FRAMED KNOTS IN 3-MANIFOLDS

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Abstract. For a fixed isotopy type K of unframed knots in S^3 there are infinitely many isotopy classes of framed knots that correspond to K when we forget the framing.

We show that the same fact is true for all the isotopy types of unframed knots in a closed oriented 3-manifold M, provided that $M \neq (S^1 \times S^2)\#M'$. On the other hand for any $M = (S^1 \times S^2)\#M'$ we construct examples of isotopy classes of unframed knots in M that correspond to only two isotopy classes of framed knots.

2000 Mathematics Subject Classification. Primary: 57M27

Keywords: framed knots, self-linking

1. Main Results

Throughout this paper M is a smooth oriented connected 3-dimensional Riemannian manifold, unless the opposite is explicitly stated.

A *curve* in a manifold M is an immersion of S^1 into M. A *framed curve* in M is a curve equipped with a continuous unit normal vector field.

A *knot (resp. framed knot)* in M is a smooth embedding (resp. framed embedding) of S^1. Ordinary and framed knots are studied up to the corresponding isotopy equivalence relation.

It is well known that for a fixed isotopy class K of ordinary knots in S^3, there are infinitely many isotopy classes of framed knots that realize K when we forget the framing. (These isotopy classes of framed knots are enumerated by the self-linking number.)

Intuitively one expects this fact to be true for all knots in all 3-manifolds. Clearly it is true for zero-homologous knots, since for such knots there is a self-linking number that is invariant under framed isotopy.

However to show that this is true for knots that are not zero homologous is much harder, and in fact the following Theorem provides examples of unframed knots K such that there are only finitely many isotopy classes of framed knots that correspond to K when we forget the framing. (We constructed the first examples of this sort in [19] Theorem 3.1.2.a and in [20].)

Theorem 1.0.1. *Let M be an oriented (not necessarily compact) 3-manifold that is a connected sum $(S^1 \times S^2)\#M'$ and let $K \subset M$ be a knot that crosses only once one of the spheres $t \times S^2 \subset (S^1 \times S^2)\#M'$. Then there are only two isotopy classes of framed knots in M that realize the isotopy class of K when we forget the framing.*

For the Proof of Theorem 1.0.1 see Section 2.1.

Remark 1.0.2. Let K be an unframed knot, and let K_f be a framed knot that coincides with K as an unframed curve. For $i \in \mathbb{Z}$ put K_f^i to be a framed knot that coincides with K_f everywhere except of a small arc where

$$\begin{cases} K^i_f \text{ has } i \text{ positive extra twists of the framing,} & \text{provided that } i > 0; \\ K^i_f \text{ has } |i| \text{ negative extra twists of the framing,} & \text{otherwise.} \end{cases}$$

Clearly a framed knot that corresponds to K under forgetting of the framing is isotopic to one of the knots K^i_f, $i \in \mathbb{Z}$. Intuitively one expects K^i_f and K^j_f to be nonisotopic for $i \neq j$. However Theorem 1.0.1 provides an example where K^0_f is isotopic to K^2_f. One can show that K^i_f and K^j_f belong to the same component[1]of the space of framed curves if and only if $i-j$ is even. Thus the minimal $i \in \mathbb{N}$ such that K^0_f and $K^{\pm i}_f$ can be isotopic is equal to two.

We are not aware of any examples where K^0_f is not isotopic to K^2_f but there exist $i \neq j$ such that K^i_f and K^j_f are isotopic (i.e. in all the examples known to us where one can add by isotopy a nonzero number of extra twists of the framing it is possible to add by isotopy two extra twists of the framing).

The following Theorem says that it is impossible to construct examples of unframed knots to which correspond only finitely many isotopy classes of framed knots, provided that the ambient manifold M is oriented closed and is not a sum of $S^1 \times S^2$ and another manifold M'. Thus for almost all closed oriented 3-manifolds M every isotopy class of unframed knots in M corresponds to infinitely many isotopy classes of framed knots. (In [19], Theorems 3.0.6, 3.0.9, and Remark 3.0.10, we have shown that this is true for all knots in closed orientable hyperbolic 3-manifolds and in the orientable total spaces of locally-trivial S^1-bundles over a surface $F \neq S^2, \mathbb{R}P^2$.)

Theorem 1.0.3. *Let M be a closed oriented 3-manifold. Then* **1** *is equivalent to* **2***.*

- **1:** *To every isotopy class of unframed knots in M correspond infinitely many isotopy classes of framed knots in M.*
- **2:** *M is not realizable as a connected sum of $S^1 \times S^2$ and another manifold M'.*

For the Proof of Theorem 1.0.3 see Section 2.4.

Remark 1.0.4. It is well known that M is realizable as a connected sum of $S^1 \times S^2$ and another manifold M' if and only if there exists an embedded two-sphere in M that does not separate M into two parts. This gives another formulation of statement **2** of Theorem 1.0.3.

From the proof of Theorem 1.0.3 it follows that if to an isotopy class of an unframed knot K correspond infinitely many isotopy classes of framed knots, then an even stronger fact is true. Namely, K^i_f and K^j_f (see 1.0.2) are not isotopic as framed knots for $i \neq j$.

Let M be a closed oriented manifold that is a connected sum $\#M_i$ of irreducible 3-manifolds. (A manifold is said to be *irreducible* if every two-sphere embedded

[1] Every oriented 3-dimensional manifold M is parallelizable, and hence it admits a spin-structure. A framed curve K in M represents a loop in the principal $SO(3)$-bundle of TM. The 3-frame corresponding to a point of K is the velocity vector, the framing vector, and the unique third vector of unit length such that the 3-frame defines the positive orientation of M. One can show that for every connected component of the space of curves in an orientable manifold M there are exactly two components of the space of framed curves that project to it. The two components of the space of framed curves are distinguishable by the values of a spin-structure on the loops in the principle $SO(3)$-bundle that correspond to framed curves.

into M bounds a ball.) The following fact can be easily deduced from the proof of Theorem 1.0.3.

1: Let K be a knot that is not free homotopic as a loop to a knot contained in one of the manifolds M_i. Then for every generic homotopy γ connecting K^i_f and K^j_f the sum of the signs of passages under γ through singular knots with one transverse double point is equal to $j-i$.

2: Let K be a knot that is free homotopic as a loop to a knot contained in one of the manifolds M_i. Then for every generic homotopy γ connecting K^i_f and K^j_f the sum of the signs of passages under γ through singular knots with one transverse double point that separates the singular knot into two loops such that one of them is contractible is equal to $j-i$.

1.0.5. Framed knots in nonclosed oriented 3-manifolds. Let M' be a nonclosed oriented 3-manifold. Since an isotopy of framed knots that connects K^i_f and K^j_f is contained in a compact submanifold of M', we get that M' contains an unframed knot that corresponds to only finitely many isotopy classes of framed knots if and only if there is a compact submanifold of M' with the same property.

Let M' be a compact (nonclosed) oriented 3-manifold that is a submanifold of a closed oriented $\tilde{M}$ such that $\tilde{M}$ is not realizable as a sum $\tilde{M}=(S^1\times S^2)\#\tilde{M}'$. Then Theorem 1.0.3 implies that every isotopy class of unframed knots from M' corresponds to infinitely many isotopy classes of framed knots in M'.

A closed oriented manifold $\tilde{M}$ that contains M' is obtained by gluing M' to another compact manifold M'', with $\partial M'$ and $\partial M''$ been homeomorphic, via an orientation reversing homeomorphism of the boundaries.

It seems to us that it should be possible to prove that if the compact oriented M' is not realizable as a connected sum of $S^1\times S^2$ and another compact oriented manifold, then there exists an oriented closed $\tilde{M}$ that contains M' that is also not realizable as a connected sum of $S^1\times S^2$ and another closed oriented manifold. Then Theorem 1.0.3 would imply that a (not necessarily compact) oriented M' contains a knot that corresponds to only finitely many isotopy classes of framed knots in M' if and only if M' is realizable as a connected sum of $S^1\times S^2$ and another oriented 3-manifold.

At least it is rather easy to prove that if M' is a compact oriented 3-manifold with incompressible boundary (i.e. $\pi_1(\partial M')\to\pi_1(M')$ is injective for all the connected components of $\partial M'$) such that it is not realizable as a connected sum of $S^1\times S^2$ and another compact oriented 3-manifold, then the closed oriented manifold obtained by gluing the two copies of M' along the identity homeomorphism of the boundary also is not realizable as a connected sum of $S^1\times S^2$ and another closed manifold.

Thus for such manifolds M' every isotopy class of unframed knots in M' corresponds to infinitely many isotopy classes of framed knots in M'.

1.0.6. Framed knots in nonorientable 3-manifolds. Straightforward geometric considerations show that if M is a nonorientable 3-manifold, then every unframed knot that realizes an orientation reversing loop corresponds to exactly two isotopy classes of framed knots in M.

It is also clear that if an unframed knot in M that realizes an orientation preserving loop corresponds to only finitely many isotopy classes of framed knots, then its lifting to the orientation double cover $\tilde{M}\to M$ also corresponds to only finitely many isotopy classes of framed knots in $\tilde{M}$. Thus if M is a nonorientable closed 3-manifold such that its orientation double cover is not realizable as a connected sum of $S^1 \times S^2$ and another closed oriented 3-manifold, then every unframed knot in M that realizes an orientation preserving loop corresponds to infinitely many isotopy classes of framed knots in M.

2. Proofs

2.1. **Proof of Theorem 1.0.1.** Put K_f to be a framed knot that coincides with K as an unframed curve, and put K_f^i, $i \in \mathbb{Z}$, to be the framed knots defined in 1.0.2.

Let $t \times S^2$ be the sphere that crosses K_f at exactly one point, and let $N = [0,1] \times S^2$ be the tubular neighborhood of $t \times S^2$. Fix $x \in S^2$ (below called the North pole) and the direction in $T_x S^2$ (below called the zero meridian). We can assume that the knot K_f inside $N = [0,1] \times S^2$ looks as follows: it intersects each $y \times S^2 \subset [0,1] \times S^2$ at the North pole of the corresponding sphere, and the framing of the knot is parallel to the zero meridian.

Consider an automorphism $\nu : M \to M$ that is identical outside of $N = [0,1] \times S^2$ such that it rotates each $y \times S^2 \in [0,1] \times S^2$ by $4\pi y$ around the North pole – South pole axis in the clockwise direction. Clearly under this automorphism K_f gets two negative extra twists of the framing and $\nu(K_f) = K_f^{-2}$.

On the other hand it is easy to see that ν is diffeotopic to the identity, since it corresponds to the contractible loop in $SO(3) = \mathbb{R}P^3$. Hence we see that K_f and K_f^{-2} are isotopic framed knots.

Thus the knots $K_f^0, K_f^{\pm 2}, K_f^{\pm 4}, \dots$ are isotopic framed knots. Similarly the knots $K_f^1, K_f^{1\pm 2}, K_f^{1\pm 4}, \dots$ are isotopic framed knots. Clearly any framed knot that is isotopic to K when we forget the framing is isotopic (as a framed knot) to one of $K_f^i, i \in \mathbb{Z}$. Thus there are at most two isotopy classes of framed knots that correspond to the isotopy class of K when we forget the framing.

Every oriented 3-dimensional manifold M is parallelizable, and hence it admits a spin-structure. A framed curve $\bar{K}_f$ in M represents a loop in the principal $SO(3)$-bundle of TM. The 3-frame corresponding to a point of $\bar{K}_f$ is the velocity vector, the framing vector, and the unique third vector of unit length such that the 3-frame defines the positive orientation of M. Clearly the values of a spin structure on the loops in the principal $SO(3)$-bundle of TM that correspond to K_f^i and K_f^j are different, provided that $i - j$ is odd. Since the value of a spin structure depends only on the connected component of the space of framed curves, we get that the framed curves K_f^i and K_f^j belong to different components of the space of framed curves, provided that $i - j$ is odd. Thus there are at least two different isotopy classes of framed knots that correspond to K. This finishes the proof of Theorem 1.0.1. □

2.2. **The covering** $\mathrm{pr} : \mathcal{F} \to \mathcal{C}$**.** In this subsection the manifold M is oriented but not necessarily compact.

Let $\mathcal{C}$ be a component of the space of unframed curves in M. Let $\mathcal{F}'$ be a connected component of the space of framed curves in M, such that the curves from $\mathcal{F}'$ realize curves from $\mathcal{C}$ if we forget the framing. Put $\mathrm{pr}' : \mathcal{F}' \to \mathcal{C}$ to be the forgetting of the framing mapping.

Let $\mathcal{F}$ be the quotient space of $\mathcal{F}'$ by the following equivalence relation: $f_1' \sim f_2'$ if there exists a path $I : [0,1] \to \mathcal{F}'$ connecting f_1' and f_2' such that $\mathrm{Im}(\mathrm{pr}'(I)) =$

$\mathrm{pr}'(f_1') = \mathrm{pr}'(f_2')$. (This means that we identify two framed curves if they coincide as unframed curves and we can change one of them into another by a continuous change of framing without changing the underlying unframed curve.)

Put $q : \mathcal{F}' \to \mathcal{F}$ to be the quotient map and put $\mathrm{pr} : \mathcal{F} \to \mathcal{C}$ to be the mapping such that $\mathrm{pr} \circ q = \mathrm{pr}'$. (Clearly pr is continuous.)

Definition 2.2.1 (of isotopic knots from $\mathcal{F}$)**.** Let $K_0, K_1 \in \mathcal{F}$ be such that $\mathrm{pr}(K_0)$ and $\mathrm{pr}(K_1)$ are knots (embedded curves). Then K_0 and K_1 are said to be *isotopic* if there exists a path $p : [0,1] \to \mathcal{F}$ such that $p(0) = K_0, p(1) = K_1$, and $\mathrm{pr} \circ p$ is an isotopy (of unframed knots) that changes $\mathrm{pr}(K_0)$ into $\mathrm{pr}(K_1)$.

Lemma 2.2.2 implies that framed knots $\bar{K}_0, \bar{K}_1 \in \mathcal{F}'$ are framed isotopic if and only if $q(\bar{K}_0) \in \mathcal{F}$ and $q(\bar{K}_1) \in \mathcal{F}$ are isotopic.

Lemma 2.2.2. $\mathrm{pr} : \mathcal{F} \to \mathcal{C}$ *is a normal covering, i.e. it is a covering such that* $\mathrm{Im}(\mathrm{pr}_*(\pi_1(\mathcal{F})))$ *is a normal subgroup of* $\pi_1(\mathcal{C})$.

Below we give the proof of Lemma 2.2.2. The pull-back of the framing of a framed curve gives a nonzero section of the pull-back of the normal bundle of the curve. For two framed curves c_{1f} and c_{2f} that are the same as unframed curves there is a $\mathbb{Z}$-valued obstruction for the corresponding sections of the pull-back of the normal bundle to be homotopic in the class of nonzero sections. We put $o(c_{1f}, c_{2f}) \in \mathbb{Z}$ to be the value of this obstruction. (The first nonzero section of the pull-back $\nu : E \to S^1$ of the normal bundle allows us to identify E with $S^1 \times \mathbb{R}^2$. The compactification $\bar{E}$ of E is then identified with $S^1 \times D^2$. The other section gives rise to a loop on $\partial(S^1 \times D^2) = S^1 \times S^1$ and $o(c_{1f}, c_{2f})$ is the degree of the projection of the loop to the second S^1-component. In particular if c_{1f}, c_{2f} are zero-homologous framed knots, then $o(c_{1f}, c_{2f})$ is the difference of the self-linking numbers of c_{1f} and of c_{2f}.)

Proposition 2.2.3. *Let* $\mathcal{F}'$ *be a connected component of the space of framed curves, let* $q : \mathcal{F}' \to \mathcal{F}$ *be the quotient mapping, let* $\mathrm{pr}' : \mathcal{F}' \to \mathcal{C}$ *be the forgetting of the framing mapping, and let* $\mathrm{pr} : \mathcal{F} \to C$ *be such that* $\mathrm{pr} \circ q = \mathrm{pr}'$.

- **a:** *Let* $c_{1f}, c_{2f} \in \mathcal{F}'$ *be such that* $\mathrm{pr}'(c_{1f}) = \mathrm{pr}'(c_{2f})$. *Then* $o(c_{1f}, c_{2f}) = 0$ *if and only if* $q(c_{1f}) = q(c_{2f})$.
- **b:** *Let* $c_{1f}, c_{2f}, c_{3f} \in \mathcal{F}'$ *be such that* $\mathrm{pr}'(c_{1f}) = \mathrm{pr}'(c_{2f}) = \mathrm{pr}'(c_{3f})$. *Then* $o(c_{1f}, c_{2f}) + o(c_{2f}, c_{3f}) = o(c_{1f}, c_{3f})$.
- **c:** *Let* $c_{1f}, c_{2f} \in \mathcal{F}'$ *be such that* $\mathrm{pr}'(c_{1f}) = \mathrm{pr}'(c_{2f})$. *Then* $o(c_{1f}, c_{2f}) = -o(c_{2f}, c_{1f})$.
- **d:** *Let* $c_{1f}, c_{2f}, c'_{1f}, c'_{2f} \in \mathcal{F}'$ *be such that*
  - **1:** $\mathrm{pr}'(c_{1f}) = \mathrm{pr}'(c_{2f})$;
  - **2:** $\mathrm{pr}'(c'_{1f}) = \mathrm{pr}'(c'_{2f})$;
  - **3:** c_{1f} *is homotopic to* c'_{1f} *as a framed curve and the underlying homotopy of unframed curves is* C^1*-small;*
  - **4:** c_{2f} *is homotopic to* c'_{2f} *as a framed curve and the underlying homotopy of unframed curves is* C^1*-small.*

  Then $o(c_{1f}, c_{2f}) = o(c'_{1f}, c'_{2f})$.

The proof of statements **a, b, c** is straightforward. To get statement **d** we observe that it suffices to prove it in the case where $M = S^1 \times \mathbb{R}^2$ and $\mathrm{pr}'(c_{1f}) = \mathrm{pr}'(c_{2f}) = S^1 \times (0,0) \subset S^1 \times \mathbb{R}^2$. (One identifies $S^1 \times \mathbb{R}^2$ with the total space of the pull-back of the normal bundle of $\mathrm{pr}'(c_{1f}) \subset M$, and pulls back the curves

c'_{1f}, c'_{2f} via the exponential mappings of the 2-planes in TM that are orthogonal to $\mathrm{pr}'(c_{1f})$.) After this observation the proof of **d** is also clear.

The following Proposition is obvious.

Proposition 2.2.4. *Let $c \in \mathcal{C}$ and $c_f \in \mathcal{F}'$ be such that $\mathrm{pr}'(c_f) = c$. Then for any path $p : [0,1] \to \mathcal{C}$ such that $p(0) = c$ there exists a path $p_f : [0,1] \to \mathcal{F}'$ such that $p_f(0) = c_f$ and $\mathrm{pr}'(p_f(t)) = p(t)$, for all $t \in [0,1]$.*

Propositions 2.2.3 and 2.2.4 imply that $\mathrm{pr} : \mathcal{F} \to \mathcal{C}$ is a covering.

Further for $f_1, f_2 \in \mathcal{F}$ we put $o(f_1, f_2)$ to be $o(f'_1, f'_2)$ for any $f'_1, f'_2 \in \mathcal{F}'$ such that $q(f'_1) = f_1, q(f'_2) = f_2$. (Proposition 2.2.3 implies that $o(f_1, f_2)$ does not depend on the choice of $f'_1, f'_2 \in \mathcal{F}'$.)

To show that $\mathrm{pr} : \mathcal{F} \to \mathcal{C}$ is a normal covering we prove the following proposition.

Proposition 2.2.5. *Let $f \in \mathcal{F}, c \in \mathcal{C}$ be such that $\mathrm{pr}(f) = c$. For a loop $\alpha : [0,1] \to \mathcal{C}$ such that $\alpha(0) = \alpha(1) = c$, put $\bar{\alpha} : [0,1] \to \mathcal{F}$ to be the path that covers α such that $\bar{\alpha}(0) = f$. Then $\delta : \pi_1(\mathcal{C}, c) \to \mathbb{Z}$ that maps the class $[\alpha] \in \pi_1(\mathcal{C}, c)$ of a loop α to $o(f, \bar{\alpha}(1)) \in \mathbb{Z}$ is a homomorphism.*

2.2.6. *Proof of Proposition 2.2.5.* We start by showing that if $f_1, f_2 \in \mathcal{F}$ are such that $\mathrm{pr}(f_1) = \mathrm{pr}(f_2) = c$ and $\alpha_1, \alpha_2 : [0,1] \to \mathcal{F}$ are liftings of α such that $\alpha_1(0) = f_1, \alpha_2(0) = f_2$, then $o(f_1, \alpha_1(1)) = o(f_2, \alpha_2(1))$. To show this take $f'_1 \in \mathcal{F}'$ such that $q(f'_1) = f_1$ and put $f'_2 \in \mathcal{F}'$ to be a curve that is the same as f'_1 everywhere except of a small piece where f'_2 has $o(f_1, f_2)$ positive extra twists of the framing. (Here and below we assume that $o(f_1, f_2) \geq 0$. If $o(f_1, f_2) < 0$ one should substitute here and below the words positive extra twists by the words negative extra twists.) Clearly $q(f'_2) = f_2$. Let $\alpha'_1 : [0,1] \to \mathcal{F}'$ be a path such that $q(\alpha'_1(t)) = \alpha_1(t)$, $\forall t \in [0,1]$. Put $\alpha'_2 : [0,1] \to \mathcal{F}'$ to be a path such that $\alpha'_2(0) = f'_2$ and such that $\forall t \in [0,1]$ the framed curve $\alpha'_2(t)$ is the same as $\alpha'_1(t)$ everywhere except of a small piece that contains $o(f_1, f_2)$ positive extra twists of the framing. Clearly $q(\alpha'_2(t)) = \alpha_2(t)$, $\forall t \in [0,1]$. And we get that $o(f'_1, f'_2) = o(\alpha'_1(1), \alpha'_2(1))$ (since both these numbers are equal to $o(f_1, f_2)$) and hence by 2.2.3.b we have $o(f'_1, \alpha'_1(1)) = o(f'_2, \alpha'_2(1))$. Thus $o(f_1, \alpha_1(1)) = o(f_2, \alpha_2(1))$.

After this the statement of the Proposition follows immediately from Proposition 2.2.3.b. □

Remark 2.2.7. From the proof of Proposition 2.2.5 it is clear that for $\alpha \in \pi_1(\mathcal{C}, c)$ the value of $\delta(\alpha) \in \mathbb{Z}$ does not depend on the choice of f such that $\mathrm{pr}(f) = c$. Moreover since δ is a $\mathbb{Z}$-valued homomorphism it is clear that δ can be viewed also as a homomorphism $\delta : H_1(\mathcal{C}) \to \mathbb{Z}$.

Let $f \in \mathcal{F}$ and $c \in \mathcal{C}$ be such that $\mathrm{pr}(f) = c$. Clearly $\mathrm{Im}(\mathrm{pr}_*(\pi_1(\mathcal{F}, f))) < \pi_1(\mathcal{C}, c)$ consists exactly of the classes of loops α that are liftable to loops $\bar{\alpha}$ in $\mathcal{F}$ with $\bar{\alpha}(0) = \bar{\alpha}(1) = f$. Thus, see 2.2.3 and 2.2.5, it consists of those $[\alpha] \in \pi_1(\mathcal{C}, c)$ that are in the kernel of the homomorphism $\delta : \pi_1(\mathcal{C}, c) \to \mathbb{Z}$, and hence $\mathrm{Im}(\mathrm{pr}_*(\pi_1(\mathcal{F}, f)))$ is a normal subgroup of $\pi_1(\mathcal{C}, c)$.

This finishes the proof of Lemma 2.2.2. □

2.3. *h*-principle and other facts needed for the proof of Theorem 1.0.3.

2.3.1. *h-principle for curves in M.*

Put STM to be the manifold obtained by the fiber-wise spherization of the tangent bundle of M, and put $p : STM \to M$ to be the corresponding locally trivial S^2-fibration. The h-principle [6] says that the space of curves in M is weak homotopy equivalent to the space of free (continuous) loops ΩSTM in STM. The weak homotopy equivalence is given by mapping a curve C to a loop $\vec{C} \in \Omega STM$ that sends a point $t \in S^1$ to the point of STM corresponding to the direction of the velocity vector of C at $C(t)$.

A loop $\alpha \in \pi_1(\Omega STM, \vec{K})$ is a mapping $\mu_\alpha : T^2 = S^1 \times S^1 \to STM$, with $\mu_\alpha\big|_{1\times S^1} = \vec{K}$ and $\mu_\alpha\big|_{S^1\times 1}$ being the trace of $1 \in S^1$ under the homotopy of $\vec{K}$ described by α. Put $t(\alpha) = \mu_\alpha\big|_{S^1\times 1} \in \pi_1(STM, \vec{K}(1))$. Since $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ is commutative, we get that $t : \pi_1(\Omega STM, \vec{K}) \to \pi_1(STM, \vec{K}(1))$ is a surjective homomorphism of $\pi_1(\Omega STM, \vec{K})$ onto the centralizer $Z(\vec{K})$ of $\vec{K} \in \pi_1(STM, \vec{K}(1))$. ($t$ is surjective because if $\gamma \in Z(\vec{K})$, then we can take $\mu_\alpha : S^1 \times S^1 \to STM$ to be such that $\mu_\alpha\big|_{1\times S^1} = \vec{K}$, $\mu_\alpha\big|_{S^1\times 1} = \gamma$ and μ_α maps the 2-cell of $S^1 \times S^1$ according to the commutation relation between $\vec{K}$ and γ. Clearly $t(\alpha) = \gamma$ for $\alpha \in \pi_1(\Omega STM, \vec{K})$ that corresponds to μ_α.)

If for the loops $\alpha_1, \alpha_2 \in \pi_1(\Omega STM, \vec{K})$ we have $t(\alpha_1) = t(\alpha_2) \in \pi_1(STM, \vec{K}(1))$, then the mappings μ_{α_1} and μ_{α_2} of T^2 corresponding to these loops can be deformed to be identical on the 1-skeleton of T^2. Clearly the obstruction for μ_{α_1} and μ_{α_2} to be homotopic as mappings of T^2 (with the mapping of the 1-skeleton of T^2 fixed under homotopy) is the element of $\pi_2(STM)$ obtained by gluing together the two 2-cells of the two tori along the common boundary. (In particular we get the Proposition of V. L. Hansen [7] that $t : \pi_1(\Omega X, \omega) \to Z(\omega) < \pi_1(X, \omega(1))$ is an isomorphism, provided that $\pi_2(X) = 0$.)

Since every oriented 3-dimensional manifold is parallelizable, we can identify STM with $S^2 \times M$ and $p : STM \to M$ with $p : S^2 \times M \to M$. Thus using the h-principle we can view t as a surjective homomorphism $t : \pi_1(\mathcal{C}, K) \to Z(K) < \pi_1(M, K(1))$.

Since $\pi_2(S^2 \times M) = \mathbb{Z} \oplus \pi_2(M)$ we get that if $\alpha_1, \alpha_2 \in \pi_1(\mathcal{C}, K)$ are such that $t(\alpha_1) = t(\alpha_2) \in Z(K) < \pi_1(M, K(1))$, then the obstruction for α_1 and α_2 to be equal in $\pi_1(\mathcal{C}, K)$ is an element of $\mathbb{Z} \oplus \pi_2(M)$.

2.3.2. *Homomorphisms β and $\bar{\beta}$.*

A transverse double point of a singular knot can be resolved in two essentially different ways. We say that a resolution of the double point is positive (resp. negative) if the tangent vector to the first strand, the tangent vector to the second strand, and the vector from the second strand to the first form the positive 3-frame (this does not depend on the order of the strands).

Let $\mathcal{C}$ be a connected component of the space of curves in M. Let α be a generic closed path in $\mathcal{C}$. Put $\bar{J}_\alpha$ to be set of instances when the knot becomes singular under the deformation α. At these instances α crosses the discriminant in $\mathcal{C}$. (The discriminant is a subspace of $\mathcal{C}$ formed by singular knots.) Since α is generic, at these instances the knot has one transverse double point, that separates the knot into two loops. Put $J_\alpha \subset \bar{J}_\alpha$ to be set of instances when one of the two loops of the singular knot is contractible.

Put σ_j, $j \in \bar{J}_\alpha$, to be the signs of the corresponding crossings of the discriminant. (The sign of the crossing is the sign of the resolution of the double point that occurs during the crossing.) Put $\Delta_\beta(\alpha) = \sum_{j\in J_\alpha} 2\sigma_j$. Put $\Delta_{\bar\beta}(\alpha) = \sum_{j\in \bar{J}_\alpha} 2\sigma_j$.

The codimension two stratum of the discriminant consists of knots with two distinct transverse double points. It is easy to see that $\Delta_\beta(\alpha') = \Delta_{\bar\beta}(\alpha') = 0$ for any small generic loop α' going around the codimension two stratum.

This implies (cf. [1]) that if α is a generic loop in $\mathcal{C}$ that starts at a nonsingular knot K, then $\Delta_\beta(\alpha)$ and $\Delta_{\bar\beta}(\alpha)$ depend only on the element of $\pi_1(\mathcal{C}, K)$ realized by a generic loop α. Now it is clear that $\beta : \pi_1(\mathcal{C}, K) \to \mathbb{Z}$ that maps the class of a generic loop α (starting at K) to $\Delta_\beta(\alpha)$ is a homomorphism. Similarly $\bar\beta$: $\pi_1(\mathcal{C}, K) \to \mathbb{Z}$ that maps the class of a generic loop α (starting at K) to $\Delta_{\bar\beta}(\alpha)$ is also a homomorphism.

Clearly if for $\alpha \in \pi_1(\mathcal{C}, K)$ $\beta(\alpha) \neq 0$ or $\bar\beta(\alpha) \neq 0$, then α is not realizable by an isotopy of the knot K.

From the construction of the homomorphisms β and $\bar\beta$ it is clear that they can be viewed as homomorphisms $\beta, \bar\beta : H_1(\mathcal{C}) \to \mathbb{Z}$.

Proposition 2.3.3. *Let M be a connected sum $M_1\#M_2$, and let K be a loop in M.*

- **1:** *If K is not free homotopic to a loop K' that is contained either in $M_1 \setminus B_3 \subset M_1\#M_2$ or in $M_2 \setminus B_3 \subset M_1\#M_2$, then the centralizer $Z(K)$ of $K \in \pi_1(M, K(1))$ is an infinite cyclic group that contains $K \in \pi_1(M, K(1))$.*
- **2:** *If K is a noncontractible loop that is contained in $M_1 \setminus B_3 \subset M_1\#M_2$, then the centralizer $Z(K)$ of $K \in \pi_1(M_1\#M_2, K(1))$ is the centralizer of $K \in \pi_1(M_1 \setminus B_3, K(1)) = \pi_1(M_1, K(1))$.*

To prove this Proposition one observes that by the Theorem of Van Kampen $\pi_1(M_1\#M_2)$ is the free product of $\pi_1(M_1)$ and of $\pi_1(M_2)$. After this the proof of the Proposition is an exercise in group theory.

The following Proposition is an immediate consequence of the Sphere Theorem, see [10].

Proposition 2.3.4. *Let M be a closed oriented manifold that is a connected sum $\#_{i=1}^k M_i$ of irreducible closed 3-manifolds and let $\phi_i : S^2 \to M$, $i = 1, \dots, k$ be the embedded spheres with respect to which M is the connected sum. Then $\pi_2(M, x)$ as the π_1 module is generated by the classes of spheres ϕ_i, i.e. every $s \in \pi_2(M, x)$ can be written as a product $s = \prod_{i\in I} s_i^{\pm 1}$ of the spheroids s_i such that*

- **a:** *s_i maps the the lower hemisphere of S^2 to a path connecting x to the south pole of $\phi_j(S^2)$, for some $j \in \{1, \dots, k\}$;*
- **b:** *s_i maps the upper hemisphere of S^2 as it is described by $\phi_j(S^2)$. (The upper hemisphere with all the points of the equator glued together is naturally identified with S^2.)*

2.3.5. *Loops γ_1, γ_2 and γ_s.* Let $\mathcal{C}$ be a connected component of the space of curves in M and let $K \in \mathcal{C}$ be a knot. Let γ_1 be the isotopy of K to itself that is the sliding of K along itself induced by the full rotation of the parameterizing circle.

Let γ_2 be the deformation of K described in Figure 1.

Let $s \in \pi_2(M, K(1))$ be an element that is realizable by a mapping $s : S^2 \to M$ such that

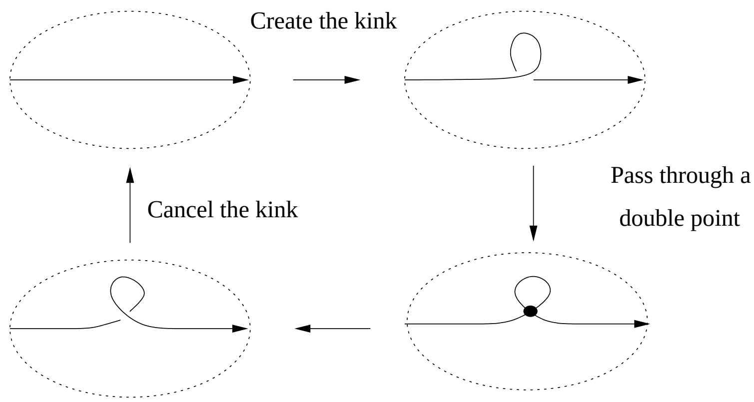

FIGURE 1. The loop γ_2.

a: s maps the the lower hemisphere of S^2 to a path ρ;
b: s restricted to the upper hemisphere of S^2 is an embedding. (The upper hemisphere with all the points of the equator glued together is naturally identified with S^2.)

Put $\gamma_s \in \pi_1(\mathcal{C}, K)$ to be a loop, under which the knot K does not move anywhere except of a small arc located close to $1 \in S^1$. The points of the arc first slide along the boundary of the tubular neighborhood of the path ρ. Then the arc reaches the embedded sphere (that is s restricted to the upper hemisphere) and slides around the sphere, see Figure 2. Finally the points of the arc slide back along the boundary of the tubular neighborhood of the path ρ.

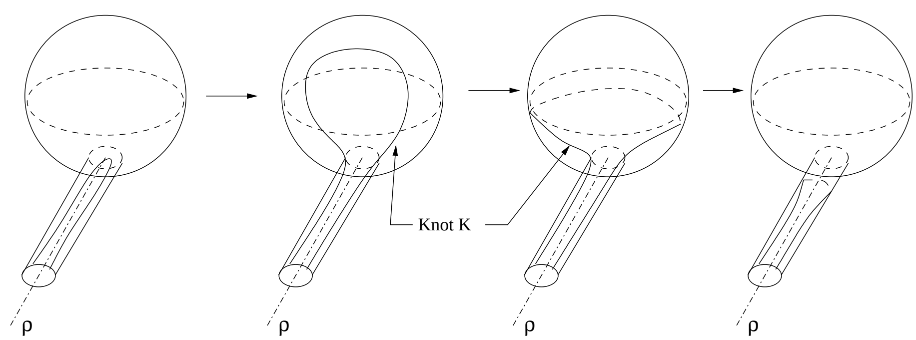

FIGURE 2. The loop γ_s.

Let $t : \pi_1(\mathcal{C}, K) \to Z(K) < \pi_1(M, K(1))$ be the homomorphism described in 2.3.1. Let $\alpha_1, \alpha_2 \in \pi_1(\mathcal{C}, K)$ be such that $t(\alpha_1) = t(\alpha_2) \in \pi_1(M, K(1))$. Let $m \oplus \epsilon \in \mathbb{Z} \oplus \pi_2(M) = \pi_2(S^2 \times M) = \pi_2(STM)$ be the obstruction for α_1 and α_2 to be equal in $\pi_1(\mathcal{C}, K)$, see 2.3.1. The Sphere Theorem [10] says that $\epsilon \in \pi_2(M)$ can be realized as a product $\epsilon = \prod_{i=1}^k s_i^{\sigma i}$ of the spheroids of the type described above ($\sigma_i = \pm 1$). Straightforward geometric considerations show that $t(\alpha_1 \prod_{i=1}^k \gamma_{s_i}^{\sigma i}) = t(\alpha_2)$ and the obstruction for $\alpha_1 \prod_{i=1}^k \gamma_{s_i}^{\sigma i}$ and α_2 to be equal elements in $\pi_1(\mathcal{C}, K)$ is an element $m' \oplus 0 \in \mathbb{Z} \oplus \pi_2(M) = \pi_2(STM)$. Finally one verifies that $t(\alpha_1(\prod_{i=1}^k \gamma_{s_i}^{\sigma i})\gamma_2^{m'}) = t(\alpha_2)$ and the obstruction for $\alpha_1(\prod_{i=1}^k \gamma_{s_i}^{\sigma i})\gamma_2^{m'}$ and α_2 to be homotopic vanishes, i.e. they realize the same elements of $\pi_1(\mathcal{C}, K)$.

Thus we get the following Lemma.

Lemma 2.3.6. *Let M be an oriented (not necessarily) closed 3-manifold, let $\mathcal{C}$ be a connected component of the space of curves in M, and let $K \in \mathcal{C}$ be a knot. Let $\alpha_1, \alpha_2 \in \pi_1(\mathcal{C}, K)$ be such that $t(\alpha_1) = t(\alpha_2) \in \pi_1(M, K(1))$.*

1: *Then there exist spheroids $s_i \in \pi_2(M, K(1))$, $i = 1, \dots k$, of the type described above, $\sigma_i = \pm 1$, and $m \in \mathbb{Z}$, such that $\alpha_1(\prod_{i=1}^k \gamma_{s_i}^{\sigma i})\gamma_2^m = \alpha_2 \in \pi_1(\mathcal{C}, K)$.*

2: *If $\pi_2(M) = 0$, then $\alpha_1\gamma_2^m = \alpha_2$, for some $m \in \mathbb{Z}$.*

Lemma 2.3.7. *Let M be (a not necessarily closed) oriented manifold that is not $(S^1 \times S^2)\#M'$. Let $\gamma_1, \gamma_2, \gamma_s$ be the loops described in 2.3.5. Then*

1: *$\delta(\gamma_1) = \beta(\gamma_1) = \bar{\beta}(\gamma_1) = 0$, and $\delta(\gamma_2) = \beta(\gamma_2) = \bar{\beta}(\gamma_2) = 2$ (see 2.2.5 and 2.3.2 for the definitions of $\delta, \beta, \bar{\beta}$);*

2: *$\delta(\gamma_s) = \bar{\beta}(\gamma_s) = 0$;*

3: *Let $M \neq (S^1 \times S^2)\#M'$ be closed and oriented. Let K be a knot that is contained in one the irreducible summands of M, then $\beta(\gamma_s) = 0$.*

2.3.8. *Proof of Lemma 2.3.7.* The proofs of statement **1** and of the identity $\delta(\gamma_s) = 0$ are obtained by straightforward geometric considerations.

Let s' be the embedded sphere used to construct γ_s. To get $\bar{\beta}(\gamma_s) = 0$ we observe that $\bar{\beta}(\gamma_s)$ is equal to the intersection index of $K \in H_1(M)$ and $s' \in H_2(M)$. (The moments when the knot becomes singular are those when the branch of K that slides around s' passes through a branch of K that intersect s'.) On the other hand, since $M \neq (S^1 \times S^2)\#M'$, every embedded sphere separates M into two disjoint parts, and thus the intersection index of K and s' is zero.

To get statement **3** we observe that since K lies in the irreducible summand of M, Proposition 2.3.6 implies that we can assume that the restriction of the spheroid s to an upper hemisphere is one of the embedded spheres s' from the realization of M as a connected sum of irreducible manifolds. Thus the only crossings of the discriminant that occur under γ_s are those when the branch that slides around the sphere s' passes through the small neighborhoods of the crossings of the path ρ that was used to construct γ_s and the sphere s'. Each small neighborhood of the crossing of ρ and s' gives rise to two crossings of the discriminant under γ_s. A straightforward verification show that the signs of the two crossings are opposite, and that if one appearing singular knot in such a pair has a contractible loop, then so does the other one. Thus $\beta(\gamma_s) = 0$. □

2.4. **Proof of Theorem 1.0.3.** Theorem 1.0.1 implies that statement **1** implies statement **2**. Below we show that statement **2** implies statement **1**.

It is well known that for zero homologous framed knots there is a well-defined self-linking number, that is invariant under isotopy of framed knots. Hence if the unframed knot is zero-homologous it is clear that there are infinitely many isotopy classes of framed knots that correspond to it when we forget the framing. *For this reason below in the proof we will assume that the knot K realizes a not contractible loop.* (If it is contractible, then it is zero-homologous.)

2.4.1. *The main idea of the proof.* Let K to be a nonsingular knot in M, let $\mathcal{C}$ be a connected component of the space of curves in M that contains K, let $\mathcal{F}'$ be a connected component of the space of framed curves in M that corresponds to $\mathcal{C}$ if we forget the framing, and let $\mathrm{pr}:\mathcal{F}\to\mathcal{C}$ be the normal covering introduced in 2.2.

Let $K_1\in\mathcal{F}$ be such that $\mathrm{pr}(K_1)=K$. For a loop $\alpha:[0,1]\to\mathcal{C}$ such that $\alpha(0)=\alpha(1)$ put $\bar\alpha:[0,1]\to\mathcal{F}$ to be the path that covers α such that $\bar\alpha(0)=K_1$. Then as it was proved in Proposition 2.2.5, $\delta:\pi_1(\mathcal{C},K)\to\mathbb{Z}$ that maps the class $[\alpha]\in\pi_1(\mathcal{C},K)$ of the loop α to $o(K_1,\bar\alpha(1))\in\mathbb{Z}$ is a homomorphism. Clearly to prove the Theorem it suffices to show that $\delta(\alpha)=0$, for any $\alpha\in\pi_1(\mathcal{C},K)$ that is realizable by an isotopy of the knot K. As it was explained in 2.3.2 if $\beta(\alpha)\neq 0$ or $\bar\beta(\alpha)\neq 0$, then α is not realizable by an isotopy.

Thus to prove the Theorem it suffices to show that for any $\alpha\in\pi_1(\mathcal{C},K)$ if $\delta(\alpha)\neq 0$, then one of $\beta(\alpha)$ and $\bar\beta(\alpha)$ is nonzero.

Let $K',K\in\mathcal{C}$ be knots. From 2.2.7 and 2.3.2 it is clear that the following two fact are equivalent:

1: for any $\alpha\in\pi_1(\mathcal{C},K)$ if $\delta(\alpha)$ is nonzero, then one of $\beta(\alpha)$ and $\bar\beta(\alpha)$ is nonzero;
2: for any $\alpha\in\pi_1(\mathcal{C},K')$ if $\delta(\alpha)$ is nonzero, then one of $\beta(\alpha)$ and $\bar\beta(\alpha)$ is nonzero.

Thus in our proof we can assume that K is any knot from $\mathcal{C}$.

Since $\alpha,\beta,\bar\beta$ are $\mathbb{Z}$-valued homomorphisms and $\mathbb{Z}$ is torsion free, to prove the theorem it suffices to show that for any $\alpha\in\pi_1(\mathcal{C},K)$ there exists a nonzero $i\in\mathbb{Z}$ such that if $\delta(\alpha^i)\neq 0$, then one of $\beta(\alpha^i)$ and $\bar\beta(\alpha^i)$ is nonzero;

It is well-known that if a closed connected oriented M can not be presented as a connected sum $(S^1\times S^2)\#M'$, then M can be presented as a connected sum $\#_{i=1}^k M_i$ of irreducible closed 3-manifolds M_i.

Clearly the following Lemma implies Theorem 1.0.3.

Lemma 2.4.2. *Let $M=\#M_i$ be a connected sum of oriented closed irreducible manifolds, and let $\mathcal{C}$ be a connected component of the space of curves in M that consists of noncontractible curves.*

1: *If none of the knots from $\mathcal{C}$ is contained in $(M_i\setminus B^3)\subset M$, for some $i\in\{1,\dots,k\}$, then for any $\alpha\in\pi_1(\mathcal{C},K)$ there exists a nonzero $i\in\mathbb{Z}$ such that $\delta(\alpha^i)=\bar\beta(\alpha^i)$.*
2: *If there exists a knot $K\in\mathcal{C}$ such that K is contained in $(M_i\setminus B^3)\subset M$, for some $i\in\{1,\dots,k\}$, then for any $\alpha\in\pi_1(\mathcal{C},K)$ there exists a nonzero $i\in\mathbb{Z}$ such that $\delta(\alpha^i)=\beta(\alpha^i)$.*

This finishes the Proof of Theorem 1.0.3 modulo Lemma 2.4.2.

2.4.3. *Proof of Lemma 2.4.2.* We start by proving statement **1** of Lemma 2.4.2. Let $t:\pi_1(\mathcal{C},K)\to Z(K)<\pi_1(M,K(1))$ be the homomorphism described in 2.3.1. Clearly $t(\gamma_1)=K\in\pi_1(M,K(1))$ for the loop γ_1 introduced in 2.3.5. Take $\alpha\in\pi_1(\mathcal{C},K)$. Proposition 2.3.3.1 implies that that the centralizer $Z(K)$ of $K\neq 1\in\pi_1(M,K(1))$ is an infinite cyclic group. Thus there exists a nonzero $i\in\mathbb{Z}$ such that $t(\alpha^i)=t(\gamma_1^k)$, for some $k\in\mathbb{Z}$. By Lemma 2.3.6 $\alpha^i=\gamma_1^k\prod\gamma_{s_i}^{\sigma_i}\gamma_2^m$, for some $m\in\mathbb{Z}$, $\sigma_i=\pm1$, and spheroids s_i of the type described in 2.3.5.

By Lemma 2.3.7 we have $\delta(\gamma_1^i)=\bar\beta(\gamma_1^i)=0$, $\delta(\prod\gamma_{s_i}^{\sigma_i})=\bar\beta(\prod\gamma_{s_i}^{\sigma_i})=0$, and $\delta(\gamma_2^m)=\bar\beta(\gamma_2^m)=2m$. This implies that for out choice of $i\neq 0$ $\delta(\alpha^i)=\bar\beta(\alpha^i)$ and this finishes the proof of statement **1**.

Below we prove statement **2**. Let M_1 be the irreducible summand of M that contains K. Proposition 2.3.3.2 implies that the centralizer of $K\in\pi_1(M,K(1))$ is canonically isomorphic to the centralizer of $K\in\pi_1(M_1,K(1))$. Thus the h-principle, see 2.3.1, implies that for any $\alpha\in\pi_1(\mathcal{C},K)$ there exists $\bar\alpha\in\pi_1(\mathcal{C},K(1))$

with $t(\alpha)=t(\bar\alpha)\in\pi_1(M,K(1))$ such that $\bar\alpha$ is realizable by a deformation of K (in the space curves) that does not take K outside of $M_1\setminus B^3$.

Lemma 2.3.6 says that $\alpha=\bar\alpha(\prod\gamma_{s_i}^{\sigma_i})\gamma_2^m$, for some $m\in\mathbb{Z},\sigma_i=\pm1$, and spheroids s of the type described in 2.3.5.

By Lemma 2.3.7 we get that $\delta(\alpha^i)=2mi+\delta(\bar\alpha^i)$ and $\beta(\alpha^i)=2mi+\beta(\bar\alpha^i)$, for any nonzero $i\in\mathbb{Z}$. Now statement **2** follows immediately from the following Theorem 2.4.4. (This finishes the Proof of Lemma 2.4.2 and of Theorem 1.0.3 modulo the Proof of Theorem 2.4.4.) □

Theorem 2.4.4. *Let M be a closed oriented irreducible 3-manifold, let $\mathcal{C}$ be a connected component of the space of curves in M that consists of not contractible curves, and let $K\in\mathcal{C}$ be a knot. Then for any $\alpha\in\pi_1(\mathcal{C},K)$ there exists a nonzero $i\in\mathbb{Z}$ such that $\delta(\alpha^i)=\beta(\alpha^i)$.*

2.5. **Proof of Theorem 2.4.4.** By the Sphere Theorem [10] $\pi_2(M)=0$. Let $t:\pi_1(\mathcal{C},K)\to Z(K)<\pi_1(M,K(1))$ be the homomorphism introduced in 2.3.1.

If $\pi_1(M)$ is finite, then for any $\alpha\in\pi_1(\mathcal{C},K)$ there exists $i\neq0$ such that $t(\alpha^i)=1=t(1)$. Lemma 2.3.6.2 says that there exists $m\in\mathbb{Z}$ such that $\alpha^i=\gamma_2^m$. Now Lemma 2.3.7 implies that $\delta(\alpha^i)=2m=\beta(\alpha^i)$. This finishes the proof in the case of $\pi_1(M)$ been finite, and *below in the proof we assume that $\pi_1(M)$ is infinite.*

Since $\pi_1(M)$ is infinite and M is orientable and irreducible, the result of Epstein, see [3] or [10] Corollary 9.9, implies *that $\pi_1(M)$ is torsion free.*

Let $p:STM=S^2\times M\to M$ be the S^2-fibration. To a loop α in $\mathcal{C}$ such that $\alpha(1)=K$ we correspond a mapping $\mu_\alpha:S^1\times S^1\to STM$ as it is described in 2.3.1. Put $\bar\mu_\alpha=p(\mu):S^1\times S^1\to M$.

Definition 2.5.1. Let M' be a compact oriented 3 manifold and let $F\neq S^2$ be a surface. A map $\Psi:F\to M'$ is *essential* if $\ker(\Psi_*:\pi_1(F)\to\pi_1(M'))=1$.

Lemma 2.5.2. *Let M' be a submanifold of an oriented closed irreducible M with infinite $\pi_1(M)$ such that $\pi_2(M')=0$ and the inclusion $i:M'\to M$ induces the injective homomorphism $i_*:\pi_1(M')\to\pi_1(M)$. Let α be a loop in the space of curves in M such that* $\operatorname{Im}\bar\mu_\alpha\subset M'$ *and $\bar\mu_\alpha:S^1\times S^1\to M'$ is not essential. Then* $\beta(\alpha)=\delta(\alpha)$ *(for the homomorphisms β and δ defined with respect to the ambient manifold M).*

2.5.3. *Proof of Lemma 2.5.2.* Since $\ker\bar\mu_{\alpha*}\neq1$ and $\pi_1(S^1\times S^1)=\mathbb{Z}\oplus\mathbb{Z}$ is generated by $1\times S^1$ and $S^1\times1$ we have that $K^j=(\bar\mu_{\alpha*}(1\times S^1))^j=(\bar\mu_{\alpha*}(S^1\times 1))^i=t(\alpha)^i\in\pi_1(M',K(1))$, for some $i,j\in\mathbb{Z}$. Since $\pi_1(M)$ is torsion free and $i_*:\pi_1(M')\to\pi_1(M)$ is injective, we have that $\pi_1(M')$ is torsion free. Thus both i and j are nonzero.

Clearly $t(\gamma_1)=K\in\pi_1(M,K(1))$ for the loop γ_1 introduced in 2.3.5. Since $\pi_2(M')=0$, Lemma 2.3.6.2 implies that there exists $m\in\mathbb{Z}$ such that $\alpha^i=\gamma_1^j\gamma_2^m$. Lemma 2.3.7 says that $\beta(\gamma_1)=\delta(\gamma_1)=0$ and $\beta(\gamma_2)=\delta(\gamma_2)=2$. Thus $\beta(\alpha^i)=\delta(\alpha^i)$ and since $i\neq0$ we get the statement of the Lemma. □

Lemma 2.5.2 proves the Theorem for nonessential $\bar\mu_\alpha$. *Below in the Proof we assume that $\bar\mu_\alpha$ is an essential mapping.*

Definition 2.5.4. A surface $F\neq S^2$, properly embedded in a 3-manifold M' (or embedded into $\partial M'$), is *compressible,* if there exists a disc $D\subset M$ such that

$D \cap F = \partial D$ and ∂D is not homotopically trivial in F; otherwise F is *incompressible* in M. A compact orientable irreducible 3-manifold is *Haken* if it contains a two-sided incompressible surface.

Definition 2.5.5. Let (μ, ν) be a pair of relatively prime integers. Let $D^2 = \{(r, \theta); 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ be the unit disc with polar coordinates. A *fibered torus of type* (μ, ν) is the quotient of $D^2 \times [0,1]$ via $((r, \theta), 1) = ((r, \theta + \frac{2\pi\nu}{\mu}), 0)$. The fibers are the closed curves that are the unions $((r, \theta) \times [0,1]) \cup ((r, \theta + \frac{2\pi\nu}{\mu}) \times [0,1]) \cup ((r, \theta + \frac{4\pi\nu}{\mu}) \times [0,1]) \cup \ldots$ etc. for fixed (r, θ). If $|\mu| > 1$, then the solid torus is said to be *exceptionally fibered* and the core of the torus is the *exceptional fiber.* Otherwise the torus is *regularly fibered* and each fiber is a *regular fiber.*

An orientable 3-manifold M' is called *Seifert fibered* if it is a union of pairwise disjoint closed curves (fibers), such that each fiber has a neighborhood consisting of fibers that is homeomorphic to a fibered solid torus via a fiber preserving homeomorphism. A fiber of a Seifert manifold M' is exceptional if its neighborhood is homeomorphic to an exceptionally fibered solid torus via a fiber preserving homeomorphism.

The quotient space obtained from M' via identifying all points in the fiber for all the fibers is the *orbit space* and the images of the exceptional fibers are the *cone points.*

Definition 2.5.6 (of a characteristic submanifold)**.** A codimension zero submanifold S of a closed oriented manifold M is called a *characteristic submanifold* if

- **1:** each component X of S admits a structure of a Seifert fibered space or of a total space of a $[0,1]$-bundle;
- **2:** if W is a nonempty codimension zero submanifold of M that consists of components of $\overline{M \setminus S}$, then $S \cup W$ does not satisfy 1;
- **3:** if S' is a codimension zero submanifold of M that satisfies 1 and 2, then S' can be deformed into S by a proper isotopy.

2.5.7. *Let us reduce the proof of the Theorem in the case of essential $\bar{\mu}_\alpha$, to the case where the homotopy α happens inside a Seifert-fibered submanifold S of M with $\pi_2(S) = 0$.*

The Torus Theorem by Casson-Jungreis [2] and Gabai [5] says that since $\bar{\mu}_\alpha$ is essential, then either M contains an embedded incompressible torus, and thus is Haken, or it is a Seifert fibered space. If it is Seifert-fibered then we have made the reduction, so consider the case where M is Haken.

The results by Jaco and Shalen [12] and Johannson [13], Proposition 9.4, say that M has a well-defined characteristic submanifold S. The Enclosing Theorem Jaco and Shalen [12] and Johannson [13], Theorem 12.5, says that since M is Haken there exists a mapping $\lambda : S^1 \times S^1 \to S \subset M$ such that λ is homotopic to $\bar{\mu}_\alpha$. There only two oriented total spaces of $[0,1]$-bundles over surfaces that admit essential mappings of $S^1 \times S^1$. They are the $[0,1] \times S^1 \times S^1$ and the unique $[0,1]$-bundle over the Klein-bottle with oriented total space. Both these two spaces admit the structure of a Seifert-fibered space. (In the case of a $[0,1]$-bundle over the Klein-bottle it is a Seifert fibered space over a Moebius strip.) Thus we get that λ is a map into a Seifert-fibered component of S.

Let $\Gamma : (S^1 \times S^1) \times [0,1] \to M$ be the homotopy such that $\Gamma\big|_{(S^1 \times S^1) \times 0} = \lambda$ and $\Gamma\big|_{(S^1 \times S^1) \times 1} = \bar{\mu}_\alpha$.

By the h-principle we can assume that $\lambda\big|_{1\times S^1}$ is a knot $K'\in\mathcal{C}$. Put $\mathcal{C}_S$ to be the connected component of the space of curves in S that contains K'. Since K' commutes with $\lambda\big|_{S^1\times 1}$ in $\pi_1(S,K'(1))$ we get that there exists a loop α' in $\mathcal{C}_S$ that starts at K' and has $t(\alpha')=\lambda\big|_{S^1\times 1}\in\pi_1(S,K'(1))$. Let $q:[0,1]\to\mathcal{C}$ be a generic path such that $q(0)=K'$, $q(1)=K$, and for all $t\in[0,1]$ the curve $q(t)$ maps $1\in S^1$ to $\Gamma(1\times 1\times t)$.

Consider the loop $q^{-1}\alpha q$ in $\mathcal{C}$ that starts at K'. Since the contributions into $\beta(q^{-1}\alpha q)$ and into $\delta(q^{-1}\alpha q)$ that correspond to q and q^{-1} cancel out, we observe that to prove the Theorem is suffices to show that $\beta(q^{-1}\alpha q)=\delta(q^{-1}\alpha q)$. By construction of q we have $t(q^{-1}\alpha q)=t(\alpha')\in\pi_1(\mathcal{C},K')$. Thus by Lemma 2.3.6.2 $\alpha'=q^{-1}\alpha q\gamma_2^m$, for some $m\in\mathbb{Z}$. Lemma 2.3.7 says that $\beta(\gamma_2)=\delta(\gamma_2)=2$. Thus the fact that $\delta(\alpha)=\beta(\alpha)$ is equivalent to the fact that $\beta(\alpha')=\delta(\alpha')$. Since α' is a loop in the space of curves in S it suffices to prove the Theorem under the assumptions that the deformation α is inside a Seifert-fibered 3-dimensional submanifold $S\subset M$ (that can coincide with M).

Finally to complete the reduction we observe that clearly the value of $\delta(\alpha')$ does not depend on whether we regard δ as a homomorphism $\delta:\pi_1(\mathcal{C},K')\to\mathbb{Z}$ or as $\delta:\pi_1(\mathcal{C}_S,K')\to\mathbb{Z}$. The inclusion of every component of S into M induces a monomorphism of fundamental groups, (see [13] remark on p.27 and 8.2). Thus the value of $\beta(\alpha')$ also does not depend on whether we regard β as a homomorphism $\beta:\pi_1(\mathcal{C},K')\to\mathbb{Z}$ or as $\beta:\pi_1(\mathcal{C}_S,K')\to\mathbb{Z}$. Also since the homotopy α' happens inside a Seifert fibered component of the characteristic submanifold S, we can assume that $\pi_2(S)=0$.

Thus we have reduced the proof of the Theorem to the case where M is an oriented connected compact (not necessarily closed) Seifert-fibered manifold, with $\pi_2(M)=0$, $\pi_1(M)$ torsion free, μ_α is an essential torus in M (see Lemma 2.5.2), and K is a noncontractible knot in M.

Definition 2.5.8. Let $q:S\to F$ be a Seifert-fibration, a mapping $\lambda:S^1\times S^1\to S$ is said to be *vertical* with respect to q if $q^{-1}(q\lambda(S^1\times S^1))=\lambda(S^1\times S^1)$ and $\lambda(S^1\times S^1))$ does not contain exceptional fibers of $q:S\to M$.

The following three Lemmas prove the Theorem in the case where $\bar\mu_\alpha$ is homotopic to a vertical mapping of $S^1\times S^1$.

Lemma 2.5.9. *Let $p:S\to Y$ be a locally trivial S^1-fibration of an oriented manifold S over a (not necessarily orientable) manifold Y. Let $f\in\pi_1(S)$ be the class of an oriented S^1-fiber of p, and let α be an element of $\pi_1(S)$. Then:*

- **a:** *$\alpha f=f\alpha\in\pi_1(S)$, provided that $p(\alpha)$ is an orientation preserving loop in Y.*
- **b:** *$\alpha f=f^{-1}\alpha\in\pi_1(S)$, provided that $p(\alpha)$ is an orientation reversing loop in Y.*

2.5.10. *Proof of Lemma 2.5.9.* If we move an oriented fiber along the loop $\alpha\in S$, then in the end it comes to itself either with the same or with the opposite orientation. It is easy to see that it comes to itself with the opposite orientation if and only if $p(\alpha)$ is an orientation reversing loop in Y. □

Lemma 2.5.11. *Let $\tilde p:N\to G$ be a Seifert fibration, let $F\subset G$ be a connected submanifold with boundary that does not contain cone points. Let $M=\tilde p^{-1}(F)\subset$*

N, and let $p : M \to F$ be the corresponding locally trivial S^1-fibration. Let α be a homotopy of a noncontractible knot K such that $\bar{\mu}_\alpha \subset M$, then $\beta(\alpha) = \delta(\alpha)$.

2.5.12. *Proof of Lemma 2.5.11.* Put f to be the class of the oriented fiber of $p : M \to F$. Let $t : \pi_1(\mathcal{C}, K) \to Z(K) < \pi_1(M, K(1))$ be the surjective homomorphism onto the centralizer $Z(K)$ of $K \in \pi_1(M, K(1))$, introduced in 2.3.1.

Consider the case of $p(K) \neq 1 \in \pi_1(F)$**.** Clearly $p_* t(\alpha) \in Z(p_*(K)) < \pi_1(F)$. Since $\partial F \neq \emptyset$, we have that $\pi_1(F)$ is a free group. Since $p(K) \neq 1 \in \pi_1(F)$ we get that there exists $i \neq 0$ and j such that $p_* t(\alpha^i) = p_*(K^j) \in \pi_1(F)$. Using Lemma 2.5.9 and the fact that f generates $\ker p_*$, we obtain that there exists k such that

$$t(\alpha^i) = K^j f^k \in \pi_1(M, K(1)). \tag{1}$$

Consider the case of $p(K)$ been an orientation reversing loop on F. Since $t(\alpha^i)$ commutes with K in $\pi_1(M)$ and by Lemma 2.5.9 $fK = Kf^{-1}$, we get that $f^{2k} = 1$. Since $\pi_2(M) = 0$, we have that f has infinite order. Thus $k = 0$ and $t(\alpha^i) = K^j$

Let γ_1 be the isotopy of K to itself introduced in 2.3.5. Clearly $t(\gamma_1) = K$. Thus by Lemma 2.3.6 we get that $\alpha^i = \gamma_1^j \gamma_2^s$, for some $s \in \mathbb{Z}$. Using Lemma 2.3.7 we get that $\beta(\alpha^i) = \delta(\alpha^i)$. Since i was chosen to be nonzero we get that $\beta(\alpha) = \delta(\alpha)$.

Consider the case of $p(K)$ been an orientation preserving loop on F. Since M is orientable, we get that the S^1-fibration over S^1 (parameterizing the knots) induced from p by $p \circ K : S^1 \to F$ is trivializable. Hence we can coherently orient the fibers of the induced fibration $\bar{p} : S^1 \times S^1 \to S^1$. The orientation of the S^1-fiber over $t \in S^1$ induces the orientation of the S^1-fiber of p that contains $K(t)$. Let γ_3 be the homotopy of K that slides every point $K(t)$ of K inside the fiber that contains $K(t)$ with unit velocity in the direction specified by the orientation of the fiber of $\bar{p}$ over $t \in S^1$. Clearly $t(\gamma_3) = f$. Thus $\alpha^i = \gamma_1^j \gamma_3^k \gamma_2^s$ for some $s \in \mathbb{Z}$.

Let us show that $\beta(\gamma_3) = 0$. The only singular knots that arise under γ_3 are those that have a double point projecting to a double point d of $p(K)$. Every double point d of $p(K)$ separates $p(K)$ into two loops (that may intersect each other). Since $p(K)$ is orientation preserving either both of these loops are orientation reversing or both are orientation reversing.

If the two loops of $p(K)$ separated by d are orientation preserving then the two points of K over d induce the same orientation of the fiber. Thus under γ_3 the two branches of K over d slide in the same direction and such double points d do not correspond to any input into β.

If d separates K into two orientation reversing loops, then the two points of K over d induce the opposite orientations of the fiber over d. Thus the two branches of K over d slide in the opposite directions under the deformation γ_3, and such d correspond to singular knots arising under γ_3. However both loops of the arising singular knots project to orientation reversing loops on F and hence to orientation reversing loops on G. Thus they are not contractible in $\tilde{M}$, and we get that $\beta(\gamma_3) = 0$.

If one considers a framing of K that is nowhere tangent to the fibers of p, then it becomes clear that $\delta(\gamma_3) = 0$. Since $\delta(\gamma_1) = \beta(\gamma_1) = 0$ and $\delta(\gamma_3) = \beta(\gamma_3) = 2$, we get that $\beta(\alpha^i) = \delta(\alpha^i)$. Since i was taken to be nonzero we get that $\beta(\alpha) = \delta(\alpha)$.

Consider the case of $p(K) = 1 \in \pi_1(F)$**.** Since $K \neq 1 \in \tilde{M}$ we have that $K = f^k$, for some $k \neq 0$. Let $K' \subset F$ be a knot such that $p\big|_{K'}$ is injective, $p(K')$ is a small circle, and $K' = f^k \in \pi_1(M)$. (This K' is the $(1, i)$ toric knot on the torus that is $p^{-1}p(K')$.) Clearly K' and K are homotopic in the space of curves in M. Let ρ be a path in the space of curves in M that describes the homotopy of K' to K. Consider the loop $\rho^{-1}\alpha\rho$. Clearly the inputs of ρ^{-1} and of ρ into $\beta(\rho^{-1}\alpha\rho)$ cancel out. Similarly their inputs into $\delta(\rho^{-1}\alpha\rho)$ also cancel out. Since $\bar{\mu}_{\rho^{-1}\alpha\rho} \subset M$, we get that when proving the Lemma we can assume that K is K'.

Clearly $p_* t(\alpha^2)$ is an orientation preserving loop on F. Let γ be the isotopy of $K = K'$ such that at every moment of time x the knot $p(\gamma(x))$ is a small circle, $\gamma(x)$ is the $(1, i)$ toric knot on the torus that is $p^{-1}p(\gamma(x))$, and $t(\gamma) = \alpha^2$. (It is easy to verify that such an isotopy really does exist.) Since γ is an isotopy we have $\beta(\gamma) = 0$.

Thus $\alpha^2 = \gamma\gamma_2^s$ for some $s \in \mathbb{Z}$. If one considers the framing of K such that the projections of the framing vectors to F are orthogonal to the circle $p(K)$, it is easy to verify that $\delta(\gamma) = 0$. Since $\beta(\gamma_2) = \delta(\gamma_2) = 2$ and $\beta(\gamma) = 0$, we have $\beta(\alpha^2) = \delta(\alpha^2)$. Thus $\beta(\alpha) = \delta(\alpha)$.

This finishes the proof of the Lemma for all the cases. □

Lemma 2.5.13. *Let $q : M \to F$ be a compact oriented (not necessarily closed) Seifert-fibered manifold with $\pi_2(M) = 0$. Let $\bar{\mu}_\alpha : S^1 \times S^1 \to M$ be homotopic to a vertical torus $\lambda : S^1 \times S^1 \to M$. Then $\beta(\alpha) = \delta(\alpha)$.*

2.5.14. *Proof of Lemma 2.5.13.* Let $M' \subset M$ be a thin neighborhood of $\mathrm{Im}(\lambda)$ such that it is locally-trivially S^1-fibered over a thin neighborhood of $q(\lambda) \subset F$. Let $q' : M' \to F'$ be the corresponding locally-trivial S^1-fibration. (Since M' is thin we can assume that F' has nonempty boundary.)

Let $\mathcal{C}$ be a connected component of the space of curves in M that contains K. By the h-principle there exists a knot $K' \in \mathcal{C}$ such that $K' \subset M'$ and there exists a closed loop α' in the space of curves in M' such that $\bar{\mu}_{\alpha'}$ is homotopic to λ inside M'. (Since M' is locally trivially S^1-fibered over $F' \neq S^2, \mathbb{R}P^2$, we have $\pi_2(M') = 0$.) Let $\mathcal{C}_{M'} \subset \mathcal{C}$ be the connected component of the space of curves in M' that contains K'.

Let $\Gamma : S^1 \times S^1 \times [0, 1] \to M$ be a homotopy such that $\Gamma(S^1 \times S^1 \times 0) = \bar{\mu}_{\alpha'}$ and $\Gamma(S^1 \times S^1 \times 1) = \bar{\mu}_\alpha$. Let $\rho : [0, 1] \to \mathcal{C}$ be a path such that $\rho(0) = K'$, $\rho(1) = K$, and $\rho(t)(1) = \Gamma(1 \times 1 \times t)$.

Consider the loop $\rho^{-1}\alpha\rho$ in $\mathcal{C}$. Clearly $t(\rho^{-1}\alpha\rho) = t(\alpha') \in \pi_1(M)$. Thus the obstruction for $\rho^{-1}\alpha\rho$ and α' to be equal in $\pi_1(\mathcal{C}, K')$ is an element of $\mathbb{Z} \oplus \pi_2(M)$. Since $\pi_2(M) = 0$ the standard arguments imply that $\alpha' = \rho^{-1}\alpha\rho\gamma_2^m \in \pi_1(\mathcal{C}, K')$, for some $m \in \mathbb{Z}$. Since the inputs of ρ and of ρ^{-1} into both $\beta(\alpha')$ and $\delta(\alpha')$ cancel out, and since $\delta(\gamma_2) = \beta(\gamma_2) = 2$, we get that to prove the Lemma it suffices to show that $\beta(\alpha') = \delta(\alpha')$. Now the statement of the Lemma immediately follows from Lemma 2.5.11. □

Thus we have reduced the Proof of Theorem 2.4.4 to the case where $\bar{\mu}_\alpha$ is an essential torus inside of an oriented compact (not necessarily closed) Seifert fibered space M with $\pi_2(M) = 0$. (In particular $\pi_1(M)$ has to be infinite.) And the arguments above give the proof of the Theorem in the case where $\bar{\mu}_\alpha$ is homotopic to a vertical torus.

2.5.15. As it is shown in [13], see Propositions 5.13 and 7.1, for most Seifert-fibered manifolds all essential tori are homotopic to vertical ones for some choice of Seifert-fibration structure.

The only compact Seifert fibered manifolds where this statement is not proved in [13] are:

1: Seifert fibered spaces with $\mathbb{R}P^2$ as an orbit space and at most one exceptional fiber;

2: Seifert fibered spaces with l exceptional fibers and the orbit space been S^2 with m holes, $l+m \leq 3$.

If M is Seifert fibered over $\mathbb{R}P^2$ with at most one exceptional fiber, then the orientation cover $S^2 \to \mathbb{R}P^2$ induces a two fold cover $\tilde{M} \to M$. The manifold $\tilde{M}$ admits a structure of a Seifert fibration over S^2 with at most two exceptional fibers. As it is shown by Orlik [17] such $\tilde{M}$ is a lens space. Since $\pi_2(M) = 0$, we get that $\pi_2(\tilde{M}) = 0$. Thus $\pi_1(\tilde{M})$ and $\pi_1(M)$, are finite and we get the proof for this case.

Below we consider the case where M is a Seifert fibration over S^2 and $l+m \leq 3$.

Consider the case of $m \neq 0$. If $l = 0$, then the proof follows from Lemma 2.5.11. If $l = 1$ and $m = 1$, then $\pi_1(M) = \mathbb{Z}$ and M does not contain essential tori.

If $l = 1$ and $m = 2$, then $\pi_1(M) = \{f, g \big| f^i g = g f^i\}$ where g is the element that projects to the core of the annulus that is the base space of M, f is the class of the exceptional fiber, and i is its multiplicity.

If $l = 2$ and $m = 1$, then $\pi_1(M) = \{f_1, f_2 \big| f_1^{i_1} = f_2^{i_2}\}$, where $f_1, f_2 \in \pi_1(M)$ are the classes of the exceptional fibers and i_1, i_2 are their multiplicities.

In both cases $l = 1, m = 2$ and $l = 2, m = 1$ the quotient group of $\pi_1(M)$ by the normal subgroup generated by the class of a regular fiber is isomorphic to the free product of cyclic groups.

An exercise in group theory shows that for any groups G_1, G_2 and $g \in G_1 \star G_2$:

a: the centralizer of g is isomorphic to the infinite cyclic group, provided that g is not conjugate to an element of $G_1 \star \{1\} < G_1 \star G_2$ or to an element of $\{1\} \star G_2 < G_1 \star G_2$;

b: if $g \neq 1$ and there exists $\tilde{g}$ such that $\tilde{g}^{-1} g \tilde{g} = g' \star \{1\} \in G_1 \star G_2$, then the centralizer of g is $\tilde{g}^{-1}(Z(g') \star \{1\})\tilde{g}$, where $Z(g')$ is the centralizer of $g' \in G_1$.

Using this and the fact that the class of the regular fiber is in the center of $\pi_1(M)$ we get that every essential torus in M is homotopic to a vertical one. This finishes the proof in the case of $m \neq 0$.

Below we assume that $m = 0$. If $m = l = 0$, then since M is irreducible we have that $\pi_1(M) = \mathbb{Z}_i$, where $i \in \mathbb{Z} = H^2(S^2)$ is the Euler class of the locally-trivial S^1-bundle $M \to S^2$. (If $i = 0$, then $M = S^1 \times S^2$.) Since in these cases $\pi_1(M)$ is finite, we get the proof.

If $m = 0$ and $l = 1$, or $m = 0$ and $l = 2$ then M is a lens space, see for example Orlik [17] p. 99. Since M is irreducible it has a finite fundamental group and we have finished the proof in these cases.

Consider the case of $m = 0$ *and* $l = 3$. Let $r, s, t \in \mathbb{N}$ be the multiplicities of the fibers. The quotient group of $\pi_1(M)$ by the subgroup generated by the regular fiber of q is the triangle group $\Delta(r, s, t)$, where r, s, t are the multiplicities of the exceptional fibers.

The result that can be found in [17] pp.100–101 says that if $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} > 1$, then $\pi_1(M)$ is finite and thus M does not contain essential tori.

The results of Hass, see [8] Theorem 1 and Lemma 2, (some of these results were independently obtained by Gao [4]) imply that every essential mapping of a torus to a closed irreducible Seifert fibered manifold over an orientable surface is homotopic to either a vertical immersed one or to a horizontal immersed one. (A mapping is *horizontal* if it is everywhere transverse to the fibers of the fibration.) It is easy to see from the work of Scott [18] that in the hyperbolic triangle group case, $\frac{1}{r}+\frac{1}{s}+\frac{1}{t}<1$, if the mapping of a torus is horizontal then the torus has a negative curvature metric. Since this is impossible, we get that every essential mapping of a torus in the hyperbolic triangle group case is homotopic to a vertical one.

The only Euclidean triangle groups are $\Delta(2,3,6)$, $\Delta(2,4,4)$, and $\Delta(3,3,3)$. Since we have already proved the Theorem for all the cases where essential tori are homotopic to vertical tori, we get from the work of Hass [8] that the only manifolds corresponding to Euclidean triangle groups we have to consider are those that contain horizontal immersed tori. Thus the Euler number of the Seifert fibering should be zero, see for example [9].

Following the work of Kirk and Livingston [15] we observe the only Seifert fibered manifolds that correspond to the Euclidean triangle groups and have zero Euler number are:

1: $M_{(2,3,6)}$ with Seifert invariants of the fibers $\{(2,1),(3,-1),(6,-1)\}$;
2: $M_{(2,4,4)}$ with Seifert invariants of the fibers $\{(2,1),(4,-1),(4,-1)\}$; and
3: $M_{(3,3,3)}$ with Seifert invariants of the fibers $\{(3,1),(3,1),(3,-2)\}$.

These spaces have a structure of torus bundles over a circle with a finite order monodromy (see [15], or [9] for a more detailed explanation). The monodromy maps for $M_{(2,3,6)}$, $M_{(2,4,4)}$, and $M_{(3,3,3)}$ are given respectively by the linear maps $\mathbb{R}^2/\mathbb{Z}^2\to\mathbb{R}^2/\mathbb{Z}^2$ determined by the matrices

$$A=\begin{pmatrix}0&-1\\1&1\end{pmatrix},\quad B=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad C=\begin{pmatrix}0&-1\\1&-1\end{pmatrix}.$$

Thus their finite covering is $S^1\times S^1\times S^1$.

To prove the Theorem for these three manifolds we will need the following Lemma.

Lemma 2.5.16. *Let N be an oriented 3-dimensional manifold, and let $\tilde{q}:\tilde{N}\to N$ be a finite covering with connected $\tilde{N}$ such that $\operatorname{Im}\tilde{q}_*(\pi_1(\tilde{N}))$ is a normal subgroup of $\pi_1(N)$. Let $\mathcal{C}$ be a connected component of the space of curves in N that consists of curves that can be lifted to $\tilde{N}$ (i.e. the elements of $\pi_1(N)$ realized by curves from $\mathcal{C}$ are in $\operatorname{Im}(\tilde{q}_*\pi_1(\tilde{N}))$. Let $\tilde{\mathcal{C}}$ be a connected component of the space of curves in $\tilde{N}$ that consists of lifted curves from $\mathcal{C}$. Let $K\in\mathcal{C}$ be a knot and $\tilde{K}\in\tilde{\mathcal{C}}$ be its lifting. Then $\beta(\alpha)=\delta(\alpha)$ for every $\alpha\in\pi_1(\mathcal{C},K)$, provided that $\beta(\tilde{\alpha})=\delta(\tilde{\alpha})$ for every $\tilde{\alpha}\in\pi_1(\tilde{\mathcal{C}},\tilde{K})$.*

2.5.17. *Proof of Lemma 2.5.16.* The lifting of a loop $\alpha\subset\mathcal{C}$ to $\tilde{\mathcal{C}}$ connects $\tilde{K}$ to another lifting of K. Since the covering $\tilde{q}:\tilde{N}\to N$ is finite there exists nonzero i such that α^i lifts to a homotopy $\tilde{\alpha^i}$ of $\tilde{K}$ to itself. Clearly $\delta(\tilde{\alpha^i})=i\delta(\alpha)$.

On the other hand each singular knot (with one double point) that occurs during α and has one of the two loops contractible corresponds to exactly i singular knots that occur during $\tilde{\alpha^i}$ and have one of the two loops contractible. Also all such singular knots arising under $\tilde{\alpha^i}$ correspond to one of such singular knots arising under α. Thus $\beta(\tilde{\alpha^i})=i\beta(\alpha)$.

Since $\beta(\tilde{\alpha^i})=\delta(\tilde{\alpha^i})$ by the assumptions of the Lemma, we get that $i\delta(\alpha)=i\beta(\alpha)$. Thus $\delta(\alpha)=\beta(\alpha)$. □

For all the three manifolds $M \in \{M_{(2,3,6)}, M_{(2,4,4)}, M_{(3,3,3)}\}$ the fundamental group of M is a semi-direct product $\pi_1(T^2) \propto \pi_1(S^1) = (\mathbb{Z}\oplus\mathbb{Z}) \propto \mathbb{Z}$. Let $m, l \in \pi_1(S^1\times S^1)$ be the classes of respectively the meridian and the longitude, and $f \in \pi_1(S^1)$ be the generator. For $a, a' \in \mathbb{Z}\oplus\mathbb{Z}$ and $f^i, f^j \in \mathbb{Z}$ the product $(a, f^i)(a', f^j) \in (\mathbb{Z}\oplus\mathbb{Z}) \propto \mathbb{Z}$ is given by $(a\xi(f^i)(a'), f^{i+j})$ where $\xi(f^i)(a') = f^i a' f^{-i} \in \pi_1(M)$.

The action $\xi(f^k)(m^i l^j)$ is calculated as follows. Let $D \in M_2(\mathbb{Z})$ be the monodromy matrix of the torus bundle $M \to S^1$. Put

$$\begin{pmatrix} i' \\ j' \end{pmatrix} = D^k \begin{pmatrix} i \\ j \end{pmatrix}.$$

Then

$$\xi(f^k)(m^i l^j) = m^{i'} l^{j'}. \tag{2}$$

We prove the Theorem for $M = M_{(3,3,3)}$. The proof of the Theorem for $M \in \{M_{(2,3,6)}, M_{(2,4,4)}\}$ is completely analogous to the case of $M = M_{(3,3,3)}$ but the calculations are a bit harder.

The matrix C that describes the monodromy of $M_{(3,3,3)} \to S^1$ has order 3. Thus $\xi(f^{3k})(a) = a$ for any $k \in \mathbb{Z}$ and $a \in \pi_1(T^2)$.

Take a knot $K \in \mathcal{C}$ and $\alpha \in \pi_1(\mathcal{C}, K)$. Then $t(\alpha^3) = (a', f^{3k'})$ and $K = (a, f^{3k+l})$, for some $a, a' \in \pi_1(T^2)$, $k, k' \in \mathbb{Z}$, and $l \in \{0,1,2\}$. Since $t(\alpha^3)$ commutes with K in $\pi_1(M)$ we have that

$$(a\xi(f^{3k+l})(a'), f^{3k+3k'+l}) = (a'\xi(f^{3k'})(a), f^{3k+3k'+l}). \tag{3}$$

Since $\pi_1(T^2)$ is commutative and $\xi(f^{3k'})(a) = a$, we get that

$$\xi(f^{3k+l})(a') = a'. \tag{4}$$

Consider the case of $l \in \{1,2\}$. Then one uses (2) to verify that $a' = 1$ (provided that $l \in \{1,2\}$). Thus $t(\alpha^3) = (1, f^{3k'}) \in (\mathbb{Z}\oplus\mathbb{Z}) \propto \mathbb{Z}$. A straightforward calculation shows that $K^3 = (a, f^{3k+l})^3 = (1, f^{3(3k+l)}) \in \pi_1(M_{(3,3,3)}, K(1))$, thus $K^{3k'} = (1, f^{3k'(3k+l)}) = (1, f^{3k'})^{3k+l} = t(\alpha^{3(3k+l)}) \in \pi_1(M_{(3,3,3)}, K(1))$.

Let $\gamma_1 \in \pi_1(\mathcal{C}, K)$ be the loop introduced in 2.3.5. Then $t(\gamma_1) = K$ and $t(\gamma_1^{3k'}) = t(\alpha^{3(3k+l)})$. Thus $\alpha^{3(3k+l)} = \gamma_1^{3k'}\gamma_2^s$, for some $s \in \mathbb{Z}$. Since $\beta(\gamma_1) = \delta(\gamma_1) = 0$ and $\beta(\gamma_2) = \delta(\gamma_2) = 2$, we get that $\beta(\alpha^{3(3k+l)}) = \delta(\alpha^{3(3k+l)})$. Since $3(3k+l) \neq 0$, we have $\beta(\alpha) = \delta(\alpha)$ and this finishes the proof for $M = M_{(3,3,3)}$ and $K = (a, f^{3k+l}) \in \pi_1(M)$ with $l \in \{1,2\}$.

Consider the case where $l = 0$. Then K is liftable to the total space of the three fold covering $S^1\times S^1\times S^1 \to M_{(3,3,3)}$. We have already proved that $\beta(\tilde\alpha) = \delta(\tilde\alpha)$ for any closed path $\tilde\alpha$ in the space of curves in $S^1\times S^1\times S^1$. Now the statement of the Theorem for $M = M_{(3,3,3)}$ and $K = (a, f^{3k+0}) \in \pi_1(M, K(1))$ immediately follows from Lemma 2.5.16.

This finishes the proof of Theorem 2.4.4 for $M = M_{(3,3,3)}$, and thus for all the remaining cases. This is also the end of the Proof of Theorem 1.0.3. □

Acknowledgments. I am very grateful to I. Fedorova, Yu. Rudyak, and A. Vylejneva for many enlightening discussions. I am thankful to A. Cattaneo, P. Kirk, C. Livingston, and P. Scott for the useful mathematical discussions and suggestions.

The ideas of some parts of the proof are inspired by the works of V. Arnold [1], E. Kalfagianni [14], and P. Kirk and C. Livingston [15] and [16]

This paper was written during my stay at the Institute for Mathematics, Zürich University, and I would like to thank the staff of the Institute for providing the excellent working conditions.

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