Notes on Symplectic Analysis and Geometric Quantization

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Contents

1 General structures

1. Symplectic form, canonical transformations, and Poisson brackets

We shall first consider the formulation of theories in the symplectic language. The question of deriving this from an action formulation will be discussed later.

In the analytical formulation of classical physics, one starts with a phase space, i.e., a smooth even-dimensional manifold $\mathcal{M}$ endowed with a symplectic structure Ω. Ω is a differential two-form on $\mathcal{M}$ which is closed and nondegenerate. This means that $d\Omega = 0$, and further, for any vector field ξ on $\mathcal{M}$, if $i_\xi \Omega = 0$, then ξ must be zero. In local coordinates q^μ, on $\mathcal{M}$, we can write

$$\Omega \;=\; \frac{1}{2}\, \Omega_{\mu\nu} dq^\mu \wedge dq^\nu \tag{1}$$

The condition $d\Omega = 0$ becomes

$$\begin{aligned} d\Omega &= \frac{1}{2}\frac{\partial \Omega_{\mu\nu}}{\partial q^\alpha} dq^\alpha \wedge dq^\mu \wedge dq^\nu \\ &= \frac{1}{3!}\left[\frac{\partial \Omega_{\mu\nu}}{\partial q^\alpha} + \frac{\partial \Omega_{\alpha\mu}}{\partial q^\nu} + \frac{\partial \Omega_{\nu\alpha}}{\partial q^\mu}\right] dq^\alpha \wedge dq^\mu \wedge dq^\nu \\ &= 0 \end{aligned} \tag{2}$$

The interior contraction of Ω with a vector field $\xi = \xi^\mu (\partial/\partial q^\mu)$ is given by

$$i_\xi \Omega = \xi^\mu \Omega_{\mu\nu} dq^\nu \tag{3}$$

The vanishing of this is the condition $\xi^\mu \Omega_{\mu\nu} = 0$. Thus if Ω is degenerate, $\Omega_{\mu\nu}$, considered as a matrix, has a zero mode ξ^μ. Nondegeneracy of Ω is thus equivalent to the invertibility of $\Omega_{\mu\nu}$ as a matrix. When needed, we denote the inverse of $\Omega_{\mu\nu}$ by $\Omega^{\mu\nu}$, i.e.,

$$\Omega_{\mu\nu}\; \Omega^{\nu\alpha} = \delta_\mu^\alpha \tag{4}$$

(We will consider Ω's which are nondegenerate. If Ω has zero modes, one has to eliminate them by constraining the variables, or equivalently, one has to project Ω to a smaller space where there is no zero mode and use this smaller space for setting up the quantum theory. Such a situation occurs in gauge theories. In general, zero modes of Ω indicate the existence of gauge symmetries.)

With the structure Ω defined on it, $\mathcal{M}$ is a symplectic manifold.

Since Ω is closed, at least locally we can write

$$\Omega = d\mathcal{A} \tag{5}$$

The one-form so defined is called the canonical one-form or symplectic potential. There is an ambiguity in the definition of $\mathcal{A}$, since $\mathcal{A}$ and $\mathcal{A} + d\Lambda$ will give the same Ω for any function Λ on $\mathcal{M}$. As we shall see shortly, this corresponds to the freedom of canonical transformations.

There are two types of features associated with the topology of the phase space which are apparent at this stage. If the phase space $\mathcal{M}$ has nontrivial second cohomology, i.e., if $\mathcal{H}^2(\mathcal{M}) \neq 0$, then there are possible choices for Ω for which there is no globally defined potential. The action, as we shall see later, is related to the integral of $\mathcal{A}$, so if Ω belongs to a nontrivial cohomology class of $\mathcal{M}$, then the definition of the action requires auxiliary variables or dimensions. Such cases do occur in physics and correspond to the Wess-Zumino terms discussed in Chapter 17. They are intrinsically related to anomalies and also to central (and other) extensions of the algebra of observables.

Even when $\mathcal{H}^2(\mathcal{M}) = 0$, there can be topological problems in defining $\mathcal{A}$. If $\mathcal{H}^1(\mathcal{M}) \neq 0$, then there can be several choices for $\mathcal{A}$ which differ by elements of $\mathcal{H}^1(\mathcal{M})$. There are inequivalent $\mathcal{A}$'s for the same Ω. One can consider the integral of $\mathcal{A}$ around closed noncontractible curves on $\mathcal{M}$. The values of these integrals or holonomies will be important in the quantum theory as vacuum angles. The standard θ-vacuum of nonabelian gauge theories is an example. We take up these topological issues in more detail later.

Given the above-defined geometrical structure, transformations which preserve Ω are evidently special; these are called canonical transformations. In other words, a canonical transformation is a diffeomorphism of $\mathcal{M}$ which preserves Ω. Infinitesimally, canonical transformations are generated by vector fields ξ such that $L_\xi \Omega = 0$, where L_ξ denotes the Lie derivative with respect to ξ. This gives

$$\begin{aligned} L_\xi \Omega &\equiv (d\, i_\xi \;+\; i_\xi\, d)\, \Omega \\ &= d\, (i_\xi \Omega) \\ &= 0 \end{aligned} \tag{6}$$

where we have used the closure of Ω. For canonical transformations, $i_\xi\Omega$ is closed. If the first cohomology of $\mathcal{M}$ is trivial, we can write

$$i_\xi\Omega = -df \tag{7}$$

for some function f on $\mathcal{M}$. In other words, to every infinitesimal canonical transformation, we can associate a function on $\mathcal{M}$. If $\mathcal{H}^1(\mathcal{M}) \neq 0$, then there is the possibility that for some transformations ξ, the corresponding $i_\xi\Omega$ is a nontrivial element of $\mathcal{H}^1(\mathcal{M})$ and hence there is no globally defined function f for this transformation. As mentioned before this is related to the possibility of vacuum angles in the quantum theory. For the moment, we shall consider the case $\mathcal{H}^1(\mathcal{M}) = 0$. Notice that for every function f we can always associate a vector field by the correspondence

$$\xi^\mu = \Omega^{\mu\nu}\partial_\nu f \tag{8}$$

Thus when $\mathcal{H}^1(\mathcal{M}) = 0$ there is a one-to-one mapping between functions on $\mathcal{M}$ and vector fields corresponding to infinitesimal canonical transformations. A vector field corresponding to an infinitesimal canonical transformation is often referred to as a Hamiltonian vector field. The function f defined by (7) is called the generating function for the canonical transformation corresponding to the vector field.

Let $\xi,\ \eta$ be two Hamiltonian vector fields, so that $L_\xi\Omega = L_\eta\Omega = 0$ and let their generating functions be f and g, respectively. Since $L_\xi L_\eta - L_\eta L_\xi = L_{[\xi,\eta]}$, the Lie bracket of ξ and η so defined is also a Hamiltonian vector field. The Lie bracket of ξ and η is given in local coordinates by

$$[\xi,\eta]^\mu = \xi^\nu\partial_\nu\eta^\mu - \eta^\nu\partial_\nu\xi^\mu \tag{9}$$

We must therefore have a function corresponding to $[\xi,\eta]$. This is called the Poisson bracket of g and f and is denoted by $\{g,f\}$. (There is a minus sign in this correspondence; $\xi \leftrightarrow f,\ \eta \leftrightarrow g$ and $[\xi,\eta] \leftrightarrow \{g,f\}$.) We define the Poisson bracket as

$$\begin{aligned}\{f,g\} &= i_\xi i_\eta\Omega = \eta^\mu\xi^\nu\Omega_{\mu\nu}\\ &= -i_\xi dg = i_\eta df\\ &= \Omega^{\mu\nu}\partial_\mu f\partial_\nu g\end{aligned} \tag{10}$$

Because of the antisymmetry of $\Omega_{\mu\nu}$ we have the property

$$\{f,g\} = -\{g,f\} \tag{11}$$

From the definition of the Poisson bracket, we can write, using local coordinates,

$$
\begin{aligned}
2\,\partial_\alpha\{f,g\} &= \partial_\alpha(\eta\cdot\partial f-\xi\cdot\partial g)\\
&= \partial_\alpha\eta^\mu\partial_\mu f+\eta^\mu(\partial_\mu\partial_\alpha f)-\partial_\alpha\xi^\mu\partial_\mu g-\xi^\mu(\partial_\mu\partial_\alpha g)\\
&= \partial_\alpha\eta^\mu\partial_\mu f-\partial_\alpha\xi^\mu\partial_\mu g+\eta\cdot\partial(\xi^\mu\Omega_{\alpha\mu})-\xi\cdot\partial(\eta^\mu\Omega_{\alpha\mu})\\
&= \partial_\alpha\eta^\mu\partial_\mu f-\partial_\alpha\xi^\mu\partial_\mu g+(\xi\cdot\partial\eta-\eta\cdot\partial\xi)^\mu\Omega_{\mu\alpha}\\
&\qquad +\eta^\mu\xi^\nu(\partial_\mu\Omega_{\alpha\nu}+\partial_\nu\Omega_{\mu\alpha})\\
&= [\xi,\eta]^\mu\Omega_{\mu\alpha}+\partial_\alpha(\eta^\mu\xi^\nu\Omega_{\mu\nu})+\eta^\mu\xi^\nu(\partial_\mu\Omega_{\alpha\nu}+\partial_\nu\Omega_{\mu\alpha}+\partial_\alpha\Omega_{\nu\mu})\\
&= [\xi,\eta]^\mu\Omega_{\mu\alpha}+\partial_\alpha\{f,g\}+\eta^\mu\xi^\nu(\partial_\mu\Omega_{\alpha\nu}+\partial_\nu\Omega_{\mu\alpha}+\partial_\alpha\Omega_{\nu\mu})
\end{aligned}
\tag{12}
$$

In local coordinates, the closure of Ω is the statement $\partial_\mu\Omega_{\alpha\nu}+\partial_\nu\Omega_{\mu\alpha}+\partial_\alpha\Omega_{\nu\mu}=0$. We then see that

$$-d\{g,f\}=i_{[\xi,\eta]}\;\Omega \tag{13}$$

which shows the correspondence stated earlier.

Consider the change in a function F due to a canonical transformation generated by a Hamiltonian vector field ξ corresponding to the function f. This is given by the Lie derivative of F with respect to ξ. Thus

$$\delta F=\xi^\mu\partial_\mu F=\{F,f\} \tag{14}$$

The transformation is given by the Poisson bracket of F with the generating function f corresponding to ξ.

An important property of the Poisson bracket is the Jacobi identity for any three functions f,g,h,

$$\{f,\{g,h\}\}+\{h,\{f,g\}\}+\{g,\{h,f\}\}=0 \tag{15}$$

This can be verified by direct computation from the definition of the Poisson bracket. In fact, if ξ,η,ρ are the Hamiltonian vector fields corresponding to the functions f,g,h, then

$$\{f,\{g,h\}\}+\{h,\{f,g\}\}+\{g,\{h,f\}\}=-i_\xi i_\eta i_\rho d\Omega \tag{16}$$

and so the Jacobi identity follows from the closure of Ω. This result is also equivalent to (13, 14). Acting on a function F

$$
\begin{aligned}
(L_\xi L_\eta-L_\eta L_\xi)F &= \{\{F,g\},f\}-\{\{F,f\},g\}\\
&= -\{\{g,f\},F\}\\
&= L_{[\xi,\eta]}\;F
\end{aligned}
\tag{17}
$$

where we have used the Jacobi identity in the second step.

The change of the symplectic potential $\mathcal{A}$ under an infinitesimal canonical transformation can be worked out as

$$\begin{aligned} \delta\mathcal{A} &= L_\xi \mathcal{A} \\ &= d(i_\xi \mathcal{A} - f) \end{aligned} \tag{18}$$

where we used the definition $L_\xi = (d\ i_\xi\ +\ i_\xi\ d)$ and equation (7). Thus under a canonical transformation $\mathcal{A} \to \mathcal{A} + d\Lambda$, $\Lambda = i_\xi \mathcal{A} - f$. Evidently $d\mathcal{A} = \Omega$ is unchanged under such a transformation. This suggests that we may think of $\mathcal{A}$ as a $U(1)$ gauge potential and Ω as the corresponding field strength. The transformation $\mathcal{A} \to \mathcal{A} + d\Lambda$ is a gauge transformation.

Finally we note that the symplectic two-form defines a volume form on the phase space $\mathcal{M}$ by

$$\begin{aligned} d\sigma(M) &= c\frac{\Omega \wedge \Omega \wedge \cdots \wedge \Omega}{(2\pi)^n} \\ &= c\sqrt{\det\left(\frac{\Omega}{2\pi}\right)}\ d^{2n}q \end{aligned} \tag{19}$$

where we take the n-fold product of Ω's for a $2n$-dimensional phase space. (c is a constant which is undetermined at this stage.) If the dimension of the phase space is infinite, then a suitable regularized form of the determinant of $\Omega_{\mu\nu}$ has to be used. The volume measure defined by equation (19) is called the Liouville measure.

Finally, notice from the last line of equation (10) that for the phase space coordinates we have $\{q^\mu, q^\nu\} = \Omega^{\mu\nu}$. This is often interpreted as saying that the basic Poisson brackets are the inverse of the symplectic structure.

2. Darboux′s theorem

A useful result concerning the symplectic form is Darboux's theorem which states that in the neighborhood of a point on the phase space it is possible to choose coordinates p_i, x_i, $i = 1, 2, \cdots, n$, (which are functions of the coordinates q^μ we start with) such that the symplectic two-form is

$$\Omega = dp_i \wedge dx_i \tag{20}$$

Evidently from the above equation, we see that the Poisson brackets in terms of this set of coordinates are

$$\{x_i, x_j\} \;=\; 0$$

$$\begin{aligned} \{x_i, p_j\} &= \delta_{ij} \\ \{p_i, p_j\} &= 0 \end{aligned} \tag{21}$$

This is the standard separation of the phase space coordinates into momenta and configuration space coordinates; Darboux's theorem is clearly an important result. The standard proof of the theorem is by induction; an outline is as follows. (This is very elegantly discussed in Arnold's book.) Let B denote the point on phase space in the neighbourhood of which we want to reduce Ω to the Darboux form. As the first coordinate p_1, we can take any nonconstant function of the coordinates q such that its differential dp_1 is not zero at B. One can assume, without loss of generality, that $p_1 = 0$ at the point B. Associated to $p_1(q)$ there is a Hamiltonian vector field

$$P_1^\mu = \Omega^{\mu\nu} \frac{\partial p_1}{\partial q^\nu} \tag{22}$$

We now choose a $(2n-1)$-dimensional surface Σ transverse to the vector P_1 and passing through the point B. The equation

$$\frac{dq^\mu}{d\tau} = -\Omega^{\mu\nu} \frac{\partial p_1}{\partial q^\nu} \tag{23}$$

defines the flow due to the vector field P_1; these flow lines intersect the surface Σ transversally. Consider any point q near Σ but not necessarily on it. We can solve (23) with q as the initial point and choose the direction such that the motion is toward the surface Σ. At some value τ determined by the initial point q, this motion arrives at Σ. This particular value of τ, viewed as a function of the initial point q, we denote by x_1. Equation (23) shows that for this function $x_1(q)$, we have

$$\{x_1, p_1\} = 1 \tag{24}$$

(Since we are considering τ as a function of the initial value q, there is an additional minus sign in differentiations. q^μ on the left-hand side of (23) is the moving point, the derivative $\partial\tau/\partial q^\mu_{initial}$ is what we want for the Poisson bracket. This eliminates the minus sign in (23).) p_1, x_1 give the first canonical pair of coordinates. Notice that $x_1 = 0$ for points on Σ.

Consider the surface Σ^* defined by $p_1 = 0$, $x_1 = 0$. The differentials dp_1, dx_1 are linearly independent since their Poisson bracket is nonzero. In fact if X_1 is the vector field corresponding to x_1, we have $i_{P_1} i_{X_1} \Omega = -1$. Let

ξ be any vector field which induces a flow along (tangential to) Σ^*. Such a vector cannot change the value of p_1, x_1, so we have $i_\xi dp_1 = 0$, $i_\xi dx_1 = 0$. Equation (24) and this result show that we can write

$$\Omega = \Omega^* \ + \ dp_1 \wedge dx_1 \tag{25}$$

Ω^* does not involve differentials dp_1 or dx_1. Evidently, $d\Omega^* = 0$. Further the contraction with Ω^* of any vector tangential to Σ^* is the same as its contraction with Ω. Thus Ω^* must be invertible for vectors tangential to Σ^*. Ω^* therefore defines a symplectic two-form on Σ^*, which is $(2n-2)$-dimensional. The problem is reduced to the question of choosing Darboux coordinates on the lower dimensional space. We can now proceed in a similar manner, starting with Σ^* and Ω^*, constructing another canonical pair p_2, x_2, obtaining a reduction to a $(2n-4)$-dimensional subspace, and so on inductively, to prove the theorem.

2 Classical dynamics

The time-evolution of any quantity is a particular canonical transformation generated by a function H called the Hamiltonian; this is the essence of the Hamiltonian formulation of dynamics. If F is any function on $\mathcal{M}$, we then have

$$\frac{\partial F}{\partial t} = \{F, H\} \tag{26}$$

Specifically for the local coordinates on $\mathcal{M}$ we can write

$$\begin{aligned} \frac{\partial q^\mu}{\partial t} &= \{q^\mu, H\} \\ &= \Omega^{\mu\nu} \frac{\partial H}{\partial q^\nu} \end{aligned} \tag{27}$$

Since Ω is invertible, we can also write this equation as

$$\Omega_{\mu\nu} \frac{\partial q^\nu}{\partial t} = \frac{\partial H}{\partial q^\mu} \tag{28}$$

At this point we can relate this to an action and a variational principle. We define the action as

$$\mathcal{S} = \int_{t_i}^{t_f} dt \ \left(\mathcal{A}_\mu \frac{dq^\mu}{dt} \ - \ H \right) \tag{29}$$

where $q^\mu(t)$ gives a path on $\mathcal{M}$. Under a general variation of the path $q^\mu(t) \rightarrow q^\mu(t) + \xi^\mu(t)$, the action changes by

$$\begin{aligned} \delta\mathcal{S} &= \int dt \left(\frac{\partial \mathcal{A}_\nu}{\partial q^\mu} \frac{dq^\nu}{dt} \xi^\mu + \mathcal{A}_\mu \frac{d\xi^\mu}{dt} - \frac{\partial H}{\partial q^\mu} \xi^\mu \right) \\ &= \mathcal{A}_\mu \xi^\mu \Big]_{t_i}^{t_f} + \int dt \left(\Omega_{\mu\nu} \frac{dq^\nu}{dt} - \frac{\partial H}{\partial q^\mu} \right) \xi^\mu \end{aligned} \tag{30}$$

The variational principle says that the equations of motion are given by the extremization of the action, i.e., by $\delta\mathcal{S} = 0$, for resricted set of variations with the boundary data (initial and final end point data) fixed. From the above variation, we see that this gives the Hamiltonian equations of motion (28). There is a slight catch in this argument because q^μ are phase space coordinates and obey first-order equations of motion. So we can only specify the initial value of q^μ. However, the Darboux theorem tells us that one can choose coordinates on neighborhoods of $\mathcal{M}$ such that the canonical one-form $\mathcal{A}$ is of the form $p_i dx^i$ for each neighborhood. Therefore, instead of specifying initial data for all q^μ, we can specify initial and final data for the x^i's. The ξ^μ in the boundary term is thus just δx^i. Since the boundary values are kept fixed in the variational principle $\delta\mathcal{S} = 0$, we may set $\delta x^i = 0$ at both boundaries and the equations of motion are indeed just (28).

We have shown how to define the action if Ω is given. However, going back to the general variations, notice that the boundary term is just the canonical one-form contracted with ξ^μ. Thus if we start from the action as the given quantity, we can identify the canonical one-form and hence Ω from the boundary term which arises in a general variation. In fact

$$\delta\mathcal{S} = i_\xi \mathcal{A}(t_f) - i_\xi \mathcal{A}(t_i) + \int dt \left(\Omega_{\mu\nu} \frac{dq^\nu}{dt} - \frac{\partial H}{\partial q^\mu} \right) \xi^\mu \tag{31}$$

We have used this equation in Chapter 3 to identify the canonical one-form and carry out the canonical quantization.

3 Geometric quantization

In the quantum theory, the algebra of observables is an operator algebra with a Hilbert space which provides an irreducible unitary representation of this algebra. The allowed transformations of variables are then unitary transformations. There are thus two essential points to quantization: 1) a correspondence between canonical transformations and unitary transformations and 2) ensuring that the representation of unitary transformations on the Hilbert space is irreducible. Since functions on phase space generate canonical transformations and hermitian operators generate unitary transformations, we get a correspondence between functions on phase space and operators on the Hilbert space. The algebra of Poisson brackets will be replaced by the algebra of commutation rules. The irreducibility leads to the necessity of choosing a polarization for the wave functions.

Before considering different aspects of the operator approach, we shall start with the notion of the wave function. In the geometric approach to the wave function, the first step is the so-called prequantum line bundle. This is a complex line bundle on the phase space with curvature Ω. Sections of this line bundle form the prequantum Hilbert space. In more practical terms, this means that we consider complex functions $\Psi(q)$ on open neighbourhoods in $\mathcal{M}$, with a suitably defined notion of covariant derivatives such that the commutator of two covariant derivatives gives Ω. Since Ω is closed, at least locally we can write $\Omega = d\mathcal{A}$, where $\mathcal{A}$ is the symplectic potential. Under a canonical transformation, Ω does not change, but the symplectic potential transforms as $\mathcal{A} \to \mathcal{A}' = \mathcal{A} + d\Lambda$, as we have seen in (18). In other words, $\mathcal{A}$ undergoes a $U(1)$ gauge transformation. The statement that Ψ's are sections of a line bundle means that locally they are complex functions which transform as

$$\Psi \to \Psi' = \exp(i\Lambda)\ \Psi \tag{32}$$

As mentioned before, one may think of $\mathcal{A}$ as a $U(1)$ gauge potential; Ψ's are then like matter fields. The above equation is equivalent to the requirement of canonical transformations being implemented as unitary transformations. The transition rules for the Ψ's from one patch on $\mathcal{M}$ to another are likewise given by exponentiating the transition function for $\mathcal{A}$. The functions Ψ's so defined, which are also square-integrable, form the prequantum Hilbert space; the inner product is given by

$$(1|2) \ = \ \int d\sigma(\mathcal{M})\ \Psi_1^*\ \Psi_2 \tag{33}$$

where $d\sigma(\mathcal{M})$ is the Liouville measure on the phase space, defined by Ω.

One can, at the prequantum level, introduce operators corresponding to various functions on the phase space. A function $f(q)$ on the phase space generates a canonical transformation which leads to the change $\Lambda = i_\xi \mathcal{A} - f$ in the symplectic potential. The corresponding change in Ψ is thus

$$\begin{aligned} \delta\Psi &= \xi^\mu \partial_\mu \Psi \; - \; i(\, i_\xi \mathcal{A} - f)\Psi \\ &= \xi^\mu \left(\partial_\mu - i\mathcal{A}_\mu\right)\Psi + i f \Psi \\ &= \left(\xi^\mu \mathcal{D}_\mu + i f\right)\;\Psi \end{aligned} \tag{34}$$

where the first term gives the change in Ψ considered as a function and the second term compensates for the change of $\mathcal{A}$. In the last expression $\mathcal{D}_\mu$ are covariant derivatives $\partial_\mu \; - \; i\mathcal{A}_\mu$; these are the appropriate derivatives to consider in view of the $U(1)$ gauge symmetry on the wave functions. The gauge potential to be used is indeed $\mathcal{A}$, so that the curvature is Ω. Based on (34), we define the prequantum operator corresponding to $f(q)$ by

$$\begin{aligned} \mathcal{P}(f) &= -i(\xi \cdot \mathcal{D} \; + \; i f) \\ &= -i\xi \cdot \mathcal{D} \; + \; f \end{aligned} \tag{35}$$

We have seen that if the Hamiltonian vector fields for $f,\ g$, are ξ and η, respectively, then the vector field corresponding to the Poisson bracket $\{f, g\}$ is $-[\xi, \eta]$. From the definition of the prequantum operator above, we then find

$$\begin{aligned} [\mathcal{P}(f), \mathcal{P}(g)] &= [-i\xi \cdot \mathcal{D} + f, -i\eta \cdot \mathcal{D} + g] \\ &= -\left[\xi^\mu \mathcal{D}_\mu, \eta^\nu \mathcal{D}_\nu\right] - i\xi^\mu [\mathcal{D}_\mu, g] + i\eta^\mu [\mathcal{D}_\mu, f] \\ &= i\xi^\mu \eta^\nu \Omega_{\mu\nu} - (\xi^\mu \partial_\mu \eta^\nu)\mathcal{D}_\nu + (\eta^\mu \partial_\mu \xi^\nu)\mathcal{D}_\nu - i\xi^\mu \partial_\mu g + i\eta^\mu \partial_\mu f \\ &= i\left(-\xi^\mu \eta^\nu \Omega_{\mu\nu} + i[\xi, \eta] \cdot \mathcal{D}\right) \\ &= i\left(-i\;(i_{[\eta,\xi]}\mathcal{D}) + \{f, g\}\right) \\ &= i\mathcal{P}(\{f, g\}) \end{aligned} \tag{36}$$

In other words, the prequantum operators form a representation of the Poisson bracket algebra of functions on phase space.

The prequantum wave functions Ψ depend on all phase space variables. The representation of the Poisson bracket algebra on such wave functions, given by the prequantum operators, is reducible. A simple example is sufficient to illustrate this. Consider a point particle in one dimension, with the symplectic two-form $\Omega = dp \wedge dx$. We can choose $\mathcal{A} = pdx$. The vector fields corresponding to x and p are $\xi_x = -\partial/\partial p$ and $\xi_p = \partial/\partial x$. The corresponding prequantum operators are

$$
\begin{aligned}
\mathcal{P}(x) &= i\frac{\partial}{\partial p} + x \\
\mathcal{P}(p) &= -i\frac{\partial}{\partial x}
\end{aligned} \tag{37}
$$

which obey the commutation rule

$$
[\mathcal{P}(x), \mathcal{P}(p)] = i \tag{38}
$$

We clearly have a representation of the algebra of $\mathcal{P}(x)$, $\mathcal{P}(p)$ in terms of prequantum functions $\Psi(x,p)$. But this is reducible. For if we consider the subset of functions on the phase space which are independent of p, namely, those which obey the condition

$$
\frac{\partial \Psi}{\partial p} = 0, \tag{39}
$$

then the prequantum operators become

$$
\begin{aligned}
\mathcal{P}(x) &= x \\
\mathcal{P}(p) &= -i\frac{\partial}{\partial x}
\end{aligned} \tag{40}
$$

which obey the same algebra (38). Thus we are able to obtain a representation of the algebra of observables on the smaller space of Ψ's obeying (39), showing that the previous representation (37) is reducible.

In order to obtain an irreducible representation, one has to impose subsidiary conditions which restrict the dependence of the prequantum wave functions to half the number of phase space variables. This is the choice of polarization and generally leads to an irreducible representation of the Poisson algebra. For the implementation of this we need to choose a set of n vector fields $P_i^\mu,\ i = 1, 2 \ldots n$, so that

$$
\Omega_{\mu\nu} P_i^\mu P_j^\nu = 0 \tag{41}
$$

and impose the condition

$$
P_i^\mu \, \mathcal{D}_\mu \, \Psi = 0 \tag{42}
$$

The vectors P_i^μ define the polarization. The wave functions so restricted are the true wave functions.

The next step is to define an inner product, and restrict to square-integrable functions, so that these wave functions form a Hilbert space. While the volume element and the notion of inner product can be defined on the phase space in terms of Ω, generally, there is no natural choice of inner product once we impose the restriction of polarization. However, there is one case where there is a natural inner product on the Hilbert space. This happens when the phase space is also Kähler and Ω is the Kähler form or some multiple thereof. In this case we can introduce local complex coordinates and write

$$\Omega \;=\; \Omega_{a\bar{a}} dx^a \wedge dx^{\bar{a}} \tag{43}$$

$a, \bar{a} \;= 1, 2 ... n$. The covariant derivatives are of the form

$$\begin{aligned} \mathcal{D}_a &= \partial_a \;-\; i\mathcal{A}_a \\ \mathcal{D}_{\bar{a}} &= \partial_{\bar{a}} \;-\; i\mathcal{A}_{\bar{a}} \end{aligned} \tag{44}$$

For a Kähler manifold, there is a Kähler potential K defined by

$$\begin{aligned} \mathcal{A}_a &= -\frac{i}{2}\partial_a K \\ \mathcal{A}_{\bar{a}} &= \frac{i}{2}\partial_{\bar{a}} K \end{aligned} \tag{45}$$

In this case, one can choose the holomorphic polarization

$$\mathcal{D}_{\bar{a}}\Psi = (\partial_{\bar{a}} + \frac{1}{2}\partial_{\bar{a}} K)\Psi = 0 \tag{46}$$

which gives

$$\Psi = \exp(-\tfrac{1}{2}K) \quad F \tag{47}$$

where F is a holomorphic function on $\mathcal{M}$. The wave functions are thus holomorphic, apart from the prefactor involving the Kähler potential. In this case the inner product of the prequantum Hilbert space can then be retained, up to a constant of proportionality, as the inner product of the Hilbert space; specifically

$$\langle 1|2\rangle = \int d\sigma(\mathcal{M}) \; e^{-K} \quad F_1^* F_2 \tag{48}$$

Almost all the cases of interest to us are of this type.

Once the polarized wave functions are defined, the idea is to represent observables as linear operators on the wave functions as given by the prequantum differential operators. Let ξ be the Hamiltonian vector field corresponding to a function $f(q)$. If the commutator of ξ with any polarization vector field P_i is proportional to P_i itself, i.e., $[\xi, P_i] = C_i^j P_j$ for some functions C_i^j, then, evidently, ξ does not change the polarization. $\xi\Psi$ obeys the same polarization condition as Ψ. In this case the operator corresponding to $f(q)$ is given by $\mathcal{P}(f)$. For operators which do not preserve the polarization, the situation is more involved. Since operators corresponding to observables are generators of unitary transformations, we must construct directly infinitesimal unitary transformations whose classical limit is the required canonical transformation and identify the quantum operator from the result. This will require, in general, a pairing between wave functions obeying different polarization conditions.

4 Topological features of quantization

We now turn to some of the topological features of phase space and their effects on quantization. As we have mentioned before two of the key topological problems have to do with the first and second cohomology of the phase space.

1. The case of nontrivial $\mathcal{H}^1(\mathcal{M}, \mathbf{R})$

Consider first the case of $\mathcal{H}^1(\mathcal{M}, \mathbf{R}) \neq 0$. In this case for a given symplectic two-form Ω, we can have different symplectic potentials. $\mathcal{A}$ and $\mathcal{A}+A$ lead to the same Ω if A is closed, i.e., if $dA = 0$. If A is exact so that $A = dh$ for some globally defined function h on $\mathcal{M}$, then the function h is a canonical transformation, physical results are unchanged, and so this is equivalent to $A = 0$ upon carrying out a canonical transformation. If A is closed but not exact, which is to say that if it is a nontrivial element of the cohomology $\mathcal{H}^1(\mathcal{M}, \mathbf{R})$, then we cannot get rid of it by a canonical transformation. Locally we can write $A = df$ for some f, but f will not be globally defined on M. Classical dynamics, which is defined by Ω as in the equations of motion (27), will not be affected by this ambiguity in the choice of the symplectic potential. In the quantum theory such A's do make a difference. We see this immediately in terms of the action. The action for a path C, parametrized as $q^\mu(t)$ from a point a to a point b is

$$\mathcal{S} = \int dt \, \left(\mathcal{A}_\mu \frac{dq^\mu}{dt} - H \right) \; + \; \int_a^b A_\mu dq^\mu \tag{49}$$

The action depends on the path but the contribution from A is topological. If we change the path slightly from C to C' with the end points fixed, we find, using Stokes' theorem,

$$\begin{aligned} \int_C A - \int_{C'} A &= \oint_{C-C'} A \\ &= \int_\Sigma dA \\ &= 0 \end{aligned} \tag{50}$$

where $C - C'$ is the path where we go from a to b along C and back from b to a along C'. (Since we are coming back the orientation is reversed, hence, the minus sign.) Σ is a surface in $\mathcal{M}$ with $C - C'$ as the boundary. The above result shows that the contribution from A is invariant under small changes of the path. (This is not true for the other terms in the action.) In particular, the value of the integral is zero for closed paths so long as they are contractible; for then we can make a sequence of small deformations of the path (which do not change the value) and eventually contract the path to zero. If there are noncontractible loops, then there can be nontrivial contributions. If $\mathcal{H}^1(\mathcal{M}, \mathbf{R}) \neq 0$, then there are noncontractible loops. In the quantum theory, it is $e^{i\mathcal{S}}$ which is important, so we need $e^{i\int A}$. Assume for simplicity that $\mathcal{H}^1(\mathcal{M}, \mathbf{R}) = \mathbf{Z}$ so that there is only one topologically distinct noncontractible loop apart from multiple traversals of the same. Let $A = \theta\alpha$, where θ is a constant and α is normalized to unity along the noncontractible loop for going around once. For all paths which include n traversals of the loop, we find

$$\exp\left(i \oint A \right) = \exp\left(i\theta \oint \alpha \right) = \exp\left(i\theta n \right) \tag{51}$$

Notice that a shift $\theta \rightarrow \theta + 2\pi$ does not change this value, so that we may restrict θ to be in the interval zero to 2π. Putting this back into the action (49), we see that, considered as a function on the set of all paths, the action has an extra parameter θ. Thus the ambiguity in the choice of the symplectic potential due to $\mathcal{H}^1(\mathcal{M}, \mathbf{R}) \neq 0$ leads to an extra parameter θ which is needed to characterize the quantum theory completely. If $\mathcal{H}^1(\mathcal{M}, \mathbf{R})$ is not just $\mathbf{Z}$

and there are more distinct paths possible, then there can be more such parameters.

It is now easy to see these results in terms of wave functions. The relevant covariant derivatives are of the form $\mathcal{D}_\mu \Psi = (\partial_\mu - i\mathcal{A}_\mu - iA_\mu)\Psi$. We can write

$$\Psi(q) = \exp\left(i\int_a^q A\right) \quad \Phi(q) \tag{52}$$

where the lower limit of the integral is some fixed point a. By using this in the covariant derivative, we see that A is removed from $\mathcal{D}_\mu$ in terms of action on Φ. This is like a canonical transformation, except that the relevant transformation $\exp\left(i\int_a^q A\right)$ is not single valued. As we go around a closed noncontractible curve, it can give a phase $e^{i\theta}$. Since Ψ is single valued, this means that Φ must have a compensating phase factor; Φ is not single valued but must give a specific phase labeled by θ. Thus we can get rid of A from the covariant derivatives and hence the various operator formulae, but diagonalizing the Hamiltonian on such Φ's can give results which depend on the angle θ.

The θ-parameter in a nonabelian gauge theory is an example of this kind of topological feature. The description of anyons or particles of fractional statistics in two spatial dimensions is another example.

2. The case of nontrivial $\mathcal{H}^2(\mathcal{M}, \mathbf{R})$

We now turn to the second topological feature, the case of $\mathcal{H}^2(\mathcal{M}, \mathbf{R}) \neq 0$. In this case there are closed two-forms on $\mathcal{M}$ which are not exact. Correspondingly, there are closed two-surfaces which are not the boundaries of any three-dimensional region, i.e., noncontractible closed two-surfaces. In general, elements of $\mathcal{H}^2(\mathcal{M}, \mathbf{R})$ integrated over such noncontractible two-surfaces are not zero. If Ω is some nontrivial element of $\mathcal{H}^2(\mathcal{M}, \mathbf{R})$, then the symplectic potential cannot be globally defined. We see this easily as follows. Consider the integral of Ω over a noncontractible closed two-surface Σ,

$$I(\Sigma) = \int_\Sigma \Omega \tag{53}$$

If Σ' is a small deformation of Σ, then

$$I(\Sigma) - I(\Sigma') = \int_{\Sigma - \Sigma'} \Omega = \int_V d\Omega = 0 \tag{54}$$

where V is a three-dimensional volume with the two surfaces Σ and Σ' as the boundary. Thus the integral of Ω is a topological invariant, invariant under small deformations of the surface on which it is integrated. If we can write Ω as $d\mathcal{A}$ for some $\mathcal{A}$ globally defined on Σ then clearly $I(\Sigma)$ is zero by Stokes' theorem. Thus if $I(\Sigma)$ is nonzero, then $\mathcal{A}$ cannot be globally defined on Σ. We have to use different functions to represent $\mathcal{A}$ in different coordinate patches, then have transition functions relating the $\mathcal{A}$'s in overlap regions. Even though we may have different definitions of $\mathcal{A}$ in an overlap region corresponding to the different patches which are overlapping, Ω is the same, and so the transition functions must be canonical transformations. As an example, consider a closed noncontractible two-sphere, or any smooth deformation of it, which is a subspace of M. We can cover it with two coordinate patches corresponding to the two hemispheres, denoted N and S as usual. The symplectic potential is represented by $\mathcal{A}_N$ and $\mathcal{A}_S$, respectively. On the equatorial overlap region, they are connected by

$$\mathcal{A}_N = \mathcal{A}_S + d\Lambda \tag{55}$$

where Λ is a function defined on the overlap region. It gives the canonical transformation between the two $\mathcal{A}$'s.

Since $\mathcal{A}$ is what is used in setting up the quantum theory and since, in particular, the canonical transformations are represented as unitary transformations on the wave functions, we see that we must have a Ψ_N for the patch N and a Ψ_S for the patch S. On the equator they must be related by the canonical transformation, which from (32), is given as

$$\Psi_N = \exp(i\Lambda) \quad \Psi_S \tag{56}$$

We now consider the integral of $d\Lambda$ over the equator E, which is a closed curve being the boundary of either N or S. From (55) this is given as

$$\begin{aligned} \Delta\Lambda = \oint_E d\Lambda &= \int_E \mathcal{A}_N - \int_E \mathcal{A}_S \\ &= \int_{\partial N} \mathcal{A}_N \ + \ \int_{\partial S} \mathcal{A}_S \\ &= \int_N \Omega \ + \ \int_S \Omega \\ &= \int_\Sigma \ \Omega \end{aligned} \tag{57}$$

In the second step, we reverse the sign for the S-term because E considered as the boundary of S has the opposite orientation compared to itself considered as the boundary of N. The above equation shows that the change of Λ as we go around the equator once, namely, $\Delta\Lambda$, is nonzero if $I(\Sigma)$ is nonzero; Λ is not single valued on the equator. But the wave function must be single valued. From (56), we see that this can be achieved if $\exp(i\Delta\Lambda) = 1$ or if $\Delta\Lambda = 2\pi n$ for some integer n. Combining with (57), we can state that single-valuedness of wave functions in the quantum theory requires that

$$\int_\Sigma \Omega = 2\pi n \tag{58}$$

The integral of the symplectic two-form on closed noncontractible two-surfaces must be quantized as 2π times an integer. We have given the argument for surfaces which are deformations of a two-sphere, but a similar argument can be made for noncontractible two-surfaces of different topology as well. The result (58) is quite general.

The typical example of this kind of topological feature is the motion of a charged particle in the field of a magnetic monopole. The condition (58) is then the Dirac quantization condition. The Wess-Zumino terms occuring in many field theories are another example.

5 A brief summary of quantization

In summary, the key features of the quantization of a system using the holomorphic polarization are the following:
1) We need a phase space which is also Kähler; the symplectic two-form being a multiple of the Kähler form.
2) The polarization condition is chosen as $\mathcal{D}_{\bar{a}}\ \Psi\ =\ 0$.
3) The inner product of the prequantum Hilbert space, which is essentially square integrability on the phase space, is retained as the inner product on the true Hilbert space in the holomorphic polarization.
4) The operator corresponding to an observable $f(q)$ which preserves the chosen polarization is given by the prequantum operator $\mathcal{P}(f)$ acting on the true (polarized) wave functions.
5) For observables which do not preserve the polarization, one has to construct infinitesimal unitary transformations whose classical limits are the required canonical transformations.
6) If the phase space $\mathcal{M}$ has noncontractible two-surfaces, then the integral of Ω over any of these surfaces must be quantized in units of 2π.

7) If $\mathcal{H}^1(\mathcal{M}, \mathbf{R})$ is not zero, then there are inequivalent $\mathcal{A}$'s for the same Ω and we need extra parameters to specify the quantum theory completely.

6 Examples

6.1 Coherent states

For a one-dimensional quantum system, $\Omega = dp \wedge dx = idz \wedge d\bar{z}$, where $(p \pm ix)/\sqrt{2} = z, \bar{z}$. Choose

$$\mathcal{A} = \frac{i}{2}(z\, d\bar{z} - \bar{z}\, dz) \tag{59}$$

The covariant derivatives are $\partial_z - \frac{1}{2}\bar{z}$ and $\partial_{\bar{z}} + \frac{1}{2}z$. Holomorphic polarization corresponds to $P = \partial/\partial\bar{z}$, leading to the condition

$$(\partial_{\bar{z}} + \tfrac{1}{2}z)\Psi = 0 \tag{60}$$

This is equation (46) for this example. The solutions are of the form

$$\Psi = e^{-\frac{1}{2}z\bar{z}}\ \varphi(z) \tag{61}$$

where $\varphi(z)$ is holomorphic in z.

The vector fields corresponding to $z, \bar{z}$ are

$$\begin{aligned} z &\leftrightarrow -i\frac{\partial}{\partial \bar{z}} \\ \bar{z} &\leftrightarrow \ \ i\frac{\partial}{\partial z} \end{aligned} \tag{62}$$

These commute with $P = \partial/\partial\bar{z}$ and so are polarization-preserving. The prequantum operators corresponding to these are

$$\begin{aligned} \mathcal{P}(z) &= -i(-i)\left(\frac{\partial}{\partial\bar{z}} + \tfrac{1}{2}z\right) + z = -\frac{\partial}{\partial\bar{z}} + \tfrac{1}{2}z \\ \mathcal{P}(\bar{z}) &= -i(\ i)\ \left(\frac{\partial}{\partial z} - \tfrac{1}{2}\bar{z}\right) + \bar{z} = \ \ \frac{\partial}{\partial z} + \tfrac{1}{2}\bar{z} \end{aligned} \tag{63}$$

In terms of their action on the functions $\varphi(z)$ in (61) corresponding to Ψ's obeying the polarization condition, these can be written as

$$\begin{aligned} \mathcal{P}(z)\ \varphi(z) &= z\ \varphi(z) \\ \mathcal{P}(\bar{z})\ \varphi(z) &= \frac{\partial\varphi}{\partial z} \end{aligned} \tag{64}$$

The inner product for the $\varphi(z)$'s is

$$
\begin{aligned}
\langle 1|2\rangle &= \int i\frac{dz\wedge d\bar{z}}{2\pi}\Psi_1^*\Psi_2 \\
&= \int i\frac{dz\wedge d\bar{z}}{2\pi}\; e^{-z\bar{z}}\;\; \varphi_1^*\,\varphi_2
\end{aligned}
\tag{65}
$$

What we have obtained is the standard coherent state (or Bargmann) realization of the Heisenberg algebra.

6.2 Quantizing the two-sphere

We consider the phase space to be a two-sphere $S^2 \sim \mathbf{CP}^1$ considered as a Kähler manifold. Using complex coordinates $z = x + iy, \bar{z} = x - iy$, the standard Kähler form is

$$
\omega = i\; \frac{dz\wedge d\bar{z}}{(1+z\bar{z})^2}
\tag{66}
$$

The metric is given by $ds^2 = e^1e^1 + e^2e^2$ where the frame fields are

$$
e^1 = \frac{dx}{1+r^2}\;, \qquad\qquad e^2 = \frac{dy}{1+r^2}
\tag{67}
$$

where $r^2 = z\bar{z}$. The Riemannian curvature is $\mathcal{R}^{12} = 4e^1\wedge e^2$, giving the Euler number

$$
\chi = \int \frac{\mathcal{R}_{12}}{2\pi} = 2
\tag{68}
$$

The phase space has nonzero $\mathcal{H}^2(\mathcal{M})$ given by the Kähler form. As we have discussed, the symplectic two-form must belong to an integral cohomology class of $\mathcal{M}$ to be able to quantize properly. We take

$$
\Omega = n\;\omega = i\;n\;\frac{dz\wedge d\bar{z}}{(1+z\bar{z})^2}
\tag{69}
$$

where n is an integer. In this case, $\int_M \Omega = 2\pi n$ as required by the quantization condition. The symplectic potential can be taken as

$$
\mathcal{A} = \frac{in}{2}\left[\frac{z\;d\bar{z} - \bar{z}\;dz}{(1+z\bar{z})}\right]
\tag{70}
$$

The covariant derivatives are given by $\partial - i\mathcal{A}$. The holomorphic polarization condition is

$$
(\partial_{\bar{z}} - i\mathcal{A}_{\bar{z}})\Psi = \left[\partial_{\bar{z}} + \frac{n}{2}\frac{z}{(1+z\bar{z})}\right]\;\Psi = 0
\tag{71}
$$

This can be solved as

$$\Psi = \exp\left(-\frac{n}{2}\log(1+z\bar{z})\right)\; f(z) \tag{72}$$

Notice that $n\log(1+z\bar{z})$ is the Kähler potential for Ω. The inner product is given by

$$\langle 1|2\rangle = i\alpha \int \frac{dz\wedge d\bar{z}}{2\pi(1+z\bar{z})^{n+2}}\; {f_1}^* f_2 \tag{73}$$

Here α is an overall constant, which can be absorbed into the normalization factors for the wave functions. Since $f(z)$ in (72) is holomorphic, we can see that a basis of nonsingular wave functions is given by $f(z) = 1,\ z,\ z^2, \cdots,\ z^n$; higher powers of z will not have finite norm. The dimension of the Hilbert space is thus $(n+1)$. We could have seen that this dimension would be finite from the semiclassical estimate of the number of states as the phase volume. Since the phase volume for $\mathcal{M} = S^2$ is finite, the dimension of the Hilbert space should be finite.

It is interesting to see this dimension in another way. The polarization condition (71) is giving the $\bar{\partial}$-closure of Ψ with a $U(1)$ gauge field $\mathcal{A}$ and curvature $\mathcal{R}_{12}$. The number of normalizable solutions to (71) is thus given by the index theorem for the twisted Dolbeault complex, i.e.,

$$\text{index}(\bar{\partial}_V) = \int_{\mathcal{M}} \text{td}(\mathcal{M}) \wedge Ch(V) \tag{74}$$

where, for our two-dimensional case, the Todd class $\text{td}(\mathcal{M})$ is $\mathcal{R}/4\pi$ and the Chern character $Ch(V) = \text{Tr}(e^{iF/2\pi}) = \text{Tr}(e^{\Omega/2\pi})$ is $\int \Omega/2\pi$ for us. We thus have

$$\begin{aligned} \text{index}(\bar{\partial}_V) &= \int_{\mathcal{M}} \frac{\Omega}{2\pi} + \int_{\mathcal{M}} \frac{\mathcal{R}}{4\pi} \\ &= n+1 \end{aligned} \tag{75}$$

Notice that, semiclassically, we should expect the number of states to be $\int \Omega/2\pi = n$. The extra one comes from the Euler number in this case.

An orthonormal basis for the wave functions may be taken to be

$$f_k(z) = \left[\frac{n!}{k!\ (n-k)!}\right]^{\frac{1}{2}}\; z^k \tag{76}$$

with the inner product, for two such functions f, g,

$$\langle f|g\rangle = i(n+1)\int \frac{dz\wedge d\bar{z}}{2\pi(1+z\bar{z})^{n+2}}\ f^*g \tag{77}$$

Notice that this is the same as (73) but with a specific choice of $\alpha = n+1$. This is the value which gives $\mathrm{Tr}1 = n+1$ as expected for an $(n+1)$-dimensional Hilbert space.

Classically the Poisson bracket of two functions F and G on the phase space is given by

$$\begin{aligned}\{F,G\} &= \Omega^{\mu\nu}\partial_\mu F\partial_\nu G\\ &= \frac{i}{n}(1+z\bar{z})^2\left(\frac{\partial F}{\partial z}\frac{\partial G}{\partial \bar{z}} - \frac{\partial F}{\partial \bar{z}}\frac{\partial G}{\partial z}\right)\end{aligned} \tag{78}$$

Consider now the vector fields

$$\begin{aligned}\xi_+ &= i\left(\frac{\partial}{\partial\bar{z}} + z^2\frac{\partial}{\partial z}\right)\\ \xi_- &= -i\left(\frac{\partial}{\partial z} + \bar{z}^2\frac{\partial}{\partial\bar{z}}\right)\\ \xi_3 &= i\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\right)\end{aligned} \tag{79}$$

It is easily verified that these are the standard $SU(2)$ isometries of the sphere. The Lie brackets of the ξ's give the $SU(2)$ algebra, up to certain factors of $\pm i$ compared to the standard form, having to do with how we have defined the ξ's; we have put in certain multiplicative constants so that the quantum algebra has the standard form. Further, the ξ's are Hamiltonian vector fields corresponding to the functions

$$\begin{aligned}J_+ &= -n\,\frac{z}{1+z\bar{z}}\\ J_- &= -n\,\frac{\bar{z}}{1+z\bar{z}}\\ J_3 &= -\frac{n}{2}\left(\frac{1-z\bar{z}}{1+z\bar{z}}\right)\end{aligned} \tag{80}$$

The prequantum operators $-i\xi\cdot\mathcal{D}+J$ corresponding to these functions are

$$\mathcal{P}(J_+) = \left(z^2\partial_z - \frac{n}{2}z\frac{2+z\bar{z}}{1+z\bar{z}}\right) - i\xi_+^{\bar{z}}\mathcal{D}_{\bar{z}}$$

$$
\begin{aligned}
\mathcal{P}(J_-) &= \left(-\partial_z - \frac{n}{2}\frac{\bar{z}}{1+z\bar{z}}\right) - i\xi_-^{\bar{z}}\mathcal{D}_{\bar{z}} \\
\mathcal{P}(J_3) &= \left(z\partial_z - \frac{n}{2}\frac{1}{1+z\bar{z}}\right) - i\xi_3^{\bar{z}}\mathcal{D}_{\bar{z}}
\end{aligned}
\tag{81}
$$

Acting on the polarized wave functions, $\mathcal{D}_{\bar{z}}$ gives zero. Writing Ψ as in (72), we can work out the action of the operators on the holomorphic wave functions $f(z)$. We get

$$
\begin{aligned}
\hat{J}_+ &= z^2\partial_z - n\ z \\
\hat{J}_- &= -\partial_z \\
\hat{J}_3 &= z\partial_z - \tfrac{1}{2}\ n
\end{aligned}
\tag{82}
$$

If we define $j = n/2$, which is therefore half-integral, we see that the operators given above correspond to a unitary irreducible representation of $SU(2)$ with $J^2 = j(j+1)$ and dimension $n+1 = 2j+1$. Notice that there is only one representation here and it is fixed by the choice of the symplectic form Ω.

From the symplectic potential (70) and from (29) we see that an action which leads to the above results is

$$
\mathcal{S} = i\frac{n}{2}\int dt\ \frac{z\dot{\bar{z}} - \bar{z}\dot{z}}{1+z\bar{z}}
\tag{83}
$$

where the overdot denotes differentiation with respect to time. This action may be written as

$$
\mathcal{S} = i\frac{n}{2}\int dt\ \ \mathrm{Tr}(\sigma_3 g^{-1}\dot{g})
\tag{84}
$$

where g is an element of $SU(2)$ written as a (2×2)-matrix, $g = \exp(i\sigma_i\theta_i)$ and σ_i, $i = 1,2,3$, are the Pauli matrices. In this action, the dynamical variable is an element of $SU(2)$. If we make a transformation $g \to g\ h$, $h = \exp(i\sigma_3\varphi)$, we get

$$
\mathcal{S} \to \mathcal{S} - n\int dt\ \dot{\varphi}
\tag{85}
$$

The extra term is a boundary term and does not affect the equations of motion. Thus classically the dynamics is actually restricted to $SU(2)/U(1) = S^2$. The choice of parametrization

$$
g = \frac{1}{\sqrt{1+z\bar{z}}}\begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix}\ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}
\tag{86}
$$

leads to the expression (83).

Even though the classical dynamics is restricted to $SU(2)/U(1)$, the boundary term in (85) does have an effect in the quantum theory. Consider choosing $\varphi(t)$ such that $\varphi(-\infty) = 0$ and $\varphi(\infty) = 2\pi$. In this case $h(-\infty) = h(\infty) = 1$ giving a closed loop in the $U(1)$ subgroup of $SU(2)$ defined by the σ_3-direction. For this choice of $h(t)$, the action changes by $-2\pi n$. $e^{i\mathcal{S}}$ remains single valued and, even in the quantum theory, the extra $U(1)$ degree of freedom is consistently removed. If the coefficient were not an integer, this would not be the case and we would have inconsistencies in the quantum theory. Thus the quantization of the coefficient to an integral value is obtained again, from a slightly different point of view.

So far we have used a local parametrization of S^2 which corresponds to a stereographic projection of the sphere onto a plane. Another more global approach is to use the homogeneous coordinates of the sphere viewed as $\mathbf{CP}^1$. We use a two-component spinor u_α, $\alpha = 1, 2$, with the identification $u_\alpha \sim \lambda u_\alpha$ for any nonzero complex number λ. We also define $\bar{u}_1 = u_2^*$, $\bar{u}_2 = -u_1^*$ or $\bar{u}_\alpha = \epsilon_{\alpha\beta} u_\beta^*$, where $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{12} = 1$. The symplectic form is

$$\Omega = -in \left[\frac{du \cdot d\bar{u}}{\bar{u} \cdot u} - \frac{\bar{u} \cdot du \; u \cdot d\bar{u}}{(\bar{u} \cdot u)^2} \right] \tag{87}$$

where the notation is $u \cdot v = u_\alpha v_\beta \epsilon_{\alpha\beta}$. It is easily checked that $\Omega(\lambda u) = \Omega(u)$; it is invariant under $u \to \lambda u$ and hence is properly defined on $\mathbf{CP}^1$ rather than $\mathbf{C}^2 - \{0\}$. The choice of $u_2/u_1 = z$ leads to the previous local parametrization. The symplectic potential is

$$\mathcal{A} = -i \frac{n}{2} \left[\frac{u \cdot d\bar{u} \; - \; du \cdot \bar{u}}{\bar{u} \cdot u} \right] \tag{88}$$

Directly from the above expression we see that

$$\mathcal{A}(\lambda u) = \mathcal{A}(u) \; - \; d \left(i \frac{n}{2} \log(\bar{\lambda}/\lambda) \right) \tag{89}$$

This means that $\mathcal{A}$ cannot be written as a globally defined one-form on $\mathbf{CP}^1$. This is to be expected because $\int \Omega \neq 0$ and hence we cannot have a globally defined potential on $\mathbf{CP}^1$. From (32), we see that the prequantum wave functions must transform as

$$\Psi(\lambda u, \bar{\lambda} \bar{u}) = \Psi(u, \bar{u}) \; \exp \left[\frac{n}{2} \log(\lambda/\bar{\lambda}) \right] \tag{90}$$

The polarization condition for the wave functions becomes

$$\left[\frac{\partial}{\partial \bar{u}_\alpha} - \frac{n}{2}\frac{u_\beta \epsilon_{\beta\alpha}}{\bar{u}\cdot u}\right] \ \Psi = 0 \tag{91}$$

The solution to this condition is

$$\Psi = \exp\left(-\frac{n}{2}\log(\bar{u}\cdot u)\right) \ f(u) \tag{92}$$

Combining this with (90), we see that the holomorphic functions $f(u)$ should behave as

$$f(\lambda u) = \lambda^n \ f(u) \tag{93}$$

$f(u)$ must thus have n u's and hence is of the form

$$f(u) = \sum_{\alpha' s} C^{\alpha_1\cdots\alpha_n} \ \ u_{\alpha_1}\cdots u_{\alpha_n} \tag{94}$$

Because of the symmetry of the indices, there are $n+1$ independent functions, as before. There is a natural linear action of $SU(2)$ on the u, $\bar{u}$ given by

$$u'_\alpha = U_{\alpha\beta} \ u_\beta, \qquad\qquad \bar{u}'_\alpha = U_{\alpha\beta} \ \bar{u}_\beta \tag{95}$$

where $U_{\alpha\beta}$ form a (2×2) $SU(2)$ matrix. The corresponding generators are the J_a we have constructed in (81, 82).

6.3 Compact Kähler spaces of the G/H-type

The two-sphere $S^2 = SU(2)/U(1)$ is an example of a group coset which is a Kähler manifold. There are many compact Kähler manifolds which are of the form G/H, where H is a subgroup of the compact Lie group G. In particular G/H is a Kähler manifold for any compact Lie group if H is its maximal torus. The maximal torus is the subspace of G generated by the mutually commuting generators. Another set of spaces is given by $\mathbf{CP}^N = SU(N+1)/U(N)$. There are many other cases as well.

One can take the symplectic form as proportional to the Kähler form or as a combination of the generators of $\mathcal{H}^2(M)$ and quantize these spaces as we have done for the case of S^2. In general, they lead to one unitary irreducible representation of the group G, the particular choice of the representation being determined by the choice of Ω.

In most of these cases, the Kähler form can be constructed in a very simple way. As an example consider $\mathbf{CP}^2 = SU(3)/U(2)$. A general element of $SU(3)$ can be represented as a unitary (3×3)-matrix. We define a $U(1)$ subgroup by elements of the form $U = \exp(iI^8\theta)$, where

$$I^8 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{96}$$

We also define an $SU(2)$ subgroup which commutes with this by elements of the form

$$h_{SU(2)} = \begin{pmatrix} h_{2\times 2} & 0 \\ 0 & 1 \end{pmatrix} \tag{97}$$

These two together form the $U(2)$ subgroup of $SU(3)$. Consider now the one-form

$$\mathcal{A}(g) = iw \ \mathrm{Tr}(I^8 g^{-1} dg) \tag{98}$$

where g is an element of the group $SU(3)$ and w is a numerical constant. If h is an element of $U(2) \subset SU(3)$, we find

$$\mathcal{A}(gh) = \mathcal{A}(g) - wd\theta \tag{99}$$

We see that $\mathcal{A}$ changes by a total differential under the $U(2)$-transformations. Thus $d\mathcal{A}$ is defined on $\mathbf{CP}^2$. Evidently it is closed, but it is not exact since the corresponding one-form is not globally defined on $\mathbf{CP}^2$, but only on $G = SU(3)$. Thus $d\mathcal{A}$ is a nontrivial element of $\mathcal{H}^2(\mathbf{CP}^2)$. We can use $d\mathcal{A}$ as the symplectic two-form. The integrals of $d\mathcal{A}$ over nontrivial two-cycles on $\mathbf{CP}^2$ will have to be integers; this will restrict the choices for w. Alternatively, we take the action to be $\mathcal{S} = \int \mathcal{A}$; for $e^{i\mathcal{S}}$ to be well defined on $\mathbf{CP}^2$, we will have restrictions on the w. The wave functions are functions on $SU(3)$ subject to the restrictions given by the action of $SU(2)$ and $U(1)$. In other words, we can write, using the Wigner $\mathcal{D}$-functions

$$\Psi \sim \mathcal{D}^{\alpha}_{AB}(g) \tag{100}$$

Here α is a set of indices which labels the representation; A, B label the states. $\mathcal{D}^{\alpha}_{AB}(g)$ is the AB-matrix element of the group element g in the irreducible representation of G characterized by the labels α.

The groups involved in the quotient can be taken as the right action on g. The transformation law for $\mathcal{A}$ then tells us that Ψ must transform as

$$\Psi(gh) = \exp(iw\theta) \ \Psi(g) \tag{101}$$

This shows that the wave functions must be singlets under the $SU(2)$ subgroup, acting on the right of g, and carry a definite charge w under the $U(1)$ subgroup. This restricts the choice of values for the index B in (100). Further w must also be quantized so that it can be one of the allowed values in the unitary irreducible representations of $SU(3)$ in (100). Once the indices B are chosen this way, the index A is free and so the result of the quantization is to yield a Hilbert space which is one unitary irreducible representation of the group $SU(3)$. w is related to the highest weights defining this representation. (There are many representations satisfying the condition on the charges for the subgroup. But the polarization condition, which we have not discussed for this problem, will impose further restrictions and choose one representation which is a "minimal" one among the various representations allowed by the charges.) To do this in more detail, notice that $I^8 = \sqrt{3/2}\ Y$, where Y is the hypercharge; thus $\mathcal{D}_{AB}(gU) = \mathcal{D}_{AB}\exp(i\sqrt{3/2}\ Y_B\theta)$, Y_B being the hypercharge of the state B, identifying w as $\sqrt{3/2 Y_B}$. Unitary representations of $SU(3)$ have hypercharge values quantized in units of $\frac{1}{3}$; $SU(2)$ invariant states have hypercharge quantized in units of $\frac{2}{3}$. Equation (101) then tells us that w must be quantized as $n\sqrt{2/3}$, where n is an integer.

An explicit parametrization can be obtained as follows. Evaluating the trace in (98), we get

$$\mathcal{A} = -iw\sqrt{\frac{3}{2}}\ \ u^*_\alpha du_\alpha \tag{102}$$

where $g_{\alpha 3} = u_\alpha$. Evidently, $u^*_\alpha u_\alpha = 1$ and we can parametrize it as

$$u_\alpha = \frac{1}{\sqrt{1+\bar{z}\cdot z}}\begin{pmatrix}1\\ z_1\\ z_2\end{pmatrix} \tag{103}$$

The symplectic two-form corresponding to the potential (102) is

$$\begin{aligned}\Omega &= -iw\sqrt{\frac{3}{2}}\ du^*_\alpha du_\alpha \\ &= -iw\sqrt{\frac{3}{2}}\left[\frac{d\bar{z}_i\ dz_i}{(1+\bar{z}\cdot z)} - \frac{d\bar{z}\cdot z\ \bar{z}\cdot dz}{(1+\bar{z}\cdot z)^2}\right]\end{aligned} \tag{104}$$

This Ω is proportional to the Kähler two-form on $\mathbf{CP}^2$. Requiring that Ω should integrate to an integer over closed nontrivial two-surfaces in $\mathbf{CP}^2$ will lead to the same quantization condition on w. This is similar to the case of the two-sphere or $\mathbf{CP}^1$. The polarization condition will tell us that the states are functions of u_α's only, not u^*_α's, and the condition on the $U(1)$-charge will fix the number of u_α's to be n. The states are thus of the form

$$\Psi \sim u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_n} \tag{105}$$

This corresponds to the rank n symmetric representation of $SU(3)$. One can get other representations by other choices of H.

More generally one can take

$$\mathcal{A}(g) = i \sum_a w_a \mathrm{Tr}(t^a g^{-1} dg) \tag{106}$$

where t^a are diagonal elements of the Lie algebra of G and w_a are a set of numbers. H will be the subgroup commuting with $\sum_a w_a t^a$; if w_a are such that all the diagonal elements of $\sum_a w_a t^a$ are distinct, then H will be the maximal torus of G. $\mathcal{A}$ will change by a total differential under $g \to gh$, $h \in H$ and $d\mathcal{A}$ will be a closed nonexact form on G/H. If some of the eigenvalues of $\sum_a w_a t^a$ are equal, H can be larger than the maximal torus. Upon quantization, for suitably chosen w_a, we will get one unitary irreducible representation of G, and w_a will be related to the weights defining the representation.

6.4 Charged particle in a monopole field

The symplectic form for a point particle in three dimensions is

$$\Omega = dp_i \wedge dx_i \tag{107}$$

The usual prescription for introducing coupling to a magnetic field involves replacing p_i by $p_i + eA_i$. The Lagrangian has a $A_i \dot{x}_i$ added to it, giving the canonical one-form $m\dot{x}_i dx_i + eA_i dx_i$. Writing $p_i = m\dot{x}_i$, the symplectic two-form is found to be

$$\Omega = dp_i \wedge dx_i + eF \tag{108}$$

where $F = dA = \frac{1}{2} F_{ij} dx^i \wedge dx^j$ is the magnetic field strength. Thus in terms of the symplectic two-form the minimal prescription for introducing electromagnetic interactions amounts to adding eF to Ω. We can use this to discuss a charged particle in the field of a magnetic monopole which has a radial magnetic field $B_k = gx_k/r^3$, where g is the magnetic charge of the monopole and $r^2 = x_i x_i$. The monopole is taken to be at the origin of the coordinate system. Thus Ω is

$$\Omega = dp_i \wedge dx_i \ + \frac{1}{2} eg\epsilon_{ijk} \frac{x_k}{r^3} dx_i \wedge dx_j \tag{109}$$

We can identify the basic Hamiltonian vector fields. Contraction of $X_i = -\partial/\partial p_i$ with Ω gives $-dx_i$ identifying it as the vector field for x^i. The contraction of $P'_i = \partial/\partial x_i$ gives

$$i_{P'}\Omega = -dp_i \ + \ eg\epsilon_{ijk}\frac{x_k}{r^3}dx_j \tag{110}$$

P' is not a Hamiltonian vector field, but since we get dx_i from contraction of $\partial/\partial p_i$ with Ω we see that the combination

$$P_i = \frac{\partial}{\partial x_i} - eg\epsilon_{ijk}\frac{x_k}{r^3}\frac{\partial}{\partial p_j} \tag{111}$$

is a Hamiltonian vector field and corresponds to p_i. The Poisson brackets are thus

$$\begin{aligned} \{x_i, x_j\} &= -i_{X_i} dx_j = 0 \\ \{x_i, p_j\} &= -i_{X_i} dp_j = \delta_{ij} \\ \{p_i, p_j\} &= -i_{P_i} dp_j = eg\epsilon_{ijk}\frac{x_k}{r^3} \end{aligned} \tag{112}$$

It is also interesting to work out the angular momentum. Under an infinitesimal rotation by angle θ_k the change in the variables $x_i,\ p_i$ are

$$\begin{aligned} \delta x_i &= -\epsilon_{ijk} x_j \theta_k \\ \delta p_i &= -\epsilon_{ijk} p_j \theta_k \end{aligned} \tag{113}$$

The corresponding vector field is therefore

$$\xi = -\epsilon_{ijk}\theta_k \left(x_j \frac{\partial}{\partial x_i} + p_j \frac{\partial}{\partial p_i} \right) \tag{114}$$

Upon taking the interior contraction of this with Ω, we find

$$\begin{aligned} i_\xi \Omega &= -\epsilon_{ijk}\theta_k \left(-x_j dp_i - p_i dx_j\right) - \theta_k eg(\delta^m_j \delta^n_k - \delta^m_k \delta^n_j)\frac{x_j x_n}{r^3} dx^m \\ &= \theta_k d(-\epsilon_{ijk} x_i p_j) + \theta_k eg \left(\frac{dx_k}{r} - \frac{x_i x_k}{r^3} dx^i \right) \\ &= \theta_k d \left(-\epsilon_{ijk} x_i p_j + eg\frac{x_k}{r} \right) \end{aligned} \tag{115}$$

The angular momentum which is the generator of rotations is thus

$$J_i = \epsilon_{ijk} x_j p_k \; - \; eg \frac{x_i}{r} \tag{116}$$

This shows that the charged particle has an extra contribution to the angular momentum which is radial. In fact $\hat{x} \cdot J = -eg$.

By converting the Poisson brackets to commutators of operators we can set up the quantum theory. The only unusual ingredient is the following. The symplectic two-form is singular at $r = 0$. Thus we need to remove this point for a nonsingular description. If we do so, then the space has noncontractible two-spheres and so there is a quantization condition for Ω. Integrating Ω over a two-sphere around the origin (which is the location of the monopole), we get $4\pi eg$. The general quantization condition (58) becomes

$$eg = \frac{n}{2} \tag{117}$$

This is the famous Dirac quantization condition stating that the magnetic charge must be quantized in units of $1/2e$. This value $1/2e$ is the lowest magnetic charge corresponding to one monopole. The argument for quantization also follows from noting that in the quantum theory, the eigenvalues of any component of angular momentum have to be half-integral. $\hat{x} \cdot J = -eg$ then leads to the same quantization; this was first noted by Saha.

6.5 Anyons or particles of fractional spin

We consider relativistic particles in two spatial dimensions. In general, they can have arbitrary spin, not necessarily quantized, and hence they are generically referred to as anyons (or particles of any spin).

We will work out some of the theory of anyons starting with a symplectic structure. A spinless particle may be described by a set of momentum variables p^a and position variables x^a, $a = 0, 1, 2$. The canonical structure or symplectic two-form is given by

$$\Omega = g_{ab} \; dx^a \wedge dp^b \tag{118}$$

where $g_{ab} = diag(1, -1, -1)$ is the metric tensor. For a charged particle, as discussed above, the coupling to the electromagnetic field A_a by the minimal prescription is equivalent to $\Omega \to \Omega + eF$.

The motion of the relativistic charged particle is given by the (classical) Lorentz equations

$$\begin{aligned} \frac{p^a}{m} &= \frac{dx^a}{d\tau} \\ \frac{dp^a}{d\tau} &= -\frac{e}{m} F^{ab} p_b \end{aligned} \tag{119}$$

τ is the parameter for the trajectory of the particle (with mass m). We have chosen a specific parametrization or equivalently a gauge-fixing for the gauge freedom of reparametrizations of the trajectory and so the equations (119) are not invariant under reparametrizations. Equations (119) tell us that the infinitesimal change of τ is given, on the phase space, by a vector field

$$V = \frac{p^a}{m}\frac{\partial}{\partial x^a} - \frac{e}{m} F^{ab} p_b \frac{\partial}{\partial p^a} \tag{120}$$

The canonical generator of the τ-evolution, say, G, is defined by $i_V \Omega = -dG$. This gives $G = -p^2/(2m) + \text{constant}$. Anticipating the eventual value of the constant, we choose it to be $m/2$. Basically this is the *definition* of the mass. Thus

$$G = -\frac{1}{2m}(p^2 - m^2) \tag{121}$$

Since we need reparametrization invariance, the τ-evolution must be trivial. Thus we must set G_0 to be zero for the classical trajectories. Quantum theoretically, this can be implemented by

$$G\Psi = 0 \tag{122}$$

This will be the basic dynamical equation of the quantum theory.

The symplectic form may be written as

$$\begin{aligned} \Omega &= \frac{1}{2}\Omega_{AB}\ d\xi^A \wedge d\xi^B \\ \Omega_{AB} &= \begin{pmatrix} 0 & -g_{ab} \\ g_{ab} & eF_{ab} \end{pmatrix} \end{aligned} \tag{123}$$

where $\xi^A = (p^a, x^a)$ denotes both sets of phase space variables. This symplectic form leads to the commutation rules

$$\begin{aligned} [x^a,\ x^b] &= 0 \\ [p^a,\ x^b] &= ig^{ab} \\ [p^a,\ p^b] &= ieF^{ab} \end{aligned} \tag{124}$$

These relations are solved by

$$p_a = i\partial_a + eA_a \tag{125}$$

The condition of trivial τ-evolution, namely, (122), becomes

$$[(\partial_a - ieA_a)^2 + m^2]\Psi = 0 \tag{126}$$

This is the Schrödinger equation (in this case, the Klein-Gordon equation) which describes the quantum dynamics of the particle. One can easily show that the Lorentz equations (119) are quantum mechanically realized by

$$i\frac{\partial \xi^A}{\partial t} = [\xi^A, G] \tag{127}$$

In a more general situation, one can obtain the Schrödinger-type equation as follows. We start with the symplectic two-form. From the equations of motion, we find the generator of the τ-evolution. Setting this generator to zero on the wave functions gives us the equation we are seeking. To realize this as a differential equation we must solve the commutation rules in terms of a set of coordinates and their derivatives (namely, canonical variables).

For a free anyon with spin $-s$, the symplectic structure is given by

$$\Omega = dx^a \wedge dp_a + \frac{1}{2}\ s\ \epsilon_{abc}\frac{p^a dp^b \wedge dp^c}{(p^2)^{3/2}} \tag{128}$$

The commutation rules are given by

$$\begin{aligned} [x^a,\ x^b] &= is\ \epsilon^{abc}\frac{p_c}{(p^2)^{3/2}} \\ [p^a,\ x^b] &= ig^{ab} \\ [p^a,\ p^b] &= 0 \end{aligned} \tag{129}$$

Consider the Lorentz generator J^a defined by

$$\begin{aligned} [J^a,\ p^b] &= i\epsilon^{abc}p_c \\ [J^a,\ x^b] &= i\epsilon^{abc}x_c \end{aligned} \tag{130}$$

It is easy to see that J^a is given by

$$J^a = -\epsilon^{abc}x_b p_c - s\frac{p^a}{\sqrt{p^2}} \tag{131}$$

This shows that the particle has a spin $-s$, easily seen in the rest frame with $p^0 = m,\ p^1 = p^2 = 0$. The expression for J^a is analogous to (116) except for the different signature for spacetime. Because of this change of signature, there is no quantization of the coefficient of the second term in the symplectic structure (128). The value s is not quantized. Alternatively, there is no closed two surface which is not the boundary of a three-volume. We can solve the commutation rules (129) in terms of canonical variables as

$$\begin{aligned} x^a &= q^a + \alpha^a(p) \\ \alpha^a(p) &= s\epsilon^{abc}\frac{p_b\eta_c}{p^2 + \sqrt{p^2}p\cdot\eta} \end{aligned} \tag{132}$$

where $\eta^a = (1,0,0)$ and $[q^a, q^b] = 0,\ [p^a, p^b] = 0,\ [p^a, q^b] = ig^{ab}$ or $q^a = -i\frac{\partial}{\partial p_a}$.

Using (132) for x^a in (131), we can write

$$J^a = -i\epsilon^{abc}p_b\frac{\partial}{\partial p^c} - s\ \frac{p^a + \sqrt{p^2}\eta^a}{\sqrt{p^2} + p\cdot\eta} \tag{133}$$

We see that $p\cdot J + s\sqrt{p^2} = 0$. With $p^2 = m^2$, we see that the spin is indeed $-s$. Thus the symplectic structure Ω of (128) is indeed appropriate to describe the anyon.

The symplectic structure Ω for anyons in an electromagnetic field is now obtained by $\Omega \to \Omega + eF$. Thus

$$\Omega = dx^a \wedge dp_a + \tfrac{1}{2}s\epsilon_{abc}\frac{p^a dp^b \wedge dp^c}{(p^2)^{3/2}} + \tfrac{1}{2}eF_{ab}dx^a \wedge dx^b + \mathcal{O}(\partial F) \tag{134}$$

With the introduction of spin it is possible that Ω has further corrections that depend on the gradients of the field strength F. This is indicated by $\mathcal{O}(\partial F)$ in the above equation. We shall not discuss the case of anyons in an electromagnetic field in any more detail here. One can actually obtain a wave equation as indicated and show that the gyromagnetic ratio for anyons is 2.

6.6 Field quantization, equal-time, and light-cone

We consider a real scalar field φ with the action

$$\mathcal{S} = \int d^4x\ \frac{1}{2}\left[\frac{\partial\varphi}{\partial x^0}\frac{\partial\varphi}{\partial x^0} - \frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^i}\right]\ -\ U(\varphi) \tag{135}$$

where $U(\varphi) = \frac{1}{2}m^2\varphi^2 + V(\varphi)$. By considering a general variation and identifying the boundary term at the initial and final time-slices, we find, using (31),

$$\mathcal{A} = \int d^3x \; \dot{\varphi} \; \delta\varphi \tag{136}$$

where we use δ to denote exterior derivatives on the field space and $\dot{\varphi} = \partial\varphi/\partial x^0$. The time-derivative of φ must be treated as an independent variable since $\mathcal{A}$ is at a fixed time. By taking another variation we find the symplectic two-form

$$\Omega = \int d^3x \; \delta\dot{\varphi} \wedge \delta\varphi \tag{137}$$

Recalling that the Poisson brackets for the coordinates on the phase space are the inverse of the symplectic two-form as a matrix, we find

$$\begin{aligned} \{ \; \varphi(\vec{x}, x^0), \varphi(\vec{x}', x^0) \; \} &= 0 \\ \{ \; \varphi(\vec{x}, x^0), \dot{\varphi}(\vec{x}', x^0) \; \} &= \delta^{(3)}(x - x') \\ \{ \; \dot{\varphi}(\vec{x}, x^0), \dot{\varphi}(\vec{x}', x^0) \; \} &= 0 \end{aligned} \tag{138}$$

Upon replacing the variables by operators with commutation rules given by i times the Poisson brackets, we get the standard equal-time rules for quantization.

This phase space also has a standard Kähler structure. Consider the fields to be confined to a cubical box with each side of length L and volume V. With periodic boundary conditions, we can write a set of mode functions as

$$\begin{aligned} u_k(\vec{x}) &= \frac{1}{\sqrt{V}} \; \exp(-i\vec{k} \cdot \vec{x}) \\ k_i &= \frac{2\pi n_i}{L} \end{aligned} \tag{139}$$

Here n_i are integers. The fields can be expanded in modes as

$$\begin{aligned} \varphi(x) &= \sum_k q_k u_k(\vec{x}) \\ \dot{\varphi}(x) &= \sum_k p_k u_k(\vec{x}) \end{aligned} \tag{140}$$

The reality of the fields requires $q_k^* = q_{-k}, \; p_k^* = p_{-k}$. Substituting the mode expansion and simplifying, Ω becomes

$$\begin{aligned} \Omega &= \sum_k \delta p_k \wedge \delta q_{-k} \\ &= i \; \sum_k \delta a_k \wedge \delta a_k^* \end{aligned} \tag{141}$$

where we define

$$a_k = \frac{1}{\sqrt{2}}(q_{-k} - ip_k), \qquad a_k^* = \frac{1}{\sqrt{2}}(q_k + ip_{-k}) \tag{142}$$

This shows the Kähler structure and we can carry out the holomorphic quantization as in the case of coherent states.

A somewhat more interesting example which illustrates the use of the symplectic structure $\omega_{ij}(\vec{x}, \vec{x}')$ is the light-cone quantization of a scalar field. We introduce light-cone coordinates, corresponding to a light-cone in the z-direction as

$$\begin{aligned} u &= \frac{1}{\sqrt{2}}(z+t) \\ v &= \frac{1}{\sqrt{2}}(z-t) \end{aligned} \tag{143}$$

Instead of considering evolution of the fields in time t, we can consider evolution in one of the the light-cone coordinates, say, u. The other light-cone coordinate v and the two coordinates $x^T = x, y$ transverse to the light-cone parametrize the equal-u hypersurfaces. Field configurations $\varphi(u, v, x, y)$ at fixed values of u, i.e., real-valued functions of v, x, y, characterize the trajectories. They form the phase space of the theory. The action can be written as

$$\mathcal{S} = \int du\ dv\ d^2x^T\ \left[-\partial_u\varphi\partial_v\varphi - \tfrac{1}{2}(\partial_T\varphi)^2 - U(\varphi)\right] \tag{144}$$

Again from the variation of the action $\mathcal{S}$, we can identify the canonical one-form $\mathcal{A}$ as

$$\mathcal{A} = \int dv\ d^2x^T\ (-\partial_v\varphi\ \delta\varphi) \tag{145}$$

(This was denoted by Θ in Chapter 3.) The symplectic two-form is given by

$$\begin{aligned} \Omega &= \frac{1}{2}\int d\mu d\mu'\ \Omega(v, x^T, v', x'^T)\ \delta\varphi(v, x^T) \wedge \delta\varphi(v', x'^T) \\ \Omega(v, x^T, v', x'^T) &= -2\ \partial_v\delta(v-v')\delta^{(2)}(x^T - x'^T) \\ d\mu &= dv\ d^2x^T \end{aligned} \tag{146}$$

From this point on, the calculation is identical to what we did in Chapter 3. The fundamental Poisson bracket can be written down from the inverse to Ω. Writing

$$\delta(v-v')\ \delta^{(2)}(x^T - x'^T) = \int \frac{d^3p}{(2\pi)^3} e^{-ip_u(v-v') - ip^T\cdot(x^T - x'^T)} \tag{147}$$

we see that

$$
\begin{aligned}
\Omega^{-1}(v,x^T,v',x'^T) &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{ip_u}\ \exp(-ip_u(v-v')-ip^T\cdot(x^T-x'^T)) \\
&= \frac{1}{4}\epsilon(v-v')\ \delta^{(2)}(x^T-x'^T) \qquad (148)
\end{aligned}
$$

Here $\epsilon(v-v')$ is the signature function, equal to 1 for $v-v'>0$ and equal to -1 for $v-v'<0$.

The phase space is thus given by field configurations $\varphi(v,x^T)$ with the Poisson brackets

$$
\{\varphi(u,v,x^T),\varphi(u,v',x'^T)\} = \frac{1}{4}\epsilon(v-v')\ \delta^{(2)}(x^T-x'^T) \qquad (149)
$$

The Hamiltonian for u-evolution is given by

$$
H = \int dv\ d^2x^T\ \left[\frac{1}{2}(\partial_T\varphi)^2\ +\ U(\varphi)\right] \qquad (150)
$$

The Hamiltonian equations of motion are easily checked using the Poisson brackets (149).

Quantization is achieved by taking φ to be an operator with commutation rules given by i-times the Poisson bracket.

6.7 The Chern-Simons theory in 2+1 dimensions

The Chern-Simons (CS) theory is a gauge theory in two space (and one time) dimensions. The action is given by

$$
\begin{aligned}
\mathcal{S} &= -\frac{k}{4\pi}\int_{\Sigma\times[t_i,t_f]} \mathrm{Tr}\left[AdA+\frac{2}{3}A^3\right] \\
&= -\frac{k}{4\pi}\int_{\Sigma\times[t_i,t_f]} d^3x\ \epsilon^{\mu\nu\alpha}\ \mathrm{Tr}\left[A_\mu\partial_\nu A_\alpha+\frac{2}{3}A_\mu A_\nu A_\alpha\right] \qquad (151)
\end{aligned}
$$

Here A_μ is the Lie-algebra-valued gauge potential, $A_\mu = -it^aA^a_\mu$. t^a are hermitian matrices forming a basis of the Lie algebra in the fundamental representation of the gauge group. We shall consider the gauge group to be $SU(N)$ in what follows, and normalize the t^a as $\mathrm{Tr}(t^at^b) = \frac{1}{2}\delta^{ab}$. k is a constant whose precise value we do not need to specify at this stage. We shall consider the spatial manifold to be some Riemann surface Σ and we shall be using complex coordinates. The equations of motion for the theory are

$$F_{\mu\nu} = 0 \tag{152}$$

The theory is best analyzed, for our purposes, in the gauge where A_0 is set to zero. In this gauge, the equations of motion (152) tell us that $A_z = \frac{1}{2}(A_1 + iA_2)$ and $A_{\bar{z}} = \frac{1}{2}(A_1 - iA_2)$ are independent of time, but must satisfy the constraint

$$F_{\bar{z}z} \equiv \partial_{\bar{z}} A_z - \partial_z A_{\bar{z}} + [A_{\bar{z}}, A_z] = 0 \tag{153}$$

This constraint is just the Gauss law of the CS gauge theory.

In the $A_0 = 0$ gauge, the action becomes

$$\mathcal{S} = -\frac{ik}{\pi} \int dt d\mu_\Sigma \, \mathrm{Tr}(A_{\bar{z}} \partial_0 A_z) \tag{154}$$

For the boundary term from the variation of the action we get

$$\delta\mathcal{S} \;=\; -\frac{ik}{\pi} \int_\Sigma \mathrm{Tr}(A_{\bar{z}} \delta A_z) \Big]_{t_i}^{t_f} \tag{155}$$

We can now identify the symplectic potential as

$$\mathcal{A} \;=\; -\frac{ik}{\pi} \int_\Sigma \mathrm{Tr}(A_{\bar{z}} \delta A_z) \;+\; \delta\rho[A] \tag{156}$$

where $\rho[A]$ is an arbitrary functional of A. The freedom of adding $\delta\rho$ is the freedom of canonical transformations. As in the case of the scalar field, δ is to be interpreted as denoting exterior differentiation on $\tilde{\mathcal{A}}$, the space of gauge potentials on Σ. $\tilde{\mathcal{A}}$ is also the phase space of the theory before reduction by the action of gauge symmetries. (The space of gaueg potentials was denoted by $\mathcal{A}$ in Chapter 16; here we use $\tilde{\mathcal{A}}$ to avoid confusion with the symplectic potential.)

The symplectic two-form Ω is given by $\delta\mathcal{A}$, i.e.,

$$\begin{aligned} \Omega \;&=\; -\frac{ik}{\pi} \int_\Sigma \mathrm{Tr}(\delta A_{\bar{z}} \delta A_z) = \frac{k}{4\pi} \int_\Sigma \mathrm{Tr}(\delta A \wedge \delta A) \\ &=\; \frac{ik}{2\pi} \int_\Sigma \delta A^a_{\bar{z}} \delta A^a_z \end{aligned} \tag{157}$$

(We will not write the wedge sign for exterior products on the field space from now on since it is clear from the context.)

The complex structure on Σ induces a complex structure on $\tilde{\mathcal{A}}$. $A_z,\ A_{\bar{z}}$ can be taken as the local complex coordinates on $\tilde{\mathcal{A}}$. Indeed we have a Kähler structure on $\tilde{\mathcal{A}}$; Ω is k times the Kähler form on $\tilde{\mathcal{A}}$ and we can associate a Kähler potential K with Ω given by

$$K \;=\; \frac{k}{2\pi}\int_\Sigma A^a_{\bar{z}} A^a_z \tag{158}$$

Poisson brackets for $A_{\bar{z}}$, A_z are obtained by inverting the components of Ω and read

$$\begin{aligned} \{A^a_z(z), A^b_w(w)\} &= 0 \\ \{A^a_{\bar{z}}(z), A^b_{\bar{w}}(w)\} &= 0 \\ \{A^a_z(z), A^b_{\bar{w}}(w)\} &= -\frac{2\pi i}{k}\delta^{ab}\delta^{(2)}(z-w) \end{aligned} \tag{159}$$

These become commutation rules upon quantization.

Gauge transformations are given by

$$A^g = gAg^{-1} - dgg^{-1} \tag{160}$$

Infinitesimal gauge transformations are generated by the vector field

$$\xi = -\int_\Sigma \left[(D_z\theta)^a \frac{\delta}{\delta A^a_z} \;+\; (D_{\bar{z}}\theta)^a \frac{\delta}{\delta A^a_{\bar{z}}}\right] \tag{161}$$

where D_z and $D_{\bar{z}}$ denote the corresponding gauge covariant derivatives. By contracting this with Ω we get

$$i_\xi \Omega \;=\; -\,\delta\left[\frac{ik}{2\pi}\int_\Sigma F^a_{z\bar{z}}\theta^a\right] \tag{162}$$

which shows that the generator of infinitesimal gauge transformations is

$$G^a = \frac{ik}{2\pi} F^a_{z\bar{z}} \tag{163}$$

Reduction of the phase space can thus be performed by setting F to zero. This is also the equation of motion we found for the component A_0. Notice also that

$$\Omega(A^g) \;-\; \Omega(A) = \delta\left[\frac{k}{2\pi}\int_\Sigma \mathrm{Tr}(g^{-1}\delta g\ F)\right] \tag{164}$$

(In the second term F is the two-form $dA + A \wedge A$.)

The reduced set of field configurations are elements of $\tilde{\mathcal{A}}/\mathcal{G}_*$ where $\mathcal{G}_*$ denotes the group of gauge transformations, $\mathcal{G}_* \; = \; \{g(x) : \Sigma \to G\}$.

The construction of the wave functionals proceeds as follows. One has to consider a line bundle on the phase space with curvature Ω. Sections of this bundle give the prequantum Hilbert space. In other words, we consider functionals $\Phi[A_z, A_{\bar{z}}]$ with the condition that under the canonical transformation $\mathcal{A} \to \mathcal{A} \; + \; \delta\Lambda, \quad \Phi \to e^{(i\Lambda)}\Phi$. The inner product on the prequantum Hilbert space is given by

$$\langle 1|2\rangle = \int \; d\mu(A_z, A_{\bar{z}}) \; \Phi_1^*[A_z, A_{\bar{z}}] \; \Phi_2[A_z, A_{\bar{z}}] \tag{165}$$

where $d\mu(A_z, A_{\bar{z}})$ is the Liouville measure associated with Ω. Given the Kähler structure, this is just the volume $[dA_z dA_{\bar{z}}]$ associated with the metric $||\delta A||^2 \; = \; \int_\Sigma \; \delta A_{\bar{z}} \delta A_z$.

The wave functionals so constructed depend on all phase space variables. We must now choose the polarization conditions on the Φ's so that they depend only on half the number of phase space variables. This reduction of the prequantum Hilbert space leads to the Hilbert space of the quantum theory. Given the Kähler structure of the phase space, the most appropriate choice is the Bargmann polarization which can be implemented as follows. With a specific choice of $\rho[A]$ in (156), the symplectic potential can be taken as

$$\mathcal{A} \; = \; -\frac{ik}{2\pi}\int_\Sigma \; \mathrm{Tr}(A_{\bar{z}}\delta A_z \; - \; A_z \delta A_{\bar{z}}) \; = \; \frac{ik}{4\pi}\int_\Sigma (A^a_{\bar{z}}\delta A^a_z - A^a_z \delta A^a_{\bar{z}}) \tag{166}$$

The covariant derivatives with $\mathcal{A}$ as the potential are

$$\nabla \; = \; (\frac{\delta}{\delta A^a_z} + \frac{k}{4\pi}A^a_{\bar{z}}), \qquad \overline{\nabla} \; = \; (\frac{\delta}{\delta A^a_{\bar{z}}} - \frac{k}{4\pi}A^a_z) \tag{167}$$

The Bargmann polarization condition is

$$\nabla \; \Phi = 0 \tag{168}$$

or

$$\Phi \; = \; \exp\left(-\frac{k}{4\pi}\int A^a_{\bar{z}} A^a_z\right) \; \psi[A^a_{\bar{z}}] \; = \; e^{-\frac{1}{2}K} \; \psi[A^a_{\bar{z}}] \tag{169}$$

where K is the Kähler potential of (158). The states are represented by wave functionals $\psi[A^a_{\bar{z}}]$ which are holomorphic in $A^a_{\bar{z}}$. Further, the prequantum inner product can be retained as the inner product of the Hilbert space. Rewriting (165) using (169), we get the inner product as

$$\langle 1|2\rangle \;=\; \int [dA^a_{\bar z}, A^a_z]\; e^{-K(A^a_{\bar z},A^a_z)}\; \psi_1^*\; \psi_2 \tag{170}$$

On the holomorphic wave functionals,

$$A^a_z\; \psi[A^a_{\bar z}] = \frac{2\pi}{k}\frac{\delta}{\delta A^a_{\bar z}}\; \psi[A^a_{\bar z}] \tag{171}$$

As we have mentioned before, one has to make a reduction of the Hilbert space by imposing gauge invariance on the states, i.e., by setting the generator $F^a_{z\bar z}$ to zero on the wave functionals. This amounts to

$$\left(D_{\bar z}\;\frac{\delta}{\delta A^a_{\bar z}} \;-\; \frac{k}{2\pi}\partial_z A^a_{\bar z}\right)\; \psi[A^a_{\bar z}] \;= 0. \tag{172}$$

Consistent implementation of gauge invariance can lead to quantization requirements on the coupling constant k. For nonabelian groups G this is essentially the requirement of integrality of k based on the invariance of e^{iS} under homotopically nontrivial gauge transformations. We now show how this constraint arises in the geometric quantization framework. Consider first the nonabelian theory on $\Sigma = S^2$. The group of gauge transformations $\mathcal{G} = \{g(x):\; S^2 \to G\}$. Obviously $\Pi_0(\mathcal{G}) = \Pi_2(G) = 0$ and $\Pi_1(\mathcal{G}) = \Pi_3(G) = \mathbf{Z}$. Correspondingly, one has $\Pi_1(\tilde{\mathcal{A}}/\mathcal{G}) = 0$ and $\Pi_2(\tilde{\mathcal{A}}/\mathcal{G}) = \mathbf{Z}$. The nontriviality of $\Pi_2(\tilde{\mathcal{A}}/\mathcal{G})$ arises from the nontrivial elements of $\Pi_1(\mathcal{G})$. Therefore, consider a noncontractible loop C of gauge transformations,

$$\begin{aligned} C \;&=\; g(x,\lambda), \qquad 0\le\lambda\le 1\\ g(x,0) \;&=\; g(x,1) \;=\; 1 \end{aligned} \tag{173}$$

We can use this to construct a noncontractible two-surface in the gauge-invariant space $\tilde{\mathcal{A}}/\mathcal{G}$.

We start with a square in the space of gauge potentials parametrized by $0 \le \lambda, \sigma \le 1$ with the potentials given by

$$A(x,\lambda,\sigma) \;=\; (gAg^{-1} -\; dgg^{-1})\;\sigma \;+\; (1-\sigma)A \tag{174}$$

The potential is A on the boundaries $\lambda = 0$ and $\lambda = 1$ and also on $\sigma = 0$. It is equal to the gauge transform A^g of A at $\sigma = 1$. Since A^g is identified with A in the quotient space, the boundary corresponds to a single point on the quotient $\tilde{\mathcal{A}}/\mathcal{G}$ and we have a closed two-surface. This surface is noncontractible if we take $g(x,\lambda)$ to be a nontrivial element of $\Pi_3(G) = \mathbf{Z}$. We can now integrate Ω over this closed two-surface; for this calculation, we may even put A equal to zero and use $A(x,\lambda,\sigma) = -\sigma dgg^{-1}$. We then have $\delta A = -\delta\sigma dgg^{-1} - \sigma d(\delta gg^{-1}) - \sigma[\delta gg^{-1}, dgg^{-1}]$. Using this in the expression for Ω and carrying out the integration over σ, we get

$$\begin{aligned}\int \Omega &= \frac{k}{4\pi}\int \mathrm{Tr}(dgg^{-1})^2\delta gg^{-1}\\ &= \frac{k}{12\pi}\int \mathrm{Tr}(dgg^{-1})^3\\ &= -2\pi k Q[g]\end{aligned} \tag{175}$$

where, in the second step $dgg^{-1} = \partial_i gg^{-1}dx^i + \partial_\lambda gg^{-1}d\lambda$; we include differentiation with respect to the spatial coordinates and with respect to the internal coordinate λ. $Q[g]$ is the winding number (which is an integer) characterizing the class in $\Pi_1(\mathcal{G}) = \Pi_3(G)$ to which g belongs; it is given by

$$Q[g] = -\frac{1}{24\pi^2}\int \mathrm{Tr}(dgg^{-1})^3 \tag{176}$$

From general principles of geometric quantization we know that the integral of Ω over any closed noncontractible two-surface in the phase space must be an integer, see (58). Thus (175) and (176) lead to the requirement that k has to be an integer. This argument can be generalized to other choices of Σ.

The situation for an Abelian group such as $U(1)$ is somewhat different. Consider the case of $G = U(1)$ and with Σ being a torus $S^1 \times S^1$. This can be described by $z = \xi_1 + \tau\xi_2$, where ξ_1, ξ_2 are real and have periodicity of $\xi_i \to \xi_i +$ integer, and τ, which is a complex number, is the modular parameter of the torus. The metric on the torus is $ds^2 = |d\xi_1 + \tau d\xi_2|^2$. The two basic noncontractible cycles (noncontractible closed curves) of the torus are usually labeled as the α and β cycles. Further the torus has a holomorphic one-form ω with

$$\int_\alpha \omega = 1, \qquad\qquad \int_\beta \omega = \tau \tag{177}$$

Since ω is a zero mode of $\partial_{\bar z}$, we can parametrize $A_{\bar z}$ as

$$A_{\bar z} = \partial_{\bar z}\chi \;+\; i\,\frac{\pi\bar\omega}{\mathrm{Im}\tau}\,a \tag{178}$$

where a is a complex number corresponding to the value of $A_{\bar{z}}$ along the zero mode of ∂_z. This is the Abelian version of (**??**).

For this space, $\Pi_0(\mathcal{G}) = \mathbf{Z} \times \mathbf{Z}$, by virtue of gauge transformations $g_{m,n}$ with nontrivial winding numbers m, n around the two cycles. Consider one connected component of $\mathcal{G}$, say, $\mathcal{G}_{m,n}$. A homotopically nontrivial $U(1)$ transformation can be written as $g_{m,n} = e^{i\lambda}\, e^{i\theta_{m,n}}$, where $\lambda(z,\bar{z})$ is a homotopically trivial gauge transformation and

$$\theta_{m,n} \;=\; \frac{i\pi}{\mathrm{Im}\tau}\left[m\int^z(\bar{\omega}-\omega) \;+\; n\int^z(\tau\bar{\omega}-\bar{\tau}\omega)\right] \tag{179}$$

With the parametrization of $A_{\bar{z}}$ as in (178), the effect of this gauge transformation can be represented as

$$\begin{aligned} \chi &\;\rightarrow\; \chi \;+\; \lambda \\ a &\;\rightarrow\; a \;+\; m \;+\; n\tau \end{aligned} \tag{180}$$

The real part of χ can be set to zero by an appropriate choice of λ. (The imaginary part also vanishes when we impose the condition $F_{z\bar{z}} = 0$.) The physical subspace of the zero modes is given by the values of a modulo the transformation (180), or in other words,

$$\begin{aligned} \text{Physical space for zero modes} &\;\equiv\; \mathcal{C} \\ &\;=\; \frac{\mathbf{C}}{\mathbf{Z}+\tau\mathbf{Z}} \end{aligned} \tag{181}$$

This space is known as the Jacobian variety of the torus. It is also a torus, and therefore we see that the phase space $\mathcal{C}$ has nontrivial Π_1 and $\mathcal{H}^2$. In particular, $\Pi_1(\mathcal{C}) = \mathbf{Z} \times \mathbf{Z}$, and this leads to two angular parameters φ_α and φ_β which can be related to the phases the wave functions acquire under the gauge transformation $g_{1,1}$. The symplectic two-form for the zero modes can be written as

$$\begin{aligned} \Omega &\;=\; \frac{k\pi}{4}\frac{d\bar{a}\wedge da}{\mathrm{Im}\tau}\int_\Sigma \frac{\bar{\omega}\wedge\omega}{\mathrm{Im}\tau} \\ &\;=\; -i\frac{k\pi}{2}\frac{d\bar{a}\wedge da}{\mathrm{Im}\tau} \end{aligned} \tag{182}$$

Integrating the zero-mode part over the physical space of zero modes $\mathcal{C}$, we get

$$\int_{\mathcal{C}} \Omega = k\pi \tag{183}$$

showing that k must be quantized as an even integer for $U(1)$ fields on the torus due to (58). (The integrality requirement on Ω arises from the use of wave functions which are one-dimensional, i.e., sections of a line bundle. If we use more general vector bundles, this quantization requirement can be relaxed; however, the probabilistic interpretation of such wave functions is not very clear.)

The symplectic potential for the zero modes can be written as

$$\mathcal{A} = -\frac{\pi k}{4} \frac{(\bar{a} - a)(\tau d\bar{a} - \bar{\tau} da)}{(\mathrm{Im}\tau)^2} \tag{184}$$

The polarization condition then becomes

$$\left[\frac{\partial}{\partial \bar{a}} + i\frac{\pi k}{4} \frac{(\bar{a} - a)\tau}{(\mathrm{Im}\tau)^2}\right] \psi = 0 \tag{185}$$

with the solution

$$\psi = \exp\left[-i\frac{\pi k}{8} \frac{(\bar{a} - a)^2 \tau}{(\mathrm{Im}\tau)^2}\right] \quad f(a) \tag{186}$$

where $f(a)$ is holomorphic in a. Under the gauge transformation (180) we find

$$\psi(a+m+n\tau) = \exp\left[-i\frac{\pi k(\bar{a} - a)^2 \tau}{8(\mathrm{Im}\tau)^2} - \frac{\pi kn(\bar{a} - a)\tau}{2\mathrm{Im}\tau} + i\frac{\pi k\tau n^2}{2}\right] f(a+m+n\tau) \tag{187}$$

Under this gauge transformation $\mathcal{A}$ changes by $d\Lambda_{m,n}$ where

$$\Lambda_{m,n} = i\frac{\pi kn(\tau\bar{a} - \bar{\tau}a)}{2\ \mathrm{Im}\tau} \tag{188}$$

The change in ψ should thus be given by $\exp(i\Lambda_{m,n})\psi$; requiring the transformation (187) to be equal to this, we get

$$f(a + m + n\tau) = \exp\left[-i\frac{\pi kn^2\tau}{2} - \pi ikna\right] \ f(a) \tag{189}$$

This shows that $f(a)$ is a Jacobi Θ-function. On these, $\bar{a}$ is realized as $(2\ \mathrm{Im}\tau/k\pi)(\partial/\partial a) + a$. The inner product for the wave functions of the zero modes is

$$\langle f|g\rangle = \int \exp\left[-\frac{\pi k\bar{a}a}{2\ \mathrm{Im}\tau} + \frac{\pi k\bar{a}^2}{4\ \mathrm{Im}\tau} + \frac{\pi ka^2}{4\ \mathrm{Im}\tau}\right] \ \bar{f}g \tag{190}$$

It is then convenient to introduce the wave functions

$$\begin{aligned}\Psi &= \exp\left[\frac{\pi k a^2}{4\ \mathrm{Im}\tau}\right]\ f(a)\\ &= \exp\left[\frac{\pi k a^2}{4\ \mathrm{Im}\tau}\right]\ \Theta(a)\end{aligned} \tag{191}$$

On these functions, $\bar{a}$ acts as

$$\bar{a} = \frac{2\ \mathrm{Im}\tau}{\pi k}\ \frac{\partial}{\partial a} \tag{192}$$

6.8 θ-vacua in a nonabelian gauge theory

Consider a nonabelian gauge theory in four spacetime dimensions; the gauge group is some compact Lie group G. We can choose the gauge where $A_0 = 0$ so that there are only the three spatial components of the gauge potential, namely, A_i, considered as an antihermitian Lie-algebra-valued vector field. The choice $A_0 = 0$ does not completely fix the gauge, one can still do gauge transformations which are independent of time. These are given by

$$A_i \rightarrow A_i' = gA_i g^{-1} - \partial_i g\ g^{-1} \tag{193}$$

The Yang-Mills action gives the symplectic two-form as

$$\begin{aligned}\Omega &= \int d^3x\ \delta E_i^a\ \delta A_i^a\\ &= -2\int d^3x\ \mathrm{Tr}\left(\delta E_i\ \delta A_i\right)\end{aligned} \tag{194}$$

where E_i^a is the electric field $\partial_0 A_i^a$, along the Lie algebra direction labeled by a. The gauge transformation of E_i is $E_i \rightarrow gE_i g^{-1}$. The vector field generating infinitesimal gauge transformations, with $g \approx 1 + \varphi$, is thus

$$\xi = -\int d^3x\ \left[(D_i\varphi)^a \frac{\delta}{\delta A_i^a} + [E_i, \varphi]^a \frac{\delta}{\delta E_i^a}\right] \tag{195}$$

This leads to

$$i_\xi \Omega = -\delta \int d^3x\ \left[-(D_i\varphi)^a E_i^a\right] \tag{196}$$

The generator of time-independent gauge transformations is thus

$$G(\varphi) = -\int d^3x\ (D_i\varphi)^a E_i^a \tag{197}$$

For transformations which go to the identity at spatial infinity, $G(\varphi) = \int \varphi^a G^a$, $G^a = (D_i E_i)^a$. $G^a = 0$ is one of the Yang-Mills equations of motion; it is the Gauss law of the theory. In the context of quantization, this is to be viewed as a condition on the allowed initial data and enforces a reduction of the phase space to gauge-invariant variables.

As discussed in Chapter 16, the spaces of interest are

$$\tilde{\mathcal{A}} = \Big\{ \text{space of gauge potentials } A_i \Big\} \tag{198}$$

$$\mathcal{G}_* = \Big\{ \text{space of gauge transformations } g(\vec{x}) : \mathbf{R}^3 \to G \quad \text{such that } g \to 1 \text{ as } |\vec{x}| \to \infty \Big\} \tag{199}$$

The transformations $g(\vec{x})$ which go to a constant element $g_\infty \neq 1$ act as a Noether symmetry. The states fall into unitary irreducible representations of such transformations, which are isomorphic to the gauge group G, up to $\mathcal{G}_*$-transformations. The true gauge freedom is only $\mathcal{G}_*$. The physical configuration space of the theory is thus $\mathcal{C} = \tilde{\mathcal{A}}/\mathcal{G}_*$. In Chapter 16, we also noted that $\mathcal{G}_*$ has an infinity of connected components so that

$$\mathcal{G}_* = \sum_{Q=-\infty}^{+\infty} \oplus \, \mathcal{G}_{*Q} \tag{200}$$

Q is the winding number characterizing the homotopy classes of gauge transformations. The space of gauge potentials $\tilde{\mathcal{A}}$ is an affine space and is topologically trivial. Combining this with $\Pi_0(\mathcal{G}_*) = \mathbf{Z}$, we see that the configurations space has noncontractible loops, with $\Pi_1(\mathcal{C}) = \mathbf{Z}$. Our general discussion shows that there must be an angle θ which appears in the quantum theory. We can see how this emerges by writing the symplectic potential.

The instanton number $\nu[A]$ for a four-dimensional potential is given by

$$\begin{aligned} \nu[A] &= -\frac{1}{32\pi^2} \int d^4x \; \mathrm{Tr}\,(F_{\mu\nu} F_{\alpha\beta})\, \epsilon^{\mu\nu\alpha\beta} \\ &= \frac{1}{16\pi^2} \int d^4x \; E_i^a F_{jk}^a \epsilon^{ijk} \end{aligned} \tag{201}$$

The density in the above integral is a total derivative in terms of the potential A, but it cannot be written as a total derivative in terms of gauge-invariant quantities. $\nu[A]$ is an integer for any field configuration which is nonsingular up to gauge transformations. It is possible to construct configurations which have nonzero values of ν and which are nonsingular; these are the instantons, also considered briefly in Chapter 16.

We may think of configurations $A(\vec{x}, x_4)$ as giving a path in $\tilde{\mathcal{A}}$ with x_4 parametrizing the path. $\nu[A]$ can be written as

$$\begin{aligned} \nu[A] &= \oint K[A] \\ K[A] &= \int d^3x \; F^a_{jk} \delta A^a_i \epsilon^{ijk} \end{aligned} \tag{202}$$

The integral of the one-form K around a closed curve is the instanton number ν and is nonzero, in particular, for the loop corresponding to the instanton configuration. We can also see that this one-form is closed as follows:

$$\begin{aligned} \delta K[A] &= -2 \int d^3x \; \delta \mathrm{Tr} \left[F_{jk} \delta A_i\right] \epsilon^{ijk} \\ &= -4 \int d^3x \; \mathrm{Tr} \left[(D_j \delta A_k) \delta A_i\right] \epsilon^{ijk} \\ &= -4 \int d^3x \; \mathrm{Tr} \left[\partial_j \delta A_k \; \delta A_i + [A_j, \delta A_k] \delta A_i\right] \epsilon^{ijk} \\ &= 0 \end{aligned} \tag{203}$$

In the last step we have used the antisymmetry of the expression under permutation of δ's, cyclicity of the trace, and have done a partial integration. We see from the above discussion that $K[A]$ is a closed one-form which is not exact since its integral around the closed curves can be nonzero.

The general solution for the symplectic potential corresponding to the symplectic two-form in (194) is thus of the form

$$\mathcal{A} = \int d^3x \; E^a_i \delta A^a_i + \theta \; K[A] \tag{204}$$

Use of this potential will lead to a quantum theory where we need the parameter θ, in addition to other parameters such as the coupling constant, to characterize the theory. The potential $\mathcal{A}$ in (204) is obtained from an action

$$\mathcal{S} = -\frac{1}{4} \int d^4x \; F^a_{\mu\nu} F^{a\mu\nu} \; + \; \theta \; \nu[A] \tag{205}$$

Thus the effect of using (204) can be reproduced in the functional integral approach by using the action (205). Since the relevant quantity for the functional integral is $\exp(i\mathcal{S})$, we see that θ is an angle with values $0 \leq \theta < 2\pi$. Alternatively, we can see that one can formally eliminate the θ-term in $\mathcal{A}$ by making a redefinition $\Psi \to \exp(i\theta\Lambda)\Psi$, where

$$\Lambda = -\frac{1}{8\pi^2} \int \mathrm{Tr} \left(AdA + \frac{2}{3} A^3 \right) \tag{206}$$

Notice that $2\pi\Lambda$ is the Chern-Simons action (151) for $k = 1$. Λ is not invariant under homotopically nontrivial transformations. The wave functions get a phase equal to $e^{i\theta Q}$ under the winding number Q-transformation, showing that θ can be restricted to the interval indicated above. This is in agreement with our discussion after equation (52).

6.9 Current algebra for the Wess-Zumino-Witten (WZW) model

The WZW action was introduced in Chapter 17 in the context of evaluating the two-dimensional Dirac determinant. One can think of the WZW action in its own right as defining a field theory in two dimensions. For this one uses the Minkowski signature; the dynamical variables are group-valued fields $g(x^0, x^1)$. The action is given by

$$\mathcal{S}(g) \;=\; -\frac{k}{8\pi} \int_{\mathcal{M}^2} d^2x \; \mathrm{Tr}(\partial_\mu g g^{-1} \partial^\mu g g^{-1}) \;+\; \frac{k}{12\pi} \int_{\mathcal{M}^3} \mathrm{Tr}(dg g^{-1})^3 \tag{207}$$

The first term involves integration over the two-dimensional manifold $\mathcal{M}^2$;the second term, the Wess-Zumino (WZ) term, requires extension of the fields to include one more coordinate, say, s, and corresponding integration. We can take $\mathcal{M}^3$ as a space whose boundary is the two-dimensional world, or we can take $\mathcal{M}^3 = \mathcal{M}^2 \times [0, 1]$ with fields at $s = 1$ corresponding to spacetime. Different ways of extending the fields to $s \neq 1$ will give the same physical results if the coefficient k is an integer. This quantization requirement arises from the single-valuedness of the transition amplitudes or wave functions. This result was also shown in Chapter 17; we just note here that the WZ term, being a differential form, is not sensitive to the signature of the metric and so the argument presented in Chapter 17 will be valid in the Minkowski case as well.

In the Minkowski coordinates we are using here, the Polyakov-Wiegmann identity becomes

$$\mathcal{S}(hg) = \mathcal{S}(h) \;+\; \mathcal{S}(g) \;-\frac{k}{4\pi} \int d^2x \; \mathrm{Tr}(h^{-1}\partial_i h \partial_j g \; g^{-1})(\eta^{ij} + \epsilon^{ij}) \tag{208}$$

where $\eta^{ij} = diag(1,-1)$ is the two-dimensional metric and ϵ^{ij} is the Levi-Civita tensor. By taking small variations, $h \approx 1+\theta$, $\theta \ll 1$, this identity gives the equation of motion

$$(\partial_0 - \partial_1)\ \left(\partial_0 g\ g^{-1} + \partial_1 g\ g^{-1}\right) = 0 \tag{209}$$

This is also equivalent to

$$(\partial_0 + \partial_1)\ \left(g^{-1}\partial_0 g\ - g^{-1}\partial_1 g\right) = 0 \tag{210}$$

There are two commonly used and convenient quantizations of this action which correspond to the equal-time and lightcone descriptions. There is a slight difficulty in obtaining the symplectic potential from the surface term resulting from time-integration. This is because the expression for the WZ term is written for a spacetime manifold which has no boundary, so that it can be the boundary of a three-volume. The variation of of the WZ term can be integrated to give

$$\delta\Gamma_{WZ} = \frac{k}{4\pi}\int_{\mathcal{M}^2} \mathrm{Tr}(\delta g g^{-1} I^2) \tag{211}$$

where $I = dgg^{-1}$. Reintegrating this over the parameter s, we get the form of the WZ term written on $\mathcal{M}^2 \times [0,1]$,

$$\Gamma_{WZ} = \frac{k}{4\pi}\int_0^1 ds \int_{\mathcal{M}^2} \mathrm{Tr}\left[\partial_s g g^{-1} I^2\right] \tag{212}$$

We will use this form to identify the symplectic potential.

First we consider the equal-time approach. The action, separating out the time derivatives, is

$$\mathcal{S}(g) \ = \ -\frac{k}{8\pi}\int_{\mathcal{M}^2} d^2x\ \mathrm{Tr}(\partial_0 g g^{-1})^2 \ + \ \Gamma_{WZ} \tag{213}$$

If we vary this and look at the surface term from the integration over time, we get the symplectic potential as

$$\mathcal{A} = -\frac{k}{4\pi}\int dx\ \mathrm{Tr}(\xi I_0) + \frac{k}{4\pi}\int \mathrm{Tr}(\xi I^2) \tag{214}$$

Here $\xi = \delta g g^{-1}$, and the last term still has integration over s as well as x. Exterior derivatives are given by

$$\begin{aligned} \delta\xi &= \xi^2 \\ \delta I_1 &= \partial_1\xi + \xi I_1 - I_1\xi \end{aligned} \tag{215}$$

Upon taking exterior derivatives, the second term in $\mathcal{A}$ becomes a total derivative and can be integrated over s as follows. Using $I^2 = dI$,

$$\begin{aligned} \delta \frac{k}{4\pi}\int \mathrm{Tr}(\xi I^2) &= \frac{k}{4\pi}\int \mathrm{Tr}\left[\xi^2 I^2 - \xi d(d\xi + \xi I - I\xi)\right] \\ &= \frac{k}{4\pi}\int \mathrm{Tr}\left[-d(\xi^2 I)\right] \\ &= \frac{k}{4\pi}\int \mathrm{Tr}\left[-\partial_1(\xi^2 I_s) + \partial_s(\xi^2 I_1)\right] \\ &= \frac{k}{4\pi}\int \mathrm{Tr}\left[\xi^2 I_1\right] \end{aligned} \tag{216}$$

The symplectic two-form is now obtained as

$$\Omega = -\frac{k}{4\pi}\int dx\ \mathrm{Tr}\left[\xi^2 I_0 - \xi^2 I_1 - \xi\delta I_0\right] \tag{217}$$

Notice that I_0 must be considered as an independent variable, as is usually done in equal-time quantization.

Consider a vector field $V_1(\theta)$ whose interior contraction has the effect of replacing ξ by θ. If we expand $\xi = \delta g g^{-1} = (-it^a)E_{ab}\delta\chi^b$, we can explicitly write

$$V_1(\theta) = \int dx\ (E^{-1})_{bc}\theta^c \frac{\delta}{\delta\chi^b} \tag{218}$$

The action of $V_1(\theta)$ on g is to make the left translation $V_1(\theta)g = (-it^a)\theta^a g$. We also define

$$V_2(\theta) = \int dx\ f^{abc}\theta^b (I_0 - I_1)^c \frac{\delta}{\delta I_0^a} \tag{219}$$

which has the effect of replacing δI_0 by $[\theta, I_0 - I_1]$ upon taking a contraction with Ω. The contraction of $V = V_1 + V_2$ with Ω gives

$$\begin{aligned} i_V\Omega &= -\delta\int dx\ \left[\frac{k I_0^a\theta^a}{8\pi}\right] \\ &\equiv -\delta J_0(\theta) \end{aligned} \tag{220}$$

Thus V is a Hamiltonian vector field corresponding to J_0.

Another vector field of interest is

$$W(\theta) = \int dx\ \left(\partial_1\theta^a + f^{abc}\theta^b I_1^c\right)\frac{\delta}{\delta I_0^a} \tag{221}$$

This has the effect of replacing δI_0 by $\partial_1\theta + [\theta, I_1]$. Contraction with Ω gives

$$\begin{aligned} i_W\Omega &= -\delta \int dx \left[\frac{k}{8\pi} I_1^a \theta^a\right] \\ &\equiv -\delta J_1(\theta) \end{aligned} \tag{222}$$

Thus W is a Hamiltonian vector field corresponding to J_1.

The currents of interest are, once again,

$$J_\mu^a = \frac{k}{8\pi}(\partial_\mu g g^{-1})^a \tag{223}$$

For the Poisson brackets, we find

$$\begin{aligned} \{J_0(\theta), J_0(\varphi)\} &= -i_V \delta J_0(\varphi) = i_V \int dx \ \mathrm{Tr} \ \delta \left(\frac{k}{4\pi} I_0 \varphi\right) \\ &= \frac{k}{4\pi}\int dx \ \mathrm{Tr} \ [\theta, I_0 - I_1]\varphi \\ &= J_0(\theta\times\varphi) \ - \ J_1(\theta\times\varphi) \end{aligned} \tag{224}$$

$$\begin{aligned} \{J_0(\theta), J_1(\varphi)\} &= i_V \int dx \ \mathrm{Tr}\left(\frac{k}{4\pi}\delta I_1 \varphi\right) \\ &= J_1(\theta\times\varphi) - \frac{k}{8\pi}\partial_1\theta^a \ \varphi^a \end{aligned} \tag{225}$$

$$\begin{aligned} \{J_1(\theta), J_1(\varphi)\} &= -i_W \int dx \ \mathrm{Tr}\left(\frac{k}{4\pi}\delta I_1 \varphi\right) \\ &= 0 \end{aligned} \tag{226}$$

In these equations $(\theta\times\varphi)^a = f^{abc}\theta^b\varphi^c$. These can be combined to yield

$$\{J_+(\theta), J_+(\varphi)\} = J_+(\theta\times\varphi) - \frac{k}{4\pi}\partial_1\theta^a\varphi^a \tag{227}$$

where $J_+ = J_0 + J_1$. The classical Hamiltonian is given by

$$H = \frac{4\pi}{k}\int dx \, [J_0^a J_0^a + J_1^a J_1^a] \tag{228}$$

In the lightcone quantization, we introduce coordinates u, v,

$$\begin{aligned} u &= \frac{1}{\sqrt{2}}(t - x), & v &= \frac{1}{\sqrt{2}}(t + x) \\ \partial_u &= \frac{1}{\sqrt{2}}(\partial_0 - \partial_1), & \partial_v &= \frac{1}{\sqrt{2}}(\partial_0 + \partial_1) \end{aligned} \tag{229}$$

This is different from (143), but given the structure of the equations of motion (209), these are easier. The action becomes

$$\mathcal{S} = -\frac{k}{4\pi}\int \mathrm{Tr}(\partial_u g g^{-1}\partial_v g g^{-1}) + \Gamma_{WZ} \tag{230}$$

We take v as the analog of the spatial coordinate and consider evolution in u. The surface term for u-integration will arise from variations on the $\partial_u g g^{-1}$-term. This gives

$$\mathcal{A} = -\frac{k}{4\pi}\int_v \mathrm{Tr}(\xi\partial_v g g^{-1}) + \frac{k}{4\pi}\int_{v,s} \mathrm{Tr}(\xi(dgg^{-1})^2) \tag{231}$$

Once again, the last term still involves the s-integration, but upon making another variation, we can integrate this as before and get

$$\Omega = \frac{k}{4\pi}\int dv\ \mathrm{Tr}\left[\xi\partial_v\xi + 2\xi^2\ I_v\right] \tag{232}$$

The equation of motion is $\partial_u I_v = 0$. The contraction of $V_1(\theta)$ as defined in (218) gives

$$\begin{aligned} i_{V_1}\Omega &= \frac{k}{4\pi}\int dv\ \mathrm{Tr}\left[2\theta\partial_v\xi + 2\theta\xi I_v - 2\xi\theta I_v\right] \\ &= \frac{k}{2\pi}\int dv\ \mathrm{Tr}\left[\theta(\partial_v\xi + [\xi, I_v])\right] \\ &= -\delta\int dv\ \left[\frac{kI_v^a\theta^a}{4\pi}\right] \\ &\equiv -\delta J_v(\theta) \end{aligned} \tag{233}$$

The current J_v is given by

$$J_v^a = \frac{k}{4\pi}(\partial_v g g^{-1})^a \tag{234}$$

The Poisson brackets are given by

$$\begin{aligned} \{J_v(\theta), J_v(\varphi)\} &= i_{V_1}\ \frac{k}{2\pi}\int \mathrm{Tr}(\delta I_v\varphi) \\ &= \frac{k}{2\pi}\int \mathrm{Tr}\left[\partial_v\theta\varphi + \theta I_v\varphi - I_v\theta\varphi\right] \\ &= \frac{k}{4\pi}\int I_v^a(\theta\times\varphi)^a - \frac{k}{4\pi}\int \partial_v\theta^a\varphi^a \\ &= J_v(\theta\times\varphi) - \frac{k}{4\pi}\int \partial_v\theta^a\varphi^a \end{aligned} \tag{235}$$

The algebra of currents given by (227), or the light-cone version (235), is an example of a Kac-Moody algebra.

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