Strong Negation in Intuitionistic Style Sequent Systems for Residuated Lattices

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Intuitionistic Style Sequent Systems for Constructive Logic with Strong Negation


\[
\begin{align*}
\sim \sim A & \to A & A & \to \sim \sim A \\
\sim (A \to B) & \to A \land \sim B & A \land \sim B & \to \sim (A \to B) \\
\sim (A \land B) & \to \sim A \lor \sim B & \sim A \lor \sim B & \to \sim (A \land B) \\
\sim (A \lor B) & \to \sim A \land \sim B & \sim A \land \sim B & \to \sim (A \lor B) \\
\sim \forall x A & \to \exists x \sim A & \exists x \sim A & \to \sim \forall x A \\
\sim \exists x A & \to \forall x \sim A & \forall x \sim A & \to \sim \exists x A
\end{align*}
\]
Intuitionistic Style Sequent Systems for Constructive Logic with Strong Negation


\[
\begin{align*}
\sim\sim A & \rightarrow A \\
\sim (A \rightarrow B) & \rightarrow A \land \sim B \\
\sim (A \land B) & \rightarrow \sim A \lor \sim B \\
\sim (A \lor B) & \rightarrow \sim A \land \sim B \\
\sim \forall_x A & \rightarrow \exists_x \sim A \\
\sim \exists_x A & \rightarrow \forall_x \sim A \\
\forall_x \sim A & \rightarrow \sim \forall_x A \\
\exists_x \sim A & \rightarrow \sim \exists_x A
\end{align*}
\]

- David N. Yetter, Quantales and (Noncommutative) Linear Logic, *Journal of Symbolic Logic* (1990)
- V. Michele Abrusci
- Joachim Lambek
- Mirjana Isaković Ilić
- Jörg Hudelmaier, Peter Schroeder–Heister
- Nikolaos Galatos, Peter Jipsen
Intuitionistic Style Sequent Systems for Constructive Logic with Strong Negation


\[
\begin{align*}
\sim & A \rightarrow A & A & \rightarrow \sim A \\
\sim (A \rightarrow B) & \rightarrow A \land \sim B & A \land \sim B & \rightarrow \sim (A \rightarrow B) \\
\sim (A \land B) & \rightarrow \sim A \lor \sim B & \sim A \lor \sim B & \rightarrow \sim (A \land B) \\
\sim (A \lor B) & \rightarrow \sim A \land \sim B & \sim A \land \sim B & \rightarrow \sim (A \lor B) \\
\sim \forall_x A & \rightarrow \exists_x \sim A & \exists_x \sim A & \rightarrow \sim \forall_x A \\
\sim \exists_x A & \rightarrow \forall_x \sim A & \forall_x \sim A & \rightarrow \sim \exists_x A
\end{align*}
\]

- Michał Kozak, "PSNC Strong Negation in Intuitionistic Style Sequent Systems for Residuated Lattices"
Negation in Residuated Lattices

A residuated lattice is a structure of the form $\mathbb{L} = (L, \leq, \cdot, \rightarrow, \leftarrow, 1)$ such that $(L, \leq)$ is a lattice, $(L, \cdot, 1)$ is a monoid, and $\rightarrow$ and $\leftarrow$ are respectively right and left residual for $\cdot$, i.e., they satisfy the following conditions for all $x, y, z \in L$:

$$x \cdot y \leq z \iff y \leq x \rightarrow z \quad \text{and} \quad x \cdot y \leq z \iff x \leq z \leftarrow y.$$  

A full Lambek algebra or FL–algebra is a residuated lattice $\mathbb{L}$ with a constant $0$.

Negations are defined by:

$$\sim x = x \rightarrow 0 \quad -x = 0 \leftarrow x$$

Cyclic involutive FL–algebras:

$$\sim x = -x \quad \sim \sim x = x$$

Moreover, contraposition holds:

$$\sim x \rightarrow y = x \leftarrow \sim y$$

and fusion is interdefinable with residuals,

$$\sim (x \rightarrow y) = \sim y \cdot x$$

differently from what is provable in Wansing’s systems:

$$\sim (y \leftarrow x) = x \cdot \sim y$$
A residuated lattice is a structure of the form $\mathbb{L} = (L, \leq, \cdot, \to, \leftarrow, 1)$ such that $(L, \leq)$ is a lattice, $(L, \cdot, 1)$ is a monoid, and $\to$ and $\leftarrow$ are respectively right and left residual for $\cdot$, i.e., they satisfy the following conditions for all $x, y, z \in L$:

$$x \cdot y \leq z \iff y \leq x \to z \quad \text{and} \quad x \cdot y \leq z \iff x \leq z \leftarrow y.$$ 

A full Lambek algebra or FL–algebra is a residuated lattice $\mathbb{L}$ with a constant 0.

Negations are defined by:

$$\neg x = x \to 0 \quad \quad \quad \quad \quad -x = 0 \leftarrow x$$

Cyclic involutive FL–algebras:

$$\neg x = -x \quad \quad \quad \quad \quad \neg
\neg x = x$$

Moreover, contraposition holds:

$$\neg x \to y = x \leftarrow \neg y$$

and fusion is interdefinable with residuals,

$$\neg(x \to y) = \neg y \cdot x$$
$$\neg(y \leftarrow x) = x \cdot \neg y$$

A cyclic involutive FL–algebra is a structure of the form $\mathbb{L} = (L, \leq, \cdot, \to, \leftarrow, \neg, 1, 0)$ such that $(L, \leq, \cdot, \to, \leftarrow, 1)$ is a residuated lattice and contraposition, double negation and $0 = \neg 1$ hold in $\mathbb{L}$. The variety of cyclic involutive FL–algebras is denoted by CyInFL.
Intuitionistic Style Sequent System for Symmetric Constructive Logic

- Igor D. Zaslavsky,
  Symmetric Constructive Logic (*in Russian*),
  *Publishing House of Academy of Sciences of Armenia SSR* (1978)

\[
\begin{align*}
\sim A & \rightarrow A & A & \rightarrow \sim \sim A \\
\sim (A \rightarrow B) & \rightarrow A \land \sim B & & & \\
\sim (A \land B) & \rightarrow \sim A \lor \sim B & \sim A \lor \sim B & \rightarrow \sim (A \land B) \\
\sim (A \lor B) & \rightarrow \sim A \land \sim B & \sim A \land \sim B & \rightarrow \sim (A \lor B) \\
\sim \forall x A & \rightarrow \exists x \sim A & \exists x \sim A & \rightarrow \sim \forall x A \\
\sim \exists x A & \rightarrow \forall x \sim A & \forall x \sim A & \rightarrow \sim \exists x A \\
\sim A & \rightarrow (A \rightarrow 0) & (A \rightarrow 0) & \rightarrow \sim A \\
(\sim A \rightarrow B) & \rightarrow (\sim B \rightarrow A) & (\sim B \rightarrow A) & \rightarrow (\sim A \rightarrow B)
\end{align*}
\]
Intuitionistic Style Sequent System for Symmetric Constructive Logic


\[
\begin{align*}
\Gamma \Rightarrow A^+ & \quad \Gamma \Rightarrow B^- \\
\hline
\Gamma \Rightarrow (A \rightarrow B)^- \\
\end{align*}
\]

\[
\begin{align*}
\Gamma, A^+ \Rightarrow B^+ & \quad \Gamma, B^- \Rightarrow A^- \\
\hline
\Gamma \Rightarrow (A \rightarrow B)^+ \\
\end{align*}
\]

\[
\begin{align*}
\Gamma, A^+, B^-, \Delta \Rightarrow C \\
\hline
\Gamma, (A \rightarrow B)^-, \Delta \Rightarrow C \\
\end{align*}
\]

\[
\begin{align*}
\Gamma, \Delta \Rightarrow A^+ & \quad \Gamma, B^+, \Delta \Rightarrow C \\
\hline
\Gamma, (A \rightarrow B)^+, \Delta \Rightarrow C \\
\end{align*}
\]

\[
\begin{align*}
\Gamma, \Delta \Rightarrow B^- & \quad \Gamma, A^-, \Delta \Rightarrow C \\
\hline
\Gamma, (A \rightarrow B)^+, \Delta \Rightarrow C \\
\end{align*}
\]

Adapted by:
- Robert A. Bull — for relevant logic RW without distributivity
- Ross T. Brady — for full RW and other variants
- Michał Kozak — for CylInDFL and variants without contraction
Symmetric Constructive Full Lambek Calculus — **SymConFL**

\[
(id) \quad A \Rightarrow A \quad \quad \quad (~0R) \quad \Rightarrow \sim 0 \quad \quad \quad (1R) \quad \Rightarrow 1
\]
Symmetric Constructive Full Lambek Calculus — SymConFL

\[
\begin{align*}
\text{(~0L)} & \quad \frac{\Gamma, \Delta \Rightarrow A}{\Gamma, \sim0, \Delta \Rightarrow A} \\
\text{(0R)} & \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0} \\
\text{([~])} & \quad \frac{\Gamma \Rightarrow A}{\sim A, \Gamma \Rightarrow} \quad \frac{\Gamma \Rightarrow \sim A}{A, \Gamma \Rightarrow} \quad \frac{\Gamma \Rightarrow A}{\Gamma, \sim A \Rightarrow} \quad \frac{\Gamma \Rightarrow \sim A}{\Gamma, A \Rightarrow} \\
\text{(~~L)} & \quad \frac{\Gamma, A, \Delta \Rightarrow B}{\Gamma, \sim\sim A, \Delta \Rightarrow B} \\
\text{(~~R)} & \quad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \sim\sim A} \\
\text{(\&L)} & \quad \frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, A \& B, \Delta \Rightarrow C} \quad \frac{\Gamma, B, \Delta \Rightarrow C}{\Gamma, A \& B, \Delta \Rightarrow C} \\
\text{(\&R)} & \quad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \& B} \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B} \\
\text{(~~\&L)} & \quad \frac{\Gamma, \sim A, \Delta \Rightarrow C}{\Gamma, \sim(A \& B), \Delta \Rightarrow C} \quad \frac{\Gamma, \sim B, \Delta \Rightarrow C}{\Gamma, \sim(A \& B), \Delta \Rightarrow C} \\
\text{(~~\&R)} & \quad \frac{\Gamma \Rightarrow \sim A}{\Gamma \Rightarrow \sim(A \& B)} \quad \frac{\Gamma \Rightarrow \sim B}{\Gamma \Rightarrow \sim(A \& B)}
\end{align*}
\]
Symmetric Constructive Full Lambek Calculus — \textbf{SymConFL}

\[
(\lor L) \quad \frac{\Gamma, A, \Delta \Rightarrow C \quad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \lor B, \Delta \Rightarrow C}
\]

\[
(\lor R) \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \lor B}
\]

\[
(\sim \lor L) \quad \frac{\Gamma, \sim A, \Delta \Rightarrow C}{\Gamma, \sim (A \lor B), \Delta \Rightarrow C}
\]

\[
(\sim \lor R) \quad \frac{\Gamma \Rightarrow \sim A \quad \Gamma \Rightarrow \sim B}{\Gamma \Rightarrow \sim (A \lor B)}
\]

\[
(\cdot L) \quad \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \cdot B, \Delta \Rightarrow C}
\]

\[
(\cdot R) \quad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \cdot B}
\]

\[
(\sim \cdot L) \quad \frac{\Gamma, \sim A, \Delta \Rightarrow C \quad \psi \Rightarrow B}{\Gamma, \psi, \sim (A \cdot B), \Delta \Rightarrow C}
\]

\[
(\sim \cdot R) \quad \frac{\Gamma \Rightarrow \sim A \quad \Gamma, A \Rightarrow \sim B}{\Gamma \Rightarrow \sim (A \cdot B)}
\]
Symmetric Constructive Full Lambek Calculus — **SymConFL**

\[
\begin{align*}
(\rightarrow L) & \quad \frac{\Gamma, B, \Delta \Rightarrow C \quad \Psi \Rightarrow A}{\Gamma, \Psi, A \rightarrow B, \Delta \Rightarrow C} & & \frac{\Gamma, \sim A, \Delta \Rightarrow C \quad \Psi \Rightarrow \sim B}{\Gamma, A \rightarrow B, \Psi, \Delta \Rightarrow C} \\
(\rightarrow R) & \quad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} & & \frac{\Gamma, \sim B \Rightarrow \sim A}{\Delta \Rightarrow A} \\
(\sim \rightarrow L) & \quad \frac{\Gamma, \sim B, A, \Delta \Rightarrow C}{\Gamma, \sim (A \rightarrow B), \Delta \Rightarrow C} \\
(\sim \rightarrow R) & \quad \frac{\Gamma \Rightarrow \sim B \quad \Delta \Rightarrow A}{\Gamma, \Delta \Rightarrow \sim (A \rightarrow B)} \\
(\leftarrow L) & \quad \frac{\Gamma, B, \Delta \Rightarrow C \quad \Psi \Rightarrow A}{\Gamma, B \leftarrow A, \Psi, \Delta \Rightarrow C} & & \frac{\Gamma, \sim A, \Delta \Rightarrow C \quad \Psi \Rightarrow \sim B}{\Gamma, \Psi, B \leftarrow A, \Delta \Rightarrow C} \\
(\leftarrow R) & \quad \frac{\Gamma, A \Rightarrow B}{\sim B, \Gamma \Rightarrow \sim A} & & \frac{\sim B, \Gamma \Rightarrow \sim A}{\Gamma \Rightarrow B \leftarrow A} \\
(\sim \leftarrow L) & \quad \frac{\Gamma, A, \sim B, \Delta \Rightarrow C}{\Gamma, \sim (B \leftarrow A), \Delta \Rightarrow C} \\
(\sim \leftarrow R) & \quad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow \sim B}{\Gamma, \Delta \Rightarrow \sim (B \leftarrow A)}
\end{align*}
\]
Symmetric Constructive Full Lambek Calculus — SymConFL

\[(\text{cut}) \quad \frac{\Gamma, A, \Delta \Rightarrow B \quad \Psi \Rightarrow A}{\Gamma, \Psi, \Delta \Rightarrow B}\]

SymConFL enriched with

SymConFL\textsubscript{e} exchange

\[
\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, B, A, \Delta \Rightarrow C}
\]

SymConFL\textsubscript{w} weakening

\[
\frac{\Gamma, \Delta \Rightarrow B}{\Gamma, A, \Delta \Rightarrow B} \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}
\]

SymConFL\textsubscript{c} contraction

\[
\frac{\Gamma, A, A, \Delta \Rightarrow B}{\Gamma, A, \Delta \Rightarrow B}
\]
Symmetric Constructive Full Lambek Calculus — SymConFL

\[
\begin{array}{c}
\text{(cut)} \quad \frac{\Gamma, A, \Delta \Rightarrow B \quad \Psi \Rightarrow A}{\Gamma, \Psi, \Delta \Rightarrow B}
\end{array}
\]

SymConFL enriched with

\[\text{SymConFL}_e\]
- exchange

\[
\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, B, A, \Delta \Rightarrow C}
\]

\[\text{SymConFL}_w\]
- weakening

\[
\frac{\Gamma, B \Rightarrow B \quad \Gamma \Rightarrow}{\Gamma, A, \Delta \Rightarrow B \quad \Gamma \Rightarrow A}
\]

\[\text{SymConFL}_c\]
- contraction

\[
\frac{\Gamma, A, A, \Delta \Rightarrow B}{\Gamma, A, \Delta \Rightarrow B}
\]

Algebraic semantics?

Inspiration — the definition of Nelson algebras introduced by Helena Rasiowa (quasi-ordering).

We call a structure of the form \( \mathbb{I} = (L, <, \cdot, \rightarrow, \leftarrow, \sim, 1, 0) \) a **symmetric constructive residuated \( \ell \)-monoid**, if its \((L, <)\) reduct is a quasi–ordered set, the \((L, \cdot, 1)\) reduct is a monoid, the relation \( \leq \) on \( L \) defined as

\[
(po) \quad x \leq y \iff x < y \text{ and } \sim y < \sim x
\]

is a lattice–order, and the following conditions are satisfied:

\begin{align*}
(dn) & \quad \sim \sim x = x, \\
(1^*) & \quad y < x \rightarrow z \iff x \cdot y < z \text{ and } y \cdot \sim z < \sim x, \\
(2^*) & \quad x < z \leftarrow y \iff x \cdot y < z \text{ and } \sim z \cdot x < \sim y, \\
(3^*) & \quad z < \sim (x \cdot y) \iff y \cdot z < \sim x \text{ and } z \cdot x < \sim y, \\
(4^*) & \quad \sim x \cdot x < 0, \\
(5^*) & \quad x \cdot \sim x < 0, \\
(6^*) & \quad \sim (x \rightarrow y) < \sim y \cdot x, \\
(7^*) & \quad \sim y \cdot x < \sim (x \rightarrow y), \\
(8^*) & \quad \sim (y \leftarrow x) < x \cdot \sim y, \\
(9^*) & \quad x \cdot \sim y < \sim (y \leftarrow x).
\end{align*}
Symmetric Constructive FL–algebras — SymConFL

We call a structure of the form \( \mathbb{I} = (L, <, \cdot, \rightarrow, \leftarrow, \sim, 1, 0) \) a symmetric constructive residuated \( \ell–\text{monoid} \), if its \((L, <)\) reduct is a quasi–ordered set, the \((L, \cdot, 1)\) reduct is a monoid, the relation \( \leq \) on \( L \) defined as

\[(po) \quad x \leq y \iff x < y \text{ and } \sim y < \sim x\]

is a lattice–order, and the following conditions are satisfied:

\[\begin{align*}
(dn) & \quad \sim \sim x = x, \\
(1^*) & \quad y < x \rightarrow z \iff x \cdot y < z \text{ and } y \cdot \sim z < \sim x, \\
(2^*) & \quad x < z \leftarrow y \iff x \cdot y < z \text{ and } \sim z \cdot x < \sim y, \\
(3^*) & \quad z < \sim (x \cdot y) \iff y \cdot z < \sim x \text{ and } z \cdot x < \sim y, \\
(4^*) & \quad \sim x \cdot x < 0, \\
(5^*) & \quad x \cdot \sim x < 0, \\
(6^*) & \quad \sim (x \rightarrow y) < \sim y \cdot x, \\
(7^*) & \quad \sim y \cdot x < \sim (x \rightarrow y), \\
(8^*) & \quad \sim (y \leftarrow x) < x \cdot \sim y. \\
(9^*) & \quad x \cdot \sim y < \sim (y \leftarrow x).
\end{align*}\]

In Nelson algebras:

\[x < y \iff x \rightarrow y = 1\]

Here:

\(<\) is a primitive relation
Symmetric Constructive FL–algebras — SymConFL

We call a structure of the form \( \mathbb{I} = (L, <, \cdot, \rightarrow, \leftrightarrow, \sim, 1, 0) \) a symmetric constructive residuated \( \ell–\)monoid, if its \((L, <)\) reduct is a quasi–ordered set, the \((L, \cdot, 1)\) reduct is a monoid, the relation \( \leq \) on \( L \) defined as

\[
(po) \quad x \leq y \iff x < y \text{ and } \sim y < \sim x
\]
is a lattice–order, and the following conditions are satisfied:

\[
\begin{align*}
(dn) & \quad \sim \sim x = x, & (0^*) & \quad 0 = \sim 1, \\
(1^*) & \quad y < x \rightarrow z \iff x \cdot y < z \text{ and } y \cdot \sim z < \sim x, \\
(2^*) & \quad x < z \leftrightarrow y \iff x \cdot y < z \text{ and } \sim z \cdot x < \sim y, \\
(3^*) & \quad z < \sim(x \cdot y) \iff y \cdot z < \sim x \text{ and } z \cdot x < \sim y, \\
(4^*) & \quad \sim x \cdot x < 0, & (5^*) & \quad x \cdot \sim x < 0, \\
(6^*) & \quad \sim(x \rightarrow y) < \sim y \cdot x & (7^*) & \quad \sim y \cdot x < \sim(x \rightarrow y), \\
(8^*) & \quad \sim(y \leftarrow x) < x \cdot \sim y & (9^*) & \quad x \cdot \sim y < \sim(y \leftarrow x). \\
\end{align*}
\]

A symmetric constructive residuated \( \ell–\)monoid we call \( \wedge–\)monotone, respectively \( \vee–\)monotone, if it satisfies the following conditions, respectively:

\[
\begin{align*}
(10^*) & \quad z < x \text{ and } z < y \implies z < x \wedge y, \\
(11^*) & \quad x < z \text{ and } y < z \implies x \vee y < z.
\end{align*}
\]
Symmetric Constructive FL–algebras — SymConFL

A symmetric constructive residuated $\ell$–monoid which is simultaneously $\wedge$–monotone and $\vee$–monotone we call a symmetric constructive FL–algebra. The class of these algebras we denote by SymConFL.

Symmetric constructive residuated $\ell$–monoids and FL–algebras we call:

- **commutative,** if they satisfy $x \cdot y = y \cdot x$, for all $x, y$.
- **integral,** if they satisfy $x < 1$ and $0 < x$, for all $x$.
- **contractive,** if they satisfy $x < x \cdot x$, for all $x$. 
Symmetric Constructive FL–algebras — SymConFL

A symmetric constructive residuated \( \ell \)–monoid which is simultaneously \( \wedge \)–monotone and \( \vee \)–monotone we call a symmetric constructive FL–algebra. The class of these algebras we denote by SymConFL.

Symmetric constructive residuated \( \ell \)–monoids and FL–algebras we call:

- **commutative**, if they satisfy \( x \cdot y = y \cdot x \), for all \( x, y \).
- **integral**, if they satisfy \( x < 1 \) and \( 0 < x \), for all \( x \).
- **contractive**, if they satisfy \( x < x \cdot x \), for all \( x \).

Symmetric constructive xxx FL–algebras

<table>
<thead>
<tr>
<th>xxx</th>
<th>symmetric constructive FL(_e)–algebras</th>
<th>SymConFL(_e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>commutative</td>
<td>symmetric constructive FL(_w)–algebras</td>
<td>SymConFL(_w)</td>
</tr>
<tr>
<td>integral</td>
<td>symmetric constructive FL(_c)–algebras</td>
<td>SymConFL(_c)</td>
</tr>
</tbody>
</table>

Taking into account the fact that \textit{SymConFL\(_{ewc}\)} is exactly symmetric constructive logic of I.D. Zaslavsky and anticipating the completeness theorem:

SymConFL\(_{ewc}\) — symmetric constructive FL\(_{ewc}\)–algebras — Zaslavsky FL\(_{ewc}\)–algebras
Completeness **SymConFL** with respect to SymConFL

A pair \((\mathbb{I}, f)\) we call a **model**, if \(\mathbb{I}\) is a symmetric constructive FL–algebra and \(f\) is a mapping from the set of formulas \(\mathcal{F}\) into \(\mathbb{I}\) satisfying \(f(\sim A) = \sim f(A)\) and \(f(A \circ B) = f(A) \circ f(B)\), for \(\circ \in \{\land, \lor, \cdot, \to, \leftrightarrow\}\). We extend \(f\) to strings of formulas in the following ways:

\[
\begin{align*}
    f_a(\varepsilon) &= 1, \\
    f_a(A_1, A_2, \ldots, A_n) &= f_a(A_1) \cdot f_a(A_2) \cdot \ldots \cdot f_a(A_n), \\
    f_s(\varepsilon) &= 0.
\end{align*}
\]

A sequent \(\Gamma \Rightarrow A\) is said to be **true** in a model \((\mathbb{I}, f)\), if \(f_a(\Gamma) < f_s(A)\) holds in \(\mathbb{I}\).

A sequent \(\Gamma \Rightarrow A\) is said to be **valid** in \(\mathbb{I}\), if it is true in \((\mathbb{I}, f)\) for any \(f\).

A sequent \(\Gamma \Rightarrow A\) is said to be **valid** in SymConFL, if it is valid in every \(\mathbb{I}\).

**Theorem**

A sequent \(\Gamma \Rightarrow A\) is provable in **SymConFL** if and only if it is valid in SymConFL.
Completeness SymConFL with respect to SymConFL

A pair \( \langle \mathbb{L}, f \rangle \) we call a model, if \( \mathbb{L} \) is a symmetric constructive FL–algebra and \( f \) is a mapping from the set of formulas \( \mathcal{F} \) into \( \mathbb{L} \) satisfying \( f(\neg A) = \neg f(A) \) and \( f(A \circ B) = f(A) \circ f(B) \), for \( \circ \in \{\land, \lor, \cdot, \to, \leftrightarrow\} \). We extend \( f \) to strings of formulas in the following ways:

\[
\begin{align*}
    f_a(\varepsilon) &= 1, &
    f_a(A_1, A_2, \ldots, A_n) &= f_a(A_1) \cdot f_a(A_2) \cdot \ldots \cdot f_a(A_n), \\
    f_s(\varepsilon) &= 0.
\end{align*}
\]

A sequent \( \Gamma \Rightarrow A \) is said to be true in a model \( \langle \mathbb{L}, f \rangle \), if \( f_a(\Gamma) < f_s(A) \) holds in \( \mathbb{L} \).

A sequent \( \Gamma \Rightarrow A \) is said to be valid in \( \mathbb{L} \), if it is true in \( \langle \mathbb{L}, f \rangle \) for any \( f \).

A sequent \( \Gamma \Rightarrow A \) is said to be valid in SymConFL, if it is valid in every \( \mathbb{L} \).

**Theorem**

A sequent \( \Gamma \Rightarrow A \) is provable in SymConFL if and only if it is valid in SymConFL.

**Proof**

**Soundness Lemma** is monotone with respect to \(<\).

Let \( x < y \) and let \( z \) be an arbitrary element. From the reflexivity of \(<\) we have \( y \cdot z < y \cdot z \) and \( \neg(y \cdot z) < \neg(y \cdot z) \). From the latter we get \( \neg(y \cdot z) \cdot y < \neg z \) by \((3^*)\). Therefore \((2^*)\) yields \( y < y \cdot z \leftrightarrow z \). From the assumption and the transitivity of \(<\), we have \( x < y \cdot z \leftrightarrow z \), and hence \( x \cdot z < y \cdot z \) by \((2^*)\) again. Likewise we prove \( z \cdot x < z \cdot y \).
Completeness **SymConFL** with respect to SymConFL

A pair \( \langle \mathbb{L}, f \rangle \) we call a model, if \( \mathbb{L} \) is a symmetric constructive FL–algebra and \( f \) is a mapping from the set of formulas \( \mathcal{F} \) into \( \mathbb{L} \) satisfying \( f(\neg A) = \neg f(A) \) and \( f(A \circ B) = f(A) \circ f(B) \), for \( \circ \in \{ \land, \lor, \cdot, \div, \to, \leftrightarrow \} \). We extend \( f \) to strings of formulas in the following ways:

\[
\begin{align*}
    f_a(\varepsilon) &= 1, \\
    f_a(A_1, A_2, \ldots, A_n) &= f_a(A_1) \cdot f_a(A_2) \cdots \cdot f_a(A_n), \\
    f_s(\varepsilon) &= 0.
\end{align*}
\]

A sequent \( \Gamma \Rightarrow A \) is said to be true in a model \( \langle \mathbb{L}, f \rangle \), if \( f_a(\Gamma) < f_s(A) \) holds in \( \mathbb{L} \).

A sequent \( \Gamma \Rightarrow A \) is said to be valid in \( \mathbb{L} \), if it is true in \( \langle \mathbb{L}, f \rangle \) for any \( f \).

A sequent \( \Gamma \Rightarrow A \) is said to be valid in SymConFL, if it is valid in every \( \mathbb{L} \).

**Theorem**

A sequent \( \Gamma \Rightarrow A \) is provable in **SymConFL** if and only if it is valid in SymConFL.

**Proof**

**Soundness**  Lemma \((\to L)\) preserves the validity of \( \frac{\Gamma, B, \Delta \Rightarrow C}{\Gamma, \Psi \Rightarrow A}, \frac{\Gamma, \Psi, A \to B, \Delta \Rightarrow C}{\Gamma, B, \Delta \Rightarrow C} \)  Assume that \( u \cdot y \cdot v < w \) and \( z < x \). Then, \( z \cdot (x \to y) < x \cdot (x \to y) \). Moreover, from the reflexivity of \( < \) and \((1^*)\) we have \( x \cdot (x \to y) < y \), and hence \( z \cdot (x \to y) < y \). Thus, \( u \cdot z \cdot (x \to y) \cdot v < u \cdot y \cdot v \) and \( u \cdot z \cdot (x \to y) \cdot v < w \), as desired.
Completeness SymConFL with respect to SymConFL

A pair \( \langle \mathbb{L}, f \rangle \) we call a model, if \( \mathbb{L} \) is a symmetric constructive FL–algebra and \( f \) is a mapping from the set of formulas \( \mathcal{F} \) into \( \mathbb{L} \) satisfying \( f(\sim A) = \sim f(A) \) and \( f(A \circ B) = f(A) \circ f(B) \), for \( \circ \in \{ \wedge, \vee, \cdot, \div \} \). We extend \( f \) to strings of formulas in the following ways:

\[
\begin{align*}
    f_a(\varepsilon) &= 1, \\
    f_a(A_1, A_2, \ldots, A_n) &= f_a(A_1) \cdot f_a(A_2) \cdot \ldots \cdot f_a(A_n), \\
    f_s(\varepsilon) &= 0.
\end{align*}
\]

A sequent \( \Gamma \Rightarrow A \) is said to be true in a model \( \langle \mathbb{L}, f \rangle \), if \( f_a(\Gamma) < f_s(A) \) holds in \( \mathbb{L} \).

A sequent \( \Gamma \Rightarrow A \) is said to be valid in \( \mathbb{L} \), if it is true in \( \langle \mathbb{L}, f \rangle \) for any \( f \).

A sequent \( \Gamma \Rightarrow A \) is said to be valid in SymConFL, if it is valid in every \( \mathbb{L} \).

Theorem

A sequent \( \Gamma \Rightarrow A \) is provable in SymConFL if and only if it is valid in SymConFL.

Proof

Soundness Lemma \((\rightarrow L)\) preserves the validity

\[
\frac{\Gamma, \sim A, \Delta \Rightarrow C \quad \Psi \Rightarrow \sim B}{\Gamma, A \rightarrow B, \Psi, \Delta \Rightarrow C}
\]

Assume now that \( u \cdot \sim x \cdot v < w \) and \( z < \sim y \). Analogously we obtain \( (x \rightarrow y) \cdot z < (x \rightarrow y) \cdot \sim y \).

Moreover, from the reflexivity of \( < \) and \((1^*)\) we also have \( (x \rightarrow y) \cdot \sim y < \sim x \), and hence \( (x \rightarrow y) \cdot z < \sim x \). Thus, \( u \cdot (x \rightarrow y) \cdot z \cdot v < u \cdot \sim x \cdot v \) and \( u \cdot (x \rightarrow y) \cdot z \cdot v < w \), as desired.
Completeness SymConFL with respect to SymConFL

Completeness is carried out using a Lindenbaum–Tarski construction. We define binary relations $<$ and $\preceq$ on the set of $\mathcal{F}$ as follows:

$$C < D \iff C \Rightarrow D \text{ is provable in SymConFL},$$

$$C \preceq D \iff C < D \text{ and } \sim D < \sim C.$$ 

From (id) and (cut) it follows that both $<$ and $\preceq$ are quasi–order relations. Moreover, we define a binary relation $\equiv$ on this set, putting

$$C \equiv D \iff C \preceq D \text{ and } D \preceq C.$$ 

It is a well–known fact that $\equiv$ is an equivalence relation on $\mathcal{F}$ and $|C|_\equiv \leq |D|_\equiv \iff C \preceq D$ is a partial–order relation on the quotient set $\mathcal{F}/_\equiv$. Analogously, one can show that $|C|_\equiv < |D|_\equiv \iff C < D$ is a quasi–order relation on $\mathcal{F}/_\equiv$.

Moreover, it is easy to prove that $\equiv$ is also a congruence with respect to all connectives and that $(\mathcal{F}/_\equiv, \leq)$ is a lattice in which the meet and join of $|C|_\equiv$ and $|D|_\equiv$ are respectively $|C \land D|_\equiv$ and $|C \lor D|_\equiv$.

Therefore one can define, in a standard manner, the quotient structure

$$\mathbb{L}_{_\equiv \mathcal{F}} = (\mathcal{F}/_\equiv, <, \cdot, \rightarrow, \leftarrow, \sim, |1|_\equiv, |0|_\equiv)$$

and show that it is a symmetric constructive FL–algebra.

Now, defining a mapping $f: \mathcal{F} \rightarrow \mathbb{L}_{_\equiv \mathcal{F}}$ by $f(B) = |B|_\equiv$ it is easy to see that if $\Gamma \Rightarrow A$ is true in $\langle \mathbb{L}_{_\equiv \mathcal{F}}, f\rangle$ then it is provable in SymConFL.
Completeness \textbf{SymConFL} with respect to SymConFL

Completeness is carried out using a Lindenbaum–Tarski construction. We define binary relations $<$ and $\preceq$ on the set of $\mathcal{F}$ as follows:

\begin{align*}
C < D & \iff C \Rightarrow D \text{ is provable in SymConFL,} \\
C \preceq D & \iff C < D \text{ and } \neg D < \neg C.
\end{align*}

From (\text{id}) and (\text{cut}) it follows that both $<$ and $\preceq$ are quasi–order relations. Moreover, we define a binary relation $\equiv$ on this set, putting

\begin{align*}
C \equiv D & \iff C \preceq D \text{ and } D \preceq C.
\end{align*}

It is a well–known fact that $\equiv$ is an equivalence relation on $\mathcal{F}$ and $|C|_\equiv \leq |D|_\equiv \iff C \preceq D$ is a partial–order relation on the quotient set $\mathcal{F}/\equiv$. Analogously, one can show that $|C|_\equiv < |D|_\equiv \iff C < D$ is a quasi–order relation on $\mathcal{F}/\equiv$.

Moreover, it is easy to prove that $\equiv$ is also a congruence with respect to all connectives and that $(\mathcal{F}/\equiv, \leq)$ is a lattice in which the meet and join of $|C|_\equiv$ and $|D|_\equiv$ are respectively $|C \wedge D|_\equiv$ and $|C \vee D|_\equiv$.

Analogously one can show that for every subset $S$ of $\{e, w, c\}$, \textbf{SymConFL}_S is complete with respect to SymConFL$_S$. 
Cut Elimination for variants without contraction ...

... i.e. for SymConFL$_S$, where $S$ is any subset of \{e, w\}.

\[
\text{(cut)} \quad \frac{\Gamma, A, \Delta \Rightarrow B \quad \Psi \Rightarrow A}{\Gamma, \Psi, \Delta \Rightarrow B}
\]

\[
\text{(cut)$_1$_1} \quad \frac{\Gamma \Rightarrow \sim A \quad \Psi \Rightarrow A}{\Gamma, \Psi \Rightarrow} \quad \text{(cut)$_2$_2} \quad \frac{\Gamma \Rightarrow A \quad \Psi \Rightarrow \sim A}{\Gamma, \Psi \Rightarrow}
\]

**Theorem**
Any derivation containing instances of the rules (cut)$_1$, (cut)$_2$ and (cut) can be transformed into one in which they do not occur.

**Proof**
The proof proceeds via the standard double induction on the complexity of $A$ and the combined height of derivations of the premises.

- example

\[
\frac{\Gamma \Rightarrow C}{\Gamma \Rightarrow \sim\sim C} \quad \frac{\sim\sim R}{\Psi \Rightarrow \sim C} \quad \frac{\Gamma \Rightarrow \sim C}{\Gamma, \Psi \Rightarrow} \quad \frac{\Gamma \Rightarrow C \downarrow}{\Psi \Rightarrow \sim C} \quad \text{(cut)$_1$_1} \quad \text{(cut)$_2$_2}
\]
Cut Elimination for variants with contraction ...

... i.e. for SymConFl_{ec} and SymConFl_{ewc}

(for SymConFl_{c} cut elimination does not hold — like for FL_{c}).

\[
\begin{align*}
(mix) & \quad \frac{\Pi \Rightarrow B \quad \Psi \Rightarrow A}{\tilde{\Pi}_A, \Psi \Rightarrow B} \\
(mix)_1 & \quad \frac{\Omega \Rightarrow \neg A \quad \Psi \Rightarrow A}{\Omega_{\neg A}, \Psi \Rightarrow} \\
(mix)_2 & \quad \frac{\Omega \Rightarrow A \quad \Psi \Rightarrow \neg A}{\Omega_{\neg A}, \Psi \Rightarrow}
\end{align*}
\]

where \( \Pi \) has at least one occurrence of \( A \), \( \tilde{\Pi}_A \) is a sequence of formulas obtained from \( \Pi \) by deleting at least one occurrence of \( A \), \( \Omega \) is arbitrary and \( \Omega_{\neg A} \) (\( \Omega_{\neg A} \), respectively) is \( \Omega \) or \( \tilde{\Omega}_{\neg A} \) (\( \Omega \) or \( \tilde{\Omega}_{\neg A} \), respectively).

**Theorem**

Any derivation containing instances of the rules \((mix)_1\), \((mix)_2\) and \((mix)\) can be transformed into one in which they do not occur.

**Proof**

Analogously, via the standard double induction on the complexity of \( A \) and the combined height of derivations of the premises.
Cut Elimination for variants with contraction — examples

\[ \frac{\Gamma, C \cdot D, C, D, \Delta \Rightarrow B}{\Gamma, C \cdot D, C \cdot D, C \cdot D, \Delta \Rightarrow B} \quad (\cdot L) \quad \frac{\psi \Rightarrow C}{\psi, \phi \Rightarrow C \cdot D} \quad (\cdot R) \]
\[ \frac{\Gamma, C \cdot D, \Delta, \psi, \phi \Rightarrow B}{\Gamma, C \cdot D, \Delta, \psi, \phi \Rightarrow B} \quad (\cdot L) \quad \frac{\psi, \phi \Rightarrow C \cdot D}{\psi \Rightarrow C} \quad (\cdot R) \]
\[ \frac{\Gamma, C \cdot D, C, D, \Delta \Rightarrow B}{\Gamma, C \cdot D, C \cdot D, \Delta \Rightarrow B} \quad (\text{mix}) \quad \frac{\psi \Rightarrow C}{\psi, \phi \Rightarrow C \cdot D} \quad (\text{mix}) \]
\[ \frac{\Gamma, C \cdot D, \Delta, \psi, \phi \Rightarrow B}{\Gamma, C \cdot D, \Delta, \psi, \phi \Rightarrow B} \quad (\text{mix}) \quad \frac{\psi \Rightarrow C}{\psi, \phi \Rightarrow C \cdot D} \quad (\text{mix}) \]

\[ \frac{\Gamma, C \cdot D, C \cdot D, \Delta \Rightarrow \sim(C \cdot D)}{\sim(C \cdot D)} \quad ([\sim]) \quad \frac{\psi \Rightarrow C \cdot D}{\psi \Rightarrow C \cdot D} \quad (\text{mix}) \]
\[ \frac{\Gamma, C \cdot D, \Delta, \psi \Rightarrow}{\Gamma, C \cdot D, \Delta, \psi \Rightarrow} \quad (\text{mix})_1 \]

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\[ D, C \cdot D, \Gamma, C \cdot D \Rightarrow \sim C \quad \text{\text{(\sim R)}} \]
\[ C \cdot D, \Gamma, C \cdot D, C \Rightarrow \sim D \quad \text{\text{(\sim R)}} \]
\[ \Gamma, C \cdot D, \sim C \cdot D, \Gamma \Rightarrow \sim (C \cdot D) \]

\[ \Psi \Rightarrow C \quad \Phi \Rightarrow D \quad \text{\text{(\text{R})}} \]

\[ \Psi, \Phi \Rightarrow C \cdot D \quad \text{\text{(mix)}} \]

\[ \Gamma, C \cdot D, \psi, \Phi \Rightarrow \sim D \quad \text{\text{some (e)(c)}} \]

\[ \Gamma, C \cdot D, \psi, \psi, \Phi \Rightarrow \sim D \quad \text{\text{some (e)(c)}} \]

\[ \Gamma, C \cdot D, \psi, \psi, \Phi, \Phi \Rightarrow \sim D \quad \text{\text{some (e)(c)}} \]

\[ \Gamma \Rightarrow \sim (C \cdot D) \quad \text{\text{(mix)}} \]

\[ \Psi, \Phi, C \cdot D, \Gamma \Rightarrow \sim C \quad \text{\text{some (e)(c)}} \]

\[ \Psi, \Phi, C \cdot D, \Gamma \Rightarrow \sim D \quad \text{\text{some (e)(c)}} \]

\[ \Psi \Rightarrow C \quad \Phi, C \cdot D \Rightarrow D \quad \text{\text{\text{(mix)}}} \]

\[ \Psi, \Phi, C \cdot D, \Gamma \Rightarrow \sim C \quad \text{\text{\text{(mix)}}} \]
Cut Elimination for variants with contraction — examples

- \( \Gamma, \neg C, \neg(C \cdot D), \Delta \Rightarrow B \quad \neg(C \cdot D), \Phi \Rightarrow D \quad \Phi, \neg(C \cdot D), \Delta \Rightarrow B \quad \Phi \Rightarrow D \quad \Psi \Rightarrow \neg(C \cdot D) \quad \psi, C \Rightarrow \neg D \quad (\neg R) \)

- \( \Gamma, \neg(C \cdot D), \Phi, \neg(C \cdot D), \Delta \Rightarrow B \quad \neg(C \cdot D), \Phi \Rightarrow D \quad \psi \Rightarrow \neg(C \cdot D) \quad (\neg L) \)

- \( \Gamma, \Phi, \neg(C \cdot D), \Delta, \Psi \Rightarrow B \)

- \( \Gamma, \neg C, \neg(C \cdot D), \Delta \Rightarrow B \quad \Psi \Rightarrow \neg(C \cdot D) \quad (mix) \)

- \( \Gamma, \neg C, \Psi, \Delta \Rightarrow B \quad \Psi \Rightarrow \neg(C \cdot D) \quad (mix) \)

- \( \Gamma, \neg C, \Psi, \Delta \Rightarrow B \quad \Psi \Rightarrow \neg(C \cdot D) \quad \Phi \Rightarrow D \quad (mix) \)

- \( \Gamma, \Psi, \Delta \Rightarrow B \quad \psi \Rightarrow \neg(C \cdot D) \quad \Phi \Rightarrow D \quad (mix) \)

- \( \Gamma, \neg(C \cdot D), \Delta \Rightarrow C \cdot D \quad (\neg \) \)

- \( \Gamma, \Delta \Rightarrow \neg(C \cdot D), \Psi \Rightarrow \quad (mix) \)

- \( \Gamma, \Delta, \neg(C \cdot D), \Psi \Rightarrow \quad \psi \Rightarrow \neg(C \cdot D) \quad (mix) \)

- \( \Gamma, \Delta, \neg(C \cdot D), \Psi \Rightarrow \quad \psi \Rightarrow \neg(C \cdot D) \quad (mix)_2 \)

- \( \Gamma, \Delta, \Psi \Rightarrow \quad \psi \Rightarrow \neg(C \cdot D) \quad (mix) \)

- \( \Gamma, \Delta, \Psi \Rightarrow \quad \psi \Rightarrow \neg(C \cdot D) \quad (mix) \)

- \( \Gamma, \Delta, \Psi \Rightarrow \quad \Psi \Rightarrow \neg(C \cdot D) \quad \Psi \Rightarrow \neg(C \cdot D) \quad \Psi \Rightarrow \neg(C \cdot D) \quad (mix) \)

- \( \Gamma, \Delta, \Psi \Rightarrow \quad \Psi \Rightarrow \neg(C \cdot D) \quad \Psi \Rightarrow \neg(C \cdot D) \quad \Psi \Rightarrow \neg(C \cdot D) \quad (mix) \)

- \( \Gamma, \Delta, \Psi \Rightarrow \quad \Psi \Rightarrow \neg(C \cdot D) \quad \Psi \Rightarrow \neg(C \cdot D) \quad \Psi \Rightarrow \neg(C \cdot D) \quad (mix) \)
Decidability of variants without contraction

**Lemma**
If a sequent $\Gamma \Rightarrow A$ is provable in $\text{SymConFL}_S$, where $S$ is any subset of $\{e, w\}$, then there exists a cut–free proof of $\Gamma \Rightarrow A$ such that any formula appearing in it is a subformula or negation of a subformula of some formula appearing in $\Gamma \Rightarrow A$.

**Definition**
To each formula $A$ we assign an integer $c_f(A)$ as follows:

- $c_f(p) = 1$, for any variable $p$,
- $c_f(0) = c_f(1) = 1$,
- $c_f(\sim A) = c_f(A) + 1$,
- $c_f(A \circ B) = c_f(A) + c_f(B) + 2$, for $\circ \in \{\wedge, \vee, \cdot, \rightarrow, \leftarrow\}$.

We extend $c_f$ to strings of formulas in the following ways:

- $c_a(\varepsilon) = 0$,
- $c_a(A_1, A_2, \ldots, A_n) = c_a(A_1) + c_a(A_2) + \ldots + c_a(A_n)$,
- $c_s(\varepsilon) = 1$,

and define the *complexity* of a sequent $\Gamma \Rightarrow A$ as $c_a(\Gamma) + c_s(A)$.

**Lemma**
If a sequent $\Gamma \Rightarrow A$ is provable in $\text{SymConFL}_S$, where $S$ is any subset of $\{e, w\}$, then every sequent in its cut–free proof has the complexity less than or equal to the complexity of $\Gamma \Rightarrow A$.

**Theorem**
For any subset $S$ of $\{e, w\}$, $\text{SymConFL}_S$ is decidable.
Decidability of variants with contraction …

… i.e. of $\text{SymConFL}_{\text{ec}}$ and $\text{SymConFL}_{\text{ewc}}$

can be shown via the technique adequate for $\text{FL}_{\text{ec}}$ and $\text{InFL}_{\text{ec}}$ of Saul A. Kripke; later developed by Robert K. Meyer.

Decidability of variants with contraction ...

... i.e. of \( \text{SymConFL}_{ec} \) and \( \text{SymConFL}_{ewc} \)

can be shown via the technique adequate for \( \text{FL}_{ec} \) and \( \text{InFL}_{ec} \) of Saul A. Kripke; later developed by Robert K. Meyer.


**Corollaries** (seeing that \( \text{SymConFL}_{ewc} = \text{SymConFL}_{wc} \))

**Theorem**
For any subset \( S \) of \( \{ e, w, c \} \) different than \( \{ c \} \), the \( \leftarrow \)–theory of \( \text{SymConFL}_S \) (i.e. the set of all expressions of the form \( x \leftarrow y \) that are valid in \( \text{SymConFL}_S \)) is decidable.

Moreover, from the equivalence

\[
    x = y \iff x \leftarrow y \text{ and } \neg y \leftarrow \neg x \text{ and } y \leftarrow x \text{ and } \neg x \leftarrow \neg y
\]

the following holds as well.

**Theorem**
For any subset \( S \) of \( \{ e, w, c \} \) different than \( \{ c \} \), the equational theory of \( \text{SymConFL}_S \) is decidable.
Mutual interpretability of SymConFL and CyInFL

**Theorem**
CyInFL and SymConFL are mutually interpretable.

**Proof**
Assume that $\mathbb{I} = (L, <, \cdot, \rightarrow, \leftarrow, \sim, 1, 0)$ is a symmetric constructive FL–algebra. As an example we verify the one of the residuation laws:

$$x \cdot y < z \quad \text{and} \quad \sim z < \sim (x \cdot y) \iff x < z \leftarrow y \quad \text{and} \quad \sim (z \leftarrow y) < \sim x.$$

Let $x \cdot y < z$ and $\sim z < \sim (x \cdot y)$. From the latter we have $y \cdot \sim z < \sim x$ and $\sim z \cdot x < \sim y$ by (3*). Using (8*) $\sim (z \leftarrow y) < y \cdot \sim z$ and the transitivity of $<$ we get $\sim (z \leftarrow y) < \sim x$; and $x < z \leftarrow y$ we obtain from $x \cdot y < z$ and $\sim z \cdot x < \sim y$ by (2*).

Let $x < z \leftarrow y$ and $\sim (z \leftarrow y) < \sim x$. From the former we have $x \cdot y < z$ and $\sim z \cdot x < \sim y$ by (2*). The transitivity of $<$ and (9*) $y \cdot \sim z < \sim (z \leftarrow y)$ yield $y \cdot \sim z < \sim x$, hence $\sim z < \sim (x \cdot y)$ follows by (3*).
Mutual interpretability of SymConFL and CyInFL

Theorem
CyInFL and SymConFL are mutually interpretable.

Proof
Conversely, assume that $\mathbb{I} = (L, \leq, \cdot, \rightarrow, \leftarrow, \sim, 1, 0)$ is a cyclic involutive FL–algebra. It is well known fact that in any cyclic involutive FL–algebra the following equivalence holds:

$$x \leq y \iff \sim y \leq \sim x.$$ 

Now, let us define $\leq$ as the partial–order relation $\leq$:

$$x < y \iff x \leq y.$$ 

Therefore, we get the equivalence from the definition of symmetric constructive FL–algebras:

$$x \leq y \iff x < y \text{ and } \sim y < \sim x. \quad (po)$$ 

Moreover the remaining conditions of this definitions also hold in $\mathbb{I}$ — it is easy to see that (1$^*$)–(11$^*$) with $\leq$ in place of $<$ hold in any cyclic involutive FL–algebra.
Theorem
CyInFL and SymConFL are mutually interpretable.

Proof
Conversely, assume that $\mathbb{L} = (L, \leq, *, \rightarrow, \leftarrow, \sim, 1, 0)$ is a cyclic involutive FL–algebra. It is well known fact that in any cyclic involutive FL–algebra the following equivalence holds:

$$x \leq y \iff \sim y \leq \sim x.$$ 

Now, let us define $<\text{ as the partial–order relation }\leq$:

$$x < y \iff x \leq y.$$ 

Therefore, we get the equivalence from the definition of symmetric constructive FL–algebras:

$$x \leq y \iff x < y \text{ and } \sim y < \sim x.$$ 

Moreover the remaining conditions of this definitions also hold in $\mathbb{L}$ — it is easy to see that $(1^*)–(11^*)$ with $\leq$ in place of $<$ hold in any cyclic involutive FL–algebra.

Theorem
CyInFL$_e$ and SymConFL$_e$ are mutually interpretable.

Theorem
CyInFL$_w$ and SymConFL$_w$ are mutually interpretable.
Nelson FL$_{ew}$–algebras

The analogous theorem for variants with contraction is not valid. We have $x < x \cdot x$, but $\sim(x \cdot x) < \sim x$ does not hold in all symmetric constructive FL$_c$–algebras.

Nevertheless, we can show the term equivalence of SymConFL$_{ewc}$ and the class of Nelson FL$_{ew}$–algebras.
Nelson FL\textsubscript{ew}–algebras

The analogous theorem for variants with contraction is not valid. We have \( x < x \cdot x \), but \( \sim(x \cdot x) < \sim x \) does not hold in all symmetric constructive FL\textsubscript{c}–algebras.

Nevertheless, we can show the term equivalence of SymConFL\textsubscript{ewc} and the class of Nelson FL\textsubscript{ew}–algebras.

- Matthew Spinks, Robert Veroff
- Manuela Busaniche, Roberto Cignoli

A Nelson FR\textsubscript{ew}–algebra or a Nelson residuated lattice is a cyclic involutive FL\textsubscript{ew}–algebra that satisfies the Nelson identity:

\[
x \cdot x \rightarrow y \land \sim y \cdot \sim y \rightarrow \sim x = x \rightarrow y. \quad (nl)
\]
Nelson FL_{ew}–algebras

The analogous theorem for variants with contraction is not valid. We have $x < x \cdot x$, but $\sim(x \cdot x) < \sim x$ does not hold in all symmetric constructive FL_{c}–algebras.

Nevertheless, we can show the term equivalence of SymConFL_{ewc} and the class of Nelson FL_{ew}–algebras.

- Matthew Spinks, Robert Veroff
- Manuela Busaniche, Roberto Cignoli

A Nelson FL_{ew}–algebra or a Nelson residuated lattice is a cyclic involutive FL_{ew}–algebra that satisfies the Nelson identity:

$$x \cdot x \rightarrow y \land \sim y \cdot \sim y \rightarrow \sim x = x \rightarrow y. \quad (nl)$$

Substituting $\sim x$ for $y$ in $(nl)$ one can easily prove the following well-known fact. **Fact**

Any Nelson FL_{ew}–algebra is 3–potent FL–algebra, i.e. it satisfies the axiom of 3–potency:

$$x \cdot x = x \cdot x \cdot x \quad (3p)$$
Lemma (equivalent definition)
A Nelson FL\textsubscript{ew}–algebra is a cyclic involutive FL\textsubscript{ew}–algebra that satisfies the following quasi–equation:

\[
x \cdot x \leq y \text{ and } \sim y \cdot \sim y \leq \sim x \quad \Rightarrow \quad x \leq y.
\]

(qnl)

Proof

Obviously, (nl) implies (qnl). We will show the reverse implication.

From integrality we have \( x \cdot x \leq x \), and hence we obtain \( x \rightarrow y \leq x \cdot x \rightarrow y \). Similarly we obtain \( x \cdot \sim y \cdot \sim y \leq \sim y \cdot x \), using commutativity. Therefore, \( \sim(\sim y \cdot \sim y \rightarrow \sim x) \leq \sim(x \rightarrow y) \), and hence \( x \rightarrow y \leq \sim y \cdot \sim y \rightarrow \sim x \). Consequently

\[
x \rightarrow y \leq x \cdot x \rightarrow y \land \sim y \cdot \sim y \rightarrow \sim x.
\]

On the other hand, from \( x \cdot x \cdot (x \cdot x \rightarrow y) \leq y \) we obtain

\[
x \cdot x \cdot (x \cdot x \rightarrow y \land \sim y \cdot \sim y \rightarrow \sim x) \leq y.
\]

Similarly we prove

\[
\sim y \cdot \sim y \cdot (x \cdot x \rightarrow y \land \sim y \cdot \sim y \rightarrow \sim x) \leq \sim x.
\]

From the former we get

\[
x \cdot x \leq (x \cdot x \rightarrow y \land \sim y \cdot \sim y \rightarrow \sim x) \rightarrow y
\]

by commutativity and the residuation laws, and from the latter we get

\[
\sim((x \cdot x \rightarrow y \land \sim y \cdot \sim y \rightarrow \sim x) \rightarrow y) \cdot \sim((x \cdot x \rightarrow y \land \sim y \cdot \sim y \rightarrow \sim x) \rightarrow y) \leq \sim x
\]

by integrality, commutativity and identity \( x \cdot y = \sim(y \rightarrow \sim x) \). Thus, by (qnl),

\[
x \leq (x \cdot x \rightarrow y \land \sim y \cdot \sim y \rightarrow \sim x) \rightarrow y,
\]

and consequently

\[
x \cdot x \rightarrow y \land \sim y \cdot \sim y \rightarrow \sim x \leq x \rightarrow y.
\]
Term equivalence of SymConFL$_{ewc}$ and the class of Nelson FL$_{ew}$–algebras

**Theorem**
The classes of Nelson FL$_{ew}$–algebras and SymConFL$_{ewc}$ are term equivalent.

**Proof**
First we will prove that the following equivalence holds in all symmetric constructive integral contractive FL–algebras:

\[ x < y \iff x \cdot x < y \text{ and } \neg y < \neg(x \cdot x). \quad (\lt *) \]

Assume that \( x < y \). Then \( x \cdot x < y \) we get by integrality, and \( \neg y \cdot x < 0 \) and \( x \cdot \neg y < 0 \) by \((4\ast)\) and \((5\ast)\), respectively. So by integrality again, \( \neg y \cdot x < \neg x \) and \( x \cdot \neg y < \neg x \), and hence \( \neg y < \neg(x \cdot x) \), by \((3\ast)\).

To prove the converse, it is sufficient to assume \( x \cdot x < y \) and use the property of being contractive.

Assume now that \( \mathbb{L} \) is a Zaslavsky FL$_{ewc}$–algebra. We need only to prove that \((qnl)\) holds in \( \mathbb{L} \). Let us assume \( x \cdot x \leq y \) and \( \neg y \cdot \neg y \leq \neg x \). Therefore, \( x \cdot x < y \) and \( \neg y \cdot \neg y < \neg x \), and hence \( x < y \) and \( \neg y < \neg x \), by the property of being contractive.
Term equivalence of SymConFL\textsubscript{ewc} and the class of Nelson FL\textsubscript{ew}–algebras

**Theorem**
The classes of Nelson FL\textsubscript{ew}–algebras and SymConFL\textsubscript{ewc} are term equivalent.

**Proof**
Conversely, assume that $\mathbb{N} = (N, \leq, \cdot, \to, \leftarrow, \sim, 1, 0)$ is a Nelson FL\textsubscript{ew}–algebra and define $<$ like a quasi–order relation in 3–potent FL\textsubscript{w}–algebras, i.e.:

$$x < y \iff x \cdot x \leq y.$$  \hfill (ro)

It remains to show that the binary relation $R$ defined as $R(x, y)$ if and only if $x < y$ and $\sim y < \sim x$, for all $x, y \in N$, is a partial–order and that all axioms of the class SymConFL\textsubscript{ewc} hold in $\mathbb{N}$. First we will even prove that $R$ is exactly $\leq$ of $\mathbb{N}$, i.e.:

$$x \leq y \iff x \cdot x \leq y \text{ and } \sim y \cdot \sim y \leq \sim x.$$ \hfill (\leq *)

Assume that $x \leq y$. Therefore, we obtain $\sim y \leq \sim x$, and hence $x \cdot x \leq y$ and $\sim y \cdot \sim y \leq \sim x$, by integrality.

Suppose now that $x \cdot x \leq y$ and $\sim y \cdot \sim y \leq \sim x$. Therefore $x \leq y$ we get by (qln).

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Strong Negation in Intuitionistic Style Sequent Systems for Residuated Lattices
The classes of Nelson FL$_{ew}$–algebras and SymConFL$_{ewc}$ are term equivalent.

Proof
Taking into account the equivalence ($\leq *$), we straightforwardly obtain $(dn)$, $(0^*)$, commutativity and integrality. The property of being contractive follows directly from the definition of $<$ (ro). The remaining axioms of symmetric constructive FL–algebras $(1^*)$–$(11^*)$ are provable as well (the Nelson identity $(nl)$ is useful).

\[
\begin{align*}
  x < y &\iff x \cdot x < y \text{ and } \neg y < \neg (x \cdot x) & (\leq *) \\
  x \leq y &\iff x < y \text{ and } \neg y < \neg x & (po) \\
  x \leq y &\iff x \cdot x \leq y \text{ and } \neg y \cdot \neg y \leq \neg x & (\leq *) \\
  x < y &\iff x \cdot x \leq y & (ro)
\end{align*}
\]
Term equivalence of SymConFL_{ewc} and the class of Nelson FL_{ew}–algebras

**Theorem**
The classes of Nelson FL_{ew}–algebras and SymConFL_{ewc} are term equivalent.

**Proof**
Taking into account the equivalence ($\leq *$), we straightforwardly obtain $(dn)$, $(0^*)$, commutativity and integrality. The property of being contractive follows directly from the definition of $<$ ($ro$). The remaining axioms of symmetric constructive FL–algebras ($1^*$)–($11^*$) are provable as well (the Nelson identity ($nl$) is useful).

The last thing we have to prove is that the interpretations are mutually inverse. Since they differ only in the definitions of the order relations (quasi and partial), it suffices to show that ($ro$) and ($po$) are mutually inverse.

For $\mathbb{I} \in$ SymConFL_{ewc} we have:

$$ x < y \overset{(ro)}{\Rightarrow} x \cdot x \leq y \overset{(po)}{\Rightarrow} x \cdot x < y \text{ and } \sim y < \sim (x \cdot x) $$

For a Nelson FL_{ew}–algebra $\mathbb{N}$ we have:

$$ x \leq y \overset{(po)}{\Rightarrow} x < y \text{ and } \sim y < \sim x \overset{(ro)}{\Rightarrow} x \cdot x \leq y \text{ and } \sim y \cdot \sim y \leq \sim x $$

$$ x < y \iff x \cdot x < y \text{ and } \sim y < \sim (x \cdot x) \quad (< *) $$

$$ x \leq y \iff x < y \text{ and } \sim y < \sim x \quad (po) $$

$$ x \leq y \iff x \cdot x \leq y \text{ and } \sim y \cdot \sim y \leq \sim x \quad (\leq *) $$

$$ x < y \iff x \cdot x \leq y \quad (ro) $$
THANK YOU FOR YOUR ATTENTION