Solvability of Cubic Graphs and The Four Color Theorem

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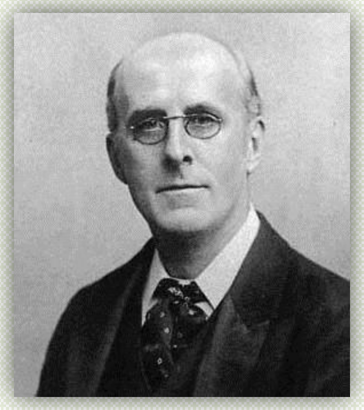
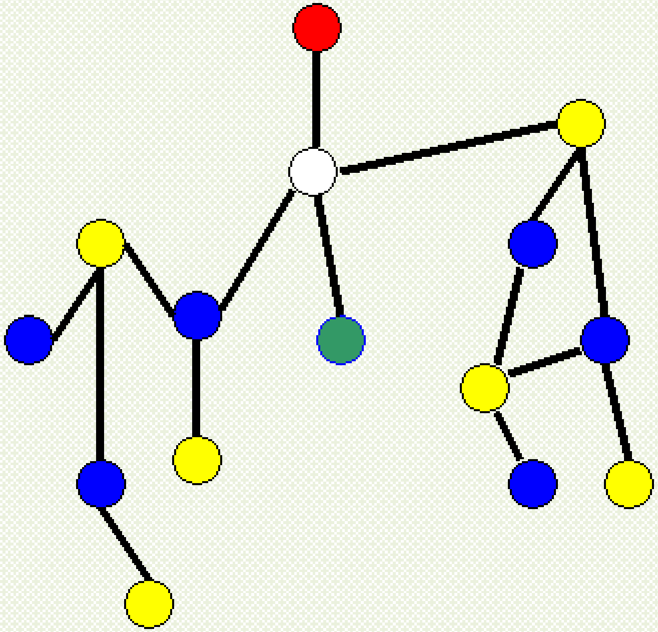
Kempe Chain

- In 1879, Kempe provided the first proof of four color theorem(4CT). Found to be flawed by Heawood in 1890.
- Introduced a technique now called Kempe chains.

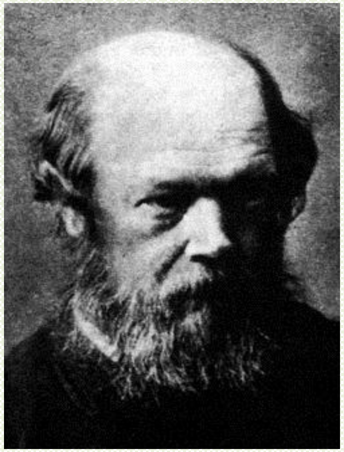


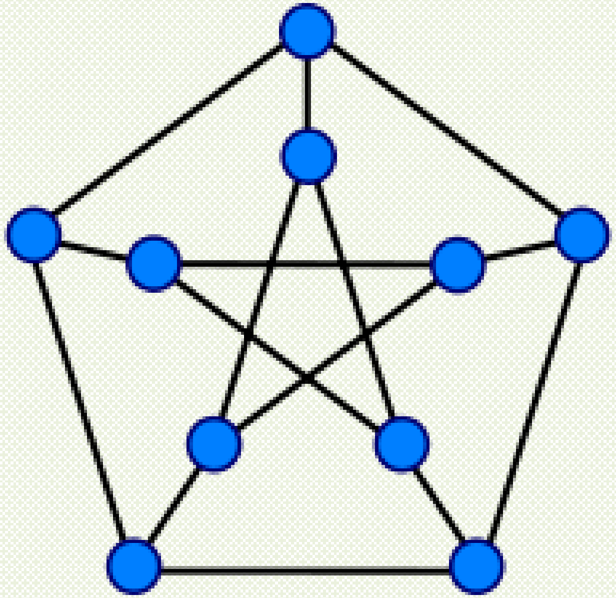


A. B. Kempe, On the Geographical Problem of Four-Colors, *Amer. J. Math*. **2** (1879), 193-200.

Tait Cycle

- Tait's proof published in 1880. Found to be flawed by Petersen in 1891.
- Found an equivalent formulation of the 4CT in terms of three-edge coloring.





P. G. Tait, Note on a theorem in geometry of position, *Trans. Roy. Soc. Edinburgh* **29** (1880), 657-660.

Computer-assisted Proof of 4CT

- Kenneth Appel and Wolfgang Haken (1976).

- Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas (1997). Another simpler proof.

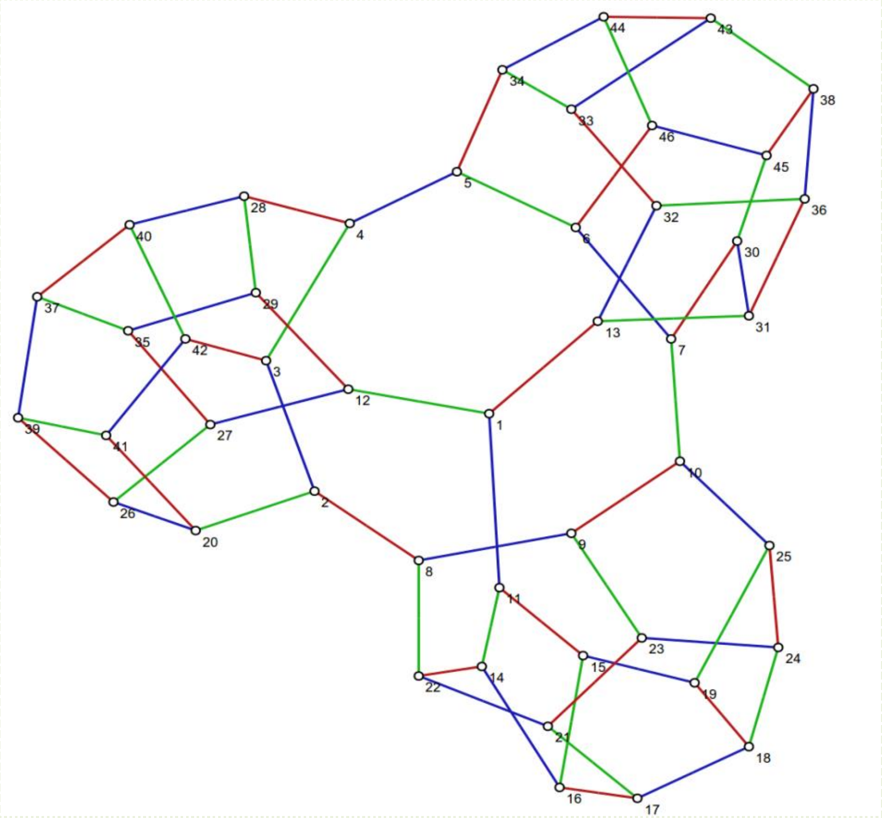
Computer-assisted Proof of 4CT

- The computer-assisted proofs of the four color theorem caused great amounts of controversy because they can not be verified by human.

- The search continues for a computer-free proof of the Four Color Theorem.

Edge Coloring

- **Edge Coloring :** an assignment of “colors” to the edges of the graph so that no two adjacent edges have the same color.



Theorems on Edge Coloring

- **Petersen's Theorem**: Every bridgeless cubic graph contains a perfect matching.
- **Vizing's Theorem**[1]: Any simple graph is either Δ- or $\Delta + 1$-edge-colorable. Chromatic Index: χ_e.
- **Holyer** [2]: Deciding Δ- or $\Delta + 1$-edge-colorable is NP-Complete, even for $\Delta = 3$.

1. V. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz*, 3(7):25–30,1964
2. I. Holyer. The NP-completeness of edge-colouring. *Siam J. Comput*, 10(4):718–720, 1981.

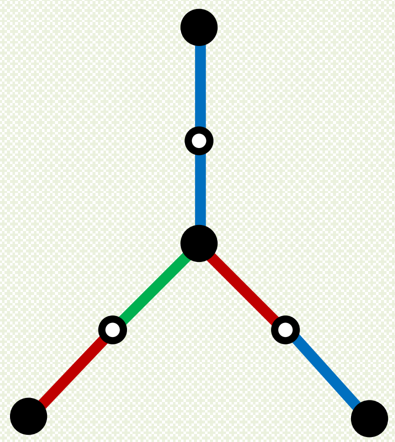
Outline

- Operations of Complex Colors
- Decomposition of Configurations
- Solvability of Configurations
- Generalized Petersen Configuration
- Three-Edge-Coloring Theorem
- Graph Theory versus Euclidean Geometry
- Conclusions

Constraints of Edge Coloring

- **Vertex constraint**

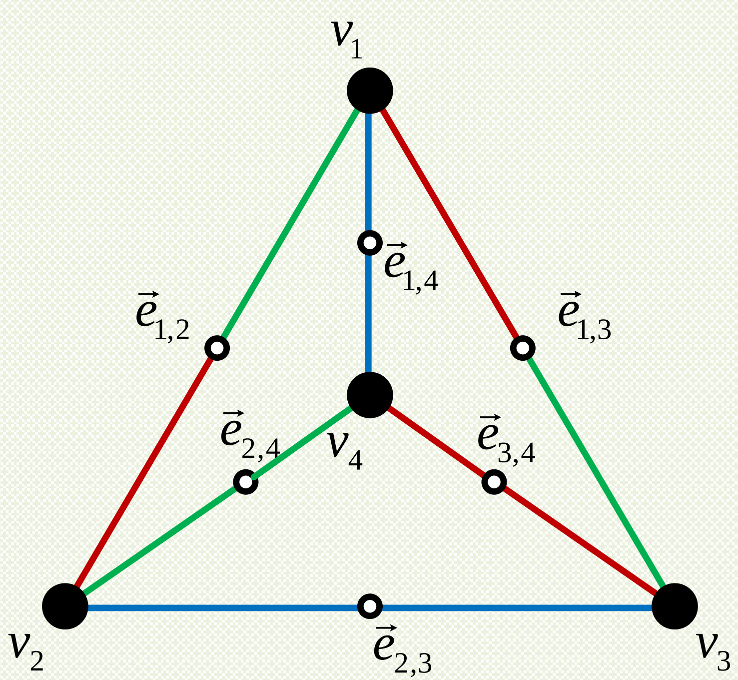
  colors assigned to links incident to the same vertex are all distinct



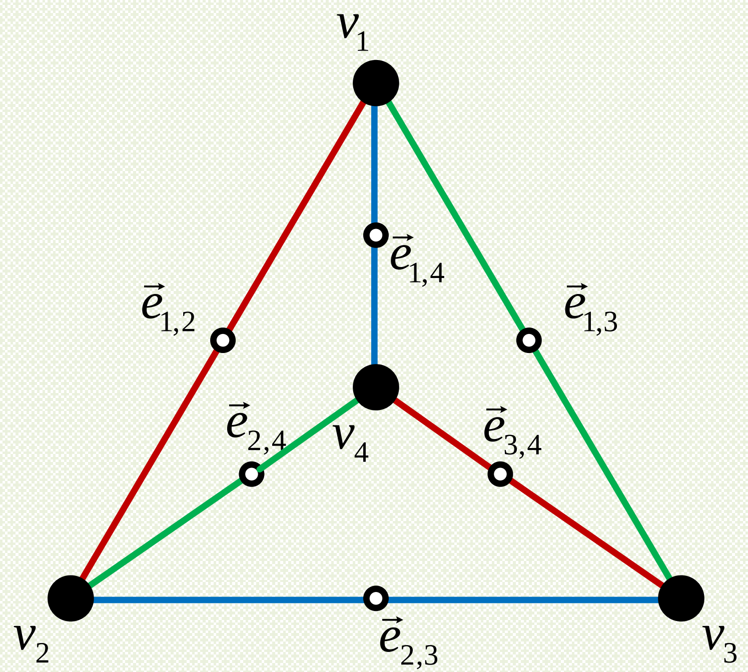
- **Edge constraint**
  - Variable-colored edge

  - Constant-colored edge

Complex Coloring of Tetrahedron



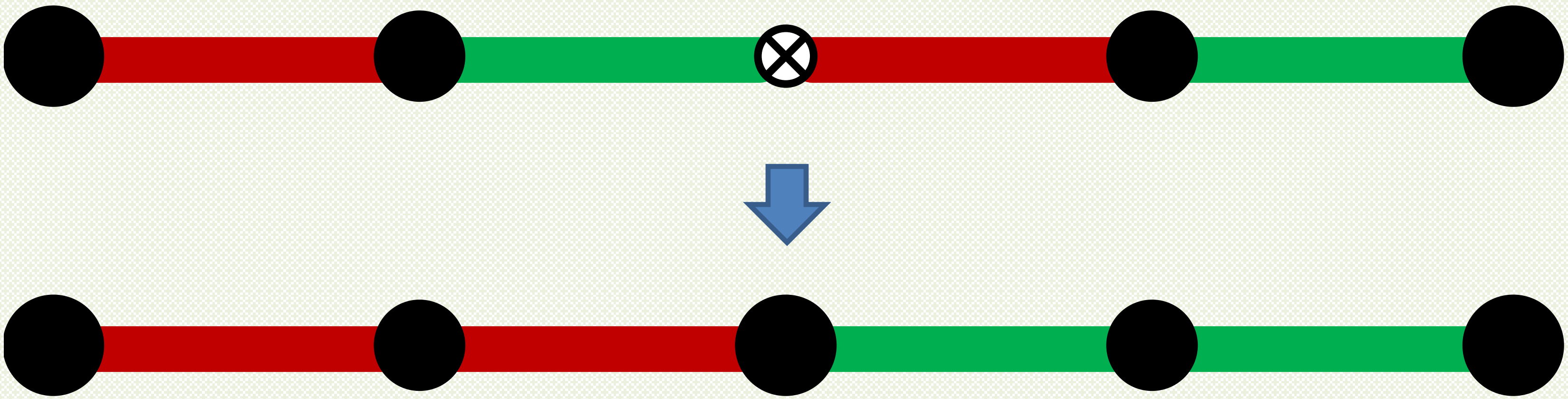
Consistent Coloring



Proper Coloring

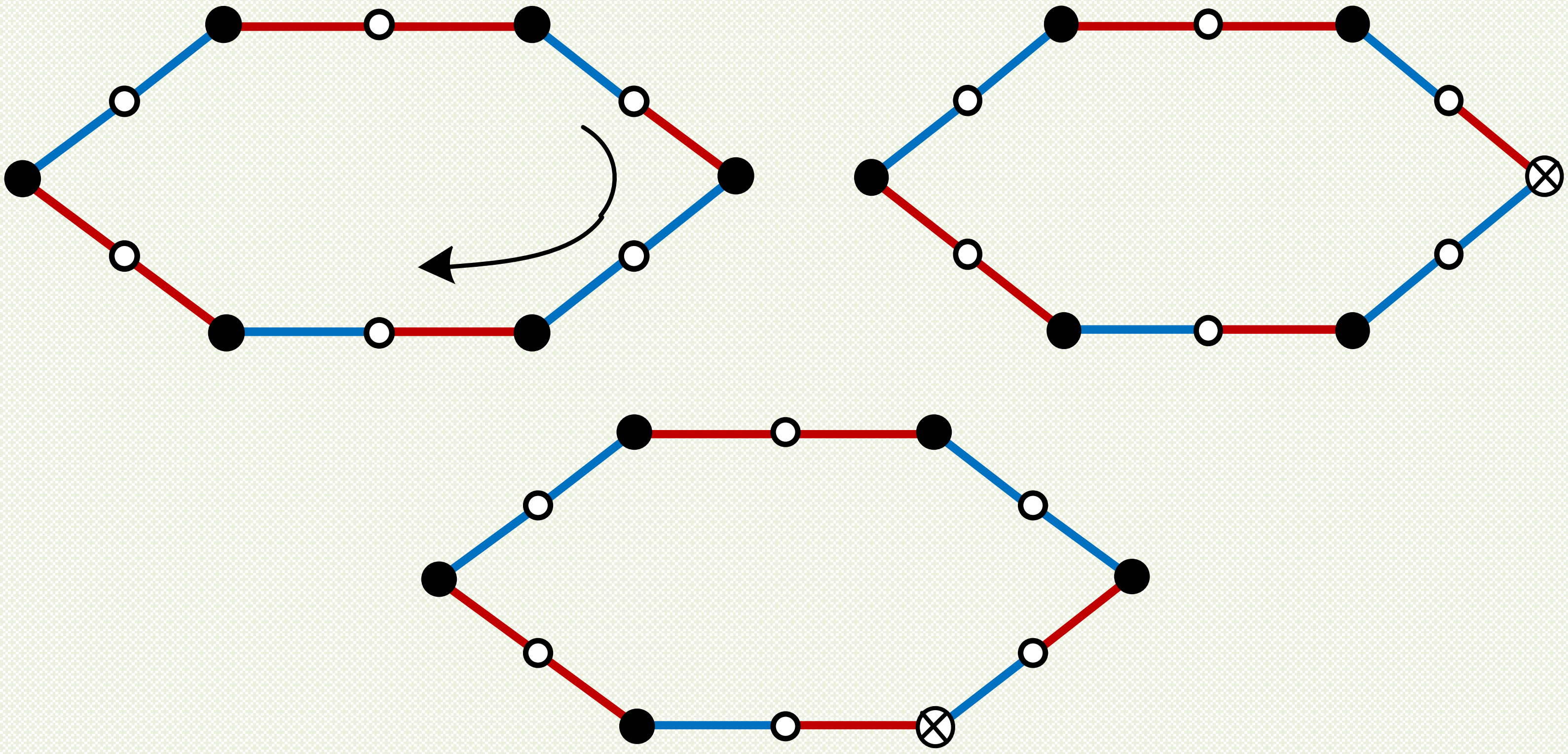
Color-Exchange Operation of Complex Colors

$$(\alpha,\beta)\otimes(\alpha,\beta)\Rightarrow(\alpha,\alpha)\circ(\beta,\beta)$$



- Color-exchange operation preserves the consistency of vertex constraint

Kempe Walks



Eliminate 2 Variables

Variable Elimination by Kempe Walks

Exhaustively eliminate variables by Kempe walks

- Proper 3-edge-coloring if no variables remaining, or
- All remaining variables are contained in odd cycles

T. T. Lee, Y. Wan, H. Guan. Randomized Δ-edge coloring via exchanges of complex colors, *International Journal of Computer Mathematics* **90** (2013), 228-245.

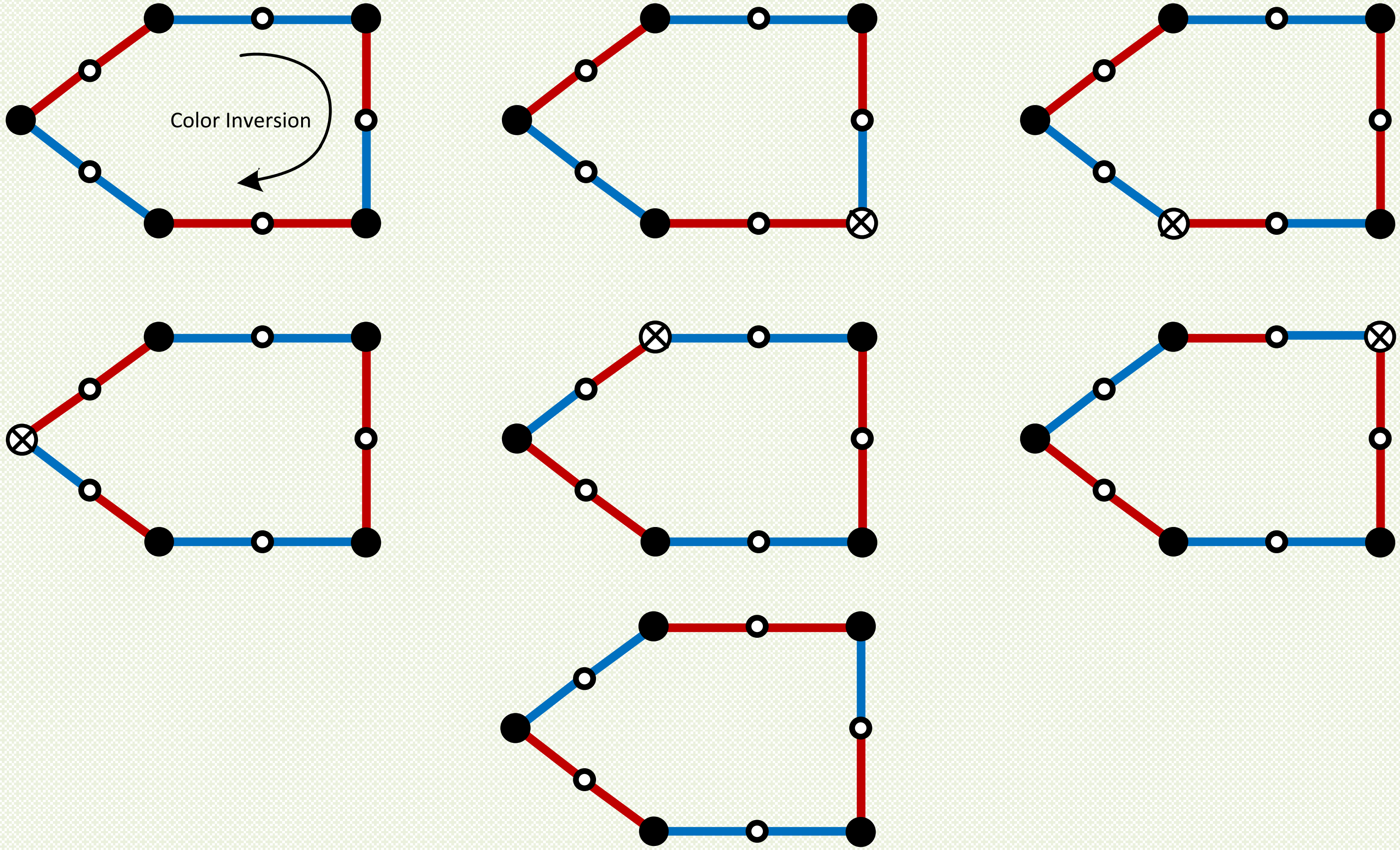
One-step move on Kempe path

Case	Next Step	Operations	Results
KW1	$(\alpha,\beta)\circ(\alpha,\alpha)$	$(\alpha,\beta)\otimes(\alpha,\alpha)\Longrightarrow(\alpha,\alpha)\circ(\beta,\alpha)$	step forward
KW2	$(\alpha,\beta)\circ(\alpha,\beta)$	$(\alpha,\beta)\otimes(\alpha,\beta)\Longrightarrow(\alpha,\alpha)\circ(\beta,\beta)$	eliminate two variables
KW3	$(\alpha,\beta)\circ(\alpha,\gamma)$	$(\alpha,\beta)\otimes(\alpha,\gamma)\Longrightarrow(\alpha,\alpha)\circ(\beta,\gamma)$	eliminate one variable
KW4	$(\alpha,\beta)\circ(\emptyset,\emptyset)$	$(\alpha,\beta)\otimes(\emptyset,\emptyset)\Longrightarrow(\alpha,\alpha)$	eliminate one variable

Limitation of Kempe Walks

- Kempe walks can only apply to two-colored sub-graphs H .
- Kempe walks cannot change the topology of any two-colored sub-graphs H .
- Variables are trapped within fixed two-colored odd cycles.

Color Inversion of Complex Colors



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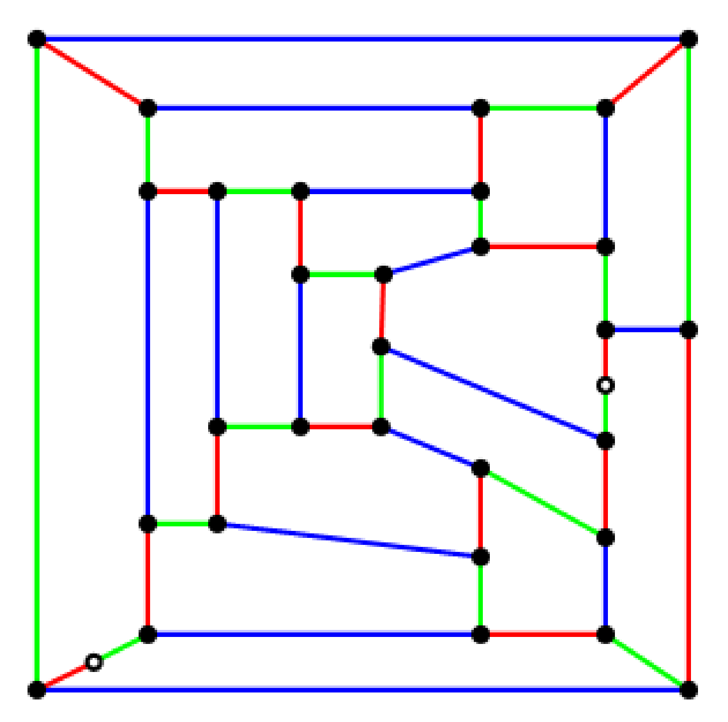
Petersen Matching and Configuration

- Edges not in the perfect matching form a set of disjoint cycles, called ***Tait cycles***.

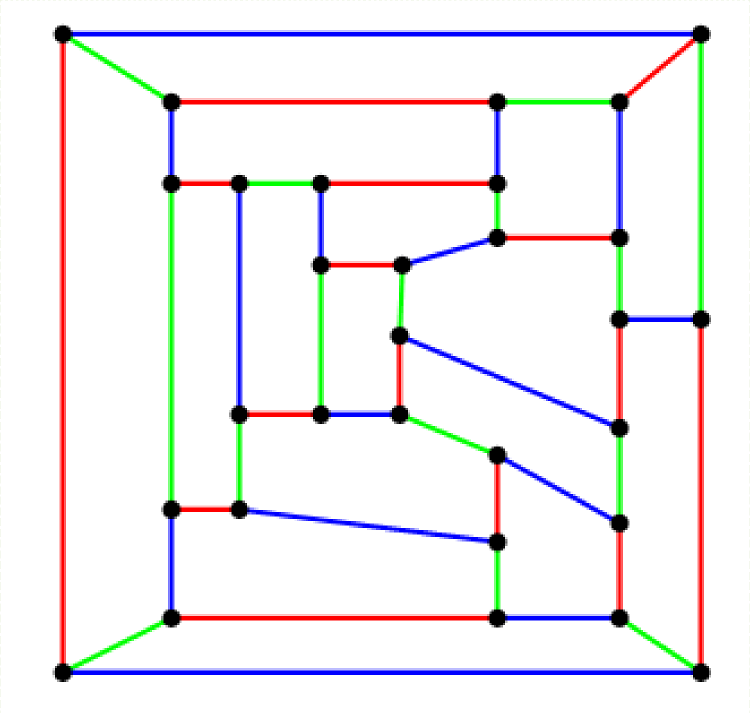
- Configuration $T(G)$: assigning color c to the edges in the perfect matching, and color a or b to the links in Tait cycles.

- Every odd (a, b) Tait cycle contains exactly one (a, b)-variable.

Color Configurations of A Cubic Plane Graph



Two disjoint odd (a, b) cycles.



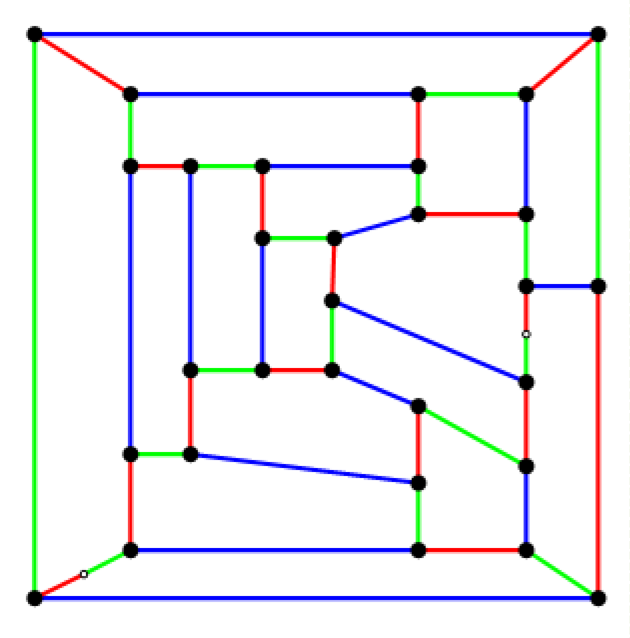
(a, b), (b, c), (a, c) even cycles.

Decomposition of Configuration

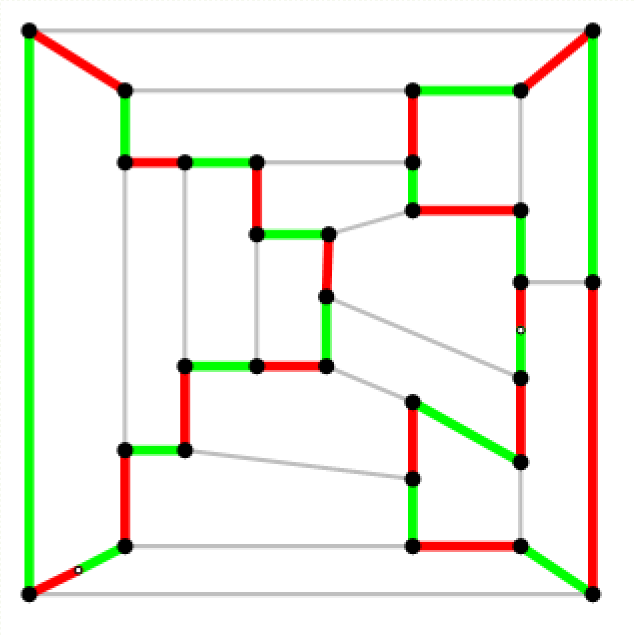
Maximal Two-Colored Sub-graphs:

- ***Locking Cycle***: odd (a, b) cycle contains one (a, b)-variable.
- ***Resolution Cycle***: (a, c) or (b, c) even cycle.
- ***Exclusive Chain***: (a, c) or (b, c) open path connecting two (a, b)-variables.

Locking Cycle

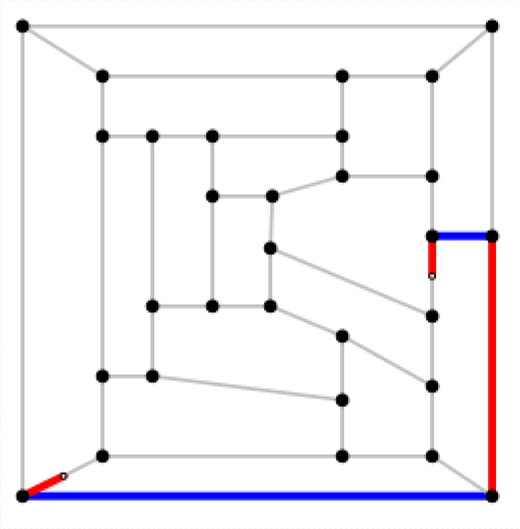


Configuration $T(G)$.

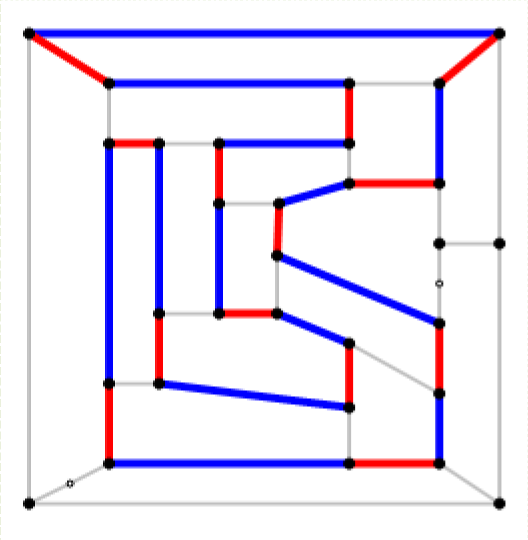


Two locking (a, b) cycles.

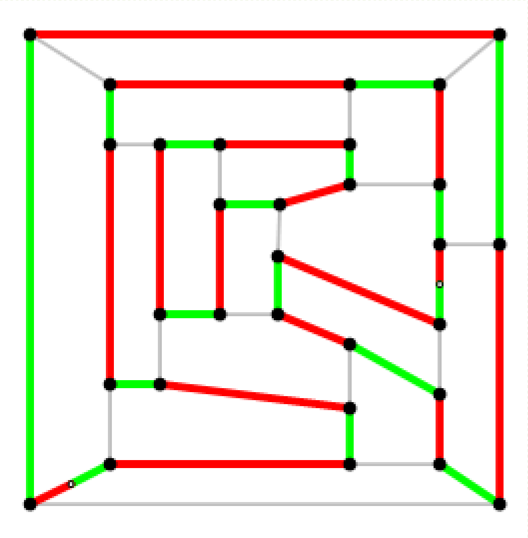
Essential Resolution Cycle



The (a, c) exclusive chain.

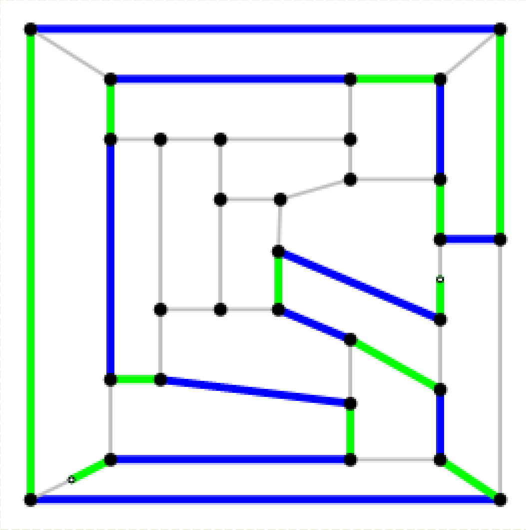


An essential (a, c) cycle.

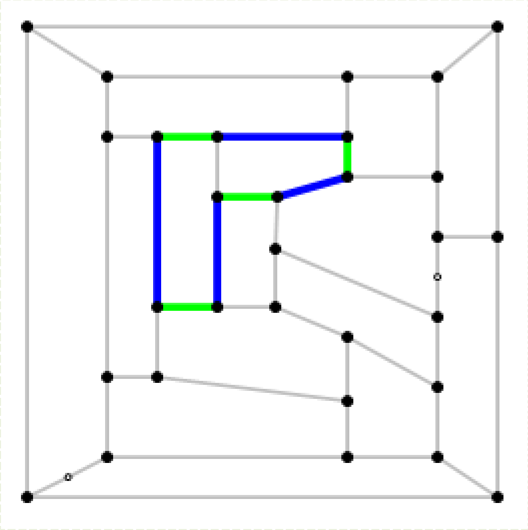


Two even (a, b) cycles after negating (a, c) cycle.

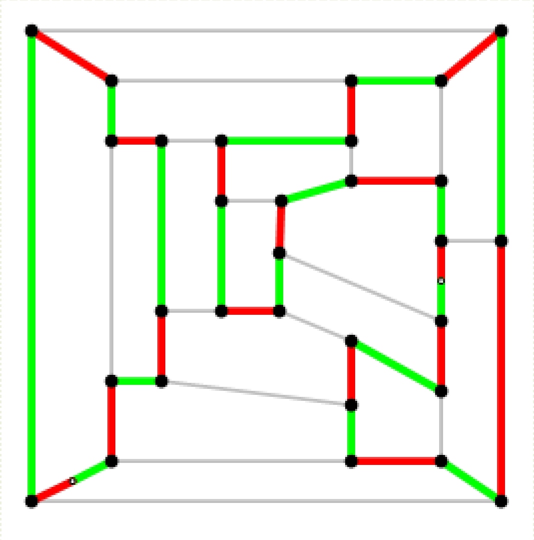
Nonessential Resolution Cycle



The (b, c) exclusive chain.



A nonessential (b, c) cycle.



Two odd (a, b) cycles after negating (b, c) cycle.

State Transitions within A Configuration $T(G)$

- State transitions of $T(G)$:
  - Negate any (a, b) cycle, either even or odd.
  - Move any (a, b)-variable within its locking cycle.

- State transitions retain the sub-graphs of all (a, b) cycles intact, but change (a, c) and (b, c) exclusive chains and resolution cycles.

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Solvability of Configurations

- A state $\xi \in S_{T(G)}$ is ***solvable*** if one of the (a, c) or (b, c) cycle in the state ξ is essential. Otherwise, the state $\xi \in S_{T(G)}$ is ***unsolvable***.

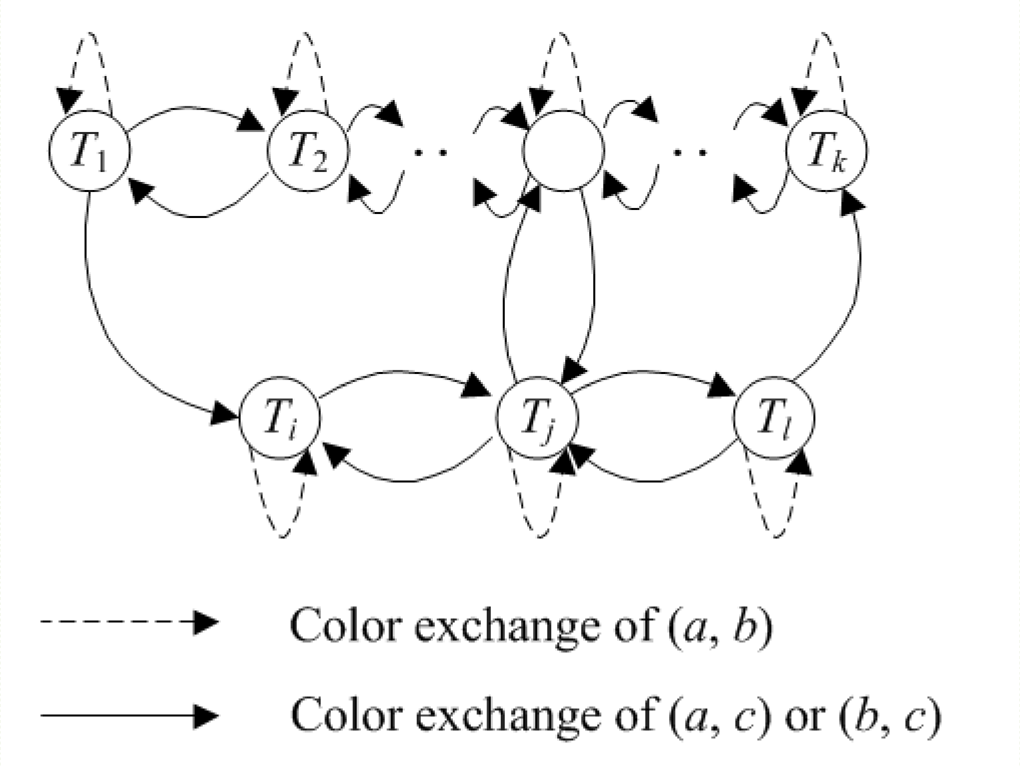
- The configuration $T(G)$ is ***solvable*** if one of the state $\xi \in S_{T(G)}$ is solvable. Otherwise, the configuration $T(G)$ is ***unsolvable*** if all states are unsolvable.

Transitions of A Configuration

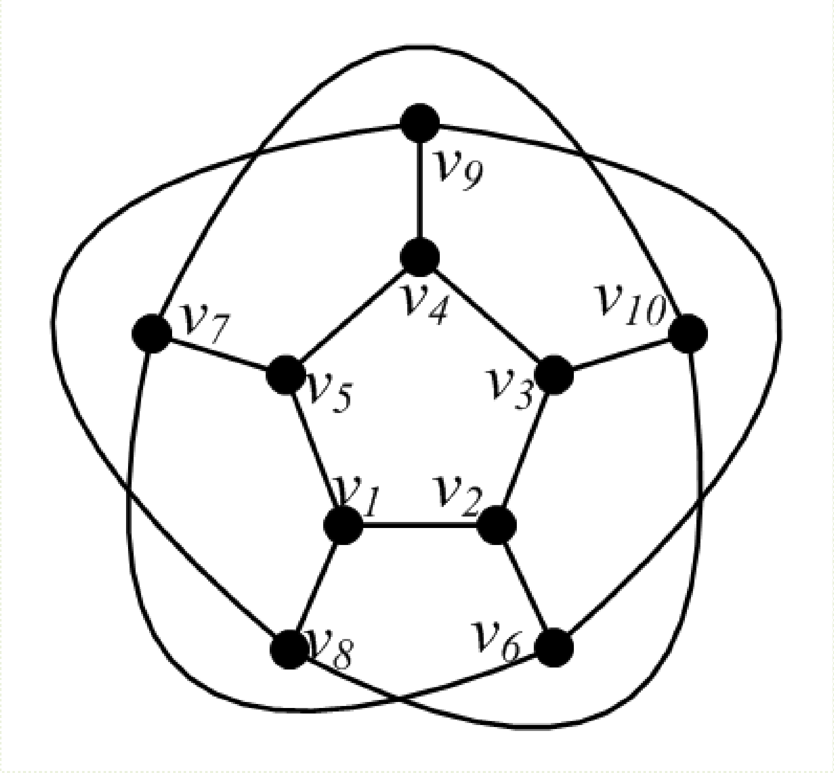
- ***Local operation***: (a, b) color exchanges, move a state to another state within the same configuration $T(G)$.

- ***Global operation***: (a, c) and (b, c) color exchanges, transform configuration $T(G)$ into another configuration $T'(G)$.

Transition Diagram of Configurations

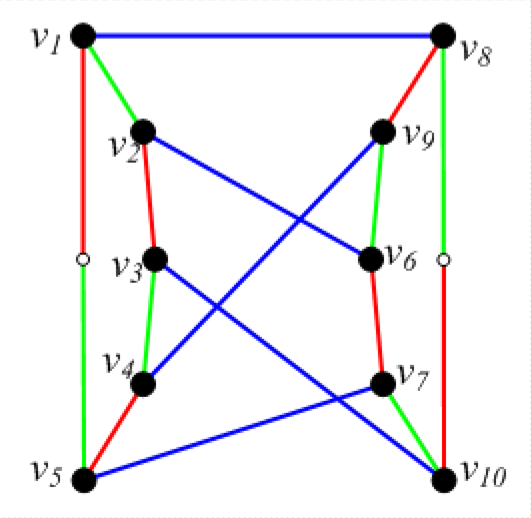


Petersen Graph

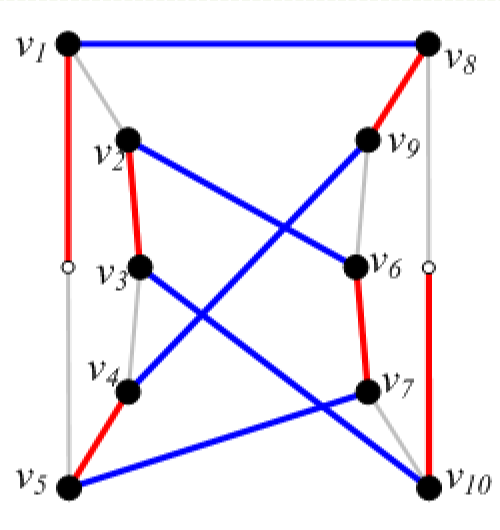


A Configuration of Petersen Graph

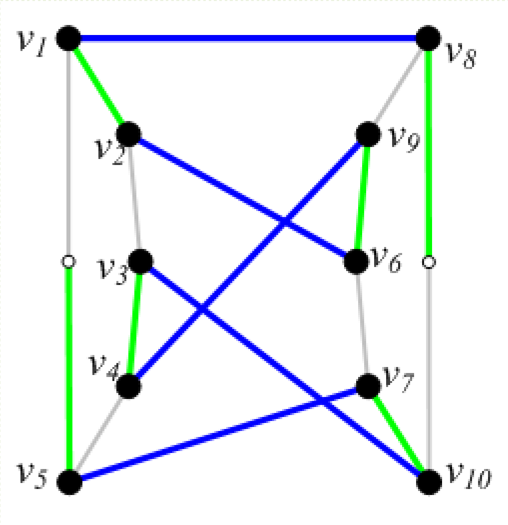
- A configuration contains 2 variables



A state ξ_1 of Petersen graph.



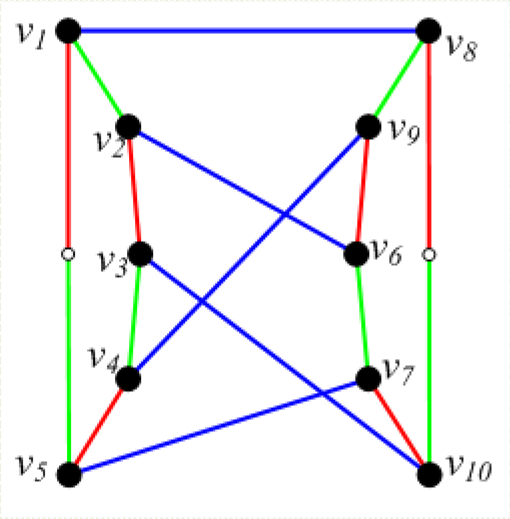
(a, c) sub-graph of ξ_1.



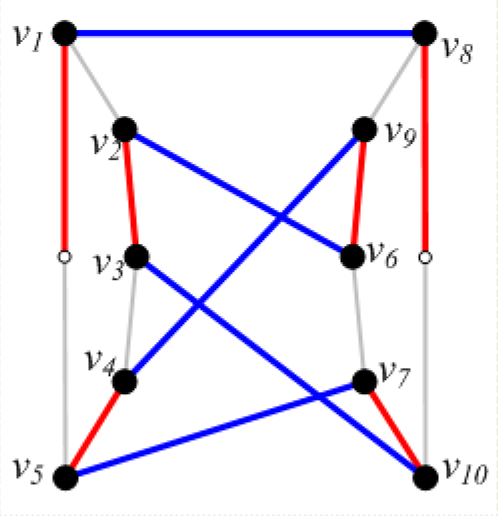
(b, c) sub-graph of ξ_1.

A Configuration of Petersen Graph

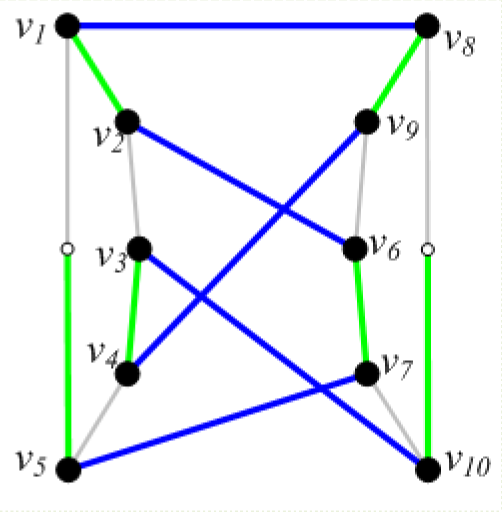
- Another state of the same configuration



A state ξ_2 of Petersen graph.



(a, c) sub-graph of ξ_2.



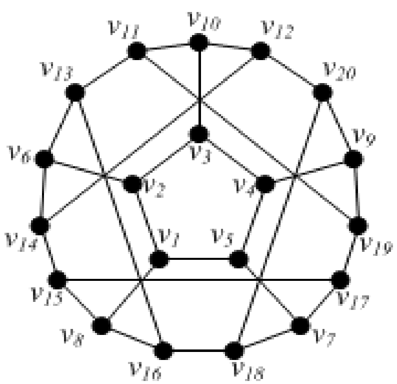
(b, c) sub-graph of ξ_2.

Tutte's Conjecture

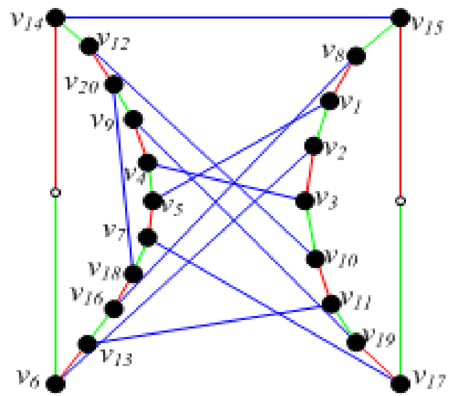
- Tutte (1966): Every snark has the Petersen graph as a graph minor.
- Neil Robertson and Robin Thomas announced in 1996 that they proved this conjecture, but did not publish the result.
- This conjecture implies the four color theorem.

W.T. Tutte. On the algebraic theory of graph colorings. *J. of Combinatorial Theory* **1** (1966), 15–50.

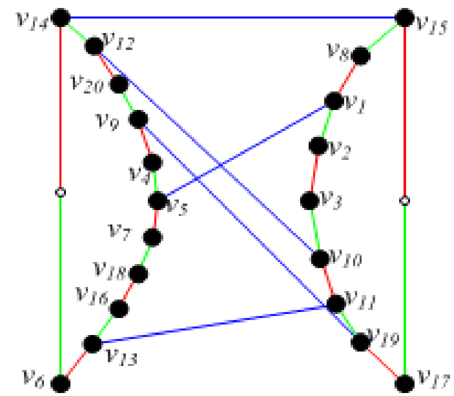
Contract Flower Snark to Petersen Graph



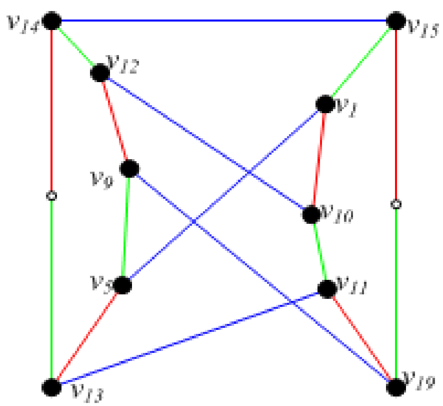
(a) Flower snark.



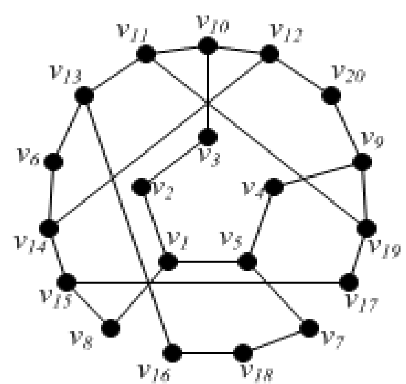
(b) A configuration of Flower snark.



(c) Find 5 chords matched with $T(G_P)$ of Petersen graph.

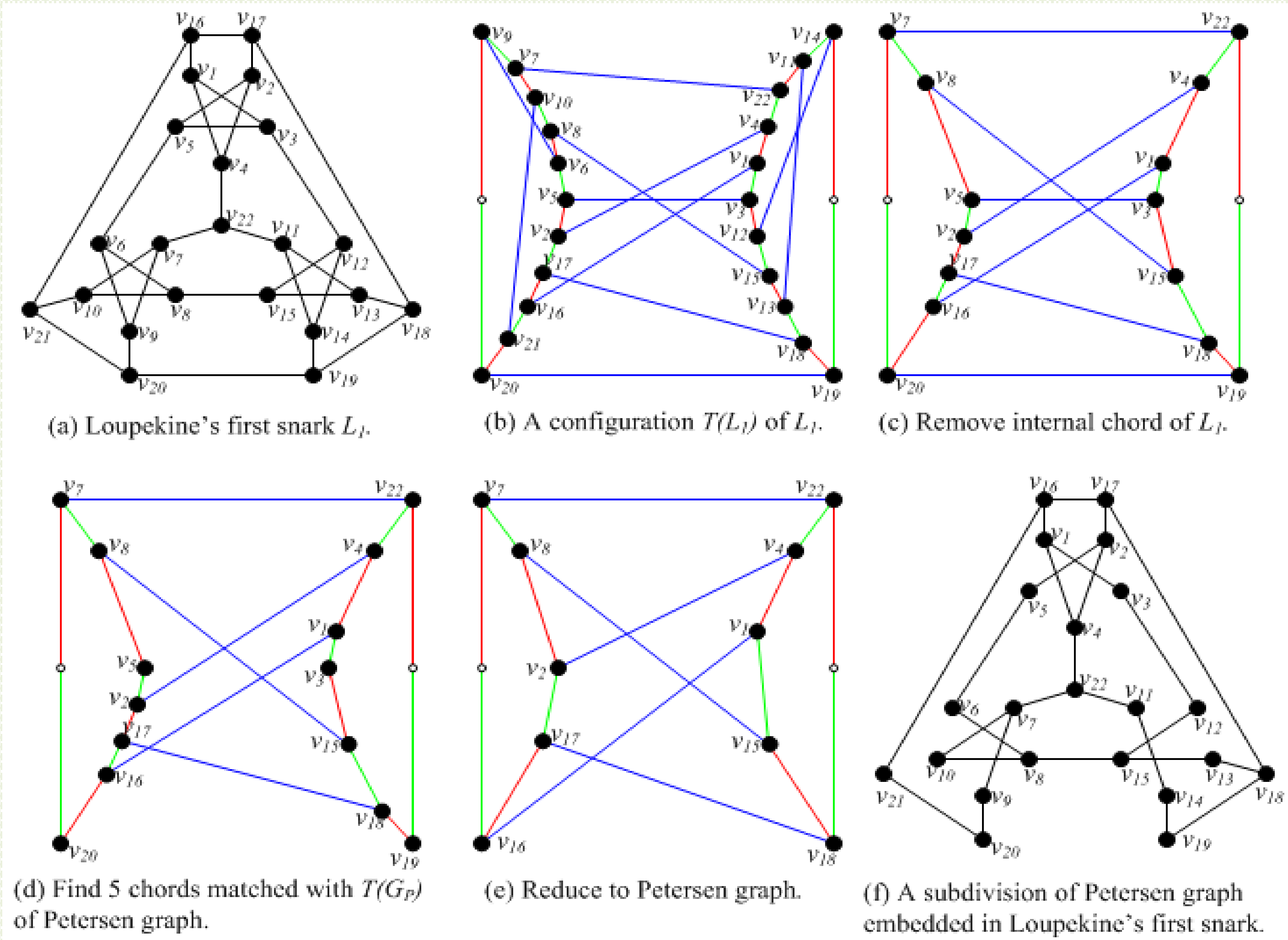


(d) Reduce to Petersen snark.



(e) A subdivision of Petersen graph embedded in Flower snark.

Contract Loupekine's First Snark



(a) Loupekine's first snark L_1.

(b) A configuration $T(L_1)$ of L_1.

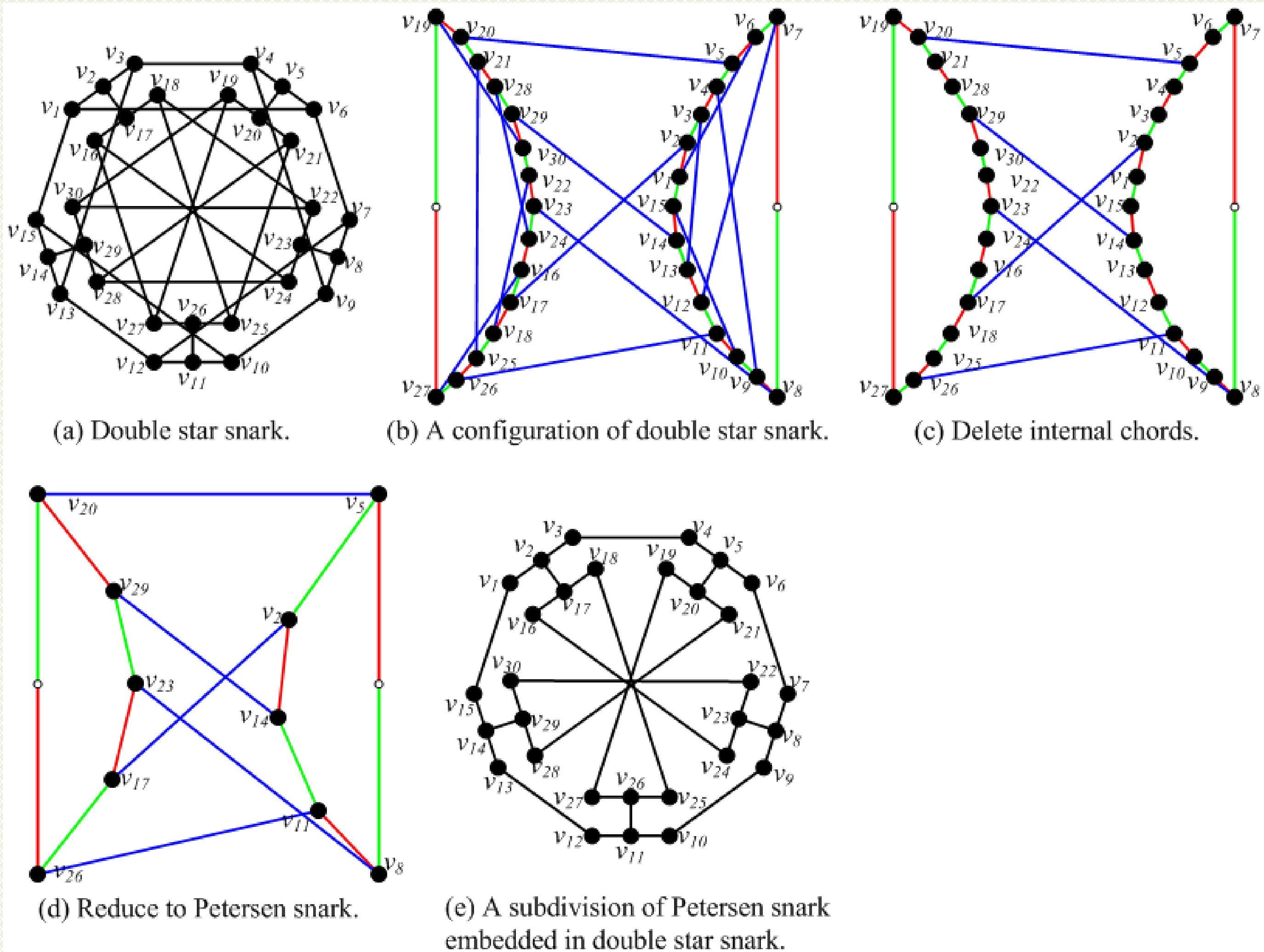
(c) Remove internal chord of L_1.

(d) Find 5 chords matched with $T(G_P)$ of Petersen graph.

(e) Reduce to Petersen graph.

(f) A subdivision of Petersen graph embedded in Loupekine's first snark.

Contract Double Star Snark



(a) Double star snark.

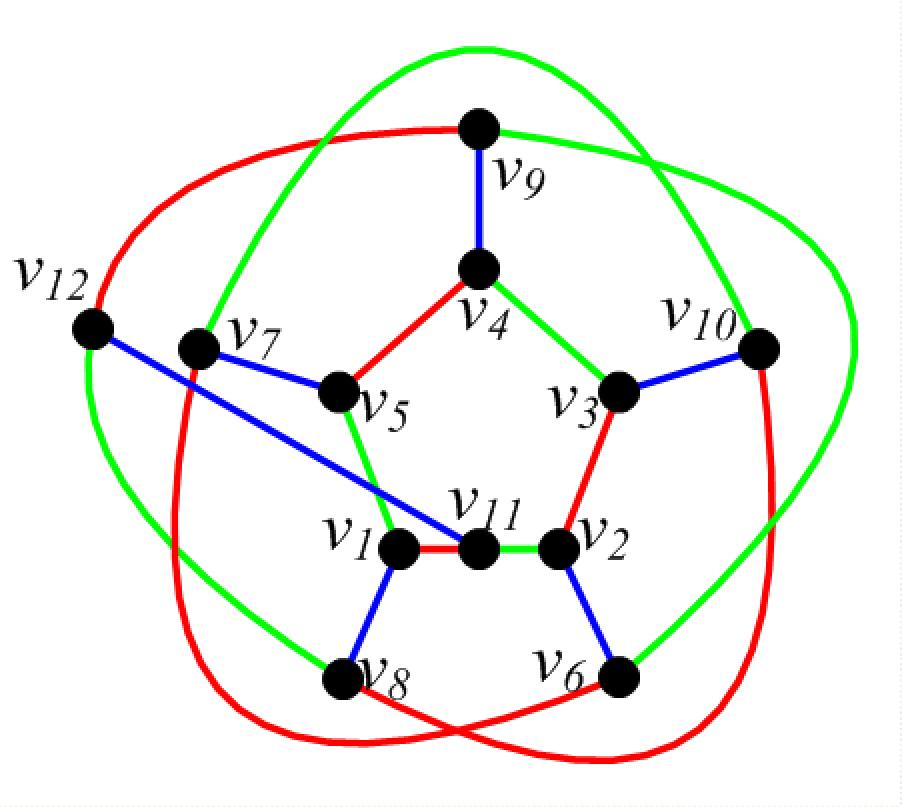
(b) A configuration of double star snark.

(c) Delete internal chords.

(d) Reduce to Petersen snark.

(e) A subdivision of Petersen snark embedded in double star snark.

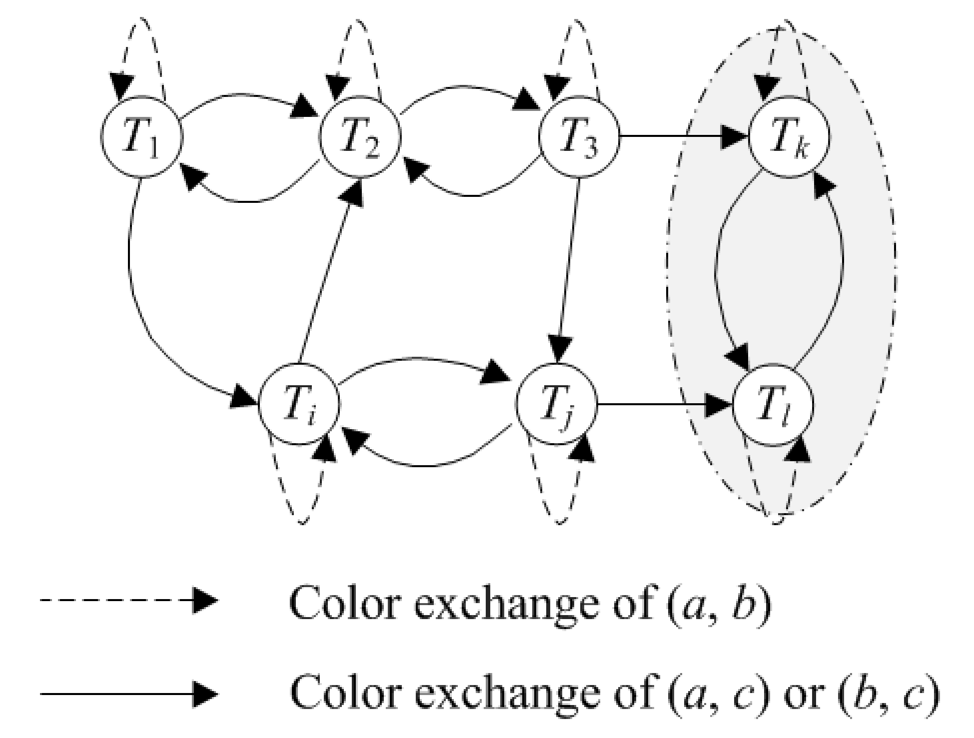
Petersen Graph as Graph Minor



Petersen graph as graph minor is NOT a characterization of a snark

Proposition of Unsolvability

- A bridgeless cubic graph $G(V, E)$ is a class 2 graph if and only if G has a closed set of unsolvable configurations.



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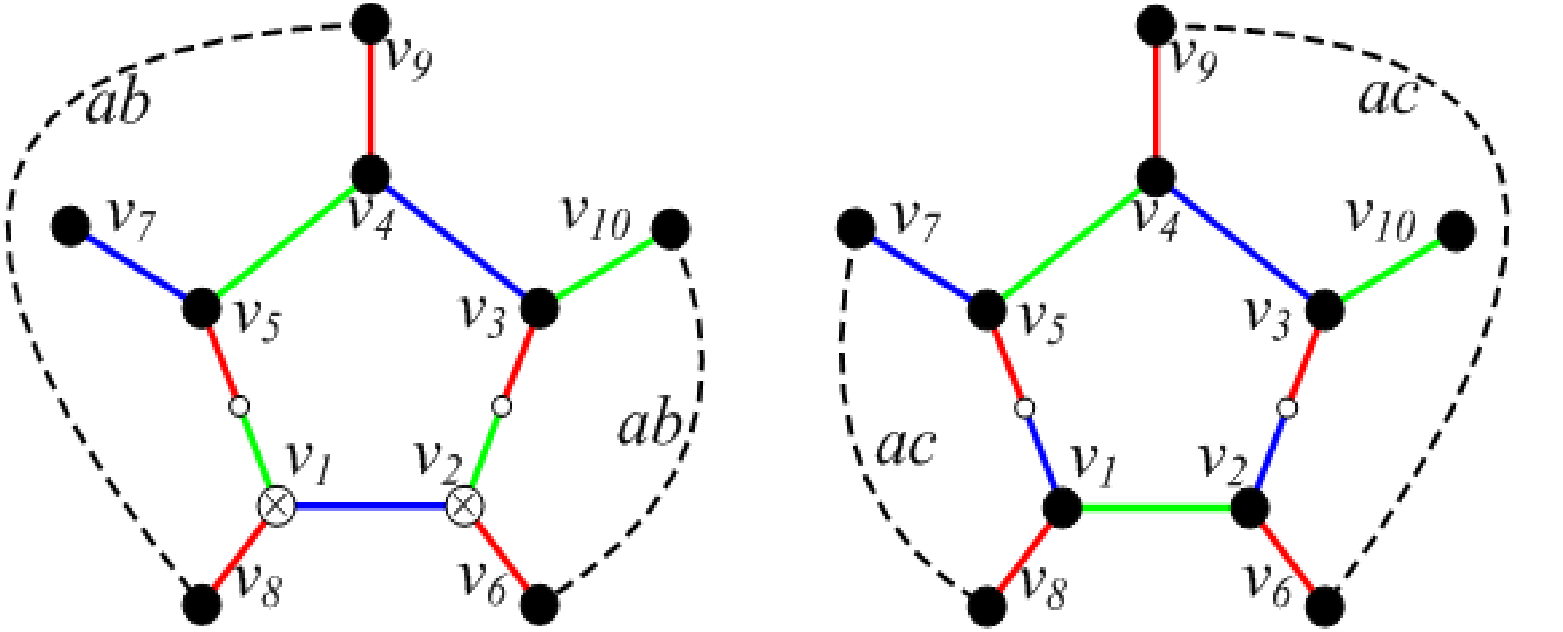
Generalized Petersen Configuration

A ***generalized Petersen configuration*** $P(G)$ satisfies:

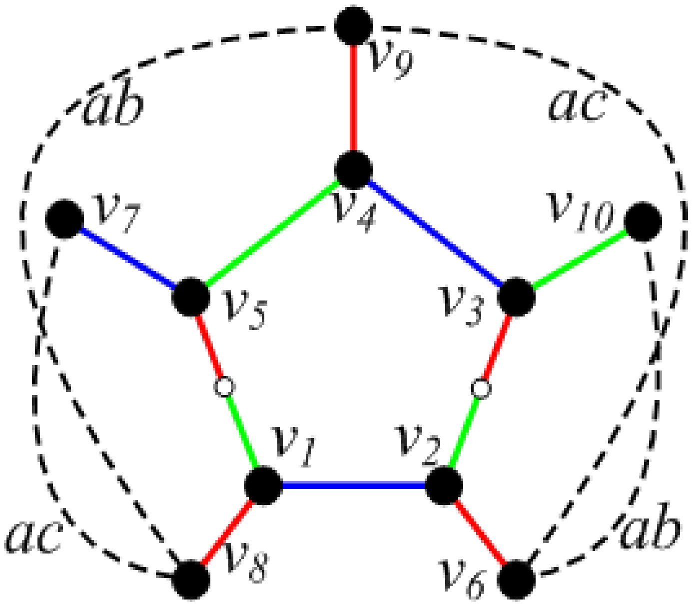
- The configuration $P(G)$ contains two (a, b)-variables.
- The two (a, b)-variables are on the boundary of a pentagon in some state ξ of $P(G)$.

Generalized Petersen Configuration

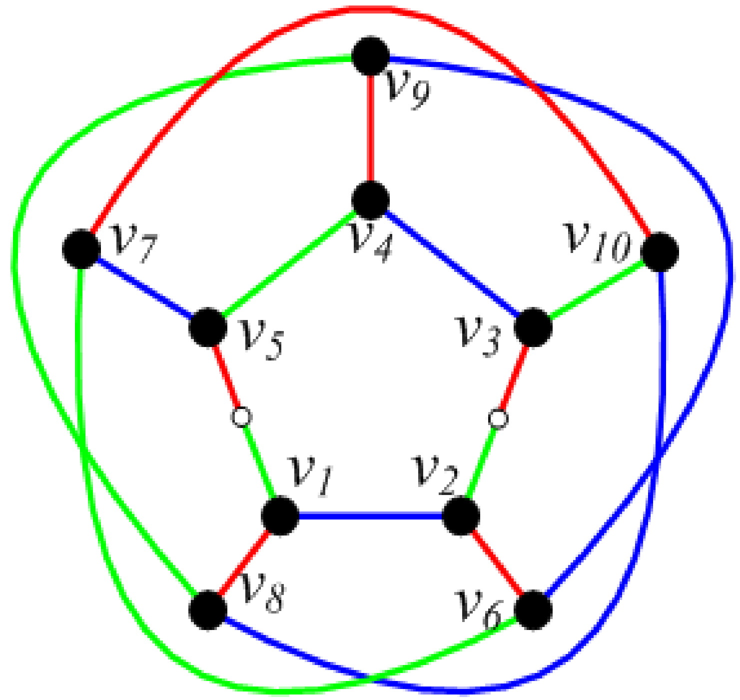


(a) Two disjoint *(a, b)*-cycles. (b) Two disjoint *(a, c)*-cycles.

Generalized Petersen Configuration

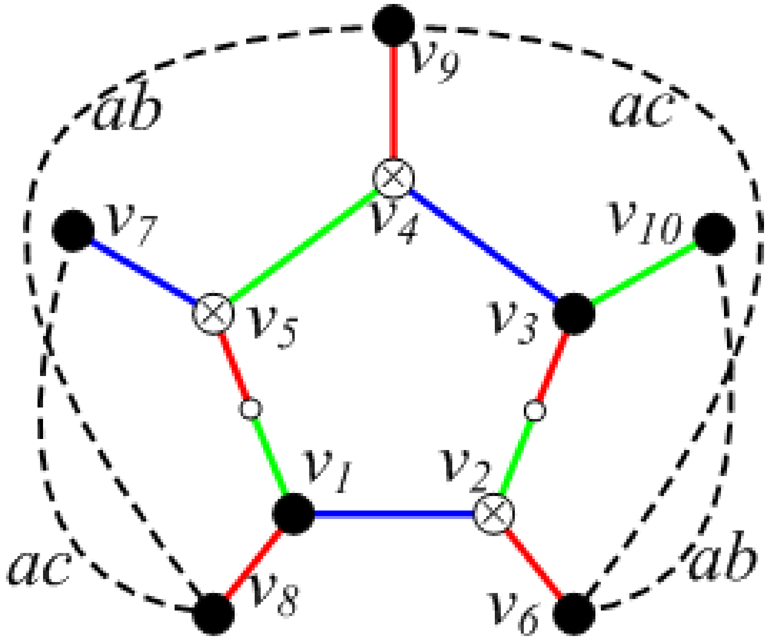


(c) The complete state ξ .

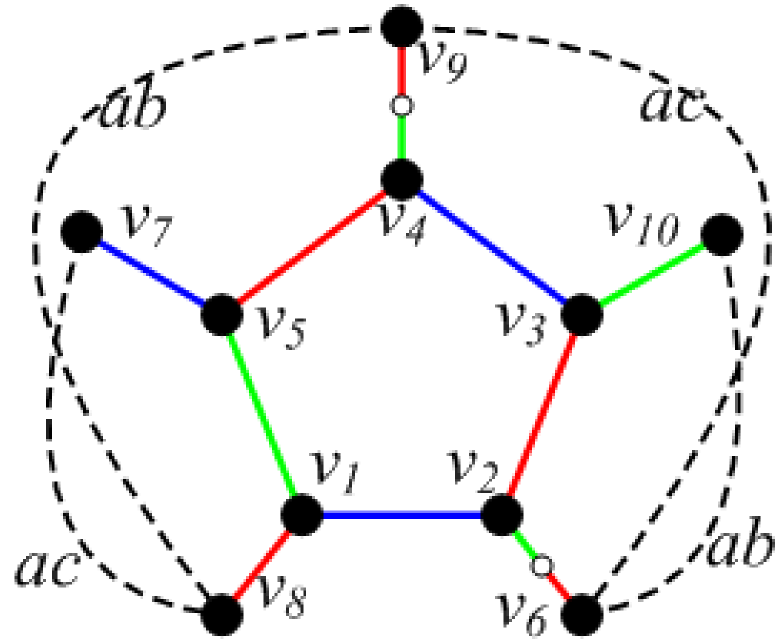


(d) The state ξ of Petersen Graph.

Resolution Cycle of Generalized Petersen Configuration



(a) *(a, b)*-color exchanges.



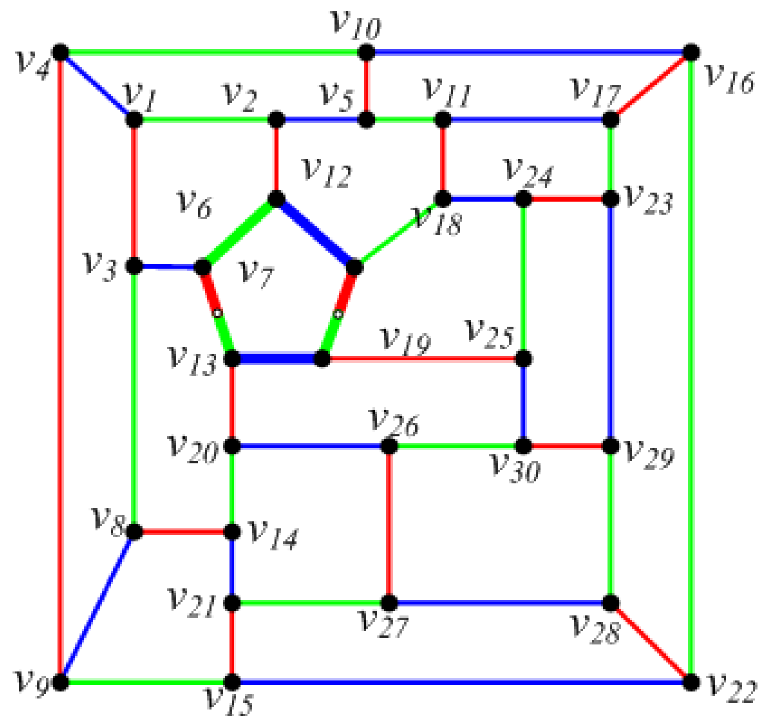
(b) *(a, c)* resolution cycle and *(a, c)* exclusive chain.

Proposition of Solvability

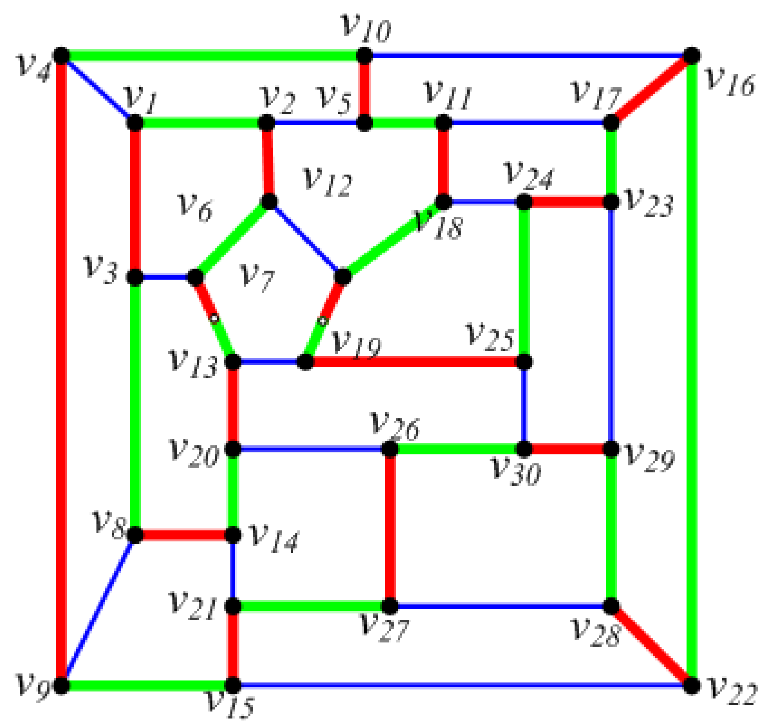
- Every generalized Petersen configuration $P(G)$ of a bridgeless cubic plane graph $G(V, E)$ is solvable.

  - Verified by more than 100,000 instances generated by computer.
  - Don't have a logical proof of this assertion.

$P(G)$ with 284 solvable states



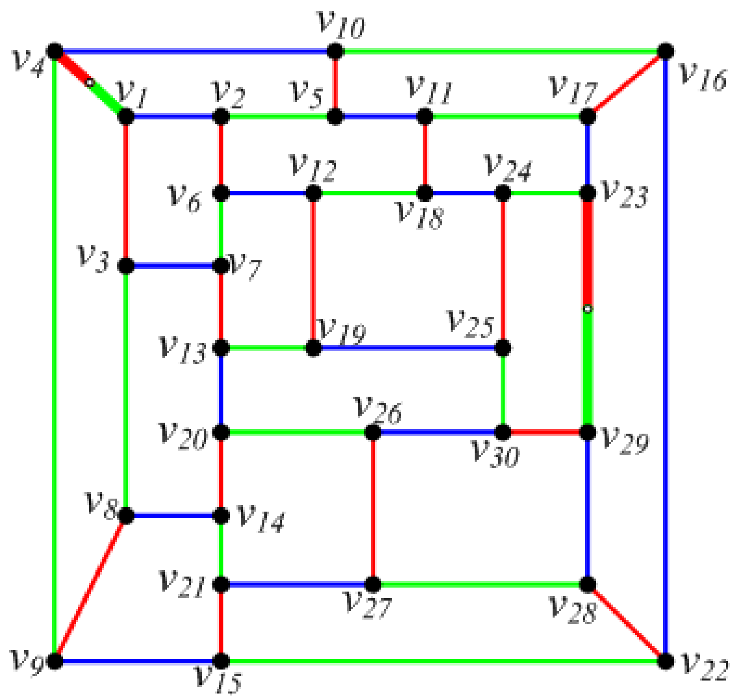
(a) A generalized Petersen configuration.



(b) Two highlighted *(a, b)* locking cycles.

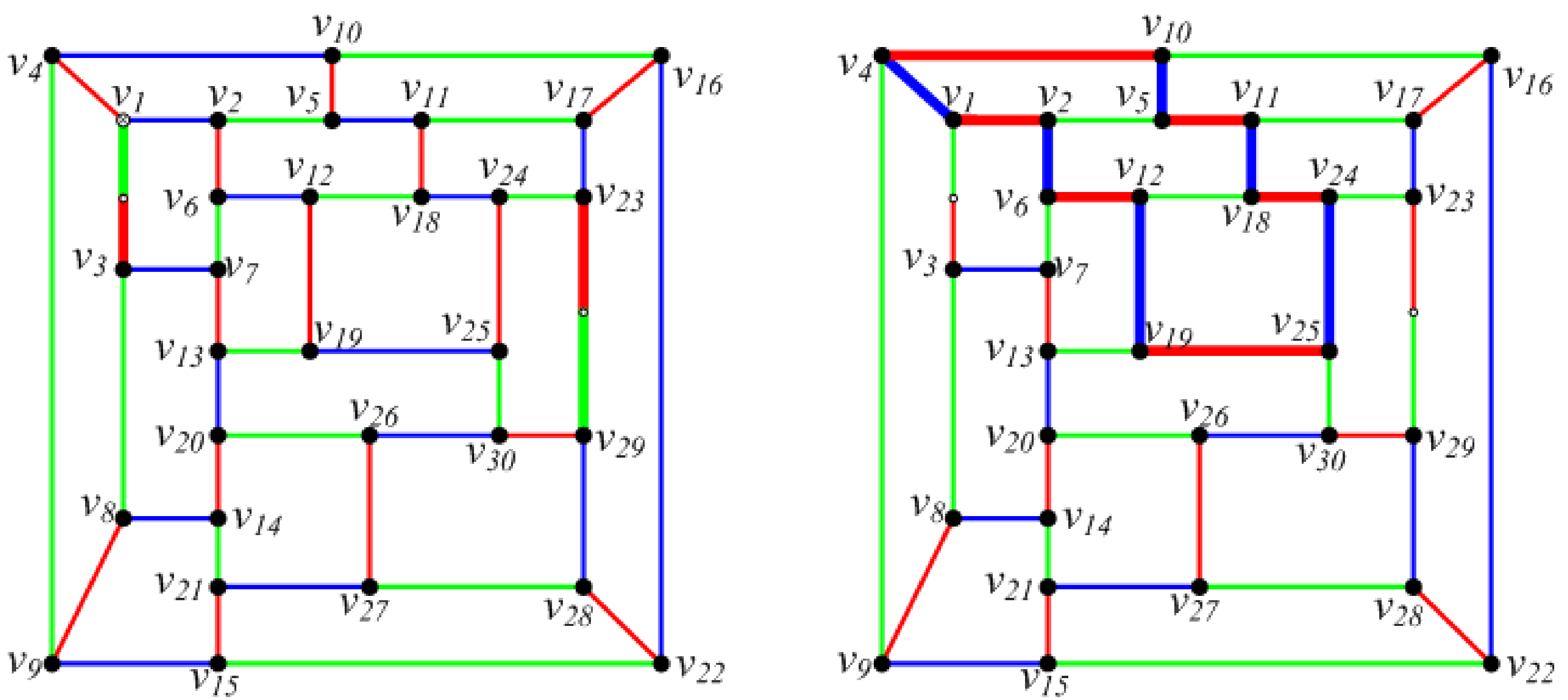
An Unsolvable Configuration

- The two (a, b)-variables are NOT on the boundary of the same pentagon in any states ξ.



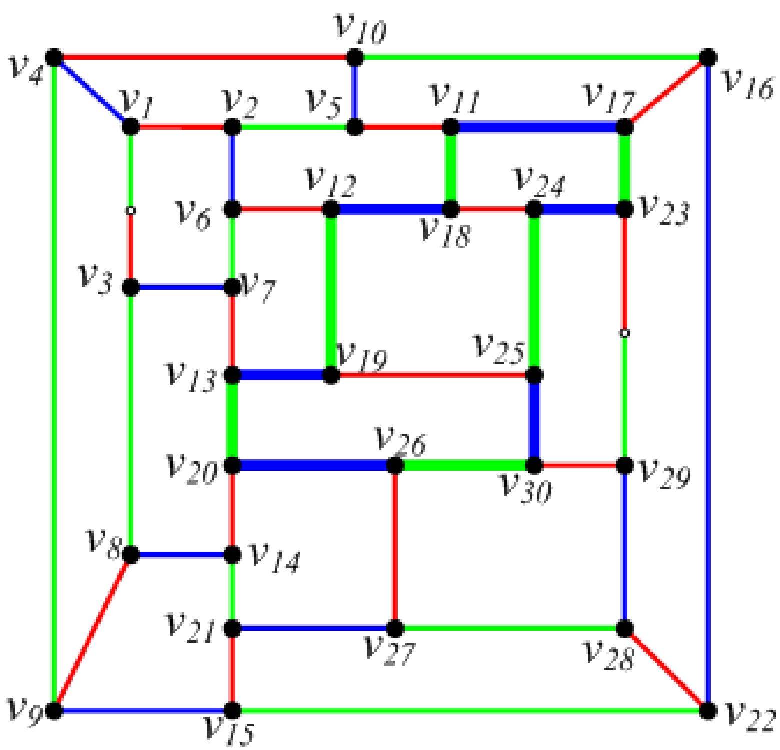
(a) An unsolvable configuration $T(G)$.

Negating (a,c) Cycle

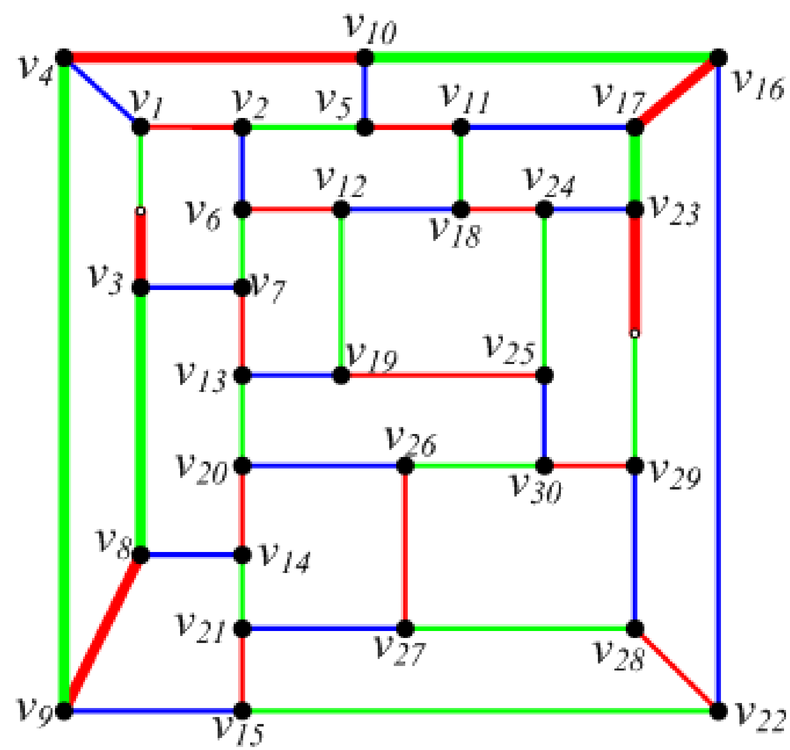


(b) Move an *(a, b)*-variable to edge (v_1, v_3). (c) Negate the highlighted *(a, c)* cycle.

Negating (b,c) Cycle



(d) Negate the highlighted *(b, c)* cycle.



(e) Two (a, b) variables are connected by a Kempe path.

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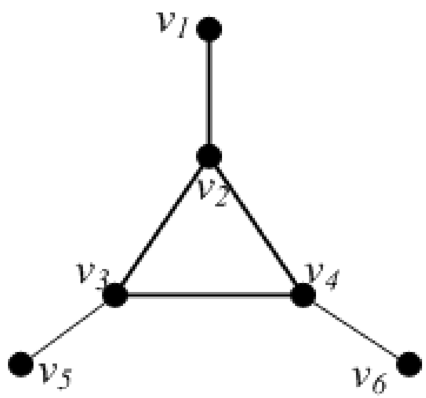
Three-edge Coloring Theorem

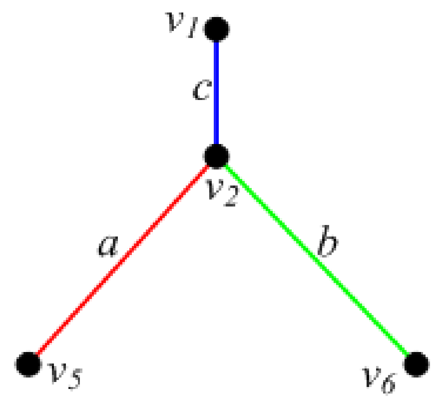
- **Lemma 1**: The girth of a bridgeless cubic plane graph $G(V,E)$ is less than or equal to 5.
- **Lemma 2**: Any face of a bridgeless cubic plane graph $G(V,E)$ has at least one admissible edge.

- **Theorem**: Every bridgeless cubic plane graph $G(V,E)$ has a 3-edge-coloring.

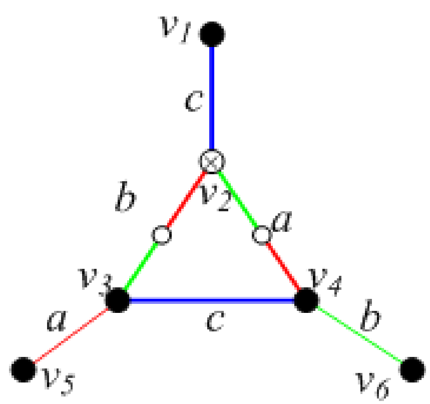
The Girth of *G* Equals 3



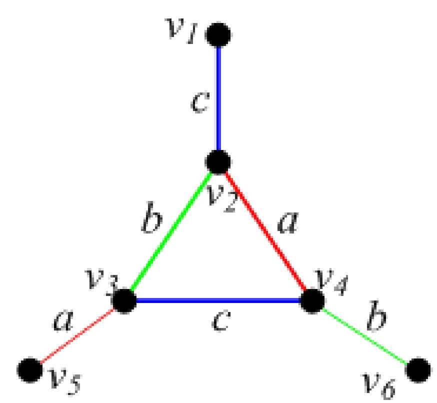
(a) A triangle in G.



(b) A 3-edge-coloring of G'.

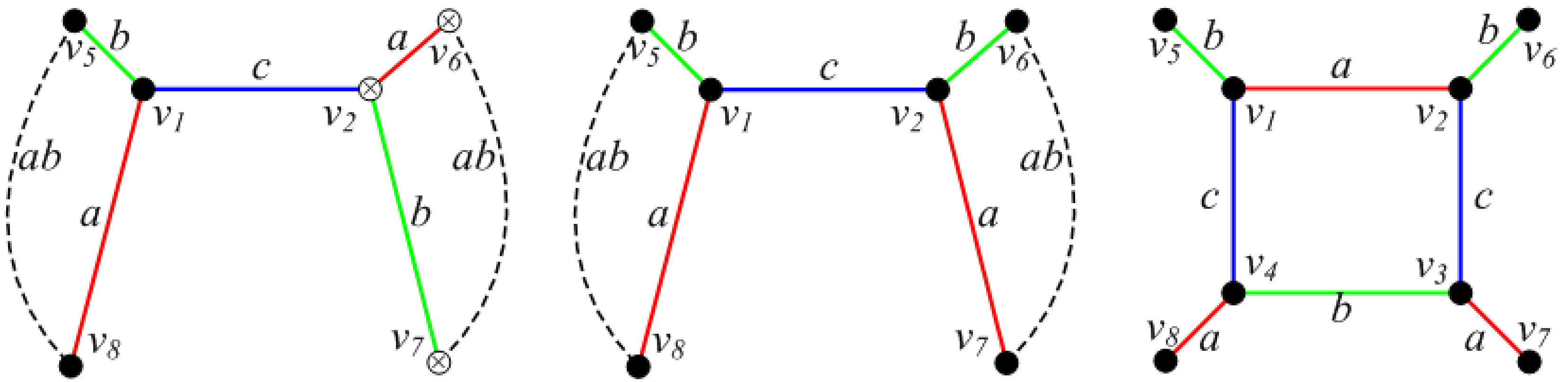


(c) A 3-edge-coloring of G.



(d) A proper 3-edge-coloring of G.

The Girth of *G* Equals 4

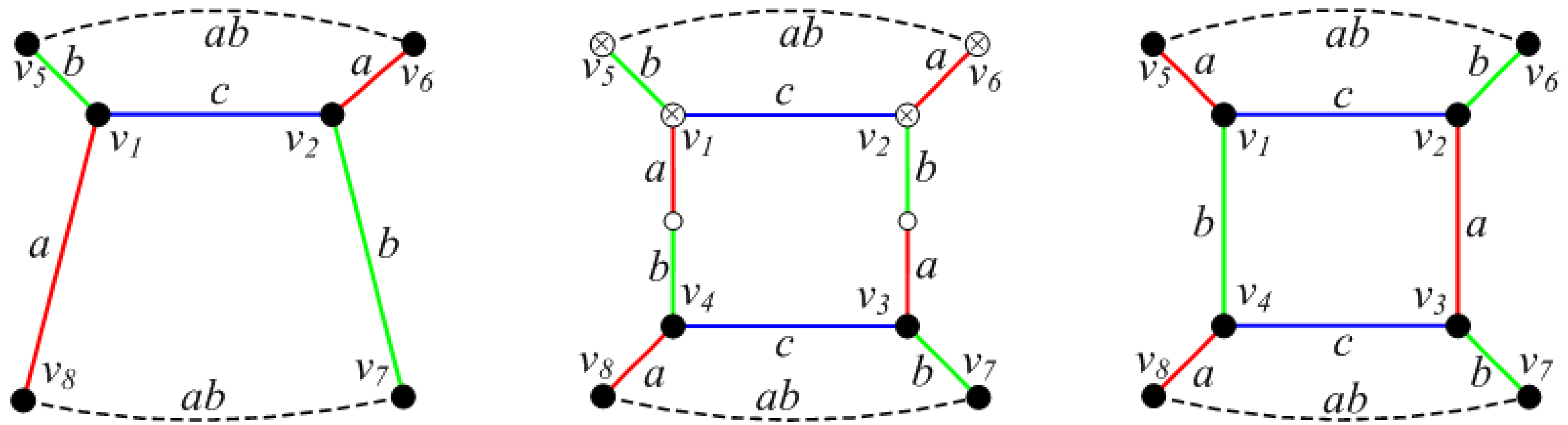


(e) A 3-edge-coloring of *G'*.

(f) Coloring of *G'* after negating one *(a, b)* cycle.

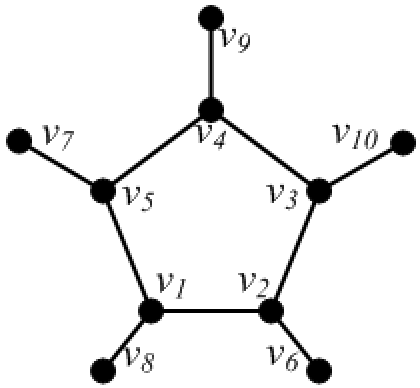
(g) A proper 3-edge-coloring of *G*.

The Girth of G Equals 4

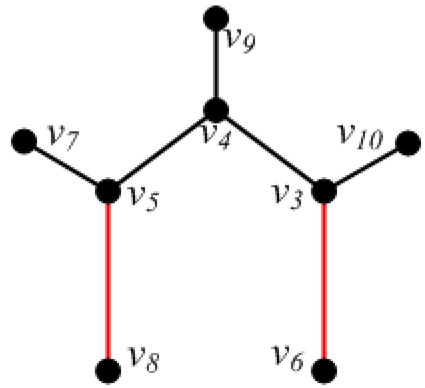


(h) A 3-edge-coloring of G'. (i) A 3-edge-coloring of G. (j) A proper 3-edge-coloring of G.

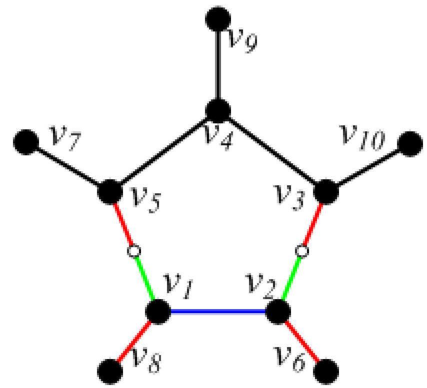
The Girth of *G* Equals 5



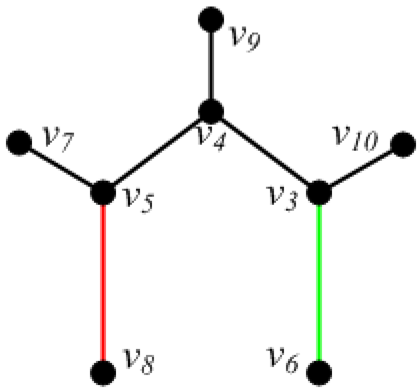
(a) A pentagon in *G*.



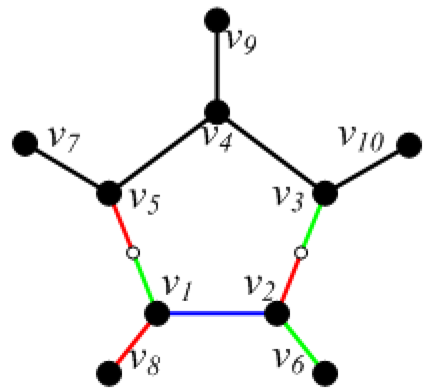
(b) A 3-edge-coloring of *G'*.



(c) A 3-edge-coloring of *G*.

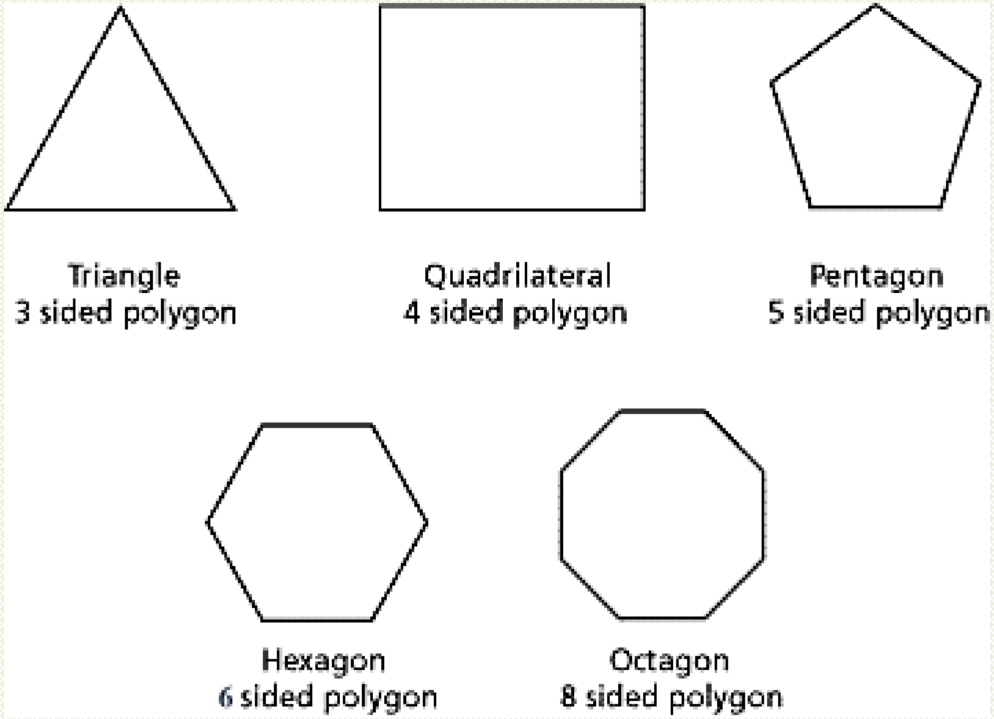


(d) A 3-edge-coloring of *G'*.



(e) A 3-edge-coloring of *G*.

Euler Formula and Solvability

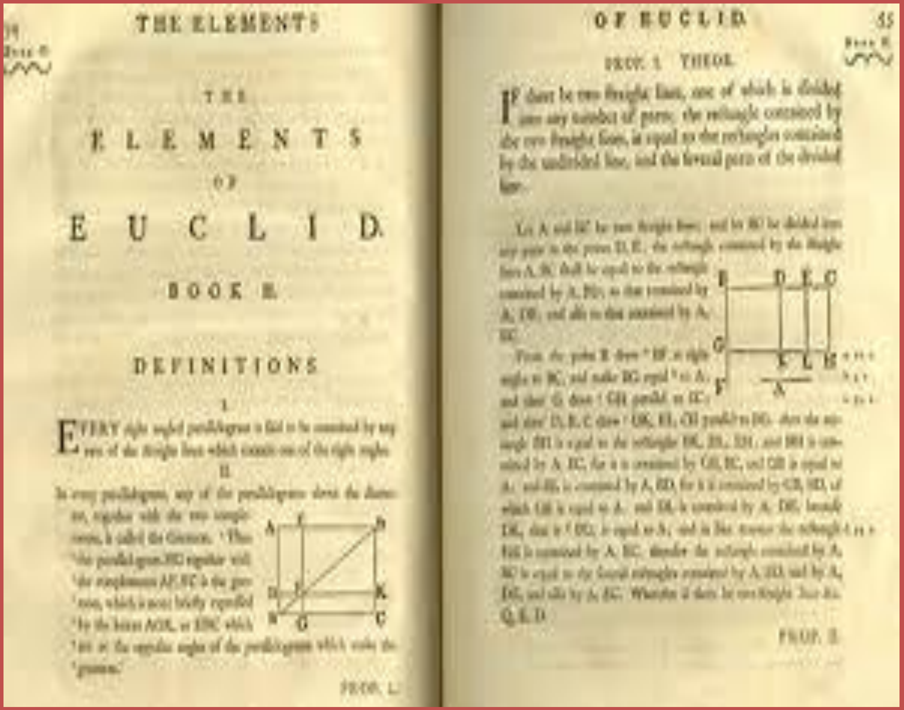


Outline

- Operations of Complex Colors
- Decomposition of Configurations
- Solvability of Configurations
- Generalized Petersen Configuration
- Three-Edge-Coloring Theorem
- Graph Theory versus Euclidean Geometry
- Conclusions

Euclid 'S *Elements*

Euclid
(325–265 BC)



THE ELEMENTS

THE
ELEMENTS
OF
EUCLID.
BOOK II.

DEFINITIONS

OF EUCLID.

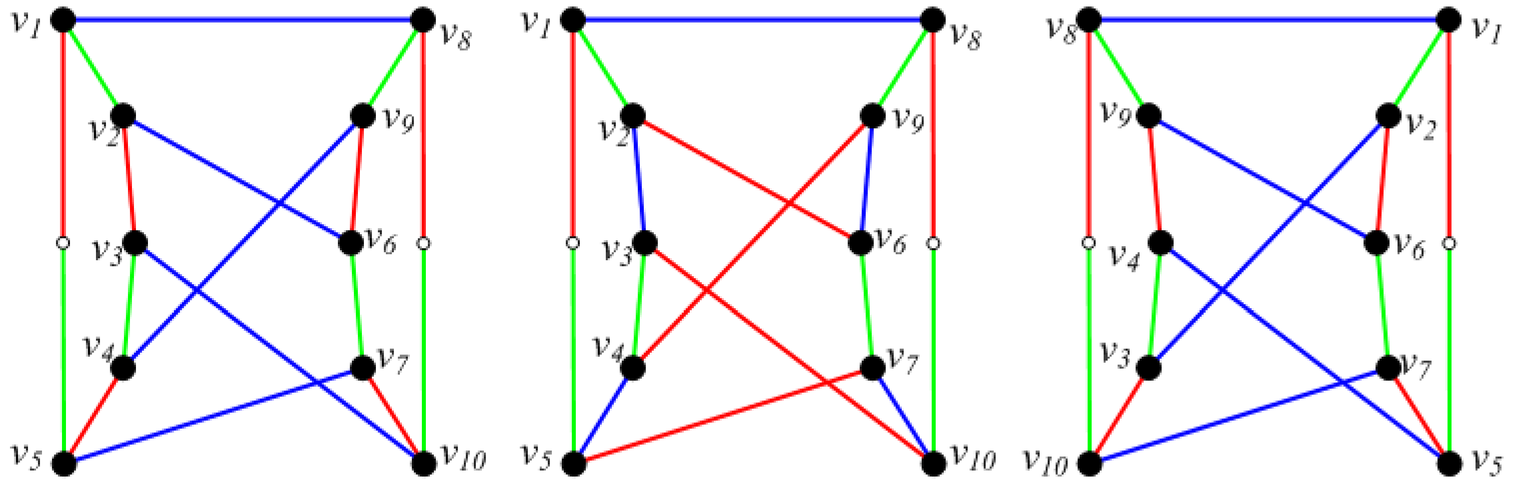
PROP. I. THEOR.

System of Linear Equations and Edge Coloring of Cubic Graphs

	System of linear equations	Edge coloring
Operations	Arithmetic Operations	Color Exchanges
Constraints	Linear Equations	Vertices and edges
Unknowns	Variables	Variable-colored edges
Algorithms	Variable Elimination	Variable Elimination
Solution	Consistency	3-colorable
No solution	Inconsistency	Snark

Isomorphic Configurations of Petersen Graph

- Negating (a, c) cycle $(v_2 - v_6 - v_9 - v_4 - v_5 - v_7 - v_{10} - v_3 - v_2)$



(a) A configuration $T(G_P)$ of Petersen graph.

(b) Negate *(a, c)* resolution cycle.

(c) Re-arrange to another configuration $T'(G_P)$.

Petersen Graph and Parallel lines

- The geometric interpretation of two inconsistent equations is two lines in parallel.

- In Petersen graph, the *two odd cycles* will never meet ; behave the same as *two parallel lines* in a Euclidean space.

Parallel Postulate of Euclidean Geometry

The parallel postulate of the *Euclid's Elements*:

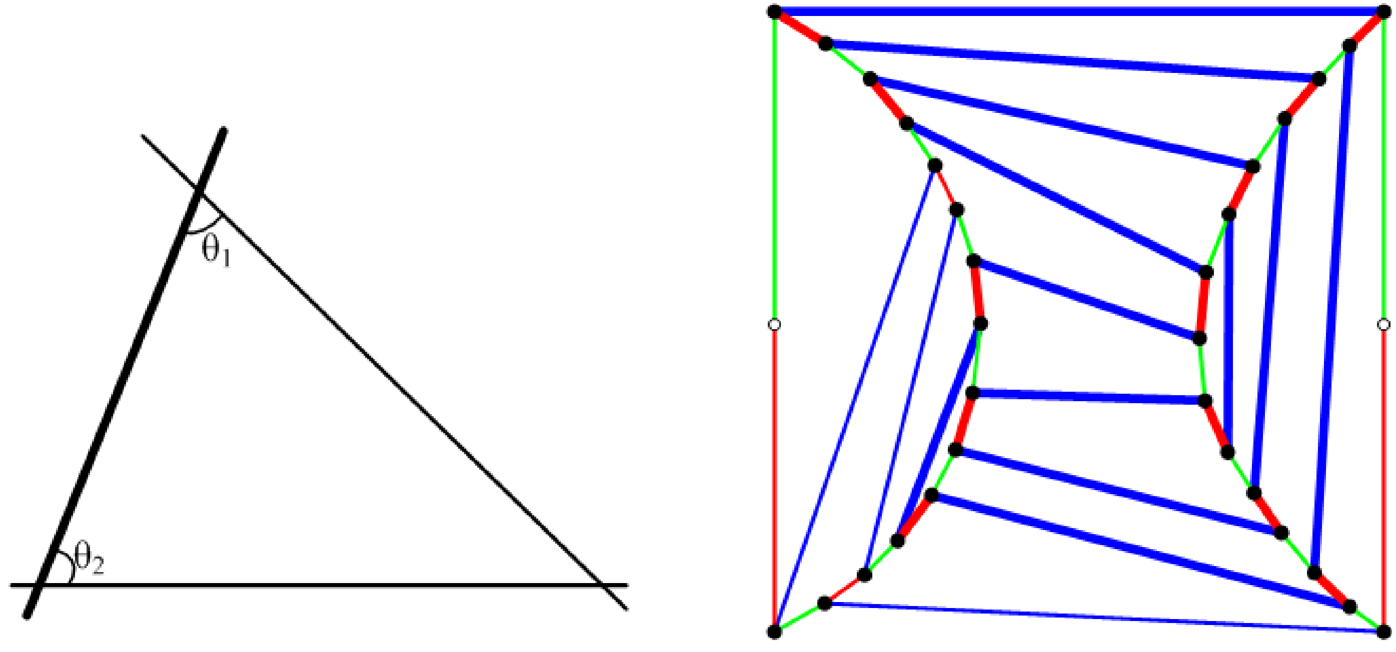
If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Solvability Conditions of the Plane

- The parallel postulate provides the solvability condition of two linear equations in the plane.

- The proposition of solvability claims that every generalized Petersen configuration $P(G)$ with two odd cycles is solvable in the plane.

Analogy Between Parallel Postulate and Proposition of Solvability



(a) Parallel postulate.

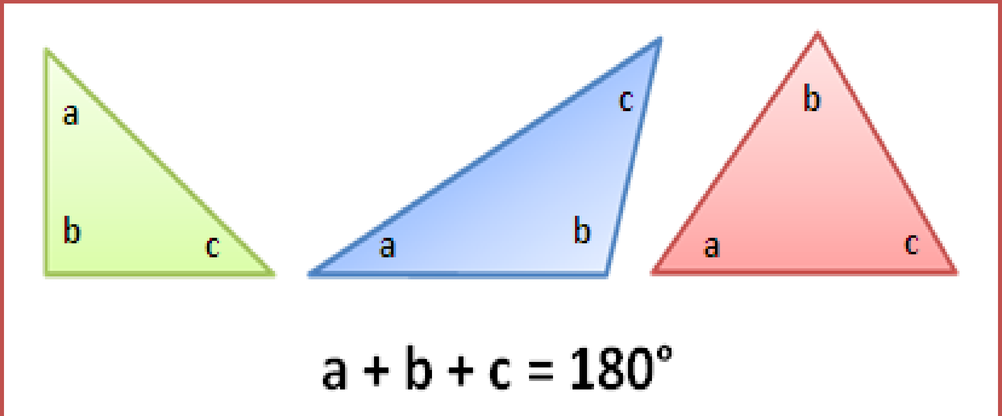
(b) Proposition of solvability.

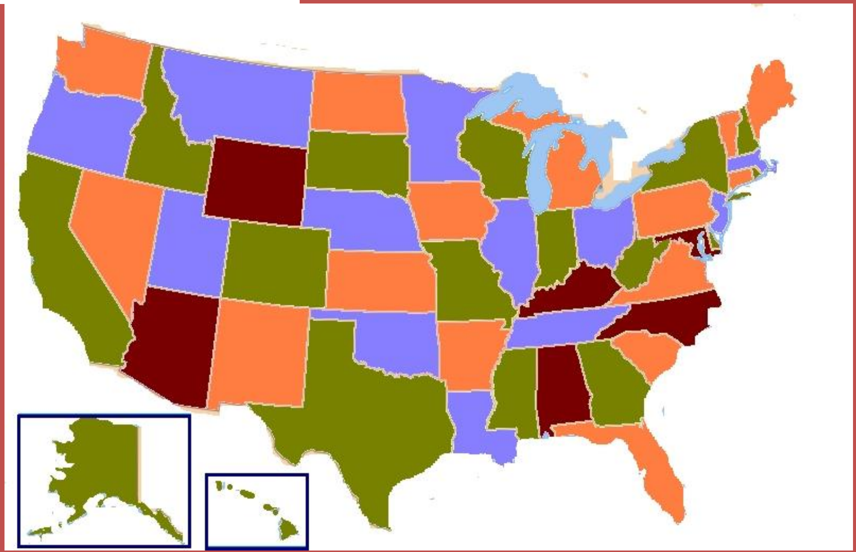
Invariants of the Plane

- **Geometric invariant**
  - The angel-sum of a triangle equals π.
  - A consequence of the parallel postulate.

- **Topological invariant**
  - The chromatic index of a bridgeless cubic plane graph equals 3.
  - A consequence of the proposition of solvability.

Invariants of the Plane





Outline

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Identification of Snarks

- Main difference between solving linear equations and edge coloring:
  - Inconsistency of a system of linear equations can be easily identified by variable eliminations in polynomial time.
  - Identifying a snark is a random walk process in the space of configurations.

Snarks and SAT

- The proposition of unsolvability is the first time that the necessary and sufficient condition of snarks can be completely specified.

- Holyer converted a Boolean expression ϕ into a cubic graph G, such that ϕ is satisfiable if and only if G is 3-edge colorable.

Snarks and SAT

- A truth assignment of the expression

$$\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_2 \vee x_3) \wedge (x_1 \vee \neg x_3)$$

is a solution of the Boolean equations:

$$x_1 \vee x_2 \vee x_3 = 1,$$
$$\neg x_2 \vee x_3 = 1,$$
$$x_1 \vee \neg x_3 = 1.$$

- What is the Cramer's rule of SAT?

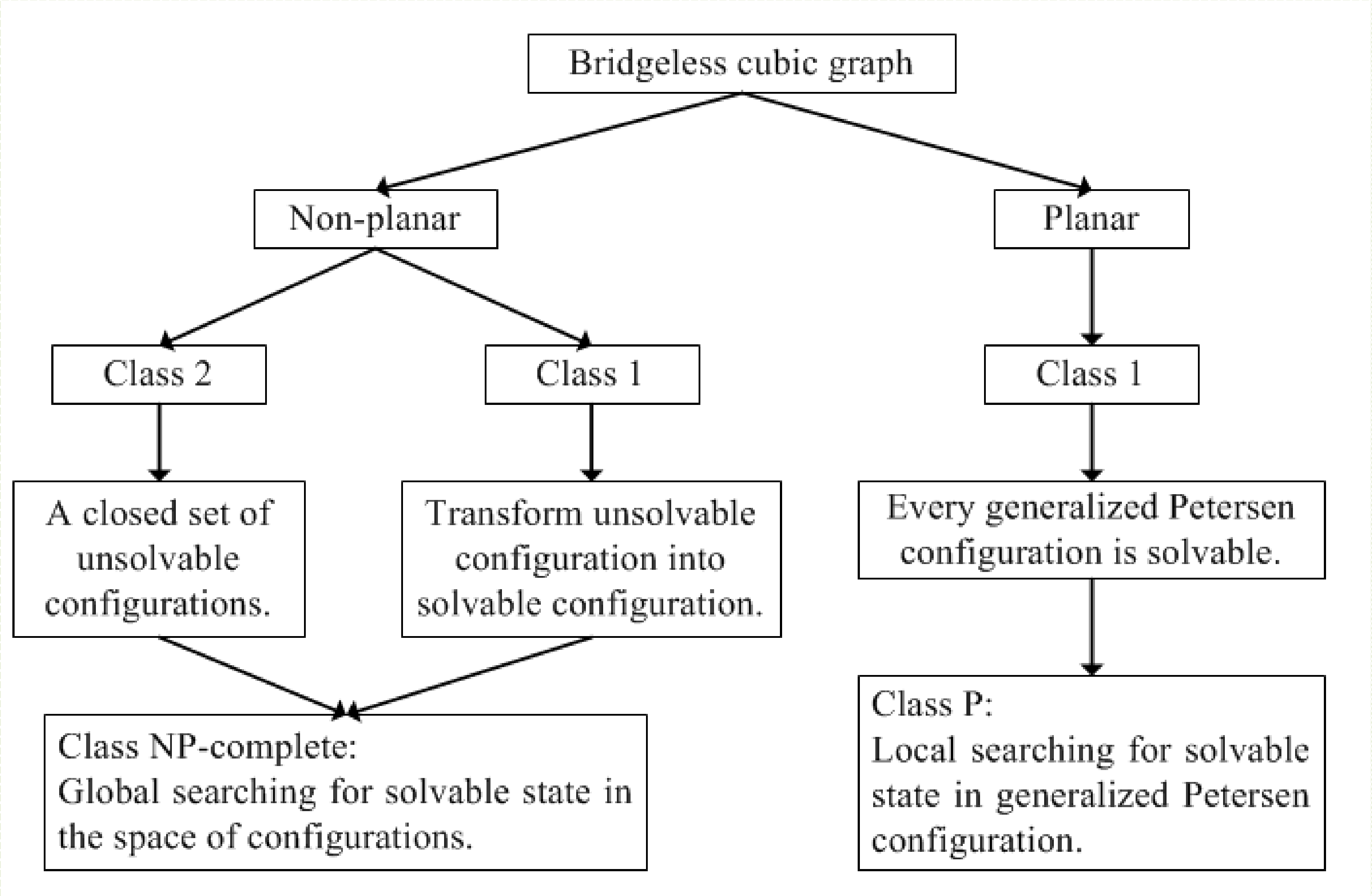
- The unsolvability condition of Snarks is also the condition for solving NP-complete problems, including SAT.

Validation of Proposition of Solvability

- If proposition of solvability is valid, then determining the chromatic index of a bridgeless cubic planar graph can be solved in polynomial time.

- Consistent with the quadratic algorithm for map coloring derived from the computer-assisted proof of the 4CT.

Classification of Complexities of Edge Coloring



Other NP-complete problem

Finding Hamiltonian cycles:

- A configuration of a simple graph *G* can be transformed into other configurations by negating maximal two-colored Tait cycles.
- A configuration is Hamiltonian if it contains a two-colored Hamiltonian cycle.
- If *G* is Hamiltonian, then a solution of a given graph *G* could be reached by random walks on the entire space of configurations.

- Title: Solvability of Cubic Graphs and the Four Color Theorem
- Authors: Tony T. Lee and Qingqi Shi
  Categories: cs.DS
- www.arxiv.org

thank you