Robust Stability of A Dynamic Traffic Assignment Model With Uncertainties

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Abstract—In this paper, a modified dynamic traffic assignment model is developed to explicitly formulate the impact of inaccuracy of cost measurement/estimation and the time-varying travel demand. The modified model is analyzed by using Lyapunov methods and robust stability results in nonlinear control theory. A robust convergence property of the model is derived, and interestingly, is closely related to Sontag’s input-to-state stability (ISS) property. Simulation results are employed to validate the main result.

I. INTRODUCTION

Severe traffic congestion has become a daily phenomenon in most metropolitan areas. Building new infrastructures is not a cost-effective, environment-friendly solution and will inevitably face the bottleneck of limited space. With this in mind, researchers and practitioners have been seeking innovative solutions based upon advances in information technology and control system theory, to make the existing transportation infrastructure more efficient, reliable and effective. Although more and more advanced software and devices become available to collect traffic information in real time, there are still significant problems in making use of the collected information to effectively improve the road transportation system. Challenges arise from the complexity of network structure, increasing travel demand, transportation delays, traffic constraints, and unpredictable accidents/ incidents. It is thus a crucial issue of practical importance that the urban transportation network is designed to be “smart” and “autonomous” to reroute traffic in the presence of unexpected network disruptions. Recently, this issue has been studied by several researchers by means of modern control engineering methods; see, for instance, [1]–[4].

The purpose of this paper is to propose a new dynamic traffic assignment model for travel cost minimization for complex traffic networks under uncertainties. The stability of traffic assignment models is an old yet very important problem for transportation networks. Related research by means of Lyapunov stability theory has appeared in [5]–[7] and references therein. While these research findings deal mainly with (internal) stability of a network equilibrium in the absence of external disturbance inputs, the issue of (external) stability of a network equilibrium in the presence of external disturbance inputs has not been studied systematically in the literature of transportation research. In this paper, we take the first step to apply robust analysis tools in modern nonlinear control in studying the robust convergence problem of road traffic networks.

Static traffic assignment models have been used to determine the network flow patterns that satisfy specified objectives such as user equilibrium (UE) objective (no driver can reduce travel time by unilaterally switching routes) and system optimal (SO) objective (total system travel time is minimized), for long-term network planning [8]. In the static models, the traffic environments are assumed to be time-invariant and the trip times and flows on network paths are constant. With an increasing desire for real-time intelligent control to avoid congestions, it has been a trend to study dynamic flow fluctuations in traffic networks, which lead to dynamic traffic assignment problem; see [9] for the basic concepts and an excellent literature review. Among various methods, day-to-day (or inter-periodic) traffic modeling methods are recognized to be very appropriate for analyzing the dynamics of traffic networks. By representing a traffic network with a directed graph, there have been two kinds of day-to-day traffic assignment modeling methods: path-based methods [6], [7], [10], [11] and, more recently, link-based methods [3], [4]. The link-based models were developed to deal with the possible path-overlapping problem arising in path-based models. In both models, differential or difference equations are employed to describe the traffic evolution. Inspired by recent developments of networked control systems within the control systems community, we will attempt to generalize the existing dynamic traffic assignment methods.

With the dynamics represented by differential or difference equations, the convergence properties of the dynamic traffic assignment models can be analyzed by means of Lyapunov functions. In [6], a dynamical system was used to model the route choice behavior and the stability problem for traffic network equilibrium was first addressed with a Lyapunov approach. More recent results can be found in [7], [10]–[12] and references therein. See also [13] for the studies based on stochastic dynamic traffic assignment models. Usually, Lyapunov functions can be defined in accordance with the traffic assignment objectives such as UE and SO. In the existing literature, the convergence to the equilibrium is usually guaranteed by proving the “asymptotic stability” of the traffic network in question, while various uncertainty...
factors such as changing travel demand and inaccuracy of travel cost estimation have not been well taken into consideration in convergence analysis. Sometimes, the time-varying travel demand can be modelled as a function of travel cost (travelling time) \([14, \text{Chapter 6}]\). For transportation systems equipped with advanced traveller information systems (ATIS), the uncertainty of cost estimation may depend on the market penetration of ATIS; see, e.g., [15]. Nonetheless, it is worth noting that stability properties may be lost in the presence of even small uncertainties and disturbance, because traffic networks are essentially nonlinear systems.

The objective of the paper is to find to what extent the previously developed route guidance methods can attenuate the influence of uncertainties. Specifically, we study the uncertainties caused by changing travel demand and inaccuracy of travel cost estimation. To this end, we first propose a modified dynamic traffic assignment model which is capable of representing the influence of changing travel demand and inaccuracy of travel cost estimation. Then, we consider the convergence problem as a robust stability problem. The robust convergence property of the modified model is derived by using the robust stability tools in modern nonlinear control. As a start of the study in this direction, the model considered in this paper is a modification of the model proposed in [6], and extensions for more general models will be carried out in our future research.

II. Preliminaries

By considering intersections as vertices and one-way links as directed edges, a road traffic network can be represented by a directed graph (digraph). We use \(G = (V, E)\) to denote the graph, where \(V = \{v_1, \ldots, v_n\}\) is the set of the vertices and \(E = \{e_1, \ldots, e_m\}\) is the set of directed edges. A route in the traffic network corresponds to a non-repeating sequence of vertices such that from each of its vertices there is a directed edge to the next vertex in the sequence. The starting vertex and the ending vertex of the route are the origin and the destination of the route, respectively. We use \(\text{OD} = \{od_1, \ldots, od_{|V|}\}\) to denote the set of all the origin-destination (OD) pairs and use \(\mathcal{P} = \{p_1, \ldots, p_{|V|}\}\) to denote the set of all the routes to be studied in the traffic network. For \(i = 1, \ldots, M\), we define \(\mu(i)\) as the set of indices of all the routes of OD pair \(i\). By default, we assume \(\mu(i) = \emptyset\) for \(i = 1, \ldots, M\). For \(j = 1, \ldots, N\), we define \(\phi(j)\) as the index of the OD pair of route \(p_j\). For related concepts of graph theory, see [16].

The pattern of traffic flow along the \(N\) routes is represented by \(x = [x_1, \ldots, x_N]^T \in \mathbb{R}^N_+\), with each \(x_i\) known as the route flow of \(p_i\). The cost of travelling along the route of \(p_i\) depends on the pattern of traffic flow \(x\), and the relation is represented by a function \(c_i: \mathbb{R}^N_+ \rightarrow \mathbb{R}_+\). Define \(c: \mathbb{R}^N_+ \rightarrow \mathbb{R}^N_+\) as \(c(x) = [c_1(x), \ldots, c_N(x)]^T\).

Denote \(M = \{1, \ldots, M\}\) and \(N = \{1, \ldots, N\}\). For any \(i, j \in N\), we use \(z_{ij}\) to represent the swapping rate from route \(p_i\) to route \(p_j\). For the routes \(p_i, p_j\) with different OD pairs, i.e., \(\phi(i) \neq \phi(j)\), we assume \(z_{ij} = 0\). Then, for each \(i \in N\), the dynamics of the path flow \(x_i\) can be represented by differential equation

\[
\dot{x}_i = \sum_{j \in N} z_{ij} - \sum_{j \in N} z_{ji}.
\] (1)

Notice that Equation (1) implies that, for each \(k \in M\),

\[
\sum_{i \in \mu(k)} \dot{x}_i = \sum_{i \in \mu(k)} \left( \sum_{j \in N} z_{ji} - \sum_{j \in N} z_{ij} \right)
= \sum_{i, j \in \mu(k)} z_{ji} - \sum_{i, j \in \mu(k)} z_{ij}
= 0.
\] (2)

This means that the summation of the traffic flows along all the routes of any specific OD pair is not changing, i.e., for each \(k \in M\), there exists a positive constant \(d_k\) such that

\[
\sum_{i \in \mu(k)} x_i = d_k.
\] (3)

Therefore, model (1) is valid only for systems with constant travel demand.

For convenience of notations, for \(i, j \in N\), we define \(\Delta_{ij} = [\delta_{ij}^1, \ldots, \delta_{ij}^N]^T\) where for \(k = 1, \ldots, N\),

\[
\delta_{ij}^k = \begin{cases} +1, & \text{if } i \neq j, \ \phi(i) = \phi(j) \text{ and } k = i; \\ -1, & \text{if } i \neq j, \ \phi(i) = \phi(j) \text{ and } k = j; \\ 0, & \text{otherwise}. \end{cases}
\] (4)

Then, we can represent the dynamics of the vector of the route flows as

\[
\dot{x} = -\sum_{i, j \in N} \Delta_{ij} z_{ij}.
\] (5)

In the case where accurate traffic information can be used for route guidance and the travel demand is time-invariant, according to [6], we may assume that the drivers swap from route \(p_i\) to route \(p_j\) according the following rule: if \(\phi(i) = \phi(j)\), then

\[
z_{ij} = x_i \varphi(c_i(x) - c_j(x)).
\] (6)

Here, \(\varphi\) is defined as \(\varphi(r) = \max\{0, r\}\) for \(r \in \mathbb{R}\).

With \(\Delta_{ij}\), \(z_{ij}\) can be rewritten in a more compact form:

\[
z_{ij} = x_i \varphi(c^T(x) \Delta_{ij}).
\] (7)

Thus, we have

\[
\dot{x} = -\sum_{i, j \in N} \Delta_{ij} x_i \varphi(c^T(x) \Delta_{ij}) := \Phi(x).
\] (8)

The equilibrium \(x^* = [x_1^*, \ldots, x_N^*]^T\) of the dynamic traffic assignment model (8) is mathematically defined by

\[
\Phi(x^*) = 0,
\] (9)

which, according to the definitions above, is equivalent to

\[
x_i^* \varphi(c_i(x^*) - c_j(x^*)) = 0,
\] (10)
and means
\[ x^*_i \neq 0 \Rightarrow c_i(x^*) \leq c_j(x^*) \quad (11) \]
for any pair of routes \( p_i, p_j \) satisfying \( \phi(i) = \phi(j) \). Note that the equilibrium defined by (10) is in accordance with the well known user equilibrium (UE) introduced by Wardrop [8]. For discussions on the existence of such equilibrium for system (8), refer to [5].

To make our paper self-contained, we give the definition of monotonicity for functions. According to [5], function \( c : \mathbb{R}^N_+ \rightarrow \mathbb{R}^N_+ \) is called monotone if and only if
\[ (c^T(x) - c^T(y))(x - y) \geq 0 \quad (12) \]
for all \( x, y \in \mathbb{R}^N_+ \). Also, a continuously differentiable function \( c \) is monotone if and only if its Jacobian matrix \( \nabla c \) is positive semidefinite. See [17, Theorem 5.4.3] and its proof therein.

If the cost function \( c \) is monotone and continuously differentiable, we define the following Lyapunov function candidate for system (8):
\[ V(x) = \sum_{i,j \in N} x_i \varphi^2(c^T(x)\Delta_{ij}). \quad (13) \]
It is easy to prove the positive definiteness of \( V \) (i.e., \( V(x^*) = 0 \) and \( V(x) > 0 \) if \( x \neq x^* \)). Also, \( V \) is continuously differentiable if \( c \) is continuously differentiable. Moreover, we can prove the asymptotic stability of \( x^* \) by proving that \( \nabla V(x)\Phi(x) < 0 \) for \( x \neq x^* \), see [5] for the details.

Notice that the definition of user equilibrium in (10) is equivalent to
\[ x_i^* \varphi^2(c_i(x^*) - c_j(x^*)) = 0. \quad (14) \]
For future reference, the asymptotic stability of \( x^* \) implies
\[ \lim_{t \to \infty} x_i(t) \varphi^2(c_i(x(t)) - c_j(x(t))) = 0. \quad (15) \]
When there are uncertainties, we have (21) instead of (15).

III. A Dynamic Traffic Assignment Model With Uncertainties

In practice, the travel demand may not be constant and the estimation/measurement of the travel costs may not be accurate. In this section, we study the convergence property of the dynamic traffic assignment model under such kinds of uncertainties.

In the case where the travel demand is time-varying, we modify equation (1) as
\[ \dot{x}_i = \sum_{j \in N} z_{ij} - \sum_{j \in N} z_{ji} + \omega_{1i} \quad (16) \]
where \( \omega_{1i} \) represents the influence of the changing travel demand. Recall that for each \( k \in \mathcal{M} \), \( \sum_{i \in \mu(k)} \dot{x}_i \) is the changing rate of the travel demand of OD pair \( \mu(k) \). Direct calculation yields: \( \sum_{i \in \mu(k)} \dot{x}_i = \sum_{i \in \mu(k)} \omega_{1i} \). In this case, it is reasonable to replace property (3) with
\[ \sum_{i \in \mu(k)} x_i \leq d_k \quad (17) \]
where \( \bar{d}_k \) is a positive constant representing an upperbound constraint on the (time-varying) travel demand of OD pair \( \mu(k) \). By the same analysis following (8), it is easily checked that (17) is equivalent to asking
\[ \sum_{i \in \mu(k)} \left( \int_0^\infty \omega_{1i}(t) dt \right) < \infty \quad (18) \]
for \( k = 1, \ldots, M \).

Because of the complexity of the traffic network, the travel costs may not be accurately measured/estimated. For \( i = 1, \ldots, N \), we use \( \hat{c}_i \) to denote the estimate of \( c_i \), and the \( z_{ij} \) defined in (7) should be modified as
\[ z_{ij} = x_i \hat{c}_i(x^* \Delta_{ij}) \quad (19) \]
where \( \hat{c}(x) = [\hat{c}_1(x), \ldots, \hat{c}_N(x)]^T \).

Under the influence of both changing travel demand and inaccuracy of travel cost estimation, the dynamics of the vector of traffic flows can be represented by
\[ \dot{x} = - \sum_{i,j \in N} \Delta_{ij} x_i \hat{c}_i(x^* \Delta_{ij}) + \omega_1 \]
\[ = - \sum_{i,j \in N} \Delta_{ij} x_i \hat{c}_i(x^* \Delta_{ij}) + \omega_1 + \omega_2 \]
\[ = \Phi(x) + \omega \quad (20) \]
where \( \omega_1 = [\omega_{11}, \ldots, \omega_{1N}]^T \) and \( \omega_2 = \sum_{i,j \in N} \Delta_{ij} x_i (\hat{c}_i(x^* \Delta_{ij}) - \varphi(\hat{c}_i(x^*) \Delta_{ij})) \). As an external input, \( \omega = \omega_1 + \omega_2 \) represents the influence of the uncertainties. When \( \omega = 0 \), system (20) is reduced to (8).

Under the influence of the uncertainties, the state \( x \) may not perfectly converge to the ideal UE at \( x^* \), and some practical convergence result would be more meaningful. In this paper, we propose a robust stability result of the dynamic traffic assignment model. Specifically, we will show that there exists a class-K function \( \chi \) such that for certain constant \( \overline{\omega} > 0 \), if \( |\omega| \leq \overline{\omega} \), then
\[ \lim_{t \to \infty} x_i(t) \varphi^2(c_i(x(t)) - c_j(x(t))) \leq \chi(\overline{\omega}). \quad (21) \]

Recall that \( \chi : \mathbb{R}_+ \to \mathbb{R}_+ \) is a class-K function if it is continuous, strictly increasing and vanishes at zero, see [18]. As a result, the smaller \( \overline{\omega} \) is the smaller upper bound in (21), and in particular, \( \chi(0) = 0 \) when \( \overline{\omega} = 0 \). Therefore, the practical convergence property given by (21) means that although the costs of different routes may not achieve an agreement, the traffic flows ultimately converge to specific ranges such that the routes with higher costs have smaller flows. Vice versa, routes with lower costs tend to allow for heavier traffic.

We analyze the practical convergence of modified model (20) through a Lyapunov-based robust analysis approach. We still use the \( V \) defined in (13) as the Lyapunov function for system (20). To analyze the convergence of \( V \) along the trajectories of system (20), we study \( \nabla V(\Phi(x)) + \omega \).

As in Section II, we still assume that \( c \) is monotone and continuously differentiable.
Direct calculation yields:
\[
\nabla V(x) = \sum_{i,j \in N} \left( 2x_i \varphi(c^T(x)\Delta_{ij}) \nabla \varphi(c^T(x)\Delta_{ij}) + (\nabla x_i) \varphi^2(c^T(x)\Delta_{ij}) \right) \\
= -2\varphi^T(x)J(x) + \sum_{i,j \in N} \varphi^2(c^T(x)\Delta_{ij}) \mathbf{1}_i^T \quad (22)
\]
where \( J(x) \) is the Jacobian matrix of \( c(x) \) and \( \mathbf{1}_i = [a_1, \ldots, a_N]^T \) with \( a_j = 0 \) for \( j \neq i \) and \( a_i = 1 \).

Due to the monotonicity of \( c \), it always holds that \( \varphi^T(x)J(x)\Phi(x) \geq 0. \) (23)

Then, we have
\[
\nabla V(x) \Phi(x) = -2\varphi^T(x)J(x)\Phi(x) + \sum_{i,j \in N} \varphi^2(c^T(x)\Delta_{ij}) \mathbf{1}_i^T \Phi(x) \\
\leq \sum_{i,j \in N} \varphi^2(c^T(x)\Delta_{ij}) \mathbf{1}_i^T \Phi(x) \\
= \sum_{i,j \in N} \varphi^2(c^T(x)\Delta_{ij}) \mathbf{1}_i^T \\
\times \left( -\sum_{i,j \in N} \Delta_{ij} x_i \varphi(c^T(x)\Delta_{ij}) \right) \\
= -\sum_{i,j \in N} \varphi^2(c^T(x)\Delta_{ij}) \mathbf{1}_i^T \Delta_{ij} x_i \varphi(c^T(x)\Delta_{ij}) \\
= -\sum_{j,i, j' \in N} \left( \sum_{i \in N} \varphi^2(c^T(x)\Delta_{ij}) \mathbf{1}_i^T \Delta_{ij'} \right) \\
\times x_i \varphi(c^T(x)\Delta_{ij'}) \\
= -\sum_{j,i, j' \in N} (\varphi^2(c^T(x)\Delta_{ij}) - \varphi^2(c^T(x)\Delta_{ij'})) \\
\times x_i \varphi(c^T(x)\Delta_{ij'}) \quad (24)
\]

From the definition in (4), it can be observed that if \( \phi(i) \neq \phi(j) \), then \( \Delta_{ij} = 0 \). Thus, if \( \phi(j) = \phi(i') = \phi(j') \) does not hold, then
\[
(\varphi^2(c^T(x)\Delta_{ij}) - \varphi^2(c^T(x)\Delta_{ij'})) x_i' \varphi(c^T(x)\Delta_{ij'}) = 0. \quad (25)
\]

Then, from (24), we have
\[
\nabla V(x) \Phi(x) \leq -\sum_{j,i, j' : \phi(i') = \phi(j')} \left( \varphi^2(c^T(x)\Delta_{ij}) - \varphi^2(c^T(x)\Delta_{ij'}) \right) \\
\times x_i' \varphi(c^T(x)\Delta_{ij'}) \quad (26)
\]

Also from the definition in (4), if \( \phi(j) = \phi(i') = \phi(j') \), then
\[
\Delta_{ij'} = \Delta_{ij} - \Delta_{ij'}, \quad (27)
\]
and thus
\[
\varphi^2(c^T(x)\Delta_{ij'}) = \varphi^2(c^T(x)\Delta_{ij}) - \varphi^2(c^T(x)\Delta_{ij'}). \quad (28)
\]

Hence, if \( \phi(j) = \phi(i') = \phi(j') \), then
\[
c^T(x)\Delta_{ij'} > 0 \Rightarrow c^T(x)\Delta_{ij} > c^T(x)\Delta_{ij'} \\
\Rightarrow \varphi^2(c^T(x)\Delta_{ij'}) > \varphi^2(c^T(x)\Delta_{ij}). \quad (29)
\]

by using the nonnegative and nondecreasing properties of \( \varphi \).

Property (29) means that every term on the right-hand side of (26) is nonpositive, and it also holds that
\[
\nabla V(x)\Phi(x) \leq -\sum_{j,i, j' : \phi(i') = \phi(j')} \left( \varphi^2(c^T(x)\Delta_{ij}) - \varphi^2(c^T(x)\Delta_{ij'}) \right) \\
\times x_i' \varphi(c^T(x)\Delta_{ij'}) \quad (30)
\]

With (17) satisfied, there exists a bounded set \( \Omega \subset \mathbb{R}_+^N \) such that \( x \in \Omega \). Because of the continuity of \( V \) with respect to \( x \), there exists a positive constant \( \bar{V} \) such that \( V(x) \leq \bar{V} \) for all \( x \in \Omega \). For \( 0 \leq s \leq \bar{V} \), define
\[
\alpha(s) = \min_{z : V(z) \geq s} \{ W(z) \}. \quad (31)
\]

Clearly, \( \alpha(0) = 0 \). For each \( 0 \leq s \leq \bar{V} \), if \( V(z) \geq s \), then at least one term of \( V(z) \), say \( z_i' \varphi^2(c^T(x)\Delta_{ij'}) \), is positive, and thus its corresponding term \( z_i' \varphi^3(c^T(x)\Delta_{ij'}) \) is positive, too. This guarantees the positive definiteness of \( \alpha \).

Also, because \( \alpha(s) \leq \min_{z : V(z) \geq s'} \{ W(z) \} = \alpha(s') \) holds for any \( s < s' \), \( \alpha \) is nondecreasing. More importantly, we have
\[
\alpha(V(x)) = \min_{z : V(z) \geq V(x)} \{ W(z) \} \leq W(x). \quad (32)
\]

Thus,
\[
\nabla V(x)\Phi(x) \leq -\alpha(V(x)). \quad (33)
\]

If \( \omega = 0 \), then it can be concluded that \( V \) converges to the origin, which guarantees the convergence of \( x \) to \( x^* \). See [18] for the Lyapunov stability results.

We now study the evolution of \( x \) under the constraint of (17) and the influence of \( \omega \). Due to the continuity of \( \nabla V \) with respect to \( x \), there exists a positive constant \( K \) such that
\[
|\nabla V(x)| \leq K \quad (34)
\]

for all \( x \in \Omega \).

For a positive definite, nondecreasing function \( \alpha \), we define \( \hat{\alpha} \) such that for \( s \in [0, \bar{V}] \), \( \hat{\alpha}(s) \) is the solution of the initial value problem
\[
\frac{d\hat{\alpha}}{ds}(s) = \epsilon(\alpha(s) - \hat{\alpha}(s)) \quad (35)
\]

with \( \hat{\alpha}(0) = 0 \), where \( \epsilon \) is a positive definite and strictly increasing function. Then, \( \hat{\alpha} \) is positive definite and strictly increasing, and \( \hat{\alpha}(s) \leq \alpha(s) \) for \( s \in [0, \bar{V}] \).
Then, we have
\[ \nabla V(x)(\Phi(x) + \omega) \leq \nabla V(x)\Phi(x) + |\nabla V(x)||\omega| \leq -\hat{\alpha}(V(x)) + K|\omega|. \] (36)

Define \( \bar{\omega} = \frac{1}{K}\hat{\alpha}(\bar{V}) \) and \( \chi(s) = \frac{1}{\hat{K}S}\chi(s) \) for \( s \in [0, \bar{\omega}] \). If \( |\omega| \leq \bar{\omega} \), then by using the asymptotic gain property of input-to-state stable (ISS) systems (see [19]), we can prove
\[ \lim_{t \to \infty} V(x(t)) \leq \lim_{t \to \infty} \chi(|\omega(t)|), \] (37)
and thus,
\[ \lim_{t \to \infty} x_i(t)\varphi^2(c_i(x(t)) - c_j(x(t))) \leq \lim_{t \to \infty} \chi(|\omega(t)|) \] (38)
for each \( i \in \mathcal{N} \).

The main result of the paper is given by Theorem 1.

**Theorem 1:** Consider the dynamic traffic assignment model (16) and (19). Under condition (18), if \( c \) is continuously differentiable and monotone, then there exist \( \chi \in \mathcal{K} \) and \( \bar{\omega} > 0 \) such that if \( |\omega| \leq \bar{\omega} \), then property (38) holds.

**Remark 1:** The dynamic traffic assignment model can only be guaranteed to be locally ISS because \( V \) is always bounded by \( \bar{V} \), and this is the reason why \( |\omega| \leq \bar{\omega} \) is required for practical convergence. In practice, the travel demand may change slowly within a large range, with which, the traffic flows may still practically converge to the desirable equilibrium. Current model cannot cover such case.

**IV. Simulation Results**

The traffic network studied in the simulation is shown in Fig. 1.

![Fig. 1. The traffic network considered in the simulation.](image)

There are two OD pairs in the traffic network with \( od_1 = (v_1, v_{12}) \) and \( od_2 = (v_3, v_{10}) \). From Fig. 1, we can see \( od_1 \) has 4 routes and \( od_2 \) has 4 routes. We make the following definitions:

- \( p_1 = (v_1, v_4, v_5, v_6, v_9, v_{12}) \)
- \( p_2 = (v_1, v_4, v_5, v_6, v_9, v_8, v_7, v_{10}, v_{11}, v_{12}) \)
- \( p_3 = (v_1, v_4, v_7, v_{10}, v_{11}, v_{12}) \)
- \( p_4 = (v_1, v_4, v_7, v_{10}, v_{11}, v_8, v_5, v_6, v_9, v_{12}) \)
- \( p_5 = (v_3, v_2, v_1, v_4, v_7, v_{10}) \)
- \( p_6 = (v_3, v_6, v_9, v_8, v_7, v_{10}) \)
- \( p_7 = (v_3, v_6, v_9, v_8, v_5, v_2, v_1, v_4, v_7, v_{10}) \)
- \( p_8 = (v_3, v_2, v_1, v_4, v_5, v_6, v_9, v_8, v_7, v_{10}) \).

Then, \( \mu(1) = \{1, 2, 3, 4\} \) and \( \mu(2) = \{5, 6, 7, 8\} \). Correspondingly, there are 17 links belonging to the routes. Denote \( e_1 = (v_2, v_1) \), \( e_2 = (v_3, v_2) \), \( e_3 = (v_1, v_4) \), \( e_4 = (v_5, v_2) \), \( e_5 = (v_3, v_6) \), \( e_6 = (v_4, v_5) \), \( e_7 = (v_5, v_6) \), \( e_8 = (v_4, v_7) \), \( e_9 = (v_6, v_9) \), \( e_{10} = (v_6, v_9) \), \( e_{11} = (v_8, v_7) \), \( e_{12} = (v_9, v_8) \), \( e_{13} = (v_7, v_{10}) \), \( e_{14} = (v_{11}, v_8) \), \( e_{15} = (v_9, v_{12}) \), \( e_{16} = (v_{10}, v_{11}) \) and \( e_{17} = (v_{11}, v_{12}) \).

We use a 17-by-8 matrix \( T = [T_{ij}] \) to represent the relation between the routes and the links. For \( i = 1, \ldots, 17 \) and \( j = 1, \ldots, 8 \), if link \( e_i \) belongs to route \( p_j \), then \( T_{ij} = 1 \); otherwise, \( T_{ij} = 0 \).

We assume that the travel cost along a link is determined by the traffic flow of the link, and the travel cost along a route is the summation of the travel costs of the links of the route. In this simulation, for each link, the travel cost is defined as \( c_L(y) = 0.01 + 0.0008y \), where \( y \) denotes the traffic flow along the link. Then,
\[ c(x) = T^T(0.01q + 0.0008Tx) \] (39)
where \( q \) equals the 17-by-1 matrix of ones. Then, its Jacobian matrix
\[ J(x) = 0.0008TT^T \] (40)
which is positive semi-definite. In fact, it can be proved that \( c \) is monotone if \( e_L \) is monotone.

The travel demands of the two OD pairs are shown in Fig. 2.

![Fig. 2. The travel demands of the OD pairs.](image)

In the simulation, we assume there are disturbances affecting the estimation of the costs along the links. For each link \( e_i \) \( (i = 1, \ldots, 17) \), the cost actually used for traffic assignment is \( e_L(y) + w_i \) with \( w_i = 0.001 \sin(t/2 + i\pi/5) \). Accordingly, the cost of each route accurately used for traffic assignment is shown in Fig. 3.

The traffic flows along the routes during the dynamic traffic assignment procedure are shown in Figs. 4 and 5. From Figs. 4 and 5, it can be seen that the traffic flows of the routes with higher costs are maintained at lower levels though they do not converge to zero. This is in accordance with our theoretical result. It should be noted that if the uncertainties caused by time-varying travel demand and inaccurate measurement of travel cost do not exist, then the traffic flows and the corresponding travel costs of the routes ultimately converge to constants.

**V. Conclusions**

In this paper, we have proposed a dynamic traffic assignment model which takes into consideration uncertainties...
caused by changing travel demand and inaccuracy of travel cost measurement/estimation. A practical convergence result has been derived for the new model by means of Lyapunov-based robust stability analysis. Specifically, if the influence of the uncertainties is not large, then the traffic flows ultimately converge to ranges such that the routes with higher costs have lighter flows. Also, routes with lower costs may have heavy flows. By considering the product of each traffic flow and its corresponding cost difference as a state, then the dynamic traffic assignment model is—to some extent—locally ISS in the sense of Sontag [19].

In the model studied in this paper, we have not considered time-delays, which may destroy the practical convergence of the model. To what extent the model can attenuate the influence of time-delays will be studied in our future work. This paper provides an analysis result, and the application of the result to robust traffic assignment will also be studied in the future.

In this paper, we used cost-flow functions $c_i$ to directly represent the relationship between traffic flows and costs. However, in the well-known Greenshield’s model, see, e.g., [20], the travel costs are not singly determined by the traffic flows. In our future work, we will consider employing Greenshields’ model to refine our model and study robust convergence properties in a more general framework of input-to-state practical stability or input-to-output practical stability [21].

REFERENCES


