EXTREMAL PROPERTIES OF LOGARITHMIC SPIRALS

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ABSTRACT. Loxodromic arcs are shown to be the maximizers of inversive arclength, which is invariant under Möbius transformations. Previously, these arcs were known to be extremals. The first result says that at any loxodromic arc, the inversive arclength functional is concave with respect to a non-trivial perturbation that fixes the circle elements at the endpoints. The second result says that among curves with monotone curvature that connect fixed circle elements, the loxodromic arcs uniquely maximize inversive arclength. These results prove a conjecture made by Liebmann in 1923.

1. Introduction

In 1923, Heinrich Liebmann [6] introduced a notion of arclength that is invariant under Möbius transformations of the complex plane. The quantity is called inversive arclength and depends on three derivatives of the parameterization. Previous authors knew the corresponding differential invariant, called inversive curvature, which depends on five derivatives of the parameterization. Taken together, these notions of arclength and curvature completely determine the inversive differential geometry of a plane curve. Together, they exemplify Klein's Erlangen program for the group $SL(2,\mathbb{C})$, and they have been of ongoing interest during much of the twentieth century. See [2, 7, 8, 10], for instance.

Before 1923, it was known that the curves with constant inversive curvature are the logarithmic spirals and their Möbius images, the loxodromes. In his paper, Liebmann showed that these curves are also the extremals of inversive arclength. (Another proof of this fact was given by Maeda in [8].) Motivated by analogous results from affine geometry, Liebmann furthermore conjectured the following.

Conjecture. *(Liebmann, 1923) Among curves connecting fixed circle elements, the loxodromic arcs* maximize *the inversive arclength.*

In this paper we prove Liebmann's conjecture.

In proving the conjecture, we establish two principal intermediate results that seem not to have been previously known and may be of independent interest. The first of these, a local result, says that perturbing a loxodromic arc results in an arc with strictly smaller inversive arclength.

Theorem. *At a loxodromic arc, the inversive arclength functional is concave with respect to any three times differentiable perturbation that fixes the circle elements at*

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the endpoints. In particular, loxodromic arcs are strict local maximizers of inversive arclength.

Our second basic geometric result is a global one; to provide a natural formulation we will first introduce a pair of invariants for a smooth arc, called respectively the Kerzman-Stein and Coxeter invariants. In part, the Kerzman-Stein invariant detects a curve's isotopy class, viewed inside the extended complex plane. Two curves are said to agree inversively to second order at the endpoints if their corresponding invariants agree.

Theorem. *Consider three times differentiable curves with monotone curvature that agree inversively to second order at the endpoints. Among them there is exactly one loxodromic arc, up to Möbius transformation, and this arc uniquely maximizes the inversive arclength.*

We mention that the analogous result in Euclidean geometry is the familiar fact that, with respect to arclength, the only extremal path between two points is a straight line segment, and this path minimizes the arclength. There is also a result in affine geometry that says that after specializing to convex curves, the parabolic arcs have constant (zero) affine curvature, and these curves uniquely maximize the affine arclength among curves that connect fixed line elements. See Blaschke [1, p.40], for instance.

The paper is structured as follows. In Section 2, we review the basic notions of inversive differential geometry, we explain the necessity of restricting to curves with monotone curvature, and we introduce a pair of invariants for a smooth arc. In Section 3, we give the precise statements of our main results, and in Section 4 and Section 5, we give their proofs. In Section 6, we record two additional facts that emerge from the proofs in the previous sections.

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2. Preliminaries

In this section, we provide a brief overview of inversive differential geometry, and we describe a pair of invariants for a smooth arc.

2.1. **Inversive arclength and curvature.**

In one dimension, inversive geometry refers to the study of geometric structures that behave invariantly with respect to the action of the Möbius group

$$SL(2,\mathbb{C}) = \left\{ \mu = \mu(z) = \frac{az+b}{cz+d} \ : \ a,b,c,d \in \mathbb{C},\ ad-bc=1 \right\}$$

on the complex plane. The group law is given by composition. Of particular interest are the integral and differential invariants of a smooth curve $\gamma \subset \mathbb{C}$, which can be described explicitly in terms of their Euclidean counterparts. We briefly recall the definitions here and refer to Cairns and Sharpe [2] and Patterson [9] for more extended treatments.

If $\kappa = \kappa(s)$ gives the Euclidean curvature of γ as a function of the arclength parameter, s, then the inversively invariant one-form is $d\lambda = |\kappa'(s)|^{1/2}\, ds$ and the inversive length of γ is $L(\gamma) = \int_\gamma d\lambda$. At times it will be helpful to use parameterizations for curves with respect to the inversive arclength parameter; for instance, $\gamma = \gamma(\lambda)$ with $|d\gamma/d\lambda| \equiv |\kappa'(s)|^{-1/2}$. Defining inversive arclength usually requires that curves are three times differentiable. Moreover, to avoid an ambiguity that occurs where κ' changes sign, it is common to restrict to curves with monotone curvature. Curves with monotone curvature have the property that their oriented osculating circles are properly nested. This means that the regions they bound (a disc, half-plane, or complement of a disc) are nested inside each other. Möbius transformations therefore preserve curves with decreasing (resp., increasing) curvature. We remark that whether a curve has increasing or decreasing curvature does not depend on its orientation.

If γ is five times differentiable, then its inversive curvature is the fifth order invariant

$$I_5 = \frac{4(\kappa''' - \kappa^2\kappa')\kappa' - 5(\kappa'')^2}{8(\kappa')^3}.$$

The curves with constant I_5 are the loxodromes, that is, the Möbius images of the logarithmic spirals. A logarithmic spiral is described most simply using $r \in \mathbb{R} \to e^{\alpha r} \in \mathbb{C}$ for some $\alpha \in \mathbb{C}$, $\mathrm{Re}\,(\alpha) \neq 0 \neq \mathrm{Im}\,(\alpha)$. Such a spiral intersects circles centered at the origin in a constant angle.

In general, the value of I_5 for a curve at a point corresponds with the angle of the loxodrome that best approximates the curve at that point. See Maeda [7] for other geometric interpretations of inversive curvature. Logarithmic spirals have one finite pole and one pole at infinity. Loxodromes generally have two finite poles, and for this reason, they are sometimes called logarithmic double spirals.

Finally, we mention that when γ has monotone curvature, it can be recovered up to Möbius transformation from its intrinsic equation, $I_5 = I_5(\lambda)$. Furthermore, the inversive curvature is infinite at a vertex, that is, a point of stationary curvature. Circles and lines have everywhere infinite inversive curvature; their inversive arclength is zero.

2.2. **The Kerzman-Stein and Coxeter invariants for a smooth arc.** We here describe a pair of first and second order invariants for a twice differentiable arc. They are expressed using distance functions on the space of line elements and circle elements, respectively, though these distance functions are not distances in the usual sense. We say γ connects circle elements (p, ϕ_p, κ_p) and (q, ϕ_q, κ_q) if its endpoints are p and q, its tangent vectors there have angle ϕ_p and ϕ_q, and its curvatures there are κ_p and κ_q, respectively. In this notation, the angles ϕ_p and ϕ_q are not unique, rather they are determined only up to a multiple of 2π.

For line elements (p, ϕ_p) and (q, ϕ_q), the Kerzman-Stein distance is the difference in angle between the vector $\exp(i\phi_p)$ at p and the vector gotten by reflecting the vector $\exp(i\phi_q)$ at q across the chord connecting p to q. It is given by

$$\theta(p, \phi_p;\, q, \phi_q) = \arg\left(\frac{q-p}{\overline{q}-\overline{p}} \cdot e^{-i(\phi_q+\phi_p)}\right). \tag{1}$$

(Kerzman and Stein encountered this angle in their study of the Cauchy kernel; see [5].) Then, for an arc γ that connects line elements (p, ϕ_p) and (q, ϕ_q), the first order invariant $\theta = \theta_\gamma$ is defined using the right hand side of (1). We choose the branch of the argument function that makes $\theta_\gamma(p, q')$ a continuous function of $q' \in \gamma$ whose value at $q' = p$ is zero. (The quantity in parentheses on the right hand side of (1) approaches 1 as $q \to p$.) In this way, the θ invariant also identifies a curve's isotopy class, viewed inside the space of line elements on the extended plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Figure 1 shows loxodromic arcs that both connect $(0, 0)$ and $(1, -\pi/4)$ but with θ invariants that differ by 2π.

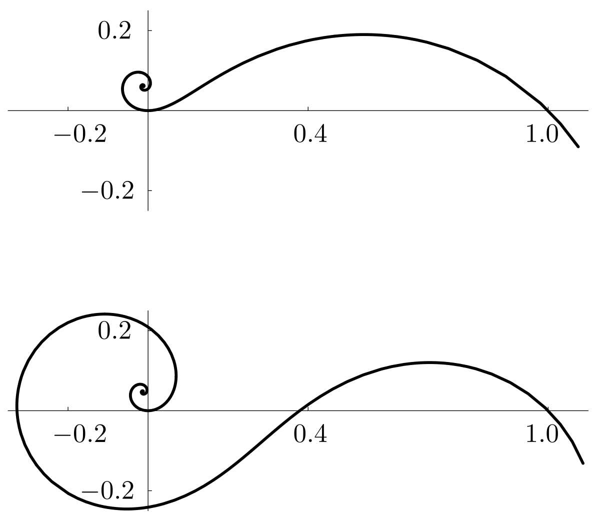

FIGURE 1. Loxodromic arcs that connect line elements $(0, 0)$ and $(1, -\pi/4)$ with θ invariants that differ by 2π.

For nonintersecting circles, the Coxeter distance (see [3]) is the quantity $\delta = \cosh^{-1}\left|(d^2 - r_1^2 - r_2^2)/(2r_1r_2)\right|$ where the circles have radius r_1 and r_2, and the distance between their centers is d. Correspondingly, for circle elements (p, ϕ_p, κ_p) and (q, ϕ_q, κ_q), the distance is

$$\delta(p, q) = \cosh^{-1}\left|\frac{\left|(p + \frac{ie^{i\phi_p}}{\kappa_p}) - (q + \frac{ie^{i\phi_q}}{\kappa_q})\right|^2 - \frac{1}{\kappa_p^2} - \frac{1}{\kappa_q^2}}{2\frac{1}{\kappa_p}\frac{1}{\kappa_q}}\right|. \tag{2}$$

In particular, for a twice differentiable curve γ connecting these two circle elements, the second order invariant $\delta = \delta_\gamma$ is defined using the right hand side of (2). It will not be necessary to simplify this expression for a general curve.

These distance functions are not distances in the usual sense since neither of them satisfies a general triangle inequality. Moreover, the Coxeter distance is zero for circles that are tangent to each other, and the Kerzman-Stein distance is zero for line elements that are tangent to a common circle. By restricting to curves with monotone curvature, however, we eliminate these degeneracies. In fact, by restricting to curves with decreasing curvature we may assume that both invariants are positive. For an explanation why the θ invariant is positive, see subsection 6.1. For the δ invariant, it is assumed that one uses the positive value of $\cosh^{-1}$ in (2).

3. Statement of Main Results

Liebmann [6] showed that the extremals of inversive arclength are the loxodromic arcs, subject to perturbations that fix the circle elements at the endpoints. Maeda also proved this fact in [8, p.256]. We show the loxodromic arcs are, in fact, maximizers of inversive arclength. Our first result is a local version of this statement.

Theorem 1. *At a loxodromic arc, the inversive arclength functional is concave with respect to any three times differentiable perturbation that fixes the circle elements at the endpoints. In particular, loxodromic arcs are strict local maximizers of inversive arclength.*

Our second result is a global version. By considering only curves with decreasing curvature, we may assume that both of a curve's invariants are positive. We mention that the endpoint circle elements only determine a curve's θ invariant up to a multiple of 2π, so by specifying the θ invariant in Theorem 2, we require that all curves belong to the same isotopy class.

Theorem 2. *Consider three times differentiable curves with decreasing curvature that connect two fixed circle elements and have the same θ invariant. Among them there is exactly one loxodromic arc, and this arc uniquely maximizes the inversive arclength.*

A Möbius transformation can send a point on a curve to the point at infinity, so it is possible that the extremal arc will pass through the point at infinity. For this reason, we also present the result in a more naturally inversive setting, without specific reference to the endpoints.

Theorem 3. *Consider three times differentiable curves with monotone curvature that agree inversively to second order at the endpoints. Among them there is exactly one loxodromic arc, up to Möbius transformation, and this arc uniquely maximizes the inversive arclength.*

In this formulation, two curves are said to *agree inversively to second order at the endpoints* if they have the same (θ, δ) invariants. We also mention that the result for curves with decreasing curvature immediately extends to curves with increasing curvature. For instance, under conjugation ($z = x + iy \longrightarrow \overline{z} = x - iy$), curves with decreasing curvature become curves with increasing curvature; meanwhile, their inversive arclength is unchanged. Loxodromes with increasing curvature are also the Möbius images of logarithmic spirals as defined in Proposition 1, for $a < 0$.

Following these observations, Theorem 3 follows directly from Theorem 2 and Lemma 6.

In the final section we provide evidence that suggests these results are optimal. For instance, when considering only arcs with the same θ invariant, it is possible to make the inversive length arbitrarily large or small, even within the family of loxodromic arcs. For this reason, it is necessary to include both invariants when formulating the problem.

4. Proof of Theorem 1

We use the variational approach to show that loxodromic arcs are strict local maximizers of inversive arclength. By using an appropriate Möbius transformation, we may assume that the loxodromic arc is an arc from a logarithmic spiral. Let $L(z; s, t)$ denote the inversive length of an arc parameterized by $r \in [s, t] \to z(r)$.

Proposition 1. *Suppose $r \in [s, t] \to z_r = z(r) = re^{ia \log r}/(1 + ia)$ parameterizes a logarithmic spiral for some $a > 0$. This is a parameterization by arclength. Consider three times differentiable functions $p : r \in [s, t] \to p_r = p(r) \in \mathbb{R}$ which satisfy*

i) $p_s = p_t = 0$,
ii) $p'_s = p'_t = 0$, *and*
iii) $p''_t/p''_s = s/t$,

and set $z^{p,\epsilon}_r = z_r + i\epsilon p_r z'_r$ for $\epsilon \in \mathbb{R}$. So $z^{p,0}_r = z_r$ for all r. Then, for each such p,

$$L(z^{p,\epsilon}; s, t) \;=\; L(z; s, t) + \epsilon^2 \cdot R_2(p; s, t) + o(\epsilon^2), \tag{3}$$

where $R_2(p; s, t) \leq 0$. There is equality if and only if $p \equiv 0$.

Proof of Theorem 1. Theorem 1 follows from Proposition 1 once we show that the conditions on p are satisfied for any perturbation that fixes the circle elements at the endpoints. The first two conditions on p say precisely that the perturbation should fix the line elements at the endpoints. Under these conditions, the last condition says it should also fix the Coxeter invariant; we omit the details of this last fact. For Theorem 1, however, it is essential to know that the third condition depends on nothing beyond the second order information at the endpoints. It is simpler, then, to verify that it says the perturbation should fix the ratio of curvatures at the endpoints. For this, let us temporarily assume the conclusion of Lemma 1. Then, to first order in ϵ, the ratio of curvatures for the perturbed curve is $(a/t + \epsilon p''_t)/(a/s + \epsilon p''_s)$. This equals the ratio of curvatures for the unperturbed curve precisely when $p''_t/p''_s = (a/t)/(a/s) = s/t$. □

Proof of Proposition 1. To simplify the notation we also write $\gamma_r = z^{p,\epsilon}_r$. Then,

$$\gamma'_r \;=\; \frac{d\gamma}{dr} \;=\; z'_r + i\epsilon p'_r z'_r + i\epsilon p_r \cdot \frac{ia}{r} \cdot z'_r \;=\; z'_r \left[1 - \epsilon\left(\frac{ap_r}{r} - ip'_r\right)\right].$$

Suppose that $u = u(r)$ is the arclength parameter for γ. Then

$$\left|\frac{d\gamma}{du}\right| \equiv 1 \equiv \left|\frac{d\gamma}{dr}\right| \cdot \frac{dr}{du} = \sqrt{(1-\epsilon\frac{ap_r}{r})^2 + (\epsilon p_r')^2} \cdot \frac{dr}{du},$$

so that

$$\frac{du}{dr} = 1 + \frac{1}{2}\left(-2\epsilon\frac{ap_r}{r} + \epsilon^2\frac{a^2p_r^2}{r^2} + \epsilon^2(p_r')^2\right) - \frac{1}{8}\left(-2\epsilon\frac{ap_r}{r}\right)^2 + o(\epsilon^2)$$

$$= 1 - \epsilon\frac{ap_r}{r} + \epsilon^2\frac{(p_r')^2}{2} + o(\epsilon^2), \tag{4}$$

and

$$\frac{dr}{du} = \left(\frac{du}{dr}\right)^{-1} = 1 + \epsilon\frac{ap_r}{r} + \epsilon^2\left(\frac{a^2p_r^2}{r^2} - \frac{(p_r')^2}{2}\right) + o(\epsilon^2).$$

From now on, we will interpret the equals sign to mean equal only up to terms of second order in ϵ. Terms of order $o(\epsilon^2)$ will be counted as zero.

Lemma 1. *Neglecting terms of order $o(\epsilon^2)$ the curvature of γ at γ_r is given by*

$$k_r = \frac{a}{r} + \epsilon\left(\frac{a^2p_r}{r^2} + p_r''\right) + \epsilon^2\left(\frac{1}{2}\frac{a(p_r')^2}{r} + \frac{2ap_rp_r''}{r} - \frac{ap_rp_r'}{r^2} + \frac{a^3p_r^2}{r^3}\right).$$

Furthermore,

$$\frac{dk}{dr} = -\frac{a}{r^2} + \epsilon\left(\frac{a^2p_r'}{r^2} - \frac{2a^2p_r}{r^3} + p_r'''\right) + \epsilon^2\left(\frac{3ap_r'p_r''}{r} - \frac{3}{2}\frac{a(p_r')^2}{r^2} + \frac{2ap_rp_r'''}{r} - \frac{3ap_rp_r''}{r^2} + \frac{2ap_rp_r'}{r^3} + \frac{2a^3p_rp_r'}{r^3} - \frac{3a^3p_r^2}{r^4}\right).$$

Proof. We first express the curvature of γ in terms of the r coordinate:

$$k_r = \frac{\frac{d}{du}\left(\frac{d\gamma}{du}\right)}{i\frac{d\gamma}{du}} = \frac{\frac{d}{dr}\left(\frac{d\gamma}{dr}\frac{dr}{du}\right)\frac{dr}{du}}{i\frac{d\gamma}{dr}\frac{dr}{du}} = \frac{\frac{d}{dr}\left(\frac{d\gamma}{dr}\right)\frac{dr}{du}}{i\frac{d\gamma}{dr}} + \frac{\frac{d}{dr}\left(\frac{dr}{du}\right)}{i}.$$

It follows that

$$k_r = \frac{\frac{d}{dr}\left[z_r'\left(1-\epsilon\left(\frac{ap_r}{r} - ip_r'\right)\right)\right] \cdot \left[1 + \epsilon\frac{ap_r}{r} + \epsilon^2\left(\frac{a^2p_r^2}{r^2} - \frac{(p_r')^2}{2}\right)\right]}{i \cdot z_r'\left[1 - \epsilon\left(\frac{ap_r}{r} - ip_r'\right)\right]} + \frac{1}{i}\frac{d}{dr}\left[1 + \epsilon\frac{ap_r}{r} + \epsilon^2\left(\frac{a^2p_r^2}{r^2} - \frac{(p_r')^2}{2}\right)\right]$$

$$= \frac{a}{r}\left[1 + \epsilon\frac{ap_r}{r} + \epsilon^2\left(\frac{a^2p_r^2}{r^2} - \frac{(p_r')^2}{2}\right)\right] - \frac{\epsilon}{i}\left[\frac{ap_r'}{r} - \frac{ap_r}{r^2} - ip_r''\right]\left[1 + \epsilon\frac{ap_r}{r}\right]\left[1 + \epsilon\left(\frac{ap_r}{r} - ip_r'\right)\right] + \frac{1}{i}\left[\epsilon\left(\frac{ap_r'}{r} - \frac{ap_r}{r^2}\right) + \epsilon^2\left[\frac{2ap_r}{r}\left(\frac{ap_r'}{r} - \frac{ap_r}{r^2}\right) - p_r'p_r''\right]\right].$$

The terms that have the factor of ϵ^1 are precisely

$$\frac{a}{r}\cdot\frac{ap_r}{r}-\frac{1}{i}\left(\frac{ap_r'}{r}-\frac{ap_r}{r^2}-ip_r''\right)+\frac{1}{i}\left(\frac{ap_r'}{r}-\frac{ap_r}{r^2}\right) = \frac{a^2p_r}{r^2}+p_r'',$$

and the terms that have the factor of ϵ^2 are precisely

$$\begin{aligned}\frac{a}{r}\left(\frac{a^2p_r^2}{r^2}-\frac{(p_r')^2}{2}\right)-\frac{1}{i}\left(\frac{ap_r'}{r}-\frac{ap_r}{r^2}-ip_r''\right)\left(\frac{2ap_r}{r}-ip_r'\right)&\\ +\frac{1}{i}\left[\frac{2ap_r}{r}\left(\frac{ap_r'}{r}-\frac{ap_r}{r^2}\right)-p_r'p_r''\right]&\\ =\frac{a^3p_r^2}{r^3}-\frac{a(p_r')^2}{2r}+p_r'\left(\frac{ap_r'}{r}-\frac{ap_r}{r^2}\right)+p_r''\left(\frac{2ap_r}{r}\right)&\\ +i\left[\frac{2ap_r}{r}\left(\frac{ap_r'}{r}-\frac{ap_r}{r^2}\right)-p_r'p_r''-\frac{2ap_r}{r}\left(\frac{ap_r'}{r}-\frac{ap_r}{r^2}\right)+p_r'p_r''\right]&\\ =\frac{1}{2}\,\frac{a(p_r')^2}{r}+\frac{2ap_rp_r''}{r}-\frac{ap_rp_r'}{r^2}+\frac{a^3p_r^2}{r^3},&\end{aligned}$$

as claimed by the lemma. The expression for dk/dr is then easy to check; we skip the few details. □

Next, the inversively invariant one form can be written

$$\left|\frac{dk}{du}\right|^{1/2} du = \left|\frac{dk}{dr}\cdot\frac{dr}{du}\right|^{\frac{1}{2}}\cdot\frac{du}{dr}\,dr = \left|\frac{dk}{dr}\cdot\frac{du}{dr}\right|^{\frac{1}{2}} dr.$$

Lemma 2. *Neglecting terms of order $o(\epsilon^2)$, we have*

$$\left|\frac{dk}{dr}\cdot\frac{du}{dr}\right|^{\frac{1}{2}} = \frac{\sqrt{a}}{r}+\epsilon\cdot A_1(r)+\epsilon^2\cdot A_2(r),$$

where

$$A_1(r) = \frac{\sqrt{a}}{2}\left(-\frac{ap_r'}{r}+\frac{ap_r}{r^2}-\frac{rp_r'''}{a}\right)$$

and

$$\begin{aligned}A_2(r) = \frac{\sqrt{a}}{2}\Bigg[&a^2\left(-\frac{p_rp_r'}{r^2}+\frac{p_r^2}{r^3}-\frac{1}{4r}\left(p_r'-\frac{p_r}{r}\right)^2\right)-\frac{1}{a^2}\frac{r^3(p_r''')^2}{4}\\ &+\left(-3p_r'p_r''+\frac{2(p_r')^2}{r}+\frac{3p_rp_r''}{r}-\frac{2p_rp_r'}{r^2}-\frac{p_rp_r'''}{2}-\frac{rp_r'p_r'''}{2}\right)\Bigg].\end{aligned}$$

Proof. Using (4) and Lemma 1, we find that

$$\frac{dk}{dr}\cdot\frac{du}{dr} = -\frac{a}{r^2}+\epsilon\cdot B_1(r)+\epsilon^2\cdot B_2(r),$$

where

$$B_1(r) = \frac{a^2p_r'}{r^2}-\frac{a^2p_r}{r^3}+p_r''',$$

and after simplifying,

$$B_2(r) = \frac{3ap_r'p_r''}{r}-\frac{2a(p_r')^2}{r^2}+\frac{ap_rp_r'''}{r}-\frac{3ap_rp_r''}{r^2}+\frac{2ap_rp_r'}{r^3}+\frac{a^3p_rp_r'}{r^3}-\frac{a^3p_r^2}{r^4}.$$

Then by writing

$$\frac{dk}{dr}\cdot\frac{du}{dr} \;=\; -\frac{a}{r^2}\left(1-\epsilon\cdot\frac{r^2}{a}\,B_1(r)-\epsilon^2\cdot\frac{r^2}{a}\,B_2(r)\right),$$

we have

$$\left|\frac{dk}{dr}\cdot\frac{du}{dr}\right|^{1/2} = \frac{\sqrt{a}}{r}\left[1-\frac{\epsilon}{2}\cdot\frac{r^2}{a}\,B_1(r)+\epsilon^2\left(-\frac{r^2}{2a}\,B_2(r)-\frac{r^4}{8a^2}B_1(r)^2\right)\right].$$

We have left then to expand and simplify $A_1(r) = -rB_1(r)/(2\sqrt{a})$ and

$$A_2(r) = -rB_2(r)/(2\sqrt{a}) - r^3B_1(r)^2/(8a^{3/2}).$$

We find that

$$-\frac{r\,B_1(r)}{2\sqrt{a}} = -\frac{r}{2\sqrt{a}}\left(\frac{a^2p_r'}{r^2}-\frac{a^2p_r}{r^3}+p_r'''\right) = \frac{\sqrt{a}}{2}\left(-\frac{ap_r'}{r}+\frac{ap_r}{r^2}-\frac{rp_r'''}{a}\right)$$

and

$$\begin{aligned}
-&\frac{r\,B_2(r)}{2\sqrt{a}}-\frac{r^3B_1(r)^2}{8a^{3/2}}\\
&\quad= \frac{\sqrt{a}}{2}\Bigg[-\frac{r}{a}\bigg(\frac{3ap_r'p_r''}{r}-\frac{2a(p_r')^2}{r^2}+\frac{ap_rp_r'''}{r}-\frac{3ap_rp_r''}{r^2}+\frac{2ap_rp_r'}{r^3}\\
&\qquad\qquad+\frac{a^3p_rp_r'}{r^3}-\frac{a^3p_r^2}{r^4}\bigg)-\frac{r^3}{4a^2}\left(\frac{a^2p_r'}{r^2}-\frac{a^2p_r}{r^3}+p_r'''\right)^2\Bigg]\\
&= \frac{\sqrt{a}}{2}\Bigg[-3p_r'p_r''+\frac{2(p_r')^2}{r}-p_rp_r'''+\frac{3p_rp_r''}{r}-\frac{2p_rp_r'}{r^2}-\frac{a^2p_rp_r'}{r^2}\\
&\qquad\qquad+\frac{a^2p_r^2}{r^3}-\frac{a^2}{4r}\left(p_r'-\frac{p_r}{r}\right)^2-\frac{1}{2}(rp_r'-p_r)p_r'''-\frac{r^3}{4a^2}(p_r''')^2\Bigg]\\
&= \frac{\sqrt{a}}{2}\Bigg[a^2\left(-\frac{p_rp_r'}{r^2}+\frac{p_r^2}{r^3}-\frac{1}{4r}\left(p_r'-\frac{p_r}{r}\right)^2\right)-\frac{1}{a^2}\,\frac{r^3(p_r''')^2}{4}\\
&\qquad+\left(-3p_r'p_r''+\frac{2(p_r')^2}{r}+\frac{3p_rp_r''}{r}-\frac{2p_rp_r'}{r^2}-\frac{p_rp_r'''}{2}-\frac{rp_r'p_r'''}{2}\right)\Bigg]
\end{aligned}$$

as claimed by the lemma. □

So far, after neglecting the terms of order $o(\epsilon^2)$,

$$L(z^{p,\epsilon};s,t) = \int_s^t\left[\frac{\sqrt{a}}{r}+\epsilon\cdot A_1(r)+\epsilon^2\cdot A_2(r)\right]dr.$$

Here, $\int_s^t \sqrt{a}/r\,dr = \sqrt{a}\,\log(t/s) = L(z;s,t)$, the inversive length of the unperturbed logarithmic spiral. Furthermore,

$$\begin{aligned}
\int_s^t A_1(r)\,dr &= \frac{\sqrt{a}}{2}\int_s^t\left(-\frac{ap_r'}{r}+\frac{ap_r}{r^2}-\frac{rp_r'''}{a}\right)dr\\
&\qquad= \frac{-a^{3/2}}{2}\int_s^t\frac{d}{dr}\left(\frac{p_r}{r}\right)dr-\frac{1}{2\sqrt{a}}\int_s^t rp_r'''\,dr.
\end{aligned}$$

The first integral in the last expression is zero since $p_s = p_t = 0$. The second integral can be evaluated using integration by parts—

$$\int_s^t r p_r''' \, dr \;=\; r p_r'' \Big|_s^t - \int_s^t p_r'' \, dr \;=\; (t p_t'' - s p_s'') - (p_t' - p_s').$$

This vanishes, too, since $p_t''/p_s'' = s/t$ and $p_s' = p_t' = 0$. It follows that $\int_s^t A_1(r) dr = 0$, and for this reason there are no first order terms on the right hand side of (3). This also confirms the already known fact that the loxodromic arcs are extremal.

We have yet then to verify that $R_2(p; s, t) \leq 0$ with equality precisely when $p \equiv 0$. Notice that both

$$\int_s^t -\frac{p_r p_r'}{r^2} + \frac{p_r^2}{r^3} \, dr \;=\; \int_s^t -\frac{1}{2} \frac{d}{dr}\left(\frac{p_r^2}{r^2}\right) dr \;=\; 0$$

and

$$\int_s^t -3 p_r' p_r'' \, dr \;=\; \int_s^t -\frac{3}{2} \frac{d}{dr} (p_r')^2 \, dr \;=\; 0$$

since $p_s = p_t = 0$ and $p_s' = p_t' = 0$. For the same reason,

$$\begin{aligned} &\int_s^t \left(\frac{2(p_r')^2}{r} + \frac{3 p_r p_r''}{r} - \frac{2 p_r p_r'}{r^2} \right) dr \\ &\qquad = \int_s^t 2 \frac{d}{dr}\left(\frac{p_r p_r'}{r}\right) dr + \int_s^t \frac{p_r p_r''}{r} \, dr \;=\; \int_s^t \frac{p_r p_r''}{r} \, dr. \end{aligned}$$

We can then write

$$\begin{aligned} R_2(p; s, t) = \int_s^t A_2(r) \, dr = \frac{\sqrt{a}}{2} \Bigg[&\int_s^t -\frac{a^2}{4r} (p_r' - \frac{p_r}{r})^2 \, dr \\ &- \int_s^t \frac{r^3 (p_r''')^2}{4a^2} dr + \int_s^t \frac{p_r p_r''}{r} - \frac{1}{2} \left(p_r p_r''' + r p_r' p_r''' \right) dr \Bigg]. \end{aligned}$$

To further simplify, we use the following.

Lemma 3.

$$\int_s^t \frac{p_r p_r''}{r} dr \;=\; -\int_s^t \frac{1}{r} \left(p_r' - \frac{p_r}{r} \right)^2 dr$$

and

$$\int_s^t p_r p_r''' + r p_r' p_r''' \, dr \;=\; -\int_s^t r (p_r'')^2 \, dr.$$

Proof. For the first integral, we first integrate by parts:

$$\int_s^t \frac{p_r p_r''}{r} dr \;=\; \frac{p_r}{r} p_r' \Big|_s^t - \int_s^t p_r' \left(\frac{p_r'}{r} - \frac{p_r}{r^2} \right) dr \;=\; -\int_s^t p_r' \left(\frac{p_r'}{r} - \frac{p_r}{r^2} \right) dr.$$

Next, define $q_r = p_r/r$ so that $q_r' = p_r'/r - p_r/r^2$. Then also $p_r' = r q_r' + p_r/r = r q_r' + q_r$. We then have

$$\begin{aligned} \int_s^t \frac{p_r p_r''}{r} dr \;&=\; -\int_s^t (r q_r' + q_r) q_r' \, dr \;=\; -\int_s^t r (q_r')^2 \, dr - \int_s^t q_r q_r' \, dr \\ &=\; -\int_s^t r (q_r')^2 \, dr - \frac{q_r^2}{2} \Big|_s^t \;=\; -\int_s^t \frac{1}{r} \left(p_r' - \frac{p_r}{r} \right)^2 dr. \end{aligned}$$

In the last step we use the fact that $q_s = q_t = 0$. For the second integral, again integrate by parts:

$$\int_s^t p_r p_r''' + r p_r' p_r''' \, dr \;=\; (p_r + r p_r') \, p_r'' \Big|_s^t - \int_s^t p_r''(2p_r' + r p_r'') \, dr$$
$$= \; 0 - (p_r')^2 \Big|_s^t - \int_s^t r (p_r'')^2 \, dr \;=\; - \int_s^t r (p_r'')^2 \, dr.$$

The lemma is then proved. □

It follows that

$$R_2(p; s,t) = \frac{\sqrt{a}}{2} \left[-\frac{a^2}{4} \int_s^t \frac{1}{r} \left(p_r' - \frac{p_r}{r} \right)^2 dr - \frac{1}{4a^2} \int_s^t r^3 (p_r''')^2 \, dr \right.$$
$$\left. + \int_s^t \left(-\frac{1}{r} \left(p_r' - \frac{p_r}{r} \right)^2 + \frac{1}{2} r (p_r'')^2 \right) dr \right]$$
$$= \frac{1}{8a^{3/2}} \left[-a^4 \cdot X + a^2(-4X + 2Y) - Z \right], \tag{5}$$

where

$$X \;=\; \int_s^t \frac{1}{r} \left(p_r' - \frac{p_r}{r} \right)^2 dr$$
$$Y \;=\; \int_s^t r \, (p_r'')^2 \, dr$$
$$Z \;=\; \int_s^t r^3 \, (p_r''')^2 \, dr.$$

In (5), the quantity in brackets is quadratic in a^2, and has discriminant $\Delta = (-4X + 2Y)^2 - 4XZ$. We *claim that the discriminant is negative when p is nonzero*; evidently it is zero when $p \equiv 0$. This suffices to prove Proposition 1 for the following reason. For p fixed, the graph of the quantity in brackets, as a function of a^2, opens downward. If the discriminant is negative, this graph never crosses the horizontal axis. So the quantity in brackets is negative for all values of a^2. As this would be true except when $p \equiv 0$, we will have established Proposition 1.

To prove the claim, we introduce new substitutions. Let $r = e^\mu$ and $dr = e^\mu d\mu$. Also, let $y = y(\mu) = p'(e^\mu)e^\mu - p(e^\mu)$. Then

$$y' = p''(e^\mu)e^{2\mu} \quad \text{and} \quad y'' = p'''(e^\mu)e^{3\mu} + 2p''(e^\mu)e^{2\mu},$$

and

$$p_r' - \frac{p_r}{r} \;=\; \frac{y}{e^\mu}, \quad p_r'' \;=\; \frac{y'}{e^{2\mu}}, \quad \text{and} \quad p_r''' \;=\; \frac{y'' - 2y'}{e^{3\mu}}.$$

We next use the following two lemmas.

Lemma 4.

$$\begin{aligned} X &= \int_{r=s}^{r=t} y^2 e^{-2\mu}\, d\mu \\ Y &= \int_{r=s}^{r=t} (y')^2 e^{-2\mu}\, d\mu \\ Z &= \int_{r=s}^{r=t} (y'')^2 e^{-2\mu}\, d\mu \end{aligned}$$

Proof. The first two integrals are immediate:

$$X = \int_s^t \frac{1}{r}\left(p_r' - \frac{p_r}{r}\right)^2 dr = \int_{r=s}^{r=t} \frac{1}{e^\mu}\left(\frac{y}{e^\mu}\right)^2 e^\mu\, d\mu = \int_{r=s}^{r=t} y^2 e^{-2\mu}\, d\mu$$

and

$$Y = \int_s^t r(p_r'')^2\, dr = \int_{r=s}^{r=t} e^\mu \left(\frac{y'}{e^{2\mu}}\right)^2 e^\mu\, d\mu = \int_{r=s}^{r=t} (y')^2 e^{-2\mu}\, d\mu.$$

For the last integral,

$$\begin{aligned} Z = \int_s^t r^3 (p_r''')^2\, dr &= \int_{r=s}^{r=t} e^{3\mu}\left(\frac{y''-2y'}{e^{3\mu}}\right)^2 e^\mu\, d\mu \\ &= \int_{r=s}^{r=t} (y'')^2 e^{-2\mu}\, d\mu - 4\int_{r=s}^{r=t} y'y'' e^{-2\mu}\, d\mu + 4\int_{r=s}^{r=t} (y')^2 e^{-2\mu}\, d\mu. \end{aligned}$$

It then suffices to show that

$$\int_{r=s}^{r=t} y'y'' e^{-2\mu}\, d\mu = \int_{r=s}^{r=t} (y')^2 e^{-2\mu}\, d\mu.$$

Again, integrate by parts:

$$\int_{r=s}^{r=t} y'y'' e^{-2\mu}\, d\mu = \frac{1}{2}(y')^2 e^{-2\mu}\Big|_{r=s}^{r=t} + \int_{r=s}^{r=t} (y')^2 e^{-2\mu}\, d\mu.$$

The boundary terms vanish since

$$(y')^2 e^{-2\mu}\Big|_{r=s}^{r=t} = (r^2 p_r'')^2 r^{-2}\Big|_{r=s}^{r=t} = (tp_t'')^2 - (sp_s'')^2 = 0,$$

so the lemma is proved. □

Lemma 5.

$$2X - Y = \int_{r=s}^{r=t} yy'' e^{-2\mu}\, d\mu$$

Proof. Starting with the expression for Y from the previous lemma, we integrate by parts. Then,

$$\begin{aligned} Y = \int_{r=s}^{r=t} (y')^2 e^{-2\mu}\, d\mu = yy'e^{-2\mu}\Big|_{r=s}^{r=t} - \int_{r=s}^{r=t} y(y''-2y')e^{-2\mu}\, d\mu \\ = -\int_{r=s}^{r=t} yy'' e^{-2\mu}\, du + \int_{r=s}^{r=t} 2yy' e^{-2\mu}\, d\mu, \end{aligned}$$

the boundary terms vanishing since $y = rp'_r - p_r = 0$ for $r = s, t$. Integrating by parts in the second integral on the right hand side gives

$$Y + \int_{r=s}^{r=t} yy'' e^{-2\mu}\, d\mu \;=\; y^2 e^{-2\mu}\Big|_{r=s}^{r=t} + \int_{r=s}^{r=t} 2y^2 e^{-2\mu}\, d\mu \;=\; 2X.$$

In the second step, the boundary terms vanish for the same reason as before, so the lemma is proved. □

Using these lemmas, we apply the Cauchy-Schwarz inequality to the functions $ye^{-\mu}$ and $y''e^{-\mu}$:

$$(2X - Y)^2 \;\leq\; \int_{r=s}^{r=t} y^2 e^{-2\mu}\, d\mu \cdot \int_{r=s}^{r=t} (y'')^2 e^{-2\mu}\, d\mu \;=\; X \cdot Z.$$

From this it follows that $\Delta = 4 \cdot [(2X - Y)^2 - XZ] \leq 0$ and we are nearly done. Cauchy-Schwarz also says there is equality only if one of the following is true:

i) $y \equiv 0$
ii) $y'' \equiv 0$
iii) $cye^{-\mu} = y''e^{-\mu}$ for all μ; that is, $y'' - cy \equiv 0$. Here, $c \neq 0$ is constant.

In each case, we must show that $p \equiv 0$.

If $y \equiv 0$, then $p'_r \cdot r - p_r = 0$ for all r, so $(p_r/r)' = 0$ and $p_r = c \cdot r$. But $p_s = p_t = 0$, so $p \equiv 0$. If $y'' \equiv 0$, then y is linear. But as before, $y = 0$ for both $r = s, t$, so then $y \equiv 0$. The argument just given implies $p \equiv 0$.

In the final case, if c is positive, there are no nontrivial solutions for y that vanish at $r = s, t$. If $c = -\lambda^2$, there are many solutions

$$y(\mu) \;=\; \sin\left[\frac{n\pi}{\log(t/s)} \cdot (\mu - \log s)\right],$$

with $\lambda = n\pi/\log(t/s)$ and $n \in \mathbb{N}$. But there remains the restriction on y that says

$$y'(\mu) \cdot \frac{1}{e^{\mu}}\Big|_{r=s}^{r=t} \;=\; p''_r \cdot r\Big|_{r=s}^{r=t} \;=\; p''_t \cdot t - p''_s \cdot s \;=\; 0.$$

For this to hold, it is necessary that $\cos(n\pi)/t - 1/s = 0$, which is impossible as $s, t > 0$ and $s \neq t$. Again, there are no nontrivial solutions, so Proposition 1 is proved. □

5. Proof of Theorem 2

To prove Theorem 2, we first determine how the circle elements at the endpoints of an arc can be normalized with respect to the curve's invariants (θ, δ). Then we show that each pair of invariants (θ, δ) is obtained exactly once, up to Möbius transformation, within the family of loxodromic arcs. Finally, we establish a context in which the logarithmic spirals are global maximizers of inversive arclength. With these facts in hand, we are then ready to prove Theorem 2.

Lemma 6. *Using a Möbius transformation, the endpoint circle elements (p, ϕ_p, κ_p) and (q, ϕ_q, κ_q) of a twice differentiable arc γ with decreasing curvature can be normalized so that $p = 0$, $\phi_p = 0$, $\kappa_p = 1$, and $q = 1$. After the normalization, the values of ϕ_q (to a multiple of 2π) and κ_q are determined by the invariants (θ, δ).*

Proof. To prove the lemma, we exhaust the six degrees of freedom that are available in $SL(2, \mathbb{C})$. We first use a translation that makes $p = 0$ and follow that with a rotation and dilation that makes $q = 1$. This uses four degrees of freedom, but we may now assume that γ is normalized with $p = 0$ and $q = 1$, and we have left the subgroup of Möbius transformations that fix $p = 0$ and $q = 1$. These Möbius transformations have the form $\mu(z) = d^{-1}z/(cz + d)$ for some $0 \neq d \in \mathbb{C}$, with $c = 1/d - d$. We claim we can choose $0 \neq d \in \mathbb{C}$ so that $\phi_p = 0$ and $\kappa_p = 1$.

Next, $\mu'(z) = (cz + d)^{-2}$ and the unit tangent vector of the curve $\mu \circ \gamma$ at $p = 0$ is $(\overline{d}/d) \exp(i\phi_p)$. After replacing d with $d \cdot \exp(i\phi_p/2)$ where $0 \neq d \in \mathbb{R}$, this tangent vector is 1. So we have also normalized $\phi_p = 0$. We may assume then that γ is normalized with $p = 0$, $\phi_p = 0$, and $q = 1$, and we have left the subgroup of transformations of the form $\mu(z) = d^{-1}z/(cz + d)$ for $0 \neq d \in \mathbb{R}$, with $c = 1/d - d$. Choosing d or $-d$ results in the same Möbius transformation, so without loss of generality, assume $d > 0$.

We claim we can choose $0 < d < \infty$ so that $\kappa_p = 1$. At this point, we may assume that $\kappa_p > 0$ else γ could never reach $q = 1$, rather it would spiral inside the circle centered at $i\kappa_p^{-1}$ with radius $|\kappa_p|^{-1}$. Suppose now that $s \to \gamma(s)$ is a parameterization by arclength, and $r = r(s)$ is defined so $r \to \mu \circ \gamma(r)$ is also a parameterization by arclength. Then the curvature of $\mu \circ \gamma$ can be expressed by

$$\frac{\dfrac{d^2\mu}{dr^2}}{i \cdot \dfrac{d\mu}{dr}} = \frac{\dfrac{d}{ds}\left(\dfrac{d\mu}{dr}\right)\dfrac{ds}{dr}}{i \cdot \dfrac{d\mu}{ds}\,\dfrac{ds}{dr}} = \frac{\dfrac{d}{ds}\left(\dfrac{d\mu}{dr}\right)}{i \cdot \dfrac{d\mu}{ds}} = \frac{\dfrac{d}{ds}\left(\dfrac{\overline{c\gamma_s + d}}{c\gamma_s + d}\,\dfrac{d\gamma}{ds}\right)}{i \cdot \dfrac{1}{(c\gamma_s + d)^2}\,\dfrac{d\gamma}{ds}}.$$

At $p = 0$, where already $\phi_p = 0$ (so $\gamma = 0$, $d\gamma/ds = 1$, and $d^2\gamma/ds^2 = i\kappa_p$), we find that the curvature of $\mu \circ \gamma$ is $d^2 \cdot \kappa_p$. Choosing $d = \kappa_p^{-1/2}$ makes this curvature equal to 1.

Finally, after the normalization, the values of ϕ_q and κ_q can be recovered from the invariants (θ, δ) by using (1) and (2). Solving (1) gives $\phi_q = -\theta$. Then solving (2) gives two possibilities for κ_q, namely $\kappa_q^{\pm} = 2(\pm \cosh \delta + \cos \phi_q + \sin \phi_q)$. Of these possibilities, only $\kappa_q = \kappa_q^{-}$ gives an (oriented) circle element $(0, \phi_q, \kappa_q)$ that is properly nested with the circle element $(0, 0, 1)$. ☐

Next we show that each pair of invariants (θ, δ) is obtained exactly once, up to Möbius transformation, in the family of loxodromic arcs.

Lemma 7. *Each pair of invariants (θ, δ) with $\theta, \delta > 0$ is obtained exactly once, up to Möbius transformation, within the family of loxodromic arcs. In particular, the map $(a,\ v) \to (\theta,\ \delta)$ determined below using (6) is both one-to-one and onto.*

Once this is proved, we may conclude from Lemma 6 and Lemma 7 the following intermediate result.

Proposition 2. *Given an arc γ with decreasing curvature, there is precisely one loxodromic arc γ^* that connects the same circle elements as γ and has the same θ invariant as γ.*

To prove Lemma 7 it suffices to consider arcs of the logarithmic spirals $z(u) = (\exp[(1+ia)u/\sqrt{a}] - 1)/(1+ia)$ for $a > 0$. Then also $z'(u) = \exp(iu\sqrt{a})$ and $z''(u) = ia\exp(-u/\sqrt{a}) \cdot z'(u)$ where the primed notation indicates differentiation with respect to arclength. For this spiral, the parameter u is the inversive arclength parameter and can be related to the arclength parameter s by $u = \sqrt{a}\log s$.

Each such spiral has a one parameter family of symmetries—a point on a logarithmic spiral can be taken to any other point on the spiral by an appropriate translation, rotation, and dilation. We choose one endpoint to be $z(0) = 0$. After a Möbius transformation, then, we consider only the arcs of logarithmic spirals

$$u \in [0, v] \to z(u) = \left(\exp\left[(1+ia)u/\sqrt{a}\right] - 1\right)/(1+ia), \tag{6}$$

that connect circle elements $(0, 0, a)$ and $(z(v), v\sqrt{a}, a\ \exp(-v/\sqrt{a}))$, for $a, v > 0$.

The Kerzman-Stein invariant for a logarithmic spiral. Here, θ is exactly the argument of the vector that is gotten by reflecting the tangent vector $z'(v)$ across the line segment connecting $z(0) = 0$ to $z(v)$. We choose the branch of the argument to be the one that makes θ into a continuous function starting with $\theta = 0$ at $v = 0$. Therefore,

$$\begin{aligned}\theta(a, v) = \arg\left(\frac{z(v)}{\overline{z(v)}} e^{-iv\sqrt{a}}\right) &= \arg\left(\frac{1-ia}{1+ia}\frac{e^{(1+ia)v/\sqrt{a}} - 1}{e^{(1-ia)v/\sqrt{a}} - 1} \cdot e^{-iv\sqrt{a}}\right) \\ &= \arg\left(\frac{1-ia}{1+ia}\frac{e^{(1+ia)v/(2\sqrt{a})} - e^{-(1+ia)v/(2\sqrt{a})}}{e^{(1-ia)v/(2\sqrt{a})} - e^{-(1-ia)v/(2\sqrt{a})}}\right) \\ &= 2 \cdot \arg\left((1-ia)\sinh\left[(1+ia)v/(2\sqrt{a})\right]\right). \end{aligned} \tag{7}$$

Using the identity $\sinh(u + iv) = \sinh u \cos v + i \cosh u \sin v$, we have

$$\begin{aligned}\theta(a, v) &= 2\tan^{-1}\left[\frac{\cosh(\frac{v}{2\sqrt{a}})\ \sin(\frac{v\sqrt{a}}{2}) - a\sinh(\frac{v}{2\sqrt{a}})\ \cos(\frac{v\sqrt{a}}{2})}{\sinh(\frac{v}{2\sqrt{a}})\ \cos(\frac{v\sqrt{a}}{2}) + a\cosh(\frac{v}{2\sqrt{a}})\ \sin(\frac{v\sqrt{a}}{2})}\right] \\ &= 2\tan^{-1}\left[\frac{\tan(\frac{v\sqrt{a}}{2}) - a\tanh(\frac{v}{2\sqrt{a}})}{\tanh(\frac{v}{2\sqrt{a}}) + a\tan(\frac{v\sqrt{a}}{2})}\right].\end{aligned} \tag{8}$$

For fixed $a > 0$, we choose the branch of $\tan^{-1}$ that makes the right hand side approach 0 when v approaches 0. Then we extend continuously for $v > 0$. □

The Coxeter invariant for a logarithmic spiral The radii of the osculating circles at the endpoints are reciprocal to the endpoint curvatures—namely, $1/a$ and

$\exp(v/\sqrt{a})/a$. The centers of these circles are i/a and $z(v) + iz'(v) \cdot \exp(v/\sqrt{a})/a$, and the distance-squared between them is

$$\begin{aligned} &\left| \frac{i}{a} - \frac{e^{(1+ia)v/\sqrt{a}} - 1}{1+ia} - i\, e^{iv\sqrt{a}} \frac{e^{v/\sqrt{a}}}{a} \right|^2 \\ &\qquad = \left| \frac{i(1+ia) - a(e^{(1+ia)v/\sqrt{a}} - 1) - i(1+ia)e^{(1+ia)v/\sqrt{a}}}{a(1+ia)} \right|^2 \\ &\qquad = \left| \frac{i - ie^{(1+ia)v/\sqrt{a}}}{a(1+ia)} \right|^2 = \frac{1 + e^{2v/\sqrt{a}} - 2e^{v/\sqrt{a}} \cos(v\sqrt{a})}{a^2(1+a^2)}. \end{aligned}$$

So we get

$$\begin{aligned} \delta(a,v) &= \cosh^{-1} \left| \frac{\frac{1 + e^{2v/\sqrt{a}} - 2e^{v/\sqrt{a}} \cos(v\sqrt{a})}{a^2(1+a^2)} - \frac{1}{a^2} - \frac{e^{2v/\sqrt{a}}}{a^2}}{2 \cdot \frac{1}{a} \cdot \frac{e^{v/\sqrt{a}}}{a}} \right| \\ &= \cosh^{-1} \left| \frac{(1 + e^{2v/\sqrt{a}} - 2e^{v/\sqrt{a}} \cos(v\sqrt{a})) - (1+a^2)(1+e^{2v/\sqrt{a}})}{2 \cdot e^{v/\sqrt{a}} \cdot (1+a^2)} \right| \\ &= \cosh^{-1} \left| \frac{-a^2(1+e^{2v/\sqrt{a}})}{(1+a^2)\, 2 \cdot e^{v/\sqrt{a}}} - \frac{\cos(v\sqrt{a})}{1+a^2} \right| \\ &= \cosh^{-1} \left| \frac{a^2 \cosh(v/\sqrt{a}) + \cos(v\sqrt{a})}{1+a^2} \right|. \end{aligned}$$

□

In Figure 2 we show contour plots for the functions $\theta = \theta(a,v)$ and $\delta = \delta(a,v)$

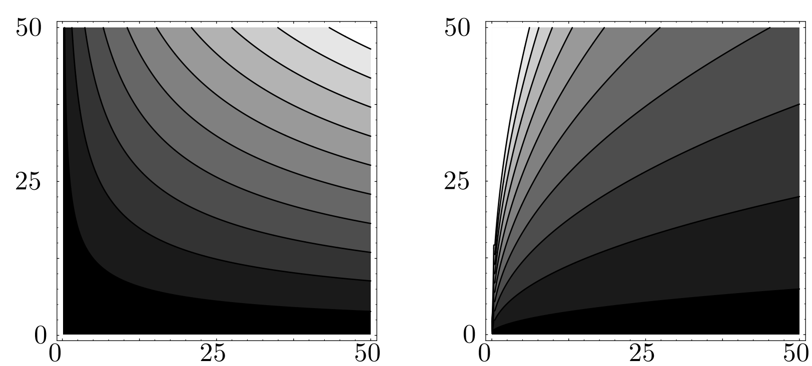

FIGURE 2. Contour plots for $\theta = \theta(a,v)$ and $\delta = \delta(a,v)$.

which were drawn for the region $0.01 \le a, v \le 50$.

To prove the injectivity of the map $(a,v) \to (\theta,\delta)$ it is enough to verify that the tangent lines to the level curves of θ have negative slope, and the tangent lines to the level curves of δ have positive slope. Then each level curve of θ intersects each level curve of δ at most once.

For the slope of the line tangent to a level curve of $\delta = \delta(a,v)$, we differentiate implicitly the quantity

$$\cosh\delta = \frac{a^2\cosh(v/\sqrt{a}) + \cos(v\sqrt{a})}{1+a^2}.$$

(It is a minor point that the quantity on the right hand side is positive for $a, v > 0$.) Then,

$$\begin{aligned}
\frac{\partial v}{\partial a}\Big|_{\delta=\text{cnst}} &= -\frac{\partial(\cosh\delta)}{\partial a}\cdot\frac{\partial v}{\partial(\cosh\delta)}\\
&= -\Bigg[\frac{2a\cosh(\frac{v}{\sqrt{a}}) - \frac{v\sqrt{a}}{2}\cdot\sinh(\frac{v}{\sqrt{a}}) - \frac{v}{2\sqrt{a}}\cdot\sin(v\sqrt{a})}{1+a^2}\\
&\qquad - \frac{2a(a^2\cosh(\frac{v}{\sqrt{a}}) + \cos(v\sqrt{a}))}{(1+a^2)^2}\Bigg]\frac{1+a^2}{a^{3/2}\sinh(\frac{v}{\sqrt{a}}) - \sqrt{a}\sin(v\sqrt{a})}\\
&= \frac{-4a^{3/2}\cosh(\frac{v}{\sqrt{a}}) + 4a^{3/2}\cos(v\sqrt{a}) + v(1+a^2)(a\sinh(\frac{v}{\sqrt{a}}) + \sin(v\sqrt{a}))}{2a(1+a^2)(a\sinh(\frac{v}{\sqrt{a}}) - \sin(v\sqrt{a}))}.
\end{aligned}$$

Likewise, for the slope of the line tangent to a level curve of $\theta = \theta(a,v)$, we differentiate implicitly the quantity

$$\tan\frac{\theta}{2} = \frac{\tan[v\sqrt{a}/2] - a\tanh[v/(2\sqrt{a})]}{\tanh[v/(2\sqrt{a})] + a\tan[v\sqrt{a}/2]} \stackrel{\text{def}}{=} \frac{f(a,v) - a\cdot g(a,v)}{g(a,v) + a\cdot f(a,v)}.$$

Then,

$$\begin{aligned}
\frac{\partial v}{\partial a}\Big|_{\theta=\text{cnst}} &= -\frac{\partial(\tan(\theta/2))}{\partial a}\cdot\frac{\partial v}{\partial(\tan(\theta/2))}\\
&= -\frac{(1+a^2)(f_a g - f g_a) - (f^2+g^2)}{(1+a^2)(f_v g - f g_v)}\\
&= -\frac{\begin{array}{r}(1+a^2)\frac{v}{8a^{3/2}}\cdot\sec^2(\frac{v\sqrt{a}}{2})\operatorname{sech}^2(\frac{v}{2\sqrt{a}})(a\sinh(\frac{v}{\sqrt{a}}) + \sin(v\sqrt{a}))\\ - \sec^2(\frac{v\sqrt{a}}{2})\operatorname{sech}^2(\frac{v}{2\sqrt{a}})(\cosh(\frac{v}{\sqrt{a}}) - \cos(v\sqrt{a}))/2\end{array}}{(1+a^2)/(4\sqrt{a})\cdot\sec^2(\frac{v\sqrt{a}}{2})\operatorname{sech}^2(\frac{v}{2\sqrt{a}})(a\sinh(\frac{v}{\sqrt{a}}) - \sin(v\sqrt{a}))}\\
&= \frac{4a^{3/2}\cosh(\frac{v}{\sqrt{a}}) - 4a^{3/2}\cos(v\sqrt{a}) - v(1+a^2)(a\sinh(\frac{v}{\sqrt{a}}) + \sin(v\sqrt{a}))}{2a(1+a^2)(a\sinh(\frac{v}{\sqrt{a}}) - \sin(v\sqrt{a}))}.
\end{aligned}$$

Some steps are omitted from the next-to-last computation—they use the hyperbolic identity as well as the trigonometric and hyperbolic double angle formulæ. It so happens that this slope is exactly opposite the slope that was gotten for δ. The author first discovered this curious fact using Mathematica.

For the injectivity, we first show that $a\sinh(v/\sqrt{a}) - \sin(v\sqrt{a}) > 0$ for $a, v > 0$. For this, substitute $y = v/\sqrt{a} > 0$ and consider $f(y) = a\sinh y - \sin(ay)$ for fixed

a. Then $f(0) = 0$, and $f'(y) = a\cosh y - a\cos(ay) \geq 0$ with equality if and only if $y = 0$. So then $f(y) > 0$ for all $y > 0$, and the assertion is proved.

To prove injectivity, then, it is enough to verify that

$$\begin{aligned} &4a^{3/2}\cosh(v/\sqrt{a}) - 4a^{3/2}\cos(v\sqrt{a}) \\ &\qquad -v(1+a^2)(a\sinh(v/\sqrt{a}) + \sin(v\sqrt{a})) \;<\; 0 \end{aligned} \tag{9}$$

for all $a, v > 0$.

Proof of inequality (9). After substituting $a = z/y$ and $v = \sqrt{yz}$ (so $y = v/\sqrt{a}$, $z = v\sqrt{a}$), and after multiplying by $-y^{5/2}/\sqrt{z}$ and rearranging terms, inequality (9) is equivalent to

$$4yz(\cos z - \cosh y) + y(y^2+z^2)\sin z + z(y^2+z^2)\sinh y > 0$$

for all $y, z > 0$. To establish this inequality, we expand the left hand side in a power series in terms of y. Then,

$$\begin{aligned} LHS &= 4yz\,\cos z - 4yz\cdot\sum_{j=0}^{\infty}\frac{y^{2j}}{(2j)!} + y(y^2+z^2)\sin z \\ &\qquad\qquad + z(y^2+z^2)\sum_{j=0}^{\infty}\frac{y^{2j+1}}{(2j+1)!} \\ &= y(4z\cos z - 4z + z^2\sin z + z^3) + y^3(-2z + \sin z + z + z^3/6) \\ &\qquad -4yz\cdot\sum_{j=2}^{\infty}\frac{y^{2j}}{(2j)!} + zy^2\cdot\sum_{j=2}^{\infty}\frac{y^{2j-1}}{(2j-1)!} + z^3\cdot\sum_{j=2}^{\infty}\frac{y^{2j+1}}{(2j+1)!}. \end{aligned}$$

Clearly the last of the five terms on the right hand side is positive. We show first that the sum of the 3rd and 4th terms is positive—

$$-4yz\cdot\sum_{j=2}^{\infty}\frac{y^{2j}}{(2j)!} + zy^2\cdot\sum_{j=2}^{\infty}\frac{y^{2j-1}}{(2j-1)!} \;=\; yz\cdot\sum_{j=2}^{\infty}\left(\frac{-4+2j}{(2j)!}\right)y^{2j} \;>\; 0.$$

To establish (9) it is enough then to verify the inequalities for $z > 0$:

$$4\cos z - 4 + z\sin z + z^2 \;>\; 0 \tag{10}$$

$$-z + \sin z + z^3/6 \;>\; 0. \tag{11}$$

To verify (11), let $f(z) = -z + \sin z + z^3/6$. Then $f'(z) = -1 + \cos z + z^2/2$ and $f''(z) = -\sin z + z$. Since $f(0) = 0$ and $f'(0) = 0$, and $f''(z) \geq 0$ for $z \geq 0$ with equality only for $z = 0$, we conclude that $f(z) > 0$ for $z > 0$. This establishes (11). To verify (10), let $f(z) = 4\cos z - 4 + z\sin z + z^2$, so $f'(z) = -3\sin z + z\cos z + 2z$. Then both $f(0) = 0$ and $f'(0) = 0$, so it is enough to show that $f'(z) > 0$ for $z > 0$. Since $|-3\sin z + z\cos z| \leq 3 + z$, it follows that $f'(z) \geq 2z - (3+z) = z - 3 > 0$ for $z > 3$. We must then check that $f'(z) > 0$ for $0 < z \leq 3$, and for this we use power series—

$$
\begin{aligned}
-3\sin z &+ z\cos z + 2z \\
&= -3\left[z + \sum_{j=1}^{\infty}\frac{(-1)^j z^{2j+1}}{(2j+1)!}\right] + z\left[1 + \sum_{j=1}^{\infty}\frac{(-1)^j z^{2j}}{(2j)!}\right] + 2z \\
&= \sum_{j=1}^{\infty}\left[-3\frac{(-1)^j}{(2j+1)!} + \frac{(-1)^j}{(2j)!}\right] z^{2j+1} = \sum_{j=2}^{\infty}(-1)^j \frac{2j-2}{(2j+1)!} z^{2j+1} \\
&= \sum_{k=1}^{\infty}\left[\frac{4k-2}{(4k+1)!} z^{4k+1} - \frac{4k}{(4k+3)!} z^{4k+3}\right] \\
&= \sum_{k=1}^{\infty}\left[(4k-2)(4k+3)(4k+2) - 4k\cdot z^2\right]\frac{z^{4k+1}}{(4k+3)!}.
\end{aligned}
$$

Here,

$$
\begin{aligned}
(4k-2)(4k+3)(4k+2) - 4k\cdot z^2 &\geq (4k-2)(4k+3)(4k+2) - 36k \\
&= 64k^3 + 48k^2 - 52k - 12 > 0
\end{aligned}
$$

since $0 < z \leq 3$ and $k \geq 1$. This establishes (10), and therefore (9) as well. □

What needs to be proved, then, is the surjectivity of $(a, v) \to (\theta, \delta)$. For this, we first show that if $a > 0$ is fixed, then $\theta(a, v)$ increases with v and attains all positive values. We then restrict to a level curve of θ, where we may assume $v = v(a)$, and we establish both

$$\lim_{a\to 0^+} \delta(a, v(a)) = +\infty \tag{12}$$

$$\lim_{a\to +\infty} \delta(a, v(a)) = 0. \tag{13}$$

This is enough to establish surjectivity, then Lemma 7 and Proposition 2 will be proved.

For the first statement, we start with (7) and find

$$
\begin{aligned}
\frac{\theta}{2} = \arg&\left[(1-ia)\sinh\left(\frac{(1+ia)v}{2\sqrt{a}}\right)\right] \\
&= -\tan^{-1} a + \tan^{-1}\left[\frac{\tan(v\sqrt{a}/2)}{\tanh(v/(2\sqrt{a}))}\right].
\end{aligned}
$$

With $a > 0$ fixed, we choose the value for $\tan^{-1} a$ that is between 0 and $\pi/2$. For $\tan^{-1}\left[\tan(v\sqrt{a}/2)/\tanh(v/(2\sqrt{a}))\right]$ we then use the value that makes the sum equal to zero when $v = 0$, and extend continuously for $v > 0$. We first show that $v \to \tan(v\sqrt{a}/2)/\tanh(v/(2\sqrt{a}))$ is an increasing function wherever it is defined.

For this we make the substitution $x = v\sqrt{a}/2$ and compute

$$\begin{aligned}\frac{d}{dx}\,\frac{\tan x}{\tanh(x/a)} &= \frac{\sec^2 x \tanh(x/a) - a^{-1}\tan x\,\mathrm{sech}^2(x/a)}{\tanh^2(x/a)}\\ &= \frac{\sec^2 x\,\mathrm{sech}^2(x/a)}{a\tanh^2(x/a)}\,[a\,\sinh(x/a)\cosh(x/a) - \sin x\,\cos x]\\ &= \frac{\sec^2 x\,\mathrm{sech}^2(x/a)}{2a\,\tanh^2(x/a)}\,[a\,\sinh(2x/a) - \sin 2x].\end{aligned}$$

In the paragraph preceding (9), we showed that the last quantity in brackets is positive, so then both $x \to \tan x/\tanh(x/a)$ and $v \to \tan(v\sqrt{a}/2)/\tanh(v/(2\sqrt{a}))$ are increasing. Furthermore, if v is large, then $\tanh(v/(2\sqrt{a})) \approx 1$, and

$$\frac{\theta(a,v)}{2} \;\approx\; -\tan^{-1} a + \tan^{-1}\big(\tan(v\sqrt{a}/2)\big) \;\approx\; -\tan^{-1} a + \frac{v\sqrt{a}}{2}.$$

The quantity on the right hand side of this estimate can be made arbitrarily large by taking v large. We have left then to verify the two limits, (12) and (13).

Proof of (12)-(13). Again we use the substitution $x = v\sqrt{a}/2$. Then on the level curve of θ, where $x = x(a)$, we have

$$\tan\big(\theta/2 + \tan^{-1} a\big) \;=\; \frac{\tan x}{\tanh(x/a)} \tag{14}$$

and

$$\cosh\delta(a, v(a)) = \frac{a^2\cosh(2x/a) + \cos 2x}{1+a^2}. \tag{15}$$

For the first limit, start with values of a that are small enough so that if $a \downarrow 0$, then $\tan(\theta/2 + \tan^{-1} a) \downarrow \tan(\theta/2)$. In case θ is an odd multiple of π, take $\tan(\theta/2) = -\infty$. Then, after restricting to a level curve of θ, we claim that $x = x(a)$ is bounded below for these small values of a. Once this is known, then as $a \to 0^+$,

$$\cosh\delta(a, v(a)) \;\approx\; \frac{a^2\exp(2x/a)/2 + \cos 2x}{1+a^2} \;\approx\; a^2\exp(2x/a)/2 \;\to\; +\infty,$$

and the first limit is proved.

To prove the claim, consider the case $0 < \theta < \pi$. If $x = x(a)$ is *not* bounded below, then there is a sequence $\{a_n\}$ with $a_n \downarrow 0$ so that $x(a_n) \to 0$. But then, since $\tan x/\tanh(x/a) = \tan(\theta/2 + \tan^{-1} a) \downarrow \tan(\theta/2) \neq 0$, using (14), it would follow that $\tanh(x(a_n)/a_n) \to 0$. This would also mean that

$$\frac{\tan x(a_n)}{\tanh(x(a_n)/a_n))} \;\approx\; \frac{x(a_n)}{x(a_n)/a_n} \;=\; a_n \;\to\; 0 \;\neq\; \tan(\theta/2),$$

a contradiction. So the claim is true for $0 < \theta < \pi$.

For the case $\theta \geq \pi$, the claim follows from the previous case and from the fact, shown above, that $\theta(a,x)$ increases with x for any fixed value of a. In particular, suppose there is a sequence $\{a_n\}$ with $a_n \downarrow 0$ and $x_\theta(a_n) \to 0$, where $x_\theta = x_\theta(a)$ is defined using the level curve corresponding to a fixed $\theta \geq \pi$. For the same sequence, one has $x_{\theta'}(a_n) < x_\theta(a_n)$ for any $0 < \theta' < \pi$, since $\theta = \theta(a,x)$ increases with x. This then means that $x_{\theta'}(a_n) \to 0$, contradicting the claim for the previous case.

For the second limit, start with values of a that are large enough so that if $a \uparrow \infty$, then $\tan(\theta/2 + \tan^{-1} a) \uparrow \tan((\theta + \pi)/2)$. In case θ is an even multiple of π, take $\tan((\theta + \pi)/2) = +\infty$. Then, on the level curve of θ, we claim $x = x(a)$ is bounded above for these large values of a. Once this is known, then as $a \to +\infty$,

$$\cosh \delta(a, v(a)) \;=\; \frac{a^2 \cosh(2x/a) + \cos 2x}{1 + a^2} \;\approx\; \frac{a^2 \cdot 1}{a^2} \;\to\; 1,$$

and the second limit is proved. The claim follows from the following observation—if $a \uparrow \infty$, the left hand side of (14) increases to $\tan((\theta + \pi)/2)$ while $x = x(a)$ varies continuously with respect to a. So for (14) to remain true as $a \uparrow \infty$, it is necessary that x remain bounded between consecutive odd multiples of $\pi/2$. (If x increases past an odd multiple of $\pi/2$, then the right hand side of (14) jumps from $+\infty$ to $-\infty$.) So $x(a)$ is bounded. $\square$

Remark: A further analysis reveals that as $a \to 0^+$ one has $x(a) = \theta/2 + \epsilon(a)$ where $\epsilon(a) \to 0$. This is also apparent (up to scaling) from the contour plot of $\theta = \theta(a, x)$ given in Figure 3. Furthermore, we find that as $a \to +\infty$ one has

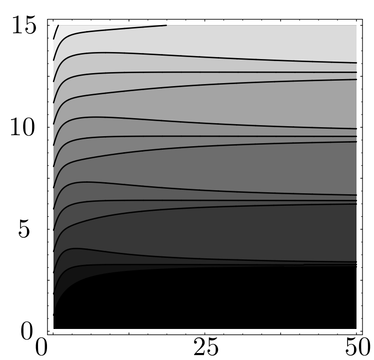

FIGURE 3. Contour plot for $\theta = \theta(a, x)$ where $x = v\sqrt{a}/2$.

$x(a) = \lceil \theta/(2\pi) \rceil \cdot \pi + \epsilon(a)$ where $\epsilon(a) \to 0$ and where $\lceil \cdot \rceil$ is the ceiling function. This, too, is apparent (up to scaling) from the plot given in Figure 3.

We now establish a context in which the logarithmic spirals are global maximizers of inversive arclength.

Proposition 3. *Suppose γ is a three times differentiable curve from p to q that has decreasing curvature, κ. Suppose the amount of winding of the tangent vector along γ is $\Delta\phi = \int_\gamma \kappa\, ds$, and suppose γ has curvature $\kappa_p > 0$ at p and $\kappa_q > 0$ at q. If $L(\gamma)$ denotes the inversive length of γ, then*

$$L(\gamma)^2 \;\le\; \Delta\phi \cdot \log(\kappa_p/\kappa_q).$$

There is equality if and only if γ is a rotated and translated image of the logarithmic spiral $r \in [s,t] \to re^{ia \log r}/(1+ia)$, where

$$a = \Delta\phi / \log(\kappa_p/\kappa_q), \quad s = a/\kappa_p, \quad \text{and} \quad t = a/\kappa_q. \tag{16}$$

Remark: Since γ has decreasing curvature and $\kappa_q > 0$, the curvature of γ must always be positive. It follows that $\Delta\phi > 0$. Moreover, since $\kappa_p > \kappa_q$, it also follows that $a > 0$ and $0 < s < t$.

Proof. We use the following version of the Hölder inequality. (See Hardy, Littlewood, and Pólya [4, p.140].) Let $0 < k < 1$ or $k < 0$, and let $1/k + 1/k' = 1$. If $f \geq 0$ and $g \geq 0$, then

$$\left(\int f^k\right)^{1/k} \left(\int g^{k'}\right)^{1/k'} \leq \int fg,$$

with equality if and only if $f^k = c \cdot g^{k'}$ for some constant c, or if f or g is identically zero. In our application, both f and g will be nonzero.

First, define constants $a, s, t > 0$ using (16), and let $r \in [s,l] \to \kappa(r)$ be the curvature function for γ using r as the arclength parameter. Assume that $r = s$ and $r = l$ correspond to $p \in \gamma$ and $q \in \gamma$, respectively. Then, applying the Hölder inequality with $k = 1/2$, $k' = -1$, $f(r) = |\kappa'(r)| = -\kappa'(r)$, and $g(r) = r$, we find

$$\begin{aligned} L(\gamma)^2 \cdot \left(\int_s^l \frac{1}{r}\, dr\right)^{-1} &\leq \int_s^l -\kappa'(r) \cdot r\, dr \;=\; -r \cdot \kappa(r)\,\Big|_s^l + \int_s^l \kappa(r)\, dr \\ &= -l \cdot \kappa(l) + s \cdot \kappa(s) + \Delta\phi \;=\; -l \cdot \frac{a}{t} + a + a \log \frac{t}{s}. \end{aligned}$$

We conclude that

$$L(\gamma)^2 \leq \log \frac{l}{s} \cdot \left(-l \cdot \frac{a}{t} + a + a \log \frac{t}{s}\right), \tag{17}$$

with equality if and only if $|\kappa'(r)|^{1/2} = c/r$ for all r and for some constant c.

Next we claim that

$$\log \frac{l}{s} \cdot \left(-\frac{l}{t} + 1 + \log \frac{t}{s}\right) \leq \left(\log \frac{t}{s}\right)^2$$

with equality if and only if $l = t$. For this we use $s \leq l < \infty$, and we replace s and t with $u = \log(l/s) \geq 0$ and $v = \log(t/s) \geq 0$. Then also $e^u = l/s$ and $e^v = t/s$, and the claim says

$$u\left(-e^{u-v} + 1 + v\right) \leq v^2,$$

with equality if and only if $u = v$. Now the function $u \to u(-e^{u-v} + 1 + v)$ has value 0 at $u = 0$, and approaches $-\infty$ as $u \to +\infty$. Its only critical point is where

$$(-e^{u-v} + 1 + v) + u(-e^{u-v}) = 0 \iff e^v(1+v) = e^u(1+u) \iff u = v.$$

Since the value of this function at $u = v$ is v^2, the claim is proved.

Combining (17) with the claim, we have

$$L(\gamma)^2 = \left(\int_s^l |\kappa'(r)|^{1/2}\,dr\right)^2 \le a\left(\log\frac{t}{s}\right)^2 = \Delta\phi\cdot\log(\kappa_p/\kappa_q), \tag{18}$$

with equality if and only if both

i) $l = t$, and
ii) $|\kappa'(r)|^{1/2} = c/r$ for constant c.

But these conditions, along with equality in (18), require first that $c = \sqrt{a}$ and then $\kappa'(r) = -a/r^2$. Since also $\kappa(s) = \kappa_p = a/s$, it follows after integrating that $\kappa(r) = a/r$ for all r. This is exactly the curvature equation for the spiral defined in the proposition, so we are done. □

We come finally to the proof of Theorem 2, which uses both Proposition 2 and Proposition 3. The only complication is to check that in a normalized setting, the θ invariant determines the amount of winding of the tangent vector. This technical aspect of the proof uses a continuity argument.

Proof of Theorem 2. Suppose that γ is a curve from p to q that has decreasing curvature and γ^* is the loxodromic arc that connects the same circle elements and has the same θ invariant as γ. After a translation and rotation, we may assume that γ starts at $p = 0$, and the tangent vector there has angle $\phi_p = 0$. Then after a further Möbius transformation we may also assume that γ^* is the logarithmic spiral that has the arclength parameterization

$$z : r \in [1, s] \to z(r) = (re^{ia\log r} - 1)/(1 + ia) \tag{19}$$

for certain $a > 0$, $s > 1$. The curvature functions for γ and γ^* are then not only decreasing, but also positive, since γ^* has positive curvature at $z(s)$. We will show that the tangent vectors of γ and γ^* have the same amount of winding. After this, Theorem 2 follows immediately from Proposition 3.

To do this, we construct a family of loxodromic arcs and use a continuity argument. Let $d = d_{q'} > 0$ be a continuous function of $q' \in \gamma$ such that $d_p = |\kappa'_p/a|^{1/4}$ and $d_q = 1$. (We will soon specify the function d.) Here, κ'_p is the derivative of the curvature function for γ taken with respect to arclength and evaluated at p. Then, for $q' \ne p$, $d = d_{q'}$ determines a loxodromic arc $\gamma^*_{q'}$ as follows:

(1) Let $\theta = \theta_{q'} > 0$ be the θ invariant of the subarc $\gamma_{q'} \subset \gamma$ that connects p to q'.
(2) Using the (fixed) value of $a > 0$ determined by γ^* in (19), the value $\theta_{q'}$ determines a value $v = v_{q'} > 0$ as in the proof of surjectivity in Lemma 7.
(3) Define $s = s_{q'} > 1$ according to $v_{q'} = \sqrt{a}\,\log s_{q'}$. The parameters a and $s_{q'}$ then determine a logarithmic spiral $z = z_{q'}$ as in (19).
(4) Let $\mu = \mu_{q'}$ be the Möbius transformation $\mu = d^{-1}z/(cz+d)$ where $d = d_{q'}$, and where $c = c_{q'} \overset{\text{def}}{=} (dq')^{-1} - dz(s)^{-1}$ for $d = d_{q'}$, $z = z_{q'}$, $s = s'_q$.
(5) Let $\gamma^*_{q'}$ be the loxodromic arc $\gamma^*_{q'} = \mu_{q'} \circ z_{q'}$. The inversive length of $\gamma^*_{q'}$ is $v_{q'}$ and its θ invariant is $\theta_{q'}$.

Using this construction, the arc $\gamma^*_{q'}$ connects the same line elements as the subarc $\gamma_{q'} \subset \gamma$, and it also has the same θ invariant. Moreover, each parameter θ, v, s, and c varies continuously with respect to $q' \in \gamma$, except that $c = c_{q'}$ may not extend continuously at $q' = p$. Moreover, as $q' \to p$, one has $\theta \to 0$, $v \to 0$, and $s \to 1$. Finally, the condition $d_p = |\kappa'_p/a|^{1/4}$ ensures the estimate $c_{q'} = o(1/|q'|)$ for q' near p, and the condition $d_q = 1$ ensures that $c_q = 0$, $\mu_q = z$, and $\gamma^*_q = \gamma^*$.

Next, for a particular choice of $d = d_{q'}$, we *claim that the amount of winding of the tangent vector for $\gamma^*_{q'}$ varies continuously with $q' \neq p$ and approaches zero as $q' \to p$.* Evidently this is true for the amount of winding for $\gamma_{q'}$, and as well, the amount of winding for $\gamma^*_{q'}$ must agree with the amount of winding for $\gamma_{q'}$ except for possibly a multiple of 2π. Once the claim is proved, then, the two amounts must be equal for all $q' \neq p$. In particular, this is true for $q' = q$, and the theorem follows.

To establish the claim, we first express the curvature of $\gamma^*_{q'}$ using a computation like the one from Lemma 6. The arclength parameter for $\gamma^*_{q'}$, call it u, is related to the arclength parameter for $z_{q'}$ according to $du/dr = |cz(r) + d|^{-2}$ for $c = c_{q'}$, $z = z_{q'}$, and $d = d_{q'}$. Then the curvature of $\gamma^*_{q'}$ at $\mu_{q'} \circ z_{q'}(r)$ is given by

$$\frac{\dfrac{d^2(\mu\circ z)}{du^2}}{i\cdot\dfrac{d(\mu\circ z)}{du}} = \frac{\dfrac{d}{dr}\left(\dfrac{\overline{cz+d}}{cz+d}\,\dfrac{dz}{dr}\right)}{i\cdot\dfrac{1}{(cz+d)^2}\,\dfrac{dz}{dr}}$$

$$= -2\,\mathrm{Im}\Big[cz'(r)\big(\overline{cz(r)+d}\big)\Big] + |cz(r)+d|^2\cdot\frac{a}{r}, \tag{20}$$

and the amount of winding of the tangent vector for $\gamma^*_{q'}$ is given by

$$\int_1^s \left(-2\,\mathrm{Im}\left[cz'(r)\big(\overline{cz(r)+d}\big)\right] + |cz(r)+d|^2\cdot\frac{a}{r}\right)\cdot\frac{du}{dr}\,dr \tag{21}$$

$$= \int_1^s \left(-2\,\mathrm{Im}\left[\frac{cz'(r)}{cz(r)+d}\right] + \frac{a}{r}\right)dr$$

$$= -2\arg\left[cz(s)+d\right] + 2\arg d + a\log s = -2\arg\left[z(s)/q'\right] + a\log s.$$

The parameters in the two integrals, namely s, c, and d, vary continuously with respect to q', so the first part of the claim is proved once we show that $cz(r)+d \neq 0$ for $1 \leq r \leq s$.

For this, we choose the function $d = d_{q'}$ that makes the curvature of $\gamma^*_{q'}$ at q' agree with the curvature of $\gamma_{q'}$ at q'. This curvature is positive, and since $\gamma^*_{q'}$ has decreasing curvature, it then follows that $\gamma^*_{q'}$ must have everywhere positive curvature. From (20), it then follows that $cz(r) + d \neq 0$ for $1 \leq r \leq s$.

To find d, let $\kappa = \kappa_{q'}$ denote the curvature of $\gamma_{q'}$ at q'. Then, after the substitutions $cz(s) + d = (dq')^{-1}z(s)$ and $c = (dq')^{-1} - dz(s)^{-1}$ in (20), we find

$$d^2_{q'} = \frac{|z(s)|^2\cdot a/s - 2\,\mathrm{Im}\left[z'(s)\overline{z}(s)\right]}{\kappa_{q'}|q'|^2 - 2\,\mathrm{Im}\left[q'\cdot z'(s)\overline{z}(s)/z(s)\right]}$$

$$= \frac{1}{1+a^2}\,\frac{2\,\sin\left(a\log s\right) + a(1/s - s)}{\kappa_{q'}|q'|^2 - 2\,\mathrm{Im}\left[q'\,e^{-i\theta}\right]}. \tag{22}$$

The numerator in the second expression for d^2 is always negative. To see this, substitute $y = \log s$ and use the inequality $-a\sinh y + \sin(ay) < 0$ for $y > 0$, established during the proof of Lemma 7. The denominator is also negative. This arises from the fact that the circle elements at the endpoints of $\gamma_{q'}$ are properly nested. In particular, the circle centered at $q' + ie^{i\phi_{q'}}/\kappa_{q'} = q' + ie^{-i\theta}q'/(\overline{q}'\kappa_{q'})$ with radius $1/\kappa_{q'}$ must enclose the origin.

We mention that since $s = s_{q'}$, $\kappa = \kappa_{q'}$, and $\theta = \theta_{q'}$ are continuous, it follows that $d = d_{q'} > 0$ defined in (22) is well-defined and continuous. Furthermore, if $q' = q$, then $z(s) = q$ and $\kappa = a/s$, since γ^* and γ connect the same circle elements. So we conclude from the first expression for d^2 that $d_q = 1$. For the behavior as $q' \to p$, we estimate

$$\begin{aligned} \frac{a\,(1/s - s) + 2\sin(a\log s)}{1+a^2} &= -\frac{a}{3}(s-1)^3 + O((s-1)^4) \\ &= -\frac{v^3}{3\sqrt{a}} + O(v^4) \\ = -\frac{(6\theta)^{3/2}}{3\sqrt{a}} + O(\theta^2) &= -\frac{(-\kappa_p')^{3/2}\tilde{s}^3}{3\sqrt{a}} + o(\tilde{s}^3). \end{aligned} \tag{23}$$

Here, the third estimate is gotten by expanding (8) to find $v = v(\theta)$, and the last estimate is gotten by expanding the right hand side of (1) in terms of the Euclidean length $\tilde{s} = \tilde{s}_{q'}$ of the arc $\gamma = \gamma_{q'}$. A similar estimate gives

$$\kappa_{q'}|q'|^2 - 2\,\mathrm{Im}\left[q'e^{-i\theta}\right] = \kappa_{q'}|q'|^2 - 2\,\mathrm{Im}\left[\overline{q}'e^{i\phi_{q'}}\right] = \frac{\kappa_p'\tilde{s}^3}{3} + o(\tilde{s}^3),$$

so that $d_{q'}^2 = |\kappa_p'/a|^{1/2} + o(1)$ as $q' \to p$.

We have yet to establish that the amount of winding in (21) approaches zero as $q' \to p$. Using the same estimates as in (23), we find that $s = s_{q'}$ and $\tilde{s} = \tilde{s}_{q'}$ are related by

$$s - 1 = \frac{v}{\sqrt{a}} + O(v^2) = \frac{\sqrt{6\theta}}{\sqrt{a}} + O(\theta) = \left|\frac{\kappa_p'}{a}\right|^{1/2}\tilde{s} + o(\tilde{s}),$$

so

$$\begin{aligned} c = \frac{1}{d}\left(\frac{1}{q'} - \frac{d^2}{z(s)}\right) = \frac{1}{d}\left(\frac{1}{\tilde{s}} - \left|\frac{\kappa_p'}{a}\right|^{1/2}\frac{1}{s-1} + o\left(\frac{1}{s-1}\right)\right) \\ = o\left(\frac{1}{\tilde{s}}\right) = o\left(\frac{1}{s-1}\right). \end{aligned}$$

Then, since $|z'(r)| = 1$ and $cz(r) = o((s-1)^{-1}) \cdot O(s-1) = o(1)$ for $1 \le r \le s$, we find

$$\begin{aligned} \left|\int_1^s \left(-2\,\mathrm{Im}\left[\frac{cz'(r)}{cz(r)+d}\right] + \frac{a}{r}\right)dr\right| &\le 2\int_1^s \left|\frac{cz'(r)}{cz(r)+d}\right| dr + a\log s \\ &= 2(s-1)\cdot o\left(\frac{1}{s-1}\right)\cdot O(1) + a\cdot O(s-1) = o(1) \end{aligned}$$

for $s \to 1^+$, and the theorem is proved. □

6. Further Observations

Here we write down two observations that are related to what has already been established.

6.1. *The function* $\theta = \theta_\gamma(p, q')$ *is an increasing (resp., decreasing) function of* $q' \in \gamma$ *from* p *to* q *provided* γ *has decreasing (resp., increasing) curvature.*

Proof. Assume that γ connects p to q and has decreasing curvature. If s is the arclength parameter for γ, then $d\theta/ds$ at $q' \in \gamma$ depends on nothing beyond the second order information of γ at q'. Let $\gamma_{q'} \subset \gamma$ be the subarc that connects p to q', and let $\gamma^*_{q'}$ be the loxodromic arc that connects the same circle elements as $\gamma_{q'}$ and has the same θ invariant as $\gamma_{q'}$. Along $\gamma^*_{q'}$, the quantity $d\theta/ds^*$ is positive, where s^* is the arclength parameter for $\gamma^*_{q'}$. Since $\gamma_{q'}$ and $\gamma^*_{q'}$ have the same second order information at q', it must then be true that $d\theta/ds$ is positive for γ at q'. So the claim is proved. □

6.2. *For arcs with monotone curvature that connect the same line elements and have the same* θ *invariant (no restriction on second order information), the inversive arclength can be made arbitrarily large or small.*

Proof. This is evident even within the family of loxodromic arcs. In particular, on a level curve of $\theta = \theta(a, v)$ we have seen that $\theta(a, v)/2 \lesssim x \lesssim \lceil \theta(a, v)/2\pi \rceil \pi$ where $x = v\sqrt{a}/2$. See Figure 3. This means that the inversive length of the loxodromic arc is comparable to $\theta/\sqrt{a}$. This can be made arbitrarily large by choosing a small, and it can be made arbitrarily small by choosing a large. So the claim is proved. □

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