same is true for all GL_n . This can be seen by using the stratification given by the Bruhat decomposition. All strata are of the form split torus times some affine space. From now on we work in the category of mixed Tate motives $DM\mathcal{T}$ introduced in 1.2.1.

3.2 Motives of some classifying spaces

Proposition 3.2.1. *There is a unique morphism*

$$b^i : \mathbb{Q}(i)[2i] \rightarrow M(B\mathbb{G}_m)$$

induced by

$$\mathbb{Q}(i)[2i] \cong (M(\mathbb{G}_m)^\sim)^{\otimes i} [i] \rightarrow M(B_i\mathbb{G}_m)[i]$$

Moreover,

$$\beta = \bigoplus b^i : \bigoplus \mathbb{Q}(i)[2i] \rightarrow M(B\mathbb{G}_m)$$

is an isomorphism.

Proof. Consider the exact triangle

$$M(\sigma_{-i+1}B\mathbb{G}_m) \rightarrow M(B\mathbb{G}_m) \rightarrow M(\rightarrow B_{i+1}\mathbb{G}_m \rightarrow B_i\mathbb{G}_m \rightarrow 0)$$

Clearly we have a map b^i to the space on the right. $M(\sigma_{-i+1}B\mathbb{G}_m)$ is an object of $D\mathcal{MT}_{[0,i-1]}$. Hence the composition of b^i with the connecting morphism vanishes by lemma 1.2.2. Then b^i lifts to a map to $M(B\mathbb{G}_m)$. Using the same argument again, we see that the lift is unique. Now we are precisely in the situation of proposition 1.3.6. It is enough to pass to the weight graded pieces of the subcomplexes $M(\sigma_{-N}B\mathbb{G}_m)$. The decomposition of $\mathrm{Gr}_w(\mu)$ is known. It determines all differentials. To compute its cohomology is a completely combinatorial question. Instead of considering the combinatorics, we can also quote the result of the computation in the Hodge realization, e.g. [Hul] 17.4.1 for $n = 1$. Either way we see that $\mathrm{Gr}_w(\beta)$ is injective and that the cokernel is a subobject of $M(B_N\mathbb{G}_m)[N]$. By 1.3.6 β is an isomorphism. \square

Definition 3.2.2. *Let m_1, \dots, m_k be simple Tate motives of the form $\mathbb{Q}(i)[2i]$. By the polynomial ring in m_1, \dots, m_k we mean the motive*

$$\mathbb{Q}[m_1, \dots, m_k] := \bigoplus_{e_1, \dots, e_k \geq 0} m_1^{\otimes e_1} \otimes \dots \otimes m_k^{\otimes e_k} .$$

It is not correct to view $\mathbb{Q}[m_1, \dots, m_k]$ as a ring. There is no multiplication but rather a comultiplication induced by the diagonal. If we apply the singular cohomology functor to it, we get a true polynomial ring in k generators.

Remark: $M(B\mathbb{G}_m)$ is the polynomial ring in the generator $b = \mathrm{Im} b^i$. The notation is consistent: the image of the map b^i is the subobject b^i .

Corollary 3.2.3. *Let $T = \mathbb{G}_m^n$ be a split torus. Then $M(BT)$ is isomorphic to the polynomial ring in b_1, \dots, b_n where b_i corresponds to the generator of the motive of the i -th factor \mathbb{G}_m in T*

Proof. We have already on the simplicial level $BT \cong (B\mathbb{G}_m)^n$. Hence $M(BT)$ is isomorphic to the n -fold tensor power of $M(B\mathbb{G}_m)$. \square

Let $c_i : \mathbb{Q}(i)[2i] \rightarrow B(T)$ be the i -th symmetric polynomial in the generating maps b_k of $M(BT)$.

Theorem 3.2.4. *The object $M(B\mathrm{GL}_n)$ in $DM_{gm-}^{eff}(\mathbb{Q}, \mathbb{Q})$ is given by the commutative polynomial ring generated by $c_i = \mathbb{Q}(i)[2i]$ for $i \leq n$.*

Proof. We have defined a map

$$\gamma : \mathbb{Q}[c_1, \dots, c_n] \rightarrow BT \rightarrow B\mathrm{GL}_n .$$

We claim that it is an isomorphism in DM_- . By proposition 1.3.6 it is enough to consider the weight graded pieces of the finite subcomplexes. Moreover, the singular realization is faithful on Tate motives of fixed weight. We know that the singular realization of γ is an isomorphism (e.g. [Du] Theorem 6.13 and Proposition 8.3). On finite subcomplexes $R_{sing}(\sigma_{-N} B\mathrm{GL}_N)$, the map $R_{sing}(\gamma)$ is not an isomorphism but the defect is direct sum of Tate motives of the form $\mathbb{Q}(i)[j]$ with $j \geq N$ (because the spectral sequence is concentrated in the first quadrant). Hence the assumptions of 1.3.6 hold and γ is an isomorphism. \square

Remark: We only need existence of β in 3.2.1 for this proof and reproof that it is an isomorphism.

Corollary 3.2.5.

$$M(B\mathrm{GL}) = \mathbb{Q}[c_1, c_2, \dots] .$$

Corollary 3.2.6. *Application of $R_{\mathcal{MR}}$ yields*

$$R_{\mathcal{MR}}(B\mathrm{GL}) = \mathbb{Q}[c_1, c_2, \dots]$$

with $c_i = \mathbb{Q}(-i)[-2i]$. The splitting is the same as the one constructed in [Hu1] 17.4.1.

Proof. Recall that $R_{\mathcal{MR}}(\mathbb{Q}(i)[2i]) = \mathbb{Q}(-i)[-2i]$. Hence the equality follows from the previous corollary. The construction of the splitting of $B \text{GL}$ is very much the same as the construction used in [Hu1] 17.3-17.4. It is enough to show that the splitting of $R_{\mathcal{MR}}(B\mathbb{G}_m)$ constructed in [Hu1] 17.3.2 is the same as ours. Note that we only have to check that the splittings agree in the l -adic realization because the splitting of the Hodge realization is unique anyway. The one in loc. cit. is induced by the Chern class of the standard line bundle on \mathbb{P}^n , ours by the cycle class of a point. That they agree is classic. \square

4 Chern classes

The aim of this chapter is to show that the higher Chern classes from higher algebraic K -theory to absolute cohomology of mixed realizations (see [Hu1]) factor over Voevodsky's motivic cohomology.

4.1 K -theory and group cohomology of $\text{GL}(X)$

We start with a review of the results in [Hu1] 18.1-18.2 in a more conceptual terminology. In this section all schemes are noetherian and regular, e.g. smooth varieties over k . We denote $\mathbf{K}(X)$ a simplicial set whose homotopy groups are the K -groups of X .

Definition 4.1.1 ([Hu1] 18.1.1). *Let U_* be a simplicial affine scheme. Assume that U_* has finite combinatorial dimension, i.e., is degenerate above some simplicial degree. Then we define*

$$\mathbf{K}(U_*) = \text{holim } \mathbf{K}(U_i) .$$

If U_* is the nerve of an open cover of X , then

$$\mathbf{K}(X) \rightarrow \mathbf{K}(U_*)$$

is a weak equivalence by the Mayer-Vietoris property of K -theory of regular schemes. In the affine case $\mathbf{K}(U)$ can be realized as $K_0(X) \times \mathbb{Z}_\infty(B \text{GL}(U))$. More generally:

Proposition 4.1.2 (Thomason, [Hu1] 18.1.5).

$$\mathbf{K}(X) \cong \varinjlim \text{Tot } \mathbb{Z} \times \mathbb{Z}_\infty(B \text{GL}(U_*))$$

where the direct limit runs through all open covers of X . In particular

$$\begin{aligned} K_0(X) &= \mathbb{Z} \oplus \varinjlim \pi_0 \operatorname{Tot} \mathbb{Z}_\infty(B \operatorname{GL}(U_*)) \\ K_i(X) &= \varinjlim \pi_i \operatorname{Tot} \mathbb{Z}_\infty(B \operatorname{GL}(U_*)) \text{ for } i \geq 1. \end{aligned}$$

Proof. The weak equivalence follows from the formula in loc. cit. because direct limits commute with homotopy groups. The explicit calculation follows from it by the spectral sequence for the total space of a simplicial space. It converges because all U_* have finite combinatorial dimension. \square

The simplicial set $\varinjlim \operatorname{Tot} \mathbb{Z}_\infty(B \operatorname{GL}(U_*))$ inherits an H -group structure from the H -group structure on $\mathbb{Z}_\infty(B \operatorname{GL}(U_i))$.

Definition 4.1.3. *Let X be a regular noetherian scheme. We put*

$$H_{\operatorname{MV}}^p(\operatorname{GL}(X), \mathbb{Q}) := H^p(|\varinjlim \operatorname{Tot} \mathbb{Z}_\infty(B \operatorname{GL}(U_*))|, \mathbb{Q})$$

where the right hand side means singular cohomology of the geometric realization. It is called Mayer Vietoris localized group cohomology of $\operatorname{GL}(X)$.

A simpler construction of the same cohomology group will be given below. Note also that in the case of $X = \operatorname{Spec} A$ this is *not* group cohomology of $\operatorname{GL}(A)$ but rather a version such that a long exact Mayer-Vietoris sequence for open covers is forced. From the definition, however, the relation to K -theory is clear:

Proposition 4.1.4. *There is a natural map*

$$K_p(X)_{\mathbb{Q}} \rightarrow H_{\operatorname{MV}}^p(\operatorname{GL}(X), \mathbb{Q}) .$$

Its image is the subgroup of primitive elements in $\bigoplus H_{\operatorname{MV}}^(\operatorname{GL}(X), \mathbb{Q})$.*

Proof. The map is nothing but the Hurewicz map from homotopy groups of a space to its cohomology. In the case of an H -space the image in rational cohomology is given by the primitive part, cf. [Lo] A.11. \square

Now we turn to the promised simpler description of our group cohomology. For a set B , we denote by $\mathbb{Q}[B]$ the \mathbb{Q} vector space with basis B . Let $U = \operatorname{Spec} A$ be an affine scheme. As in the proof of [Hu1] 18.2.4, the maps

$$\mathbb{Z}_\infty B \operatorname{GL}(U) \rightarrow \mathbb{Z} \mathbb{Z}_\infty B \operatorname{GL}(U) \leftarrow \mathbb{Z} B \operatorname{GL}(U)$$

induce isomorphisms on singular cohomology. Hence

$$H^p(|\mathbb{Z}_\infty(B \operatorname{GL}(U))|, \mathbb{Q}) = H^p(|B \operatorname{GL}(U)|, \mathbb{Q}) = H^p(\operatorname{GL}(U), \mathbb{Q})$$

is group cohomology in the usual sense. It is computed by the standard Bar complex (and this is in fact how the last equality is proved):

$$H^p(\mathrm{GL}(U), \mathbb{Q}) = H^{-p}(\mathbb{Q}[B_{-*} \mathrm{GL}(U)]) .$$

(We stick to our convention: all complexes are cohomological ones. A simplicial group is turned into a complex by putting it into negative degrees.)

Proposition 4.1.5. *Let X as in definition 4.1.3. There is a natural isomorphism*

$$H_{\mathrm{MV}}^p(\mathrm{GL}(X), \mathbb{Q}) = \varinjlim H^{-p}(\mathrm{Tot} \mathbb{Q}[B_{-*} \mathrm{GL}(U_*)])$$

where the direct system runs through all open covers of X .

Proof. Clear from the above. Note that cohomology commutes with direct limits. \square

Rather than constructing Chern classes on higher K -groups, we will construct them on group cohomology of $\mathrm{GL}(X)$.

Remark: If X itself is affine, then the natural map

$$\mathbb{Z}_\infty B \mathrm{GL}(X) \rightarrow \mathrm{Tot} \mathbb{Z}_\infty B \mathrm{GL}(U_*)$$

induces an isomorphism on all higher homotopy groups but not on π_0 . The space on the left is connected, the one on the right is not in general. They are certainly not weakly equivalent. Hence there is no reason for it to induce an isomorphism on singular cohomology. This justifies the above remark that Mayer-Vietoris group cohomology is not the same thing as group cohomology.

4.2 Chern classes into motivic cohomology

The construction of Chern classes into motivic cohomology proceeds along the same lines as for absolute realization cohomology in [Hu1] 18.2.5. The key observation is that

$$B_n \mathrm{GL}(U) = \mathrm{Hom}_{\mathrm{Sm}}(U, B_n \mathrm{GL}).$$

Theorem 4.2.1. *Let X be a smooth variety over k , some fixed ground field of characteristic zero. There is a natural transformation*

$$H_{\mathrm{MV}}^p(\mathrm{GL}(X), \mathbb{Q}) \rightarrow \mathrm{Hom}_{\mathrm{DM}_-(k, \mathbb{Q})}(M(X)[p], M(B \mathrm{GL})) .$$

Proof. We use the description of proposition 4.1.5. Let $\mathbb{Q}\text{Sm}$ be the category of smooth varieties with morphisms formal \mathbb{Q} -linear combinations of morphisms of varieties. An element of

$$\varinjlim H^{-p}(\text{Tot } \mathbb{Q}[B_{-*} \text{GL}(U_*)])$$

is represented by a morphism of complexes in $\mathbb{Q}\text{Sm}$

$$U_*[p] \rightarrow B \text{GL} .$$

Note that the functor M from proposition 1.1.6 has values in the category of complexes of Nisnevich sheaves with transfers, not only in the derived category. By functoriality, it induces a map

$$M(U_*[p]) \rightarrow M(B \text{GL})$$

in $D\mathcal{M}_-$. The natural map $M(U_*) \rightarrow M(X)$ is a quasi-isomorphism. Hence we have constructed an element in $\text{Hom}_{D\mathcal{M}_-}(M(X)[p], M(B \text{GL}))$ as claimed. Note that it is well-defined: two representatives differ by a homotopy of morphisms of complexes. \square

Corollary 4.2.2. *There is a natural map*

$$\begin{aligned} K_p(X) &\rightarrow H_{\text{MV}}^p(\text{GL}(X), \mathbb{Q}) \\ &\rightarrow \text{Hom}_{D\mathcal{M}_-}(M(X)[p], M(B \text{GL})) \rightarrow \bigoplus H_{\mathcal{M}}^{2j-p}(X, \mathbb{Q}(j)) . \end{aligned}$$

For $(j, p) \neq (0, 0)$ let

$$c_j : K_p(X) \rightarrow H_{\mathcal{M}}^{2j-p}(X, \mathbb{Q}(j)) .$$

c_0 on $K_0(X)$ is given by the above composition plus the natural map

$$K_0(X) \xrightarrow{\text{deg}} \mathbb{Z} \rightarrow H_{\mathcal{M}}^0(X, \mathbb{Q}(0))$$

mapping 1 to the structural morphism. c_j is called j -th motivic Chern class.

Proof. Recall that by corollary 3.2.5

$$M(B \text{GL}) = \mathbb{Q}[c_1, c_2, \dots]$$

with $c_j = \mathbb{Q}(j)[2j]$. The map in the corollary is nothing but the composition of the transformation in the theorem with the natural projection. \square

Note that our transformation of functors maps primitive elements to primitive elements. Hence we do not lose anything by projecting to the primitive part of $M(BGL)$ in the corollary.

Corollary 4.2.3. *The Chern class*

$$c_j : K_p(X) \rightarrow H_{\mathcal{MR}}^{2j-p}(X, \mathbb{Q}(j))$$

constructed in [Hu1] 18.2.6 factors through the motivic Chern class.

Proof. The construction of Chern classes in loc. cit. is precisely the one above with $\tilde{R}_{\mathcal{MR}}$ replacing the functor M . The compatibility is a direct consequence of functoriality of our construction. \square

Remark: For simplicity we have restricted to the case of a smooth variety in the above. Everything works directly for bounded complexes of smooth varieties, e.g. smooth simplicial varieties with finite combinatorial dimension. As in [Hu1] 18.1.3 bounded above complex of smooth varieties can be treated as the direct limit of its truncations. Singular varieties or complexes of such can be replaced by a smooth proper hypercovering.

Erratum: We have to correct an inaccuracy in [Hu1] 18.1.5: The variety has to be assumed smooth. The mistake is that Mayer-Vietoris holds for K -theory of general varieties only if we allow negative K -groups. The group $\pi_{-1}A(U)$ (notation of loc. cit.) might not be zero. As a consequence the arguments in loc. cit. work directly only for smooth simplicial varieties. However, they extend to the general case again by replacing singular varieties by smooth proper hypercoverings.

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