Component Factors with Large Components in Graphs

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Abstract

In this paper we obtain sufficient conditions using isolated vertices for component factors with each component of order at least three. In particular, we show that if a graph $G$ satisfies $iso(G - S) \leq |S|/2$ for all $S \subset V(G)$, then $G$ has a $\{K_1, 2, K_1, 3, K_5\}$-factor, where $iso(G - S)$ denotes the number of isolated vertices in $G - S$.

1 Introduction

In this paper we consider component factors of graphs, which are defined as follows. For a set $S$ of connected graphs, a spanning subgraph $F$ of a graph $G$ is called an $S$-factor of $G$ if every component of $F$ is an element of $S$. An $S$-factor is also referred as a component factor. There have been many papers on component factors of graphs, but in most cases, $S$ contains $K_2$ (i.e., a single edge), but it is relatively rare that $S$ contains no small component. In addition, it is known that if $S$ does not contain $K_2$, then in most cases finding a criterion for a graph to have an $S$-factor is very difficult since finding a maximum $S$-subgraph of a given graph is an $NP$-complete problem. In this paper we obtain several sufficient conditions in terms of the number of isolated vertices for a graph to have a component factor such that each component has order at least three.

We begin with some notation and definitions. We consider a finite simple graph $G$ with vertex set $V(G)$ and edge set $E(G)$, which has neither loops nor multiple edges. We denote by $|G|$ the order of $G$. For a subset $S \subseteq V(G)$, $G - S$ denotes the subgraph of $G$ induced by $V(G) - S$. For a vertex $v$ of $G$, the degree of $v$ and the neighborhood of $v$ in $G$ are denoted by $d_G(v)$ and $N_G(v)$, respectively. In particular, $d_G(v) = |N_G(v)|$. The minimum

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degree and the maximum degree of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Denote by $\alpha(G)$ the independence number of $G$, which is the maximum cardinality among the independent sets of vertices of $G$. Let $iso(G)$ and $Iso(G)$ denote the number of isolated vertices and the set of isolated vertices of $G$, respectively. In particular, $iso(G) = |Iso(G)|$.

For sets $X$ and $Y$, $X \subset Y$ means that $X$ is a proper subset of $Y$.

We denote the complete graph, the path and the cycle of order $n$ by $K_n$, $P_n$ and $C_n$, respectively. We denote the complete bipartite graph by $K_{n,m}$. A criterion for a graph to have a star-factor is given below.

**Theorem 1.** (Amahashi and Kano [1]) A graph $G$ has a star-factor, i.e., $\{K_{1,1}, \ldots , K_{1,n}\}$-factor, if and only if $iso(G - S) \leq n|S|$ for all $S \subset V(G)$.

A graph $R$ is called factor-critical if for every vertex $x$ of $R$, $R - x$ has a 1-factor ($K_2$-factor). A graph $H$ is called a sun if $H = K_1$, $H = K_2$ or $H$ is the corona of a factor-critical graph $R$ with order at least three, i.e., $H$ is obtained from $R$ by adding a new vertex $w = w(v)$ together with a new edge $vw$ for every vertex $v$ of $R$ (Figure 1). A sun with order at least 6 is called a big sun. The number of sum components of $G$ is denoted by $sun(G)$. The next theorem gives a criterion for a graph to have a path-factor each of whose components is of order at least three. Note that a shorter proof of the following theorem and a formula for a maximum $\{P_3, P_4, P_5\}$-subgraph of a graph was given in [3].

![Figure 1: A factor-critical graph $R$ and the sun $H$ obtained from $R$.](image)

**Theorem 2.** (Kaneko [2]) A graph $G$ has a $\{P_3, P_4, P_5\}$-factor (i.e., $P_{\geq 3}$-factor) if and only if $sun(G - S) \leq 2|S|$ for all $S \subset V(G)$.

In this paper we consider the following problem, and give partial answers to the problem.

**Problem 1.** Let $G$ be a graph and $\lambda$ be a positive rational number. If $iso(G - S) \leq \lambda|S|$ for all $\emptyset \neq S \subset V(G)$, what factor does $G$ have?

## 2 Component Factors with Large Components

In this section, we first prove the next theorem.

**Theorem 3.** If a graph $G$ satisfies

$$iso(G - S) \leq \frac{2}{3}|S|$$

for all $S \subset V(G)$,
then $G$ has a $\{P_3, P_4, P_3\}$-factor.

**Proof.** Suppose that $G$ satisfies the condition but has no $\{P_3, P_4, P_3\}$-factor. By Theorem 2, there exists a subset $S \subset V(G)$ such that $\text{sun}(G - S) > 2|S|$. Assume that there exist $a$ isolated vertices, $b$ $K_2$'s and $c$ big sun components $H_1, H_2, \ldots, H_c$, where $|H_i| \geq 6$, in $G - S$. We choose one vertex from each $K_2$ component of $G - S$, and denote the set of such vertices by $X$. Then $|X| = b$. For each $H_i$, let $R_i$ denote the factor-critical subgraph of $H_i$ and let $Y_i = V(R_i)$. Then $\text{iso}(H_i - Y_i) = |Y_i| = |H_i|/2$. Let $Y = \bigcup_{i=1}^c Y_i$. So we have

$$
\text{iso}(G - (S \cup X \cup Y)) = a + b + \sum_{i=1}^c \frac{|H_i|}{2}.
$$

Moreover, it follows that

$$
|S \cup X \cup Y| < \frac{\text{sun}(G - S)}{2} + |X| + |Y| \quad \text{(from } \text{sun}(G - S) > 2|S|\text{)}
$$

$$
= \frac{a + b + c}{2} + b + \sum_{i=1}^c \frac{|H_i|}{2}
$$

$$
\leq \frac{3}{2} \left( a + b + \sum_{i=1}^c \frac{|H_i|}{2} \right) = \frac{3}{2} \text{iso}(G - (S \cup X \cup Y)).
$$

This contradicts the condition that $\text{iso}(G - S') \leq (2/3)|S'|$ for all $S' \subset V(G)$. ■

Let $m \geq 1$ be an integer. Let $G = K_m + (2m + 1)K_2$, which is a graph obtained from $K_m$ and $(2m + 1)K_2$ by joining every vertex of $K_m$ to every vertex of $(2m + 1)K_2$. Then $G$ has no $\{P_3, P_4, P_3\}$-factor. Let $T \subseteq V(G)$ be an independent set with $|T| \geq 2$. Then $T \subseteq V((2m + 1)K_2)$ and so $|N_G(T)| = |T| + m$. If $|T| \leq 2m$, then $i(G - N_G(T)) \leq 2|N_G(T)|/3$, otherwise $i(G - N_G(T)) = 2|N_G(T)|/3 + 1 = 2m + 1$. Since $\delta(G) \geq m + 1 \geq 2$, so $i(G - S) \leq 2|S|/3 + 1$ for all $S \subseteq V(G)$. Therefore the condition of Theorem 3 is sharp.

The next lemma is known as Harlem Theorem, which is a generalization of Hall’s Theorem.

**Lemma 1.** Let $G$ be a bipartite graph with bipartition $(U, W)$, and $f : U \to \{1, 2, 3, \ldots\}$. If $|W| = \sum_{x \in U} f(x)$ and

$$
|N_G(S)| \geq \sum_{x \in S} f(x) \quad \text{for all } \emptyset \neq S \subseteq U,
$$

then $G$ has a star-factor $F$ such that each vertex $u$ of $U$ satisfies $d_F(u) = f(u)$, that is, every $u$ is the center of a star $K_{1,f(u)}$ in $F$.

We next consider graphs satisfying $\text{iso}(G - S) \leq |S|/2$ for all $S \subset V(G)$.

**Lemma 2.** If $|G| \leq 6$ and $\text{iso}(G - S) \leq |S|/2$ for all $S \subset V(G)$, then $G$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.
**Proof.** It is clear that if $G$ satisfies the condition, then $\delta(G) \geq 2$ and $|G| \geq 3$. If $|G| = 3$, then $G$ is connected and has a $K_{1,2}$-factor. If $|G| = 4$, then $\Delta(G) = 3$, which implies that $G$ has a $K_{1,3}$-factor. Assume $|G| = 5$. If $G$ has two non-adjacent vertices $x$ and $y$, then $2 = |\{x, y\}| = iso(G - \{x, y\}) \leq |V(G) - \{x, y\}|/2 = 3/2$, a contradiction. Hence $G$ is a complete graph $K_5$, and so it has a $K_5$-factor. Now we consider the case of $|G| = 6$. By Theorem 2, $G$ has a $\{P_3, P_4, P_5\}$-factor, say $F$. Then $F$ must be a $P_3$-factor, which is a $K_{1,2}$-factor. Therefore the lemma holds. ■

**Theorem 4.** If a graph $G$ satisfies

$$\text{iso}(G - S) \leq \frac{|S|}{2} \quad \text{for all} \quad S \subseteq V(G),$$

then $G$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

**Proof.** It is clear that $|G| \geq 3$ and $\delta(G) \geq 2$. Use induction on the lexicographic order of $\{|G|, |E(G)|\}$. So we assume that the theorem holds for a graph $H$ with either $|H| < |G|$ or $|H| = |G|$ and $|E(H)| < |E(G)|$. Moreover, we may assume that $G$ is connected and $|G| \geq 7$ by Lemma 2. Let

$$\beta = \min \left\{ \frac{|S|}{2} - \text{iso}(G - S) \mid S \subset V(G) \text{ and } \text{iso}(G - S) \geq 1 \right\}.$$

Then $\beta \geq 0$ as $\text{iso}(G - S) \leq |S|/2$. For a vertex $x$ with $d_G(x) = \delta(G)$, we have $\beta \leq |N_G(x)|/2 - \text{iso}(G - N_G(x))$ and so

$$\delta(G) = d_G(x) = |N_G(x)| \geq 2(\beta + \text{iso}(G - N_G(x))) \geq 2(\beta + 1). \quad (1)$$

Take a maximal vertex subset $S$ such that $|S|/2 - \text{iso}(G - S) = \beta$. Then

$$\frac{|S'|}{2} - \text{iso}(G - S') > \beta \quad \text{for all} \quad S \subset S' \subset V(G). \quad (2)$$

**Claim 1.** $G - S$ has no component of order two or three.

Assume that $G - S$ has a component $D$ isomorphic to $K_2$. Let $V(D) = \{x, y\}$. Then

$$\frac{|S \cup \{x\}|}{2} - \text{iso}(G - (S \cup \{x\})) = \frac{|S| + 1}{2} - \text{iso}(G - S) + 1 < \beta,$$

a contradiction.

Assume that $G - S$ has a component $D$ of order three. Let $V(D) = \{x, y, z\}$. Then

$$\frac{|S \cup \{x, y\}|}{2} - \text{iso}(G - (S \cup \{x, y\})) = \frac{|S| + 2}{2} - \text{iso}(G - S) + 1 = \beta,$$

...
a contradiction to the maximality of $S$.

Claim 2. Every component $D$ of $G - S$ with $|D| \geq 4$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

Let $X$ be a non-empty subset of $V(D)$. Then by (2), we have
\[
\frac{|S \cup X|}{2} - \text{iso}(G - (S \cup X)) > \beta = \frac{|S|}{2} - \text{iso}(G - S).
\]
Thus $|X|/2 > \text{iso}(D - X)$, which implies that $D$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor by the induction hypothesis.

By Claim 1, let $G - S = aK_1 \cup (D_1 \cup \cdots \cup D_e)$, where $V(aK_1) = \{u_1, \ldots, u_a\}$ and each $D_i$ is a component of $G - S$ with $|D_i| \geq 4$. It is immediate that
\[
a = \text{iso}(G - S) = |S|/2 - \beta \geq 1. \quad (3)
\]

We construct a bipartite graph $B$ with vertex set $V(B) = S \cup U$, where $U = \{u_1, u_2, \ldots, u_a\}$, such that two vertices $u_i \in U$ and $x \in S$ are adjacent in $B$ if and only if $u_i$ and $x$ are joined by an edge of $G$.

Claim 3. For every $\emptyset \neq Y \subset U$, we have $|N_B(Y)| \geq 2|Y| + 2\beta$, and $|N_B(U)| = 2|U| + 2\beta = |S|$.

It follows from (3) and the choice of $S$ that $|N_B(U)| = |S| = 2a + 2\beta = 2|U| + 2\beta$. Assume that there exists a subset $\emptyset \neq Y' \subset U$ such that $N_B(Y') < 2|Y'| + 2\beta$. Then, by the definition of $\beta$, $N_B(Y') = N_G(Y') \subset S$ satisfies
\[
|Y'| \leq \text{iso}(G - N_G(Y')) \leq \frac{|N_G(Y')|}{2} - \beta < |Y'|,
\]
a contradiction. Hence the claim holds.

Claim 4. If $\beta \geq 2$, then the theorem holds.

Assume $\beta \geq 2$. Then $\delta(G) \geq 6$ by (1). It is obvious that $G$ has an edge $e$ such that $G - e$ is connected. Let $X \subset V(G - e) = V(G)$. If $\text{iso}(G - X) \geq 1$, then
\[
\text{iso}(G - e - X) \leq \text{iso}(G - X) + 2 \leq \frac{|X|}{2} - \beta + 2 \leq \frac{|X|}{2}.
\]
If $\text{iso}(G - X) = 0$, then $\text{iso}(G - e - X) \leq 2$. Further $\text{iso}(G - e - X) \geq 1$ implies $|X| \geq 5$ as $\delta(G - e) \geq 5$. Hence if $\text{iso}(G - X) = 0$, then $\text{iso}(G - e - X) \leq 2 \leq |X|/2$. Therefore by the induction hypothesis, $G - e$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, which is of course the desired factor of $G$.

From Claim 4 and the definition of $\beta$, it remains to consider the cases of $\beta \in \{0, 1/2, 1, 3/2\}$. Note that $|S| = 2|U| + 2\beta$.

Case 1. $\beta = 0$.

Define $f : U \to \{1, 2, 3, \ldots\}$ by $f(u) = 2$ for all $u \in U$. Then by Lemma 1 and Claim 3, $B$ has a $K_{1,2}$-factor with centers in $U$. Hence by Claim 2, $G$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.
Case 2. \( \beta = 1/2 \).

In this case, \( |S| = 2|U| + 1 \). Choose a vertex \( u_1 \in U \) and define \( f : U \to \{1, 2, 3, \ldots \} \) by \( f(u_1) = 3 \) and \( f(u_i) = 2 \) for all \( u_i \in U \setminus \{u_1\} \). Then \( |N_B(Y)| \geq \sum_{x \in Y} f(x) \) for all \( Y \subseteq U \) by Claim 3. Hence by Lemma 1, \( B \) has a \( \{K_{1,2}, K_{1,3}\} \)-factor. Therefore we can obtain a \( \{K_{1,2}, K_{1,3}, K_5\} \)-factor of \( G \).

Case 3. \( \beta = 1 \).

Clearly, \( \delta(G) \geq 4 \) by (1). We consider two subcases.

Subcase 3.1. \( |U| \geq 2 \).

In this case, \( |S| = 2|U| + 2 \). Choose two vertices \( u_1, u_2 \in U \) and define \( f : U \to \{1, 2, 3, \ldots \} \) by \( f(u_1) = f(u_2) = 3 \) and \( f(u_i) = 2 \) for all \( u_i \in U \setminus \{u_1, u_2\} \). Then \( |N_B(Y)| \geq \sum_{x \in Y} f(x) \) for all \( Y \subseteq U \) by Claim 3. Hence, by Lemma 1, \( B \) has a \( \{K_{1,2}, K_{1,3}\} \)-factor and so \( G \) has a \( \{K_{1,2}, K_{1,3}, K_5\} \)-factor.

Subcase 3.2. \( |U| = iso(G - S) = 1 \).

In this case, \( |S| = 2|U| + 2 = 4 \) and \( V(G) \neq S \cup U \). Let \( U = \{u\} \) and \( S = \{s_1, s_2, s_3, s_4\} \). If \( S \cup \{u\} \) induces a complete graph \( K_5 \) in \( G \), then \( G \) has the desired \( \{K_{1,2}, K_{1,3}, K_5\} \)-factor by Claims 1 and 2. So \( S \cup \{u\} \) does not induce a complete graph \( K_5 \). Without loss of generality, we may assume that \( s_3 \) and \( s_4 \) are not adjacent in \( G \).

Considering \( G - \{s_1, u, s_2\} \), if \( iso(G - \{s_1, u, s_2\} - X) \leq |X|/2 \) for all \( X \subseteq V(G) - \{s_1, u, s_2\} \), then the result is followed by induction hypothesis. So we may assume that there exists \( \emptyset \neq R \subseteq V(G) - \{s_1, u, s_2\} \) such that \( iso(G - \{s_1, u, s_2\} - R) \geq (|R| + 1)/2 \). We choose maximal such a vertex subset \( R \). Then Claims 1 and 2 hold for \( G - \{s_1, u, s_2\} - R \) by the maximality of \( R \). Moreover,

\[
\frac{|R \cup \{s_1, u, s_2\}|}{2} \leq iso(G - \{s_1, u, s_2\} - R) \leq \frac{|R| + 3}{2} - \frac{|R| + 1}{2} = 1.
\]

Since \( \beta = 1 \), we obtain

\[
\frac{|R \cup \{s_1, u, s_2\}|}{2} - iso(G - \{s_1, u, s_2\} - R) = 1.
\]

Therefore \( |R| \) is odd. If \( |R| \geq 3 \), then \( S' = R \cup \{s_1, u, s_2\} \) satisfies \( |S'|/2 - iso(G - S') = \beta = 1 \) and \( iso(G - S') \geq 2 \). So the result is followed with the similar discussion as in Subcase 3.1.

So we assume \( |R| = 1 \) and thus \( iso(G - \{s_1, u, s_2\} - R) = 1 \). Let \( R = \{r\} \) and \( Iso(G - \{s_1, u, s_2\} - r) = \{y\} \). Since \( \delta(G) \geq 4 \), we have \( d_G(y) = 4 \) and \( N_G(y) = \{u, s_1, s_2, r\} \). Recall that \( N_G(u) = \{s_1, s_2, s_3, s_4\} = S \), so \( y \in S \), say \( y = s_3 \).

If \( r \in S \) (i.e., \( r = s_4 \)), then \( yr = s_3s_4 \) is an edge of \( G \), which contradicts the fact that \( s_3 \) and \( s_4 \) are not adjacent in \( G \). Hence \( r \notin S \). Let \( M = G - (S \cup \{u, r\}) \). Then for every \( \emptyset \neq Y \subseteq V(M) \), it follows from (2) and \( \{u\} = Iso(G - (S \cup Y \cup \{r\})) - Iso(M - Y) \) that

\[
iso(M - Y) = iso(G - (S \cup Y \cup \{r\}) - 1 < \frac{|S| + |Y| + 1}{2} - \beta = \frac{|Y| + 1}{2}.
\]
Hence $iso(M - Y) \leq |Y|/2$, and so by induction, $M$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, and this factor can be extended to a $\{K_{1,2}, K_{1,3}, K_5\}$-factor of $G$ by adding two $K_{1,2}$'s with centres $u$ and $y$.

**Case 4.** $\beta = 3/2$.

By (1), we have $\delta(G) \geq 5$. Let $uv, vw \in E(G)$. Then for every $X \subseteq V(G) - \{u, v, w\}$ with $iso(G - \{u, v, w\} - X) \geq 1$, it follows that

$$iso(G - \{u, v, w\} - X) \leq |X \cup \{u, v, w\}| - \beta \leq |X|/2.$$

If $iso(G - \{u, v, w\} - X) = 0$, then obviously $iso(G - \{u, v, w\} - X) \leq |X|/2$. Hence by the induction hypothesis, $G - \{u, v, w\}$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, which can be extended to a $\{K_{1,2}, K_{1,3}, K_5\}$-factor of $G$.

Consequently the theorem is proved.

We now show that the condition in Theorem 4 is sharp. Consider a graph $G$ given in Figure 2. Then $G$ satisfies $iso(G - S) \leq (|S| + 1)/2$ for all $S \subseteq V(G)$, but has no $\{K_{1,2}, K_{1,3}, K_5\}$-factor. Hence the condition of the theorem is sharp in this sense. The condition of Theorem 4 is sufficient but not necessary. For example, let $G = K_{1,3}$ (or $C_{3m}$, where $m \geq 2$). Then $G$ contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor but dissatisfies the condition of Theorem 4.

![Figure 2: A graph has no $\{K_{1,2}, K_{1,3}, K_5\}$-factor.](image)

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