Input-Output Finite-Time Stabilization of Linear Systems with Input Constraints

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Abstract

In this paper the problem of input-output finite-time stabilization of linear time-varying systems is dealt with. The classical definition of input-output finite-time stability (IO-FTS) is extended to that one of structured IO-FTS, which allows to incorporate, in the definition of the stabilization problem, some amplitude constraints on the control input variables. A sufficient condition and a necessary and sufficient condition for structured IO finite-time stabilization are provided in the case of $L_\infty$ and $L_2$ inputs, respectively. Such conditions require the existence of a solution to a certain differential linear matrix inequality (DLMI). The theory is applied to design the active suspension control system for a two-degree-of-freedom quarter-car model.

Keywords: Input-output finite-time stability, Differential Linear Matrix Inequalities, input constraints.

I. INTRODUCTION

In some recent papers the concept of input-output finite-time stability (IO-FTS) has been introduced: see [1], [2] in the context of linear systems, [3], [4] in the context of switching and hybrid systems, and [5] in the stochastic framework. Roughly speaking, a system is said to be IO-FTS if, given a class of norm bounded input signals defined over a specified time interval of length $T$, the outputs of the system do not exceed an assigned threshold during such time interval.

In order to correctly frame the definition of IO-FTS in the current literature, we recall that a system is said to be IO $L_p$-stable [6, Ch. 5] if for any input of class $L_p$, the system exhibits a

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corresponding output which belongs to the same class. The main differences between *classic* IO-stability and IO-FTS are that the latter involves signals defined over a finite-time interval, does not necessarily require the inputs and outputs to belong to the same class, and that *quantitative* bounds on both inputs and outputs must be specified. Therefore, IO-stability and IO-FTS are independent concepts.

While IO-stability deals with the behavior of a system within a sufficiently long (in principle infinite) time interval, IO-FTS is a more practical concept, useful to study the behavior of the system within a finite (possibly short) interval, and therefore it finds application whenever it is desired that the output variables do not exceed a given threshold during the transients, given a certain class of input signals. It is important to remark that the definition of IO-FTS given in [1] is consistent with the definition of (state) FTS (see, [7], [8], [9], [10], [11], [12] among others), where the state of a zero-input system, rather than the input and the output, is involved.

In the framework of linear systems, two sufficient conditions for IO-FTS, for the class of $L_\infty$ inputs and the class of $L_2$ inputs, respectively, have been provided in [1]. Both conditions require the solution of a feasibility problem involving differential linear matrix inequalities (DLMIs, [13]). In [2], LTV systems are seen as linear operators between the $L_2$ and the $L_\infty$ spaces, therefore IO-FTS is interpreted as the $L_2$ to $L_\infty$ gain from the input to the output on a finite-time interval. By following this approach it has been shown that, in the case of $L_2$ inputs, the condition given in [1] is also necessary for IO-FTS. The IO finite-time stabilization problem has been discussed in [1] (state feedback) and [2] (dynamic output feedback).

It should be recalled that, in the special case of a scalar output, $H_2$ control can also be interpreted as the minimization of the $L_2$ to $L_\infty$ gain\(^1\). However, in the general case, IO-FTS and $H_2$ control are completely different concepts. Indeed, $H_2$ control is based on the minimization of a system norm which is not induced by inputs and outputs signal norms. IO-FTS is not related to $H_\infty$ control either, since this is based on the minimization of the $L_2$ to $L_2$ gain (see [15]).

In practical situations, the controller should be designed with the constraint of limiting the effort of the control variables. To this regard, in this paper, we deal with the state feedback IO-FTS problem with constrained control inputs; to achieve this goal, a fictitious system

\(^1\)For the various interpretations of $H_2$ control, the interested readers can refer to [14].
is built, in which the output vector is augmented by the control input variables, which are conceptually dealt with in the same way as the actual outputs. However, since outputs and control inputs need to be constrained separately, the definition given in [1] is extended to that one of structured IO-FTS. As a by-product of this extension, we also create a framework which allows, in the general context of IO-FTS, to partition the output vector and to impose different constraints on each group of partitioned outputs. The theory is illustrated through a practical engineering control problem, namely the design of an active suspension system for a two-degree-of-freedom quarter-car model [16].

In this spirit, the contribution of this paper goes in several directions. First of all we provide a sufficient condition and a necessary and sufficient condition for structured IO-FTS in the case of $L_\infty$ and $L_2$ inputs, respectively. All conditions are provided in terms of DLMIs, which can be efficiently solved, as will be discussed in Section V (for more details, the reader is referred to [13], [2] and [17, Section 2.4]).

Then the state feedback control problem is considered; in this case, the definition of structured IO-FTS allows to take into account, in a natural way, the constraints on the control inputs in the DLM problem (a similar approach has been proposed, in a different context, in [18] and [19]). The theory is finally applied to solve the finite-time control problem of the quarter-car suspension system.

The paper is organized as follows: in Section II some preliminaries are provided, the model of the two-degree-of-freedom quarter-car suspension system is illustrated and the problem we deal with is precisely stated. In Section III a sufficient conditions ($L_\infty$ case) and a necessary and sufficient condition ($L_2$ case) for structured IO-FTS are provided. In Section IV the main results, concerning the existence of a state feedback constrained control law which IO finite-time stabilizes the closed loop system, are stated. The proposed approach is then applied to design the active suspension control system in Section V, where it is shown that the structured IO-FTS technique permits to improve the control performance, if compared with the $H_\infty$ approach proposed in [16]. Eventually, some concluding remarks are given.

II. STRUCTURED IO-FTS: A MOTIVATING EXAMPLE

In this section we first introduce the notation used throughout the paper and recall the definition of IO-FTS provided in [1]. Then, the two-degree-of-freedom quarter-car model by [16] is illustrated. Finally, the definition of structured IO-FTS and the statement of the structured IO finite-time stabilization problem via constrained state feedback are given. As
we shall see, structured IO-FTS generalizes the original concept introduced in [1]; this more general definition allows to consider additional constraints on the control variables when solving the synthesis problem.

A. Notation and preliminary definitions

In the rest of the paper, we will use the symbol \( L_p \) to denote the space of vector-valued signals whose \( p \)-th power is absolutely integrable over \([0, +\infty)\). The restriction of \( L_p \) to \( \Omega := [t_0, t_0 + T] \) is denoted by \( L_p(\Omega) \).

Given the set \( \Omega \), a symmetric positive definite matrix-valued function \( R(\cdot) \), bounded on \( \Omega \), and a vector-valued signal \( s(\cdot) \in L_p(\Omega) \), the weighted norm

\[
\left( \int_{\Omega} [s^T(\tau)R(\tau)s(\tau)]^{\frac{p}{2}}d\tau \right)^{\frac{1}{p}},
\]

will be denoted by \( \|s(\cdot)\|_{p,R} \). If \( p = \infty \)

\[
\|s(\cdot)\|_{\infty,R} = \text{ess sup}_{t \in \Omega} [s^T(t)R(t)s(t)]^{\frac{1}{2}}.
\]

When the weighting matrix \( R(\cdot) \) is time-invariant and equal to the identity matrix \( I \), we will use the simplified notation \( \|s(\cdot)\|_p \).

According to [1], [2], let us consider a linear time-varying (LTV) system in the form

\[
\dot{x}(t) = A(t)x(t) + G(t)w(t), \quad x(t_0) = 0 \tag{1a}
\]
\[
y(t) = C(t)x(t) + F(t)w(t), \tag{1b}
\]

where \( A(\cdot) : \Omega \mapsto \mathbb{R}^{n \times n}, G(\cdot) : \Omega \mapsto \mathbb{R}^{n \times r}, C(\cdot) : \Omega \mapsto \mathbb{R}^{m \times n}, \) and \( F(\cdot) : \Omega \mapsto \mathbb{R}^{m \times r} \).

In the following we assume that all the involved matrices are continuous, unless otherwise stated.

We can now recall the definition of IO-FTS (see [1]).

**Definition 1:** Given a positive scalar \( T \), a class of signals \( \mathcal{W} \) defined over \( \Omega \), a positive definite weighting matrix \( Q(\cdot) \), system (1) is said to be IO-FTS with respect to \((\mathcal{W}, Q(\cdot), \Omega)\) if

\[
w(\cdot) \in \mathcal{W} \Rightarrow y^T(t)Q(t)y(t) < 1, \quad t \in \Omega.
\]

\[\blacktriangle\]

In [2] some necessary and sufficient conditions for IO-FTS have been provided when \( \mathcal{W} \) is the set of the square integrable signals over \( \Omega \). The conditions turn out to be only sufficient when \( \mathcal{W} \) coincides with the set of the signals bounded over \( \Omega \).
The corresponding design problem refers to the following system
\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w(t), \quad x(t_0) = 0 \tag{2a}
\]
\[
y(t) = C(t)x(t) + D(t)u(t) + F(t)w(t), \tag{2b}
\]
with \(B(\cdot) : \Omega \mapsto \mathbb{R}^{n \times q}\), \(D(\cdot) : \Omega \mapsto \mathbb{R}^{m \times q}\), and where \(u(\cdot) : \Omega \mapsto \mathbb{R}^q\) is the control input and \(w(\cdot)\) is the disturbance (exogenous) input. The IO finite-time state feedback stabilization problem consists (see [1]) of designing a control law \(u(t) = K(t)x(t)\) such that the overall closed loop system is IO-FTS; note that, in this formulation, there is no constraints on the control inputs \(u(\cdot)\).

B. Quarter car suspension model

In order to introduce the theoretical problem dealt with in this paper, we consider a typical engineering case-study, namely a vehicle active suspension system. Indeed, the typical constraints that arise in this applicative field can be effectively framed in the IO-FTS context.

The scheme of a two-degree-of-freedom quarter-car model, taken by [16], is reported in Figure 1: the system comprises the sprung mass, \(M_s\), the unsprung mass, \(M_u\), the suspension damper with damping coefficient \(B_s\), the suspension spring with elastic coefficient \(K_s\), the elastic effect caused by the tire deflection, modeled by means of a spring with elastic coefficient \(K_u\), the hydraulic actuator \(S\), generating a scalar active force \(u_f\).
In the following we denote by \( x_s \) and \( x_u \) the vertical displacements of the sprung and unsprung masses, respectively, and by \( x_o \) the vertical ground displacement caused by the road unevenness. The state variables are the suspension stroke \( x_s - x_u \), the tire deflection \( x_u - x_o \), and the derivatives of \( x_s \) and \( x_u \), that is

\[
\begin{align*}
  x_1 &= x_s - x_u \\
  x_2 &= \dot{x}_s \\
  x_3 &= x_u - x_o \\
  x_4 &= \dot{x}_u.
\end{align*}
\]

The resulting open-loop dynamical model reads

\[
\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & -1 \\ \frac{K_s}{M_s} & \frac{B_s}{M_s} & 0 & \frac{B_s}{M_s} \\ 0 & 0 & 0 & 1 \\ \frac{K_u}{M_u} & \frac{B_u}{M_u} & -\frac{K_u}{M_u} & -\frac{B_u}{M_u} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{u_{\text{max}}}{M_s} \\ 0 \\ \frac{u_{\text{max}}}{M_u} \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} w(t),
\tag{3}
\]

where the normalized active force \( u(\cdot) = u_f(\cdot)/u_{\text{max}} \) is the control input and \( w(\cdot) = \dot{x}_o(\cdot) \) represents the disturbance caused by the road roughness.

When designing a controller for an active suspension system, a number of constraints should be considered ([16]). In order to ensure a firm uninterrupted contact of the wheels to the road, the dynamic tire load should not exceed the static one, that is

\[
K_u |x_3(t)| < (M_s + M_u) g, \quad t \geq 0, \tag{4}
\]

and the suspension stroke should fulfill the following constraint

\[
|x_1(t)| \leq SS, \quad t \geq 0, \tag{5}
\]

where \( SS \) is the maximum allowable value for the suspension stroke. Therefore, in order to cast the control design problem into the IO-FTS framework, we consider the following system outputs

\[
\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} \dot{x}_2(t) \\ \frac{x_1(t)}{SS} \\ \frac{K_u x_3(t)}{g(M_s + M_u)} \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} x(t) + \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} u, \tag{6}
\]
where

\[
C_1 = \begin{pmatrix}
-\frac{K_s}{M_s} & -\frac{B_s}{M_s} & 0 & \frac{B_s}{M_s}
\end{pmatrix}, \quad D_1 = \frac{u_{\text{max}}}{M_s},
\]

\[
C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \quad D_2 = 0,
\]

\[
C_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \quad D_3 = 0.
\]

We consider the following values for the model parameters [20], [16]

\[
M_s = 320 \text{ kg}, \quad K_s = 18 \frac{kN}{m},
\]

\[
B_s = 1 \frac{kN \cdot s}{m}, \quad K_u = 200 \frac{kN}{m},
\]

\[
M_u = 40 \text{ kg}, \quad u_{\text{max}} = 1.5 \text{ kN},
\]

\[
SS = 0.08 \text{ m}.
\]

In a real control problem one has to take into account that, due to the actuators saturation, the active force is bounded by \(u_{\text{max}}\), i.e. the normalized force has to satisfy

\[
|u(t)| \leq 1, \quad t \geq 0.
\]  

Eventually, the objective of the active suspension is to keep as small as possible the body acceleration \(\ddot{x}_s(\cdot) = \dot{\ddot{x}}_2(\cdot)\) on a finite-time interval.

\subsection*{C. Structured IO-FTS and problem statement}

As observed in Section II-A, the IO finite-time stabilization problem, as defined in [1], does not allow to effectively deal with constraints on the control variables as in (7). In this section, we shall show how to modify the definition of IO-FTS in order to take into account, during the design phase, such kind of control requirements.

We introduce the concept of structured IO-FTS, which generalizes the original definition of IO-FTS given in [1]. To this end, given an \(\alpha\)-tuple of integer numbers \(m_1, \ldots, m_\alpha\), where \(1 < \alpha < m\), and \(\sum_{i=1}^{\alpha} m_i = m\), we partition the output vector as

\[
y(t) = \begin{pmatrix} y_1^T(t) & \cdots & y_\alpha^T(t) \end{pmatrix}^T, \quad t \in \Omega.
\]  

Note that the output partition (8) induces a partition of the output equation matrices

\[
C(t) = \begin{pmatrix} C_1^T(t) & \cdots & C_\alpha^T(t) \end{pmatrix}^T,
\]

\[
F(t) = \begin{pmatrix} F_1^T(t) & \cdots & F_\alpha^T(t) \end{pmatrix}^T.
\]
In the original definition of IO-FTS the output weighting is a symmetric, positive definite matrix belonging to the space $\mathbb{R}^{m \times m}$. Here we consider $\alpha$ positive definite weighting matrices $Q_i(t) \in \mathbb{R}^{m_i \times m_i}, i = 1, \ldots, \alpha$; defining

$$Q(t) := \text{diag}(Q_1(t), \ldots, Q_\alpha(t)),$$

we introduce the definition of structured IO-FTS of LTV systems.

**Definition 2:** Given a positive scalar $T$, a class of signals $W$ defined over $\Omega$, the output partition (8) and the corresponding positive definite weighting matrix $Q(\cdot)$ defined in (9), system (1) is said to be structured IO-FTS with respect to $(W, Q(\cdot), \Omega)$ if

$$w(\cdot) \in W \Rightarrow y_i^T(t)Q_i(t)y_i(t) < 1, \quad t \in \Omega, \quad i = 1, \ldots, \alpha.$$ 

Given Definition 2, it is straightforward to note that the classical Definition 1 can be obtained by letting $\alpha = 1$.

In this paper we consider the following two classes of signals, which require different analysis and synthesis techniques:

i) the set $W$ is defined as

$$\mathcal{W}_\infty(\Omega, R(\cdot)) := \{w(\cdot) \in L_\infty(\Omega) : \|w\|_{\infty, R} \leq 1\},$$

which is the case of essentially bounded signals over $\Omega$, whose weighted norm is less than or equal to one.

ii) the set $W$ coincides with the set of square integrable signals over $\Omega$, $\mathbb{L}_2$

In the rest of the paper we will drop the dependency of $W$ on $\Omega$ and $R(\cdot)$ in order to simplify the notation.

The first results of this paper, namely some conditions guaranteeing that a given system is structured IO-FTS, will be given in Section III. The related design problem, concerning structured IO finite-time stabilization via state feedback, will be dealt with in Section IV. To state precisely this problem, consider system (1b) and correspondingly, given a $\beta$-tuple of integer numbers $q_1, \ldots, q_\beta$, where $\sum_{i=1}^{\beta} q_i = q$, partition the control input vector as

$$u(t) = (u_1^T(t) \cdots u_\beta^T(t))^T, \quad t \in \Omega;$$

(10)
correspondingly consider $\beta$ positive definite weighting matrices $T_i(t) \in \mathbb{R}^{q_i \times q_i}, i = 1, \ldots, \beta$; define

$$T(t) := \text{diag}(T_1(t), \ldots, T_\beta(t)).$$

(11)
Eventually, we consider the following partition of $D(\cdot)$, induced by (8)

$$D(t) = (D^T_1(t) \cdots D^T_\alpha(t))^T.$$ 

**Problem 1:** Given a positive scalar $T$, the class of signals $\mathcal{W}$ defined over $\Omega$, the output partition (8), the input partition (10) and the corresponding positive definite weighting matrices $Q(\cdot), T(\cdot)$ defined in (9) and (11) respectively, find a state feedback control law

$$u(t) = K(t)x(t),$$

where $K(\cdot) : \Omega \mapsto \mathbb{R}^{q \times n}$, such that the system

$$\dot{x}(t) = (A(t) + B(t)K(t))x(t) + G(t)w(t)$$

$$y(t) = C(t)x(t) + D(t)K(t)x(t) + F(t)w(t),$$

with

$$x(t_0) = 0, \quad w(\cdot) \in \mathcal{W} \implies y^T_i(t)Q_i(t)y_i(t) < 1, \quad i = 1, \ldots, \alpha,$$

where

$$\mathcal{W}_\infty = \{ w(\cdot) \mid w(\cdot) \in \mathcal{W}, w(\cdot) \in \mathcal{W}_{\infty} \},$$

is structured IO-FTS with respect to $(\mathcal{W}, \text{diag}(Q(\cdot), T(\cdot)), \Omega)$. ▲

### III. ANALYSIS

Let us consider the case of $\mathcal{W}_\infty$ signals first; we need the following technical lemma.

**Lemma 1:** Given system (1), the positive definite weighting matrix $Q(\cdot)$ defined in (9) and $t \in \Omega$, the condition

$$w(\cdot) \in \mathcal{W}_\infty \implies y^T_i(t)Q_i(t)y_i(t) < 1, \quad i = 1, \ldots, \alpha,$$
is satisfied if there exist a positive definite matrix-valued function \( P(\cdot) \) and \( \alpha \) scalar functions \( \theta_i(\cdot) > 1, \ i = 1, \ldots, \alpha \), such that

\[
\dot{P}(\tau) + A^T(\tau)P(\tau) + P(\tau)A(\tau) + P(\tau)G(\tau)R^{-1}(\tau)G^T(\tau)P(\tau) < 0, \quad \tau \in [t_0, t] \tag{14a}
\]

\[
\theta_i(t)R(t) - R(t) \geq 2\theta_i(t)Q_i(t)F_i(t), \quad i = 1, \ldots, \alpha \tag{14b}
\]

\[
P(t) \geq 2\theta_i(t)C_i^T(t)\tilde{Q}_i(t)C_i(t), \quad i = 1, \ldots, \alpha \tag{14c}
\]

where \( \tilde{Q}_i(t) = (t - t_0)Q_i(t) \).

**Proof:** Given \( t \in \Omega \), we have

\[
y_i(t)Q_i(t)y_i(t) = x^T(t)C_i^T(t)Q_i(t)C_i(t)x(t) + w^T(t)F_i^T(t)Q_i(t)F_i(t)w(t)
\]

\[
+ x^T(t)C_i^T(t)Q_i(t)F_i(t)w(t) + w^T(t)F_i^T(t)Q_i(t)C_i(t)x(t),
\]

for all \( i \in \{1, \ldots, \alpha\} \). Now let

\[
v_i(t) = \left( Q_i(t)^{-\frac{1}{2}}C_i(t)x(t) - Q_i(t)^{-\frac{1}{2}}F_i(t)w(t) \right),
\]

then (the time argument is omitted for brevity)

\[
v_i^Tv_i = x^TC_i^TP_iQ_iC_ix + w^TF_i^TP_iQ_iF_iw - x^TC_i^TF_iQ_iF_iw - w^TF_i^TP_iQ_iC_ix,
\]

which can be rewritten as

\[
x^TC_i^TP_iQ_iF_iw + w^TF_i^TP_iQ_iC_ix = x^TC_i^TP_iQ_iC_ix + w^TF_i^TP_iQ_iF_iw - v_i^Tv_i. \tag{16}
\]

Replacing (16) in (15), it holds

\[
y_i^TQ_iy_i = 2x^TC_i^TP_iQ_iC_ix + 2w^TF_i^TP_iQ_iF_iw - v_i^Tv_i
\]

\[
< 2 \left( x^TC_i^TP_iQ_iC_ix + w^TF_i^TP_iQ_iF_iw \right). \tag{17}
\]

Condition (17) together with (14b) and (14c) imply that

\[
y_i^TQ_iy_i < \left( \frac{1}{\theta_i} \frac{P_x}{t - t_0} + \frac{\theta_i - 1}{\theta_i} w^TRw \right). \tag{18}
\]

Assuming, for the moment, that \( t > t_0 \), in [1] it has been proven that (14a) implies

\[
x^T(t)P(t)x(t) < t - t_0. \tag{19}
\]

Exploiting (19), and recalling that \( w(\cdot) \in \mathcal{W}_\infty \) implies that \( \|w\|_{\infty,R} \leq 1 \), from (18) we obtain

\[
y_i^T(t)Q_i(t)y_i(t) < 1, \quad i \in \{1, \ldots, \alpha\}.
\]
To conclude the proof, let us now discuss the case in which the given \( t \) coincides with the initial time \( t_0 \). In this case, since the initial state \( x(t_0) \) is zero, it is straightforward to prove that condition (14b) is sufficient to conclude that

\[
y_i^T(t_0)Q_i(t_0)y_i(t_0) < 1,
\]

for \( i = 1, \ldots, \alpha \).

\[\Box\]

**Remark 1:** Note that in [1] and [2], strictly proper LTV systems are considered. Conversely, Lemma 1 deals with systems having a nonzero feedthrough matrix (namely the matrix function \( F(\cdot) \) in (1)). Therefore, when \( \mathcal{W}_\infty \) signals are dealt with, the main result of this paper recovers, even in the non-structured case, a more general result than those ones stated in the previous literature.

**Remark 2:** If, for a given \( i \), \( F_i(t) = 0 \) for all \( t \in \Omega \), it can be easily shown that the related optimization scalar function \( \theta_i(\cdot) \) is not needed, since the constraint (14b) is always fulfilled, while \( P(\cdot) \) can be scaled in such a way that inequality (14b) becomes

\[
P(t) \geq C_i^T(t)\tilde{Q}_i(t)C_i(t).
\]

Hence, in the case of \( F_i(t) = 0 \), the sufficient condition originally proposed in [1] are recovered.

In order to check structured IO-FTS of system (1), Lemma 1 would require to check the feasibility of infinitely many optimization problems (one for each \( t \) in \( \Omega \)), which is obviously an impossible task. However, by exploiting similar arguments as in [1], it is possible to prove the following theorem, which requires to check a single DLMI feasibility problem.

**Theorem 1:** Let \( \tilde{Q}_i(t) = (t - t_0)Q_i(t) \), \( i = 1, \ldots, \alpha \); if there exist a positive definite and continuously differentiable matrix-valued function \( P(\cdot) \) and \( \alpha \) scalar functions \( \theta_i(\cdot) \), \( i = 1, \ldots, \alpha \), such that the coupled DLMI/LMI

\[
\begin{pmatrix}
\dot{P}(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\
G^T(t)P(t) & -R(t)
\end{pmatrix} < 0 \tag{20a}
\]

\[
\theta_i(t)R(t) - R(t) \geq 2\theta_i(t)F_i^T(t)Q_i(t)F_i(t), \quad i = 1, \ldots, \alpha \tag{20b}
\]

\[
P(t) \geq 2\theta_i(t)C_i(t)^T\tilde{Q}_i(t)C_i(t), \quad i = 1, \ldots, \alpha, \tag{20c}
\]

are satisfied over \( \Omega \), then system (1) is structured IO-FTS with respect to \( (\mathcal{W}_\infty, Q(\cdot), \Omega) \).

Now, let us consider structured IO-FTS in presence of \( \mathcal{W}_2 \) signals. In this case we have to set \( F(\cdot) = 0 \), otherwise the concept of structured IO-FTS with respect to \( \mathcal{W}_2 \) would
be ill posed. Indeed, it is straightforward to recognize that $W_2$ includes signals that are unbounded on an interval of zero measure included in $\Omega$. When $F(\cdot) \neq 0$, in presence of such signals, it readily follows that there exists at least one time instant where the output would be unbounded. Hence, when $F(\cdot) \neq 0$, system (1) cannot be IO-FTS with respect to $W_2$. Under the assumption that $F(\cdot) = 0$, the next theorem states a necessary and sufficient condition for structured IO-FTS of system (1) when $W_2$ signals are considered.

**Theorem 2:** Given system (1) with $F(\cdot) = 0$, the class of signals $W_2$ defined over $\Omega$, the output partition (8) and the corresponding positive definite weighting matrix $Q(\cdot)$ defined in (9), system (1) is structured IO-FTS with respect to $(W_2, Q(\cdot), \Omega)$ if and only if the coupled DLMI/LMI

\[
\begin{pmatrix}
\dot{P}(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\
G^T(t)P(t) & -R(t)
\end{pmatrix} < 0
\]  

(21a)

\[P(t) \geq C_i^T(t)Q_i(t)C_i(t), \quad i = 1, \ldots, \alpha, \]  

(21b)

admits a positive definite solution $P(\cdot)$ over $\Omega$.

**Proof:** Given the output partition (8), system (1) can be considered as a collection of $\alpha$ fictitious systems with the same state equation (1a), and output equation given by

\[y_i(t) = C_i(t)x(t),\]

for each $i = 1, \ldots, \alpha$. The proof of the theorem readily follows by considering the result given in [2], for each one of the $\alpha$ fictitious systems. It is worth noticing that condition (21a) is not affected by the output partition, since it involves only the state equation. \hfill \blacksquare

**IV. MAIN RESULTS**

In this section we propose a number of results to solve Problem 1. First of all we shall deal with the case in which $F(\cdot) \neq 0$ in system (1) for $W_\infty$ signals. At the end of the section we shall let $F(\cdot) = 0$, and give a necessary and sufficient condition to solve Problem 1 with respect to $W_2$ signals. It is worth to mention that we can always consider $D(\cdot) \neq 0$, since we assume that the control action $u(t) = K(t)x(t)$ is bounded in $\Omega$.

**Theorem 3:** Given the class of signals $W_\infty$, Problem 1 is solvable if there exist a positive definite and continuously differentiable matrix-valued function $\Pi(\cdot)$, $\beta$ continuously
differentiable matrix-valued functions \( L_1(\cdot), \ldots, L_\beta(\cdot) \), and functions \( \lambda_i(\cdot), 0 < \lambda_i(t) < 1, \ i = 1, \ldots, \alpha, t \in \Omega \), such that

\[
\begin{pmatrix}
\Theta(t) & G(t) \\
G^T(t) & -R(t)
\end{pmatrix} < 0, \tag{22a}
\]

\[
R(t) - \lambda_i(t)R(t) \geq 2 F_i^T(t)Q_i(t)F_i(t), \quad i = 1, \ldots, \alpha \tag{22b}
\]

\[
\begin{pmatrix}
\Pi(t) & \Pi(t)C_i^T(t) + (L_1^T(t) \cdots L_\beta^T(t))D_i^T(t) \\
C_i(t)\Pi(t) + D_i(t)\left(L_1^T(t) \cdots L_\beta^T(t)\right)^T & \frac{\lambda_i(t)}{2} \tilde{\Xi}_i(t)
\end{pmatrix} \geq 0, \quad i = 1, \ldots, \alpha \tag{22c}
\]

\[
\begin{pmatrix}
\Pi(t) & L_j^T(t) \\
L_j(t) & \tilde{\Upsilon}_j(t)
\end{pmatrix} \geq 0, \quad j = 1, \ldots, \beta, \tag{22d}
\]

for all \( t \in \Omega \), with

\[
\Theta(t) := -\Pi(t) + \Pi(t)A^T(t) + A(t)\Pi(t) + B(t)\left(L_1^T(t) \cdots L_\beta^T(t)\right)^T + (L_1^T(t) \cdots L_\beta^T(t))B^T(t)
\]

\[
\tilde{\Xi}_i(t) := ((t - t_0)Q_i(t))^{-1}
\]

\[
\tilde{\Upsilon}_j(t) := ((t - t_0)T_j(t))^{-1}.
\]

A controller gain which solves Problem 1 for the signal class \( \mathcal{W}_\infty \) is given by (13) with \( K_j(t) = L_j(t)\Pi^{-1}(t), j = 1, \ldots, \beta \).

**Proof:** Conditions (20) for the augmented output closed-loop system (12) read

\[
\begin{pmatrix}
P(t) + A^T_d(t)P(t) + P(t)A_d(t) & P(t)G(t) \\
P^T(t)P(t) & -R(t)
\end{pmatrix} < 0, \tag{23a}
\]

\[
\theta_i(t)R(t) - R(t) \geq 2 \theta_i(t)F_i^T(t)Q_i(t)F_i(t), \quad i = 1, \ldots, \alpha \tag{23b}
\]

\[
P(t) \geq 2\theta_i(t)\left(C_i^T(t) + K_i^T(t)D_i^T(t)\right)(t - t_0)Q_i(t)\left(C_i(t) + D_i(t)K(t)\right), \quad i = 1, \ldots, \alpha \tag{23c}
\]

\[
P(t) \geq K_j^T(t)\left(t - t_0\right)T_j(t)K_j(t), \quad j = 1, \ldots, \beta. \tag{23d}
\]

Note that for the fictitious outputs \( u_j(t) = K_j(t)x(t), j = 1, \ldots, \beta \), in (12b) the only constraints to be considered are (23d), since for these outputs there is no direct link with the vector \( w(\cdot) \) (see also Remark 2).
Now, let us pre- and post-multiply inequality (23a) by \(\begin{pmatrix} \Pi(t) & 0 \\ 0 & I \end{pmatrix} > 0\), where \(\Pi(t) = P^{-1}(t)\). We obtain
\[
\begin{pmatrix}
-\dot{\Pi}(t) + \Pi(t)A^T_d(t) + A_d(t)\Pi(t) & G(t) \\
G^T(t) & -R(t)
\end{pmatrix} < 0,
\]
which turns to be equal to (22a), if we let \(L_j(t) = K_j(t)\Pi(t), j = 1, \ldots, \beta\).

Consider now condition (22b); it is easy to see that, if we let \(\lambda_i(\cdot) = \theta_i^{-1}(\cdot)\) in \(\Omega\), then (23b) is equivalent to (22b), when \(0 < \lambda_i(t) < 1, t \in \Omega\).

Similarly, inequality (23c) is equivalent to
\[
\frac{\lambda_i(t)}{2} P(t) \geq \left( C^T_i(t) + K^T(t)D^T_i(t) \right) \left( t - t_0 \right) Q_i(t) \left( C_i(t) + D_i(t)K(t) \right), \quad i = 1, \ldots, \alpha.
\]

By pre- and post-multiplying (44) and (23d) by \(\Pi(t)\), we have
\[
\begin{pmatrix}
\Pi(t) & \Pi(t)C^T_i(t) + \Pi(t)K^T(t)D^T_i(t) \\
C_i(t)\Pi(t) + D_i(t)K(t)\Pi(t) & \frac{\lambda_i(t)}{2}\Xi_i(t)
\end{pmatrix} \geq 0, \quad i = 1, \ldots, \alpha
\]
(25a)
\[
\begin{pmatrix}
\Pi(t) & \Pi(t)K^T_j(t) \\
K_j(t)\Pi(t) & \tilde{\Upsilon}_j(t)
\end{pmatrix} \geq 0, \quad j = 1, \ldots, \beta
\]
(25b)
where (25a) and (25b) are obtained by applying the Schur complements, and are equivalent to (22c) and (22d), respectively, when letting \(L_j(t) = K_j(t)\Pi(t), j = 1, \ldots, \beta\).

If \(F(\cdot) = 0\) in \(\Omega\), exploiting similar arguments as in the previous proof, it is possible to derive the following necessary and sufficient condition to solve Problem 1 in the case of \(\mathcal{W}_2\) signals.

**Theorem 4:** Given the class of signals \(\mathcal{W}_2\) and \(F(\cdot) = 0\), Problem 1 is solvable if and only if there exist a positive definite and continuously differentiable matrix-valued function \(\Pi(\cdot)\), and \(\beta\) continuously differentiable matrix-valued functions \(L_1(\cdot), \ldots, L_\beta(\cdot)\), such that inequality (22a) and
\[
\begin{pmatrix}
\Pi(t) & \Pi(t)C^T_i(t) + \left( L^T_1(t) \cdots L^T_\beta(t) \right)D^T_i(t) \\
C_i(t)\Pi(t) + D_i(t)\left( L^T_1(t) \cdots L^T_\beta(t) \right) \Xi_i(t)
\end{pmatrix} \geq 0, \quad i = 1, \ldots, \alpha
\]
(26a)
\[
\begin{pmatrix}
\Pi(t) & L^T_j(t) \\
L_j(t) & \tilde{\Upsilon}_j(t)
\end{pmatrix} \geq 0, \quad j = 1, \ldots, \beta
\]
(26b)
hold for all \( t \in \Omega \), with \( \Xi_i(t) := Q_i^{-1}(t) \), and \( \Upsilon_j(t) := T_j^{-1}(t) \). The controller gain which solves Problem 1 for the signals class \( \mathcal{W}_2 \) is given by (13) with \( K_j(t) = L_j(t)\Pi^{-1}(t) \), \( j = 1, \ldots, \beta \).

V. DESIGN OF AN ACTIVE SUSPENSION CONTROL SYSTEM USING STRUCTURED IO-FTS

In order to frame the problem of designing the active suspension control system, illustrated in Section II-B, in the context of the structured IO finite-time stabilization, according to (12b), let us rewrite the output equation as

\[
\begin{pmatrix}
y_1(t) \\
y_2(t) \\
y_3(t) \\
u(t)
\end{pmatrix} =
\begin{pmatrix}
\dot{x}_2(t) \\
\frac{1}{SS} x_1(t) \\
K(x(t)) \\
K(t) x(t)
\end{pmatrix}, \quad (27)
\]

We will design the time-varying controller \( K(t) \) trying to optimize the response to an isolated bump modeled as the \( \mathcal{W}_2 \) signal \( w(t) = \dot{b}(t) \), where \( b(t) \) describes the ground asperity

\[
b(t) = \begin{cases} 
\frac{M}{2} \left( 1 - \cos \left( \frac{2\pi V}{L} t \right) \right) , & 0 \leq t \leq \frac{L}{V} \\
0 , & t > \frac{L}{V}
\end{cases}, \quad (28)
\]

and where \( M = 0.1 \) m and \( L = 5 \) m are the bump height and width, while \( V = 45 \) km/h is the vehicle forward velocity. The ground asperity considered for the controller design is reported in Fig. 2.

Given the bump (28), our goal is to minimize the body acceleration \( y_1(t) = \dot{x}_2(t) \) fulfilling the constraints (4)-(7). In order to do that, we consider the following IO-FTS parameters

\[
T = 2 \text{ s} , \quad R = 8.
\]

Furthermore, given the selected outputs (27), the two outputs weighting matrices

\[
Q_2 = Q_3 = 1,
\]
allows to take into account the constraints (4) and (5), while the input weighting matrix is

\[
T_1 = 0.15,
\]

which allows to exploit the full scale of the control input when (28) is considered.
It turns out that, in order to minimize the body acceleration it is possible to exploit Theorem 4 and solve the following optimization problem

\[
\begin{align*}
\text{minimize } & \Xi_1 \\
\text{subject to } & (26),
\end{align*}
\]  

where \( \Xi_1 = Q_1^{-1} \).

In order to recast the optimization problem (29) in terms of an LMIs optimization problem, the DLMI condition (26) need to be recasted into the LMI framework. To this aim the matrix-valued functions \( \Pi(\cdot) \) and \( L(\cdot) \) can been assumed piecewise linear, by dividing the time interval \( \Omega \) in \( \nu = T/T_s \) subintervals, and assuming the time derivative of \( \Pi(\cdot) \) is constant in each subinterval. As an example, the structure of the matrix-valued function \( \Pi(\cdot) \) is here reported

\[
\Pi(t) = \begin{cases} 
\Pi_0 + \Psi_1 (t - t_0), & t \in [t_0, t_0 + T_s], \\
P_0 + \sum_{h=1}^{J} \Psi_h T_s + \Psi_{j+1} (t - jT_s - t_0), & t \in (t_0 + jT_s, t_0 + (j + 1)T_s] 
\end{cases}
\]  

(30)

where \( J = \max \{ j \in \mathbb{N} : j < T/T_s \} \), \( T_s \ll T \) and \( P_0, \Psi_l, l = 1, \ldots, J + 1 \) are the optimization variables. It is straightforward to recognize that such a piecewise function can approximate a generic continuous \( \Pi(\cdot) \) with adequate accuracy, provided that the length of \( T_s \) is sufficiently small. Moreover, Theorem 4 does not require any specific structure for the optimization variables; hence IO-FTS of the closed-loop systems is guaranteed as far
as there exist at least one $\Pi(\cdot)$ and one $L(\cdot)$ that fulfill conditions (26). A structure similar to (30) can be assumed also for the matrix-valued function $L(\cdot)$.

Note that, given the structure (30), $J + 1$ constraints (which are also LMIs) need to be added to (26) in order to force $\Pi(\cdot)$ to be positive definite.

For the sake of completeness, in [2] it has been shown that solving DLMIs by means of piecewise linear approximation turns to be numerically less efficient if compared with differential Lyapunov equations (which also give necessary and sufficient conditions to check IO-FTS). However, only DLMIs allows to tackle synthesis problem, and in particular the optimization problem (29). Furthermore, though less efficient, DLMIs need to be solved offline in order to design the controller, hence there is no real practical issue.

Once the problem (29) is recast in the LMIs framework with the above choice of $\Pi(\cdot)$ and $L(\cdot)$, it is possible to solve it by using off-the-shelf optimization tools such as the Matlab LMI Toolbox® [21].

In particular, by solving (29), we get $\Xi_{1\text{min}} = 7.22$ and the two matrix-valued functions $\Pi(\cdot)$ and $L(\cdot)$; the time-varying controller $K(t)$ is then given by $K(t) = L(t)\Pi(t)^{-1}$.

It is important to remark once again that the solution of the DLMI feasibility problem is performed offline during the design phase. The interpolation between two time samples of the time-varying matrix-valued function $K(t)$ is the only computation required in real-time to determine the control gain, which does not have any impact on the computational performance.

**Fig. 3.** Bump response: IO-FTS time-varying controller (–), constrained $H_{\infty}$ controller (– -).
In Figure 4 the time behavior of the constraints on $y_2(\cdot)$ and $y_3(\cdot)$ is reported.

Having proved that structured IO-FTS allows to achieve good performance, we now describe a possible practical implementation of the proposed control algorithm. Indeed, being time-varying over a finite-time horizon, the computed control law cannot be applied over an infinite time interval. In order to overcome this limitation, we propose a control scheme based on the hybrid automaton [22] reported in Fig. 5.

In particular, the considered automaton has the following two discrete states:

- $S_1$: this state corresponds to the Standard control. When the automaton is in this state, the static state-feedback

\[
K^* = \begin{pmatrix}
7.0776 & -0.8193 & 1.0865 & -0.2617
\end{pmatrix}
\]

proposed in [16] is applied. $S_1$ is also the initial state.
\[
\dot{z}(0) = 0
\]
\[
\begin{align*}
S_1 \text{ (Standard):} & \quad \dot{z}(t) = 0 \\
S_2 \text{ (Enhanced):} & \quad \dot{z}(t) = 1 \\
\end{align*}
\]
\[
\dot{z} > T | \dot{z}^+ = 0
\]

Fig. 5. Hybrid automata proposed for the real implementation of the structured IO-FTS controller for the active suspension system.

* $S_2$: when this state is active the proposed time-varying control law is applied for a time interval equal to $T$. After that time interval the automaton switches to $S_1$. State $S_2$ correspond to the Enhanced control.

As reported in Fig. 5, the state transition from $S_1$ to $S_2$ occurs when the absolute value of tire acceleration $a_t(\cdot) = \ddot{x}_t(\cdot)$ is greater than the threshold $\tilde{a}$.

Note that the stability of the overall hybrid system is guaranteed by the fact that the time-varying control law is always applied for $T$ seconds, that is the time interval used to solve Problem 1, while the constrained $\mathcal{H}_\infty$ controller $K^*$ guarantees internal stability by design.

To prove the effectiveness of the proposed hybrid approach, instead of considering the isolated bump (28), the ground asperity shown in Fig. 6 is considered. Furthermore, the threshold $\tilde{a}$ is set equal to 3 m/s.

Fig. 6. Ground asperity considered to prove the effectiveness of the hybrid controller for the active suspension system.
Fig. 7. Response to the ground asperity reported in Fig. 6. Hybrid controller (−), constrained $H_\infty$ controller (⋅⋅). The red circles denote the time instants when the discrete state $S_2$ of the hybrid automaton is activated.

The simulation results are shown in Fig. 7, where a comparison between the behavior of the proposed hybrid controller is compared with the constrained $H_\infty$ controller. Even in this case, the proposed controller, which relies on the structured IO-FTS time-varying state feedback, achieves to reduce the peaks of the body acceleration. Note that the first peak is not reduced; this is due to the fact that the tire acceleration is below the threshold, and hence the time-varying state feedback is not applied.

CONCLUSIONS

In this paper we have considered the IO-FTS problem in which the output is partitioned into sub-groups, and the groups are weighted differently each other. This leads to the concept of structured IO-FTS, which allows to tackle the important problem of the IO-FTS with control inputs constraints. We have first provided a sufficient condition ($L_\infty$ signals) and a necessary and sufficient condition ($L_2$ signals) for structured IO-FTS (open loop system). Then, a sufficient condition and a necessary and sufficient condition for structured IO finite-time stabilization has been stated in the $L_\infty$ and $L_2$ context respectively. To show the benefits of the proposed approach, the theory developed in the paper has been used to design the active suspension control system for a quarter-car model, showing that the structured IO-FTS approach allows to improve the control performance if compared with the constrained $H_\infty$
controller proposed in [16]. A practical implementation of a structured IO-FTS time-varying control law by means of an hybrid controller has been also proposed.

REFERENCES

