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Analytical fuzzy plane geometry I

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Abstract

This paper provides a detailed analysis of fuzzy point, fuzzy line segment, fuzzy distance and the angle between two fuzzy line segments. Two new concepts, same points and inverse points, are defined for this analysis. The basic properties of fuzzy distance, ideas about the containment of a fuzzy point on a fuzzy line segment and the coincidence of two fuzzy points are also described. A linear combination of two fuzzy points is introduced to define a fuzzy line segment. The fuzzy point dividing a fuzzy line segment in a given ratio is also investigated. All the discussion points are supported by suitable examples.

Keywords: Fuzzy number; Fuzzy point; Fuzzy distance; Fuzzy line segment; Same points; Inverse points; Fuzzy linear combination; Extension principle

1. Introduction

In the field of fuzzy plane geometry, Buckley and Eslami introduced a new concept to investigate the shapes of different fuzzy curves [3,4]. Yuan and Shen proved that the concepts defined in [3,4] are based on the sup-min composition of fuzzy sets [20]. This sup-min composition is similar to the extension principle. Rosenfeld published a brief review of studies on fuzzy geometry and the topology of image subsets including adjacency, separation and connectedness [16]. Prior to the work of Buckley and Eslami, Chaudhuri defined some fuzzy geometrical shapes [5], but in general their cores do not correspond to well-known shapes in classical geometry. Ideas on a fuzzy disk and fuzzy perimeter were introduced by Rosenfeld and Haber [17]; this fuzzy disk is a fuzzy point as defined in [3] with a circular base. Rosenfeld introduced concepts of the height, width and diameter of fuzzy sets using real integrals [15]. Certain ideas about the height, width and diameter of a fuzzy set were investigated by Bogomolny using a projection of the fuzzy sets onto two mutually perpendicular directions [1]. Bogomolny observed that the definitions introduced in [15,17] lack inner conformity when reduced to the corresponding customary definitions for crisp sets. To maintain this conformity, Bogomolny modified the definition given in [15]. As a result of this modification, for some situations Bogomolny obtained completely different results to those reported by Rosenfeld. For example, according to Bogomolny, ‘the area of a fuzzy set is less than or equal to its height times its width’, whereas Rosenfeld reported that ‘the area of a fuzzy set is not less than its height times its width’. The results in [1] are more meaningful, but in [1,15,17] measurements of the defined height, width, perimeter, etc. are all crisp numbers. These should be fuzzy numbers and cannot be real numbers [8] because if the region is itself ill-defined, then it is difficult to see how measurements can be precisely defined.

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The area and perimeter of fuzzy regions defined by Buckley and Eslami are fuzzy numbers [4]. The same authors investigated the basics of fuzzy plane geometrical concepts such as fuzzy points, fuzzy lines, fuzzy circles and fuzzy trigonometric functions [3,4]. This work was further elucidated by Clark and co-workers [6] and by Yuan and Shen [20] and has since been extended by other researchers [9,14]. In the application field of fuzzy geometry, Safi et al. used the geometrical concepts of Buckley and Eslami to solve fuzzy linear programming problems [18]. Li and Guo applied the concepts in the modeling of fuzzy geometrical objects [11]. Rui et al. used the concepts for location discovery in passive sensor networks [19]. Bloch performed a study on different approaches to obtain fuzzy geometrical distances and their applications in image processing [2]. Some study on fuzzy shapes may be obtained in [13].

In fuzzy geometrical shapes, a fuzzy line may be visualized as a straight, infinitely long, hazy band consisting of a group of crisp lines with varied membership grades. In other words, a fuzzy line has one (deep) crisp line at its core with a uniform and smooth transition in membership values between neighboring points because a fuzzy line can be considered as the locus of a fuzzy point along a particular direction. We propose that a fuzzy line cannot have a sudden wider spread as observed by Buckley and Eslami [3]. Furthermore, the slope of a fuzzy line cannot have more than one value for its core because otherwise it lacks inner conformity in that as it does not match the customary definition of a crisp line.

A fuzzy point can be viewed in two different ways: a collection of points with different membership values or a collection of (normal, convex) fuzzy sets along lines passing through the core of the fuzzy point. To obtain a mathematical formulation of a fuzzy line passing through two fuzzy points using the second view of fuzzy points, the following analysis is possible. Consider all the (normal and convex) fuzzy sets that lie on the supports of the two fuzzy points. If only the fuzzy sets along the same direction are combined using the extension principle, then there are two types of combinations: effective combinations (combinations of same points) and redundant combinations. We have observed that combining only effective combinations leads to a fuzzy line with an elegant formulation for which the membership values of different points on the fuzzy line can be more easily evaluated than previously [3,4].

In this paper we introduce the concepts of same and inverse points, which are obtained after identifying the redundant combinations mentioned above. The basic ideas of fuzzy reference frames, fuzzy points, fuzzy distances, fuzzy angles and linear combinations of fuzzy points are also studied.

The remainder of the paper is organized as follows. Section 2 explains the preliminaries, the extension principle and the addition operation for two fuzzy points. Linear combinations of two fuzzy points and the concepts of same and inverse points are introduced in Section 3. The distance between two fuzzy points, the coincidence of two fuzzy points, fuzzy line segments, inclusion of a fuzzy point in a fuzzy line segment and the angle between two fuzzy line segments are described with suitable examples in Section 4. In Section 5, our results are discussed and compared with existing methods. Section 6 concludes.

2. Preliminaries

2.1. Definitions and notations on fuzzy sets

The basic definitions used here are adopted from [3] with little modifications. Capital or small letters with a tilde bar ($\widetilde{A}, \widetilde{B}, \widetilde{C}, \ldots$ and $\widetilde{a}, \widetilde{b}, \widetilde{c}, \ldots$) are all fuzzy subsets of $\mathbb{R}^n,\ n = 1, 2$. The membership function of a fuzzy set $\widetilde{A}$ of $\mathbb{R}^n$ is represented by $\mu(x|\widetilde{A}),\ x \in \mathbb{R}^n$, with $\mu(\mathbb{R}^n) \subseteq [0, 1],\ n = 1, 2$. The symbols $\oplus$ and $\ominus$ represent extended addition and subtraction, respectively.

Definition 2.1 (*α-cut of a fuzzy set*). For a fuzzy set $\widetilde{A}$ of $\mathbb{R}^n,\ n = 1, 2$, its α-cut is denoted by $\widetilde{A}(\alpha)$ and is defined by

$$\widetilde{A}(\alpha) = \begin{cases} \{x : \mu(x|\widetilde{A}) \geq \alpha\} & \text{if } 0 < \alpha \leq 1, \\ closure\{x : \mu(x|\widetilde{A}) > 0\} & \text{if } \alpha = 0. \end{cases}$$

The set $\{x : \mu(x|\widetilde{A}) > 0\}$ is called the support of the fuzzy set $\widetilde{A}$.

To represent the construction of membership function of a fuzzy set $\widetilde{A}$, the notation $\bigvee\{x : x \in \widetilde{A}(0)\}$ is frequently used, which means $\mu(x|\widetilde{A}) = \sup\{\alpha : x \in \widetilde{A}(\alpha)\}$.

Definition 2.2 (*Fuzzy numbers, Buckley and Eslami [3]*). A fuzzy set $\widetilde{A}$ of $\mathbb{R}$ is called a fuzzy real number if its membership function μ has the following properties:

(i) $\mu(x|\widetilde{A})$ is upper semi-continuous,
(ii) $\mu(x|\widetilde{A}) = 0$ outside some interval $[a, d]$, and
(iii) there exist real numbers b and c so that $a \le b \le c \le d$ and $\mu(x|\widetilde{A})$ is increasing on $[a, b]$ and decreasing on $[c, d]$, and $\mu(x|\widetilde{A}) = 1$ for each x in $[b, c]$.

Since $\mu(x|\widetilde{A})$ is upper semi-continuous for a fuzzy number $\widetilde{A}$, the set $\{x : \mu(x|\widetilde{A}) \ge \alpha\}$ is closed for all α in $\mathbb{R}$. Thus, the α-cut of a fuzzy number $\widetilde{A}$ (the set $\widetilde{A}(\alpha)$) is a closed and bounded interval of $\mathbb{R}$ for all α in $[0, 1]$.

For $b = c$, letting $f(x) = \mu(x|\widetilde{A})\ \forall x \in [a, b]$ and $g(x) = \mu(x|\widetilde{A})\ \forall x \in [c, d]$, in this paper the notation $(a/c/d)_{fg}$ is used to represent the above defined fuzzy number. In particular, if $f(x)$ and $g(x)$ are linear functions, then the fuzzy number is called a triangular fuzzy number and is denoted by $(a/c/d)$.

Definition 2.3 (*Fuzzy points, Buckley and Eslami [3]*). A fuzzy point at (a, b) in $\mathbb{R}^2$, written as $\widetilde{P}(a, b)$, is defined by its membership function:

(i) $\mu((x, y)|\widetilde{P}(a, b))$ is upper semi-continuous,
(ii) $\mu((x, y)|\widetilde{P}(a, b)) = 1$ if and only if $(x, y) = (a, b)$, and
(iii) $\widetilde{P}(a, b)(\alpha)$ is a compact, convex subset of $\mathbb{R}^2$, for all α in $[0, 1]$.

Often the notations $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3, \dots$ are used to represent fuzzy points.

Example 2.3.1. Let (a, b) be a point in $\mathbb{R}^2$. Consider a right elliptical cone with elliptical base $\{(x, y) : ((x-a)/p)^2 + ((y-b)/q)^2 \le 1\}$ and vertex (a, b). This right elliptical cone can be taken as the membership function of a fuzzy point $\widetilde{P}(a, b)$ at (a, b). The mathematical form of $\mu(\cdot|\widetilde{P}(a, b))$ is

$$\mu((x, y)|\widetilde{P}(a, b)) = \begin{cases} 1 - \sqrt{\left(\frac{x-a}{p}\right)^2 + \left(\frac{y-b}{q}\right)^2} & \text{if } \left(\frac{x-a}{p}\right)^2 + \left(\frac{y-b}{q}\right)^2 \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

2.2. The extension principle

Suppose φ is a real function of n variables $x_1, x_2, \dots, x_n$. The extension principle, as stated by Zadeh, allows us to extend this function to $\widetilde{\varphi}(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n)$, which is a fuzzy set, $\widetilde{y}$ say, with membership function:

$$\mu(y|\widetilde{y}) = \begin{cases} \sup\limits_{y=\varphi(x_1,x_2,\dots,x_n)} \min\limits_{i=1,2,\dots,n} (\mu(x_i|\widetilde{x}_i)) & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ 0 & \text{if } \varphi^{-1}(y) = \emptyset. \end{cases}$$

For an increasing continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$, Dubois and Prade proved the following two lemmas [7].

Lemma 2.1. *Let φ be an increasing continuous function from $\mathbb{R}^n$ to $\mathbb{R}$, and let $f_1, f_2, \dots, f_n$ be n convex continuous functions; we suppose that f_i is strictly increasing on $(-\infty, b_i]$ and strictly decreasing on $[b_i, \infty)$ and $f_i(\mathbb{R}) = [0, 1]$ for each i. Let $(x_1, x_2, \dots, x_n)$ be an element of $\mathbb{R}^n$; assume, for instance, that $\varphi(x_1, x_2, \dots, x_n) \le \varphi(b_1, b_2, \dots, b_n)$. Then there exist $x_1^*, x_2^*, \dots, x_n^*$ such that:*

(i) *$x_i^* < b_i$ for each i,*
(ii) *$f_1(x_1^*) = f_2(x_2^*) = \cdots = f_n(x_n^*)$,*
(iii) *$\varphi(x_1^*, x_2^*, \dots, x_n^*) = \varphi(x_1, x_2, \dots, x_n)$, and*
(iv) *$\mu(\varphi(x_1, x_2, \dots, x_n)|\widetilde{\varphi}(f_1, f_2, \dots, f_n)) = f_1(x_1^*) = \cdots = f_n(x_n^*)$.*

Lemma 2.2. *Let $\mu(x_i|\widetilde{m}_i)$ be the membership function of the continuous fuzzy number $\widetilde{m}_i$, $i = 1, 2, \dots, n$. Each $\mu(x_i|\widetilde{m}_i)$ is increasing on $[a_i, b_i]$ and decreasing on $[b_i, c_i]$ for all i. (Possibly $a_i = -\infty$, $b_i = \pm\infty$, $c_i = +\infty$).*

Let x_i, $i = 1, 2, \ldots, n$, be such that $x_i \leq b_i$ and $\mu(x_i|\widetilde{m}_i) = \omega \in [0, 1]$ for all i. Then, for an increasing function φ of n variables, $\mu(\varphi(x_1, x_2, \ldots, x_n)|\widetilde{\varphi}(\widetilde{m}_1, \widetilde{m}_2, \ldots, \widetilde{m}_n)) = \omega$, where $\mu(\cdot|\widetilde{\varphi}(\widetilde{m}_1, \widetilde{m}_2, \ldots, \widetilde{m}_n))$ is the membership function of the extension of φ.

As an extension of [7], Hong proved the following theorem to allow easy application of the extended n-ary operation on continuous fuzzy numbers [10].

Theorem 2.1. *Let $\widetilde{m}_1, \widetilde{m}_2, \ldots, \widetilde{m}_n$ be continuous fuzzy numbers whose membership functions are surjective and whose supports are bounded. Let φ be a continuous increasing n-ary operation. Then the extension $\widetilde{\varphi}(\widetilde{m}_1, \widetilde{m}_2, \ldots, \widetilde{m}_n)$ is a continuous fuzzy number whose membership function is continuous and surjective from $\mathbb{R}$ to $[0, 1]$. This fuzzy number can be constructed by applying Lemma 2.2 to the increasing and decreasing parts of $\mu(x_i|\widetilde{m}_i)$ separately. This decomposition is authorized by Lemma 2.1.*

Illustration. Let $\widetilde{2}$ and $\widetilde{6}$ be two triangular fuzzy numbers; $\widetilde{2} = (1/2/3)$, $\widetilde{6} = (5/6/7)$. Here,

(i) the membership functions of $\widetilde{2}$ and $\widetilde{6}$ are continuous and surjective,
(ii) the supports of $\widetilde{2}$ and $\widetilde{6}$ are bounded, and
(iii) $\mu(x_1|\widetilde{2})$ is increasing on $[1, 2]$ and decreasing on $[2, 3]$; a similar property applies to $\mu(x_2|\widetilde{6})$.

Take φ to be the normal algebraic addition $+$, which is a continuous increasing operator.

(a) *Determination of redundant combinations*: Take $1.8 \in \widetilde{2}(0)$ and $6.7 \in \widetilde{6}(0)$. Here, $(1.8, 6.7)$ is an element of $\mathbb{R}^2$ and $\varphi(1.8, 6.7) = 8.5 > 8 = \varphi(2, 6)$. In the following, we show that although $(1.8, 6.7)$ is an element of $\mathbb{R}^2$ and $\varphi(1.8, 6.7) = 8.5$, this is a redundant combination for obtaining the value of $\mu(8.5|\widetilde{2} \oplus \widetilde{6})$ because Lemma 2.1 implies that $\exists x_1^*,\ x_2^*$ such that:

(i) they lie on the right-hand side of $\widetilde{2}(1)$ and $\widetilde{6}(1)$, respectively ($x_1^* > 2$ and $x_2^* > 6$),
(ii) they have same membership value ($\mu(x_1^*|\widetilde{2}) = \mu(x_2^*|\widetilde{6})$),
(iii) $x_1^* + x_2^* = 8.5$, and
(iv) $\mu(8.5|\widetilde{2} \oplus \widetilde{6}) = \mu(x_1^*|\widetilde{2}) = \mu(x_2^*|\widetilde{6})$.

We can find $x_1^* = 2.25$ and $x_2^* = 6.25$. Except for (x_1^*, x_2^*), there are pairs of combinations, such as (1.5, 7), (1.9, 6.6) and (1.8, 6.7), such that φ for each of them is 8.5. However, these do not offer the supremum of the computation $\mu(8.5|\widetilde{2} \oplus \widetilde{6}) = \sup_{x_1+x_2=8.5} \min\{\mu(x_1|\widetilde{2}), \mu(x_2|\widetilde{6})\}$ and do not satisfy the condition given in Lemma 2.1. Thus, such combinations may be discarded when computing $\mu(8.5|\widetilde{2} \oplus \widetilde{6})$. We call these combinations redundant or irrelevant because without considering them we can still evaluate $\mu(8.5|\widetilde{2} \oplus \widetilde{6})$. By contrast, we call the combination (2.25, 6.25) perfect or effective (or a combination of same points). In addition, (2.25, 6.25) satisfies all the conditions of Lemma 2.1.

(b) *Evaluation of* $\mu(\cdot|\widetilde{2} \oplus \widetilde{6})$: Apparently, $(\widetilde{2} \oplus \widetilde{6})(0) = [6, 10]$ and $\mu(8|\widetilde{2} \oplus \widetilde{6}) = 1$. By applying the extension principle and evaluating the membership value for each point in $[6, 10]$, the membership function of $\widetilde{2} \oplus \widetilde{6}$ can be obtained as

$$\mu(x|\widetilde{2} \oplus \widetilde{6}) = \begin{cases} \frac{x-6}{2} & \text{if } 6 \leq x \leq 8, \\ \frac{10-x}{2} & \text{if } 8 \leq x \leq 10, \\ 0 & \text{elsewhere.} \end{cases}$$

This analytical form of $\mu(x|\widetilde{2} \oplus \widetilde{6})$ can be obtained very easily by combining perfect combinations or same points as follows. Let $\alpha \in [0, 1]$. The numbers in $\widetilde{2}(0)$ with membership value α are $1 + \alpha$, $3 - \alpha$. The numbers in $\widetilde{6}(0)$ with membership value α are $5 + \alpha$, $7 - \alpha$. According to Lemma 2.2, as $1 + \alpha \leq 2$, $5 + \alpha \leq 6$ and they have membership value α, the membership value of $(1 + \alpha) + (5 + \alpha)$ in $\widetilde{2} \oplus \widetilde{6}$ will be α, i.e., $\mu((1 + \alpha) + (5 + \alpha)|\widetilde{2} \oplus \widetilde{6}) = \alpha$. Similarly, $\mu((3 - \alpha) + (7 - \alpha)|\widetilde{2} \oplus \widetilde{6}) = \alpha$. Therefore, $\mu(6 + 2\alpha|\widetilde{2} \oplus \widetilde{6}) = \alpha$ and $\mu(10 - 2\alpha|\widetilde{2} \oplus \widetilde{6}) = \alpha$. Clearly, these two functional equations will provide the analytical form of the membership function of $\widetilde{2} \oplus \widetilde{6}$, which is identical to that written above.

Therefore, we conclude that in computing $\mu(x|\widetilde{2} \oplus \widetilde{6})$ there are infinitely many pairs (x_1, x_2) with $x_1 \in \widetilde{2}(0)$, $x_2 \in \widetilde{6}(0)$ and $x_1 + x_2 = x$. However, consideration of the combinations for which either (i) $\mu(x_1|\widetilde{2}) \neq \mu(x_2|\widetilde{6})$ or (ii) $x_1 < 2$,

$x_2 > 6$ or $x_1 > 2$, $x_2 < 6$ is unnecessary. This is why these types of combinations may be called redundant or irrelevant.

The next subsection provides a process for identifying redundant combinations for addition of two fuzzy points.

2.3. Addition operation for fuzzy points

Fuzzy points may be viewed in two different ways: either as a collection of crisp points with varied membership grades or as a collection of (normal, convex) fuzzy sets along different lines passing through the core of the fuzzy point. To clarify the second view, we introduce a new concept, a fuzzy number along a line.

Definition 2.4 (*Fuzzy number along a line*). In defining a fuzzy number, conventionally a real line ($\mathbb{R}$) is taken as the universal set. Instead of a real line as the universal set, consider any line on the plane $\mathbb{R}^2$ where the x-axis represents real line, and let $\widetilde{p}$ be a fuzzy number. On the x-axis, the membership function of $\widetilde{p}$ may be written as $\mu((x,0)|\widetilde{p}) = \mu(x|\widetilde{p})\ \forall x \in \mathbb{R}$. More explicitly:

$$\mu((x,y)|\widetilde{p}) = \begin{cases} \mu(x|\widetilde{p}) & \text{if } y = 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation that includes rotation of the axes by angle θ and translation of the origin to $(ac/(a^2+b^2), bc/(a^2+b^2))$, which is the point of intersection for $ax + by = c$ and its perpendicular line through origin. T can be expressed by $T(x,y) = (x \cos\theta - y \sin\theta + ac/(a^2+b^2), x \sin\theta + y \cos\theta + bc/(a^2+b^2))$. T is a bijective transformation that transforms the x-axis to $ax + by = c$. Now, $\widetilde{p}$ may be considered as a fuzzy number on the line $ax + by = c$ and may be defined in the following way:

$$\mu((u,v)|\widetilde{p}) = \begin{cases} \mu((x,0)|\widetilde{p}) & \text{if } (u,v) = T(x,0),\ au + bv = c, \\ 0 & \text{elsewhere.} \end{cases}$$

Example 2.4.1. Let $\widetilde{2}$ be a fuzzy number with membership function:

$$\mu(x|\widetilde{2}) = \begin{cases} (1-x)^2 & \text{if } 1 \le x \le 2, \\ 2 - \frac{x}{2} & \text{if } 2 \le x \le 4, \\ 0 & \text{elsewhere.} \end{cases}$$

Here, the support of $\widetilde{2}$ is $\{x : 1 \le x \le 4\}$. This $\widetilde{2}$ can also be placed on the line $x - y = 0$ as follows. In $\mathbb{R}^2$, the x-axis can be imagined as real line. Considering the x-axis as the universal set, the fuzzy number $\widetilde{2}$ can be expressed as

$$\mu((x,y)|\widetilde{2}) = \begin{cases} (1-x)^2 & \text{if } 1 \le x \le 2,\ y = 0, \\ 2 - \frac{x}{2} & \text{if } 2 \le x \le 4,\ y = 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Here, the support of $\widetilde{2}$ is $\{(x,y) : 1 \le x \le 4, y = 0\}$. A transformation involving only 45° rotation of the axes transforms the x-axis to $x - y = 0$. This transformation is $T(x,y) = (x/\sqrt{2} - y/\sqrt{2}, x/\sqrt{2} + y/\sqrt{2})$. Now $\widetilde{2}$ on $x - y = 0$ can be expressed as

$$\mu((u,v)|\widetilde{2}) = \begin{cases} (1-\sqrt{2}u)^2 & \text{if } \frac{1}{\sqrt{2}} \le u \le \sqrt{2},\ v = u, \\ 2 - \frac{u}{\sqrt{2}} & \text{if } \sqrt{2} \le u \le 2\sqrt{2},\ v = u, \\ 0 & \text{elsewhere.} \end{cases}$$

This fuzzy number is said to be 'fuzzy number two' on the line $x - y = 0$. Here, the support of $\widetilde{2}$ is $\{(x,y) : \frac{1}{\sqrt{2}} \le x \le 2\sqrt{2},\ x - y = 0\}$.

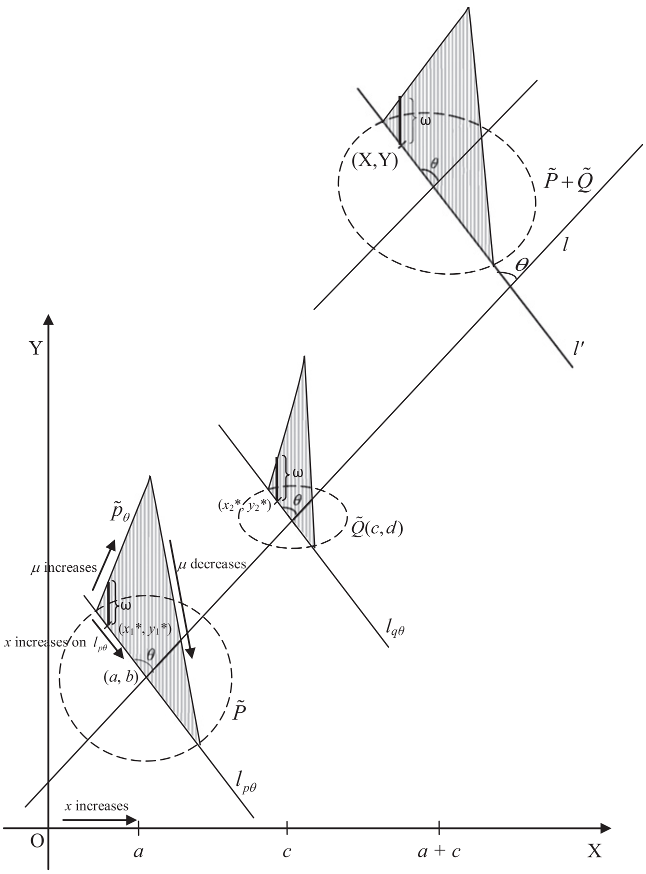

Fig. 1. Addition of two fuzzy points for the increasing and decreasing parts of $\mu(\cdot|\widetilde{P})$ along $l_{p\theta}$.

Note 1. The definition is titled 'fuzzy number along a line' because for each and every fuzzy number $\widetilde{p}$ on line $ax + by = c$ there always exists a unique fuzzy number on the real line and the converse is also true (this converse fuzzy number on the real line can be obtained by mapping T^{-1}). However, the title '(normal, convex) fuzzy set (or fuzzy point) along a line' may be more appropriate. In our discussion, the two terminologies are used interchangeably.

2.3.1. Fuzzy sets along a line on the support of a fuzzy point

Let $\widetilde{P}(a, b)$ and $\widetilde{Q}(c, d)$ be two fuzzy points. The fuzzy points may be viewed as a collection of normal, convex fuzzy sets or fuzzy numbers along different directions as, $\widetilde{P}(a, b) = \bigcup_{\theta\in[0,\pi]} \widetilde{p}_\theta$, where the membership function of $\widetilde{p}_\theta$ can be written as

$$\mu((x, y)|\widetilde{p}_\theta) = \mu_{p\theta}(x, y) = \begin{cases} \mu((x, y)|\widetilde{P}) & \text{if } (x, y) \in l_{p\theta}, \\ 0 & \text{otherwise,} \end{cases}$$

where $l_{p\theta}$ is a line passing through (a, b) (Fig. 1) with angle θ to the line, l say, joining (a, b) and (c, d). Here, for $(x, y) \in l_{p\theta}$ and $x \leq a$, $\mu((x, y)|\widetilde{P})$ gradually increases as x increases, and for $(x, y) \in l_{p\theta}$ and $x \geq a$, $\mu((x, y)|\widetilde{P})$ gradually decreases as x increases. The same reasoning may be applied to y. Similarly, $\widetilde{Q}(c, d) = \bigcup_{\theta\in[0,\pi]} \widetilde{q}_\theta$.

We define the 'addition of two fuzzy points' $\widetilde{P}$ and $\widetilde{Q}$ as $\widetilde{P} + \widetilde{Q} = \bigcup_{\theta\in[0,\pi]}(\widetilde{p}_\theta \oplus \widetilde{q}_\theta)$.

2.3.2. Redundant combinations for addition of two fuzzy points

In computing $\bigcup_{\theta\in[0,\pi]}(\widetilde{p}_\theta \oplus \widetilde{q}_\theta)$, we have attempted to capture effective combinations. Theorem 2.2 helps to separate out these effective combinations. First, we provide the following lemma, which is needed to prove the theorem.

In Lemma 2.3 and Theorem 2.2, the lines l, $l_{p\theta}$ and $l_{q\theta}$, the functions $\mu_{p\theta}$ and $\mu_{q\theta}$, and the fuzzy numbers $\widetilde{p}_\theta$ and $\widetilde{q}_\theta$ bear the same meaning as defined above.

Lemma 2.3. *Let $\widetilde{P}(a,b)$ and $\widetilde{Q}(c,d)$ be two fuzzy points. If (x_1,y_1) and (x_2,y_2) are two points in $l_{p\theta}\cap\widetilde{P}(0)$ and $l_{q\theta}\cap\widetilde{Q}(0)$ with $x_1\le a,\ x_2\le c$ and $\mu((x_1,y_1)|\widetilde{P})=\mu((x_2,y_2)|\widetilde{Q})=\omega$, then*

$$\mu((x_1+x_2,y_1+y_2)|\widetilde{P}+\widetilde{Q})=\omega.$$

Proof. Let $(x_1',y_1')\in l_{p\theta}\cap\widetilde{P}(0)$ and $(x_2',y_2')\in l_{q\theta}\cap\widetilde{Q}(0)$, where $(x_1',y_1')\neq(x_1,y_1)$ and $(x_2',y_2')\neq(x_2,y_2)$ but $(x_1'+x_2',y_1'+y_2')=(x_1+x_2,y_1+y_2)$. Four cases may arise:

Case 1: $x_1'>x_1$ and $y_1'\le y_1$, i.e., $x_2'\le x_2$ and $y_2'>y_2$.

Case 2: $x_1'>x_1$ and $y_1'>y_1$, i.e., $x_2'\le x_2$ and $y_2'\le y_2$.

Case 3: $x_1'\le x_1$ and $y_1'<y_1$.

Case 4: $x_1'\le x_1$ and $y_1'>y_1$.

In all four cases, either $x_1'\le x_1$ or $x_2'\le x_2$. Therefore, $\min(\mu((x_1',y_1')|\widetilde{P}),\mu((x_2',y_2')|\widetilde{Q}))\le\omega$ since $\mu((x,y)|\widetilde{P})$ and $\mu((x,y)|\widetilde{Q})$ are increasing with respect to the first variable x along $l_{p\theta}$ for $x\le a$ and along $l_{q\theta}$ for $x\le c$, respectively.

Thus, in any situation, $\min(\mu((x_1',y_1')|\widetilde{P}),\mu((x_2',y_2')|\widetilde{Q}))\le\omega$ and the maximum is attained for (x_1,y_1) and (x_2,y_2), which yields the result. □

Theorem 2.2. *Let $\widetilde{P}(a,b)$ and $\widetilde{Q}(c,d)$ be two continuous fuzzy points. If $(x_1,y_1)\in l_{p\theta}\cap\widetilde{P}(0)$ and $(x_2,y_2)\in l_{q\theta}\cap\widetilde{Q}(0)$ are two points such that $x_1+x_2\le a+c$, then $\exists(x_1^*,y_1^*)\in l_{p\theta}\cap\widetilde{P}(0)$ and $(x_2^*,y_2^*)\in l_{q\theta}\cap\widetilde{Q}(0)$ such that*:

(i) $x_1^*\le a,\ x_2^*\le c$,
(ii) $\mu((x_1^*,y_1^*)|\widetilde{P})=\mu((x_2^*,y_2^*)|\widetilde{Q})$,
(iii) $x_1+x_2=x_1^*+x_2^*,\ y_1+y_2=y_1^*+y_2^*$, *and*
(iv) $\mu((x_1+x_2,y_1+y_2)|\widetilde{P}+\widetilde{Q})=\mu((x_1^*,y_1^*)|\widetilde{P})=\mu((x_2^*,y_2^*)|\widetilde{Q})$.

Proof. As $\widetilde{P}$ comprises continuous fuzzy points along $l_{p\theta}$, the function $\mu_{p\theta}((x,y)|\widetilde{P})$ is increasing with respect to x for $x\le a$. Similarly, for $\widetilde{Q}$ along $l_{q\theta}$, the function $\mu_{q\theta}((x,y)|\widetilde{Q})$ is increasing with respect to x for $x\le c$. Here two cases may arise.

Case 1: In this case, we consider that $\mu_{p\theta}$ and $\mu_{q\theta}$ are strictly increasing for $x\le a$ and $x\le c$, respectively. Then, both $\mu_{p\theta}$ and $\mu_{q\theta}$ are bijective and hence $\mu_{p\theta}^{-1}$ and $\mu_{q\theta}^{-1}$ exist and they are continuous and strictly increasing on [0, 1].

Consider the function $g=\mu_{p\theta}^{-1}+\mu_{q\theta}^{-1}$. Then, obviously, g is strictly increasing and continuous on [0, 1].

Let $(X,Y)=(x_1+x_2,y_1+y_2)$ and let ω be the value of $g^{-1}(X,Y)$. For this ω, we consider the points $(x_1^*,y_1^*)=\mu_{p\theta}^{-1}(\omega)$ and $(x_2^*,y_2^*)=\mu_{q\theta}^{-1}(\omega)$.

Addition of these two points is $(x_1^*+x_2^*,y_1^*+y_2^*)=g(\omega)=(X,Y)$. Moreover, $x_1^*\le a$ since $\mu_{p\theta}^{-1}$ is strictly increasing in [0, 1] and $\mu_{p\theta}^{-1}(1)=(a,b)$. Similarly, $x_2^*\le c$.

Since (x_1^*,y_1^*) and (x_2^*,y_2^*) are two points on $l_{p\theta}$ and $l_{q\theta}$, respectively, with $x_1^*\le a$ and $x_2^*\le c$. According to Lemma 2.3, we obtain $\mu((x_1+x_2,y_1+y_2)|\widetilde{P}+\widetilde{Q})=\mu((x_1^*,y_1^*)|\widetilde{P})=\mu((x_2^*,y_2^*)|\widetilde{Q})=\omega$, which proves the theorem in this case.

Case 2: Consider another case in which $\mu_{p\theta}$ and $\mu_{q\theta}$ are not strictly increasing for $x\le a$ and $x\le c$, respectively. In other words, $\exists$ two intervals $[a_1,a_2]$ and $[c_1,c_2]$ (possibly $a_1=a_2$ and $c_1=c_2$) such that $\mu_{p\theta}$ and $\mu_{q\theta}$ are constant, ω say, for x in $[a_1,a_2]$ and $[c_1,c_2]$, respectively. In this case, we claim that $\mu((x,y)|\widetilde{P}+\widetilde{Q})=\omega$ for all $(x,y)\in(\widetilde{P}+\widetilde{Q})(0)\cap l'$ with $x\in[a_1+c_1,a_2+c_2]$, where l' is the line with angle θ to l and passing though $(a+c,b+d)$.

Let $(x_1,y_1)\in\widetilde{P}(0)\cap l_{p\theta}$ and $(x_2,y_2)\in\widetilde{Q}(0)\cap l_{q\theta}$ be two points with $x_1\in[a_1,a_2]$ and $x_2\in[c_1,c_2]$. Then $\min(\mu((x_1,y_1)|\widetilde{P}),\mu((x_2,y_2)|\widetilde{Q}))=\min(\omega,\omega)=\omega$.

Let $(x_1,y_1)\in\widetilde{P}_1(0)\cap l_{p\theta}$ and $(x_2,y_2)\in\widetilde{P}_2(0)\cap l_{q\theta}$ be such that $x_1+x_2=a_1+c_1$. If $x_1\le a_1$, then $\mu((x_1,y_1)|\widetilde{P})\le\omega$, and hence $\min(\mu((x_1,y_1)|\widetilde{P}),\mu((x_2,y_2)|\widetilde{Q}))\le\omega$. If $x_1>a_1$, then $x_2\le c_1$. Thus, $\mu((x_2,y_2)|\widetilde{Q})\le\omega$. This implies that $\min(\mu((x_1,y_1)|\widetilde{P}),\mu((x_2,y_2)|\widetilde{Q}))\le\omega$ and equality occurs for $x_1=a_1$, and $x_2=c_1$.

The same result holds true for two points $(x_1, y_1) \in \widetilde{P}_1(0) \cap l_{p\theta}$ and $(x_2, y_2) \in \widetilde{P}_2(0) \cap l_{q\theta}$ with $x_1 + x_2 = a_2 + c_2$. Hence, for $(x, y) \in (\widetilde{P} + \widetilde{Q})(0) \cap l'$ with $x \in [a_1 + c_1, a_2 + c_2]$,

$$\begin{aligned}\mu((x, y)|\widetilde{P} + \widetilde{Q}) &= \sup_{(x_1,y_1)+(x_2,y_2)=(x,y)} \min(\mu((x_1, y_1)|\widetilde{P}), \mu((x_2, y_2)|\widetilde{Q}))\\ &= \sup_{x_1+x_2=x} \min(\mu((x_1, y_1)|\widetilde{P}), \mu((x_2, y_2)|\widetilde{Q}))\\ &= \omega.\end{aligned}$$

Thus, our claim is proved and this result assures the existence of many (x_1^*, y_1^*) and (x_2^*, y_2^*) that satisfy properties (i)–(iv) in the theorem. Hence, the theorem is proved. □

Note 2. Theorem 2.2 implies that in computing $\widetilde{P} + \widetilde{Q}$, to obtain the membership value of any point $(x_1 + x_2, y_1 + y_2)$, there are many $(x_1', y_1') \in l_{p\theta} \cap \widetilde{P}(0)$ and $(x_2', y_2') \in l_{q\theta} \cap \widetilde{Q}(0)$ such that $(x_1' + x_2', y_1' + y_2') = (x_1 + x_2, y_1 + y_2)$. However, consideration of the combinations for which either (i) $\mu((x_1', y_1')|\widetilde{P}) \neq \mu((x_2', y_2')|\widetilde{Q})$ or (ii) $x_1' < a, x_2' > c$ or $x_1' > a, x_2' < c$ is unnecessary. This is why these types of combinations can be called redundant or irrelevant and combinations of the points (x_1^*, y_1^*) and (x_1^*, y_2^*) are called effective. In Definition 3.3, the points (x_1^*, y_1^*) and (x_1^*, y_2^*) are called same points with respect to $\widetilde{P}$ and $\widetilde{Q}$.

It should be mentioned that in applying a binary increasing operator for the sup-min composition on continuous fuzzy sets, while taking the minimum of two membership values of two different elements, the lower membership value always dominates the higher one. Thus, it is reasonable to take only combinations of elements with the same membership values (effective combinations) because otherwise higher membership values would not have any effect on the membership value. In fact, Theorems 2.1 and 2.2 suggest that this composition should be applied.

Note 3. According to the extension principle,

$$\begin{aligned}\widetilde{P} \oplus \widetilde{Q} &= \bigcup_{\theta\in[0,\pi]} \widetilde{p}_\theta \oplus \bigcup_{\theta\in[0,\pi]} \widetilde{q}_\theta\\ &= \bigcup_{\theta,\phi\in[0,\pi]} (\widetilde{p}_\theta \oplus \widetilde{q}_\phi).\end{aligned}$$

That is, for all possible values of θ and ϕ, the fuzzy sets $\widetilde{p}_\theta$ and $\widetilde{q}_\phi$ have to be added by the extension principle to obtain $\widetilde{P} \oplus \widetilde{Q}$. However, $\widetilde{p}_\theta \oplus \widetilde{q}_\phi$ is not a 'fuzzy set along a line' for $\theta \neq \phi$, as its support is not a line segment, and is a 'fuzzy set along a line' for $\theta = \phi$. To perform addition of $\widetilde{P}$ and $\widetilde{Q}$ as a collection of fuzzy sets along lines, which involves addition of fuzzy sets along lines of $\widetilde{P}$ and $\widetilde{Q}$, we considered the combinations $\widetilde{p}_\theta \oplus \widetilde{q}_\theta$ only for different values of $\theta \in [0, \pi]$. That is, instead of taking $\bigcup_{\theta,\phi\in[0,\pi]}(\widetilde{p}_\theta \oplus \widetilde{q}_\phi)$ as $\widetilde{P} \oplus \widetilde{Q}$, we define addition of $\widetilde{P}$ and $\widetilde{Q}$ in a fuzzy geometrical plane as $\bigcup_{\theta\in[0,\pi]}(\widetilde{p}_\theta \oplus \widetilde{q}_\theta)$. We denote this addition by $\widetilde{P} + \widetilde{Q}$. Thus, we need to prove that 'addition of two fuzzy points is a fuzzy point'. This is addressed in Theorem 3.1.

Note 4. Observe that combinations besides effective combinations are redundant since only effective combinations $\widetilde{P} + \widetilde{Q}$ can be combined. We call these combinations of same points. A formal definition of same points with respect to two continuous fuzzy points is given in the next section.

The next section introduces the concepts of same and inverse points. These concepts are then used to define various fuzzy geometrical ideas.

3. Same points and inverse points

Let φ be an increasing function and let $\widetilde{m}_1, \widetilde{m}_2, \ldots, \widetilde{m}_n$ be n continuous fuzzy numbers. To evaluate $\widetilde{y} = \widetilde{\varphi}(\widetilde{m}_1, \widetilde{m}_2, \ldots, \widetilde{m}_n)$ by the extension principle, we take a number $y \in \widetilde{y}(0)$. From Lemmas 2.1 and 2.2 and Theorem 2.1, it is clear that in the following two situations the combinations $(x_1, x_2, \ldots, x_n)$ with $y = \varphi(x_1, x_2, \ldots, x_n)$ are redundant:

(i) if there exist x_j, x_k such that $\mu(x_j|\widetilde{m}_j) \neq \mu(x_k|\widetilde{m}_k)$, $j, k \in \{1, 2, \ldots, n\}$, or
(ii) if there exist x_j, x_k such that $x_j > m_j$ and $x_k < m_k$, $j, k \in \{1, 2, \ldots, n\}$.

This redundancy leads us to define same points and inverse points with respect to continuous fuzzy numbers in $\mathbb{R}$ and continuous fuzzy points in the plane $\mathbb{R}^2$.

Definition 3.1 (*Same points with respect to continuous fuzzy numbers*). Let x, y be two numbers belonging to the supports of the continuous fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$, respectively. The numbers x and y are said to be same points with respect to $\widetilde{a}$ and $\widetilde{b}$ if:

(i) $\mu(x|\widetilde{a}) = \mu(y|\widetilde{b})$, and
(ii) $x \le a$ and $y \le b$, or $x \ge a$ and $y \ge b$, where a, b are midpoints of $\widetilde{a}(1), \widetilde{b}(1)$, respectively.

Example 3.1.1. Consider $\widetilde{a} = \widetilde{2}, \widetilde{b} = \widetilde{6}$ in the illustration of Section 2.2. For each particular $\alpha \in [0, 1]$, the pairs $1+\alpha, 5+\alpha$ and $3-\alpha, 7-\alpha$ are same points.

Example 3.1.2. Let $\widetilde{a} = (1/2/3)$ and $\widetilde{b}$ be defined as

$$\mu(x|\widetilde{b}) = \begin{cases} (x-4)^2 & \text{if } 4 \le x \le 5, \\ \frac{8-x}{3} & \text{if } 5 \le x \le 8, \\ 0 & \text{elsewhere.} \end{cases}$$

The pairs of numbers $\frac{5}{4}, \frac{9}{2}$ and $\frac{8}{3}, 7$ are same points, where $\frac{5}{4}, \frac{8}{3} \in \widetilde{a}(0)$ and $\frac{9}{2}, 7 \in \widetilde{b}(0)$.

Definition 3.2 (*Inverse points with respect to continuous fuzzy numbers*). Let x, y be two numbers belonging to the supports of the continuous fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$, respectively. The numbers x and y are said to be inverse points with respect to $\widetilde{a}$ and $\widetilde{b}$ if $x, -y$ are same points with respect to $\widetilde{a}$ and $-\widetilde{b}$, where $-\widetilde{b}$ is scalar multiplication of $\widetilde{b}$ by -1.

Example 3.2.1. Consider $\widetilde{a} = \widetilde{2}, \widetilde{b} = \widetilde{6}$ in the illustration of Section 2.2. For each particular $\alpha \in [0, 1]$, the pairs $1+\alpha, 7-\alpha$ and $5-\alpha, 7+\alpha$ are inverse points.

Example 3.2.2. Let $\widetilde{a}$ and $\widetilde{b}$ be the two fuzzy numbers considered in Example 3.1.2. The pairs of numbers $\frac{5}{4}, \frac{29}{4}$ and $\frac{71}{25}, \frac{22}{5}$ are inverse points with respect to $\widetilde{a}$ and $\widetilde{b}$, where $\frac{5}{4}, \frac{71}{25} \in \widetilde{a}(0)$ and $\frac{29}{4}, \frac{22}{5} \in \widetilde{b}(0)$.

Definition 3.3 (*Same points with respect to continuous fuzzy points*). Let $(x_1, y_1), (x_2, y_2)$ be two points on the supports of the continuous fuzzy points $\widetilde{P}(a, b), \widetilde{P}(c, d)$, respectively, and let L_1 be a line joining (x_1, y_1) and (a, b).

As $\widetilde{P}(a, b)$ is a fuzzy point, along L_1, a fuzzy number, $\widetilde{r}_1$ say, is situated on the support of $\widetilde{P}(a, b)$. The membership function of this fuzzy number $\widetilde{r}_1$ can be written as: $\mu((x, y)|\widetilde{r}_1) = \mu((x, y)|\widetilde{P}(a, b))$ for (x, y) in L_1, and 0 otherwise.

Similarly, along a line (L_2) joining (x_2, y_2) and (c, d), a fuzzy number, $\widetilde{r}_2$ say, will be obtained on the support of $\widetilde{P}(c, d)$. Now the points $(x_1, y_1), (x_2, y_2)$ are said to be same points with respect to $\widetilde{P}(a, b)$ and $\widetilde{P}(c, d)$ if:

(i) (x_1, y_1) and (x_2, y_2) are same points with respect to $\widetilde{r}_1, \widetilde{r}_2$, and
(ii) L_1, L_2 make the same angle with the line joining (a, b) and (c, d).

Example 3.3.1. Let $\widetilde{P}(2, 2)$ be a fuzzy point whose membership function is a right circular cone with base $\{(x, y) : (x-2)^2 + (y-2)^2 \le 2\}$ and vertex (2,2). Let $\widetilde{P}(5, 6)$ be another fuzzy point whose membership function is a right elliptical cone with base $\{(x, y) : (x-5)^2/(5/3) + (y-6)^2/(5/2) \le 1\}$ and vertex (5, 6).

The bases of these fuzzy points are depicted in Fig. 2 by the circle centered at $Q_1(2, 2)$ and the ellipse centered at $Q_2(5, 6)$.

Consider the points $P_1(1.5, 2.5)$ and $P_2(4.5, 6.5)$ from the supports of $\widetilde{P}(2, 2)$ and $\widetilde{P}(5, 6)$, respectively. The line joining P_1 and Q_1 is $L_1 : x + y = 4$. Along L_1, there exists a triangular fuzzy number, $\widetilde{r}_1$ say, on the support of $\widetilde{P}(2, 2)$. The base of $\widetilde{r}_1$ is the set $\{(x, y) : (x-2)^2 + (y-2)^2 \le 2, x + y = 4\}$. Visualizing the membership function of $\widetilde{P}(2, 2)$ as a surface in $\mathbb{R}^3$, the membership function of $\widetilde{r}_1$ may be perceived as the union of the straight line segments from (1, 3, 0) to (2, 2, 1) and from (2, 2, 1) to (3, 1, 0).

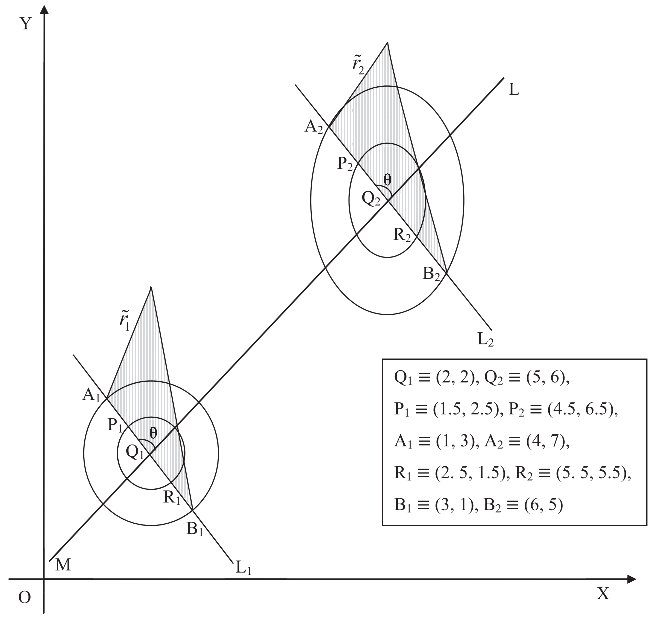

Fig. 2. Same and inverse points for two continuous fuzzy points $\widetilde{P}(Q_1)$ and $\widetilde{P}(Q_2)$.

Similarly, along the line joining P_2 and Q_2, $L_2 : x + y = 11$, there exists a triangular fuzzy number, $\widetilde{r}_2$ say, on the support of $\widetilde{P}(5, 6)$. The base of $\widetilde{r}_2$ is the set $\{(x, y) : (x - 5)^2/(5/3) + (y - 6)^2/(5/2) \leq 1, x + y = 11\}$. The membership function of $\widetilde{r}_2$ is the union of the straight line segments from (4, 7, 0) to (5, 6, 1) and from (5, 6, 1) to (6, 5, 0).

In Fig. 2, $A_1 \equiv (1, 3)$, $B_1 \equiv (3, 1)$, $A_2 \equiv (4, 7)$, $B_2 \equiv (6, 5)$. Apparently, $\widetilde{r}_1(0) = \overline{A_1 B_1}$ and $\widetilde{r}_2(0) = \overline{A_2 B_2}$.

Note that

(i) With respect to $\widetilde{r}_1$ and $\widetilde{r}_2$, the points $P_1(1.5, 2.5)$ and $P_2(4.5, 6.5)$ are same points.
(ii) $\mu((1.5, 2.5)|\widetilde{P}(2, 2)) = \mu((1.5, 2.5)|\widetilde{r}_1) = 0.29$, $\mu((4.5, 6.5)|\widetilde{P}(5, 6)) = \mu((4.5, 6.5)|\widetilde{r}_2) = 0.29$.
(iii) The line joining (2, 2) and (5, 6) is $4y - 3x = 2$. Both the lines $L_1 : x + y = 4$ and $L_2 : x + y = 11$ make the same angle $\theta = \tan^{-1}(-7)$ with $4y - 3x = 2$.

Therefore, the points (1.5, 2.5) and (4.5, 6.5) are same points with respect to the fuzzy points $\widetilde{P}(2, 2)$ and $\widetilde{P}(5, 6)$.

Definition 3.4 (*Addition of two fuzzy points*). Addition of the fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ is denoted by $\widetilde{P}_1 + \widetilde{P}_2$ and its membership function is defined by $\mu(t|\widetilde{P}_1 + \widetilde{P}_2) = \sup\{\alpha : t = x + y$, where $x \in \widetilde{P}_1(0)$ and $y \in \widetilde{P}_2(0)$ are same points with membership value $\alpha\}$. Here $x, y, t \in \mathbb{R}^2$.

Definition 3.5 (*Scalar multiplication of a fuzzy point, Muganda [12]*). Let $\lambda \in \mathbb{R}$. Scalar multiplication of a fuzzy point $\widetilde{P}(a, b)$ by λ is written as $\lambda\widetilde{P}(a, b)$ and its membership function is defined by

$$\mu((x, y)|\lambda\widetilde{P}(a, b)) = \begin{cases} \mu((x/\lambda, y/\lambda)|\widetilde{P}(a, b)) & \text{if } \lambda \neq 0, \\ \sup\limits_{(u,v)\in\mathbb{R}^2} \mu((u, v)|\widetilde{P}(a, b)) & \text{if } \lambda = 0, (x, y) = (0, 0), \\ 0 & \text{if } \lambda = 0, (x, y) \neq (0, 0). \end{cases}$$

Theorem 3.1. *If* $\widetilde{P}_1, \widetilde{P}_2$ *are two continuous fuzzy points, then*

(1) $\lambda\widetilde{P}_1$ *is a fuzzy point* $\forall\lambda \in \mathbb{R}$,
(2) $\widetilde{P}_1 + \widetilde{P}_2$ *is a fuzzy point, and*
(3) *the linear combination* $\lambda_1\widetilde{P}_1 + \lambda_2\widetilde{P}_2, \lambda_1, \lambda_2 \in \mathbb{R}$ *is also a fuzzy point.*

Proof.

(1) This obviously follows from Definition 3.5.
(2) Let $\alpha \in [0, 1]$. First, we argue that the set $\mathbb{A}(\alpha) := \{z : \mu(z|\widetilde{P}_1 + \widetilde{P}_2) \geq \alpha\}$ is convex and bounded. Let $z_1, z_2 \in \mathbb{A}(\alpha)$. Thus, there exist same points x_1, y_1 and x_2, y_2 such that $z_1 = x_1 + y_1$, $z_2 = x_2 + y_2$ with each of $\mu(x_1|\widetilde{P}_1)$, $\mu(x_2|\widetilde{P}_1)$, $\mu(y_1|\widetilde{P}_2)$, $\mu(y_2|\widetilde{P}_2)$ is greater than or equal to α. Let $\lambda \in \mathbb{R}$. Note that $\lambda z_1 + (1-\lambda)z_2$ can be expressed as $(\lambda x_1 + (1-\lambda)x_2) + (\lambda y_1 + (1-\lambda)y_2)$. Since $x_1, x_2 \in \widetilde{P}_1(\alpha)$ and $\widetilde{P}_1(\alpha)$ is convex, $\lambda x_1 + (1-\lambda)x_2 \in \widetilde{P}_1(\alpha)$. Similarly, $\lambda y_1 + (1-\lambda)y_2 \in \widetilde{P}_2(\alpha)$. Thus, whether or not $\lambda x_1 + (1-\lambda)x_2$ and $\lambda y_1 + (1-\lambda)y_2$ are same points, $\mu(\lambda z_1 + (1-\lambda)z_2|\widetilde{P}_1 + \widetilde{P}_2)$ is at least α. Therefore, $\lambda z_1 + (1-\lambda)z_2 \in \mathbb{A}(\alpha)$, and hence $\mathbb{A}(\alpha)$ is convex. As $\widetilde{P}_1(\alpha)$, $\widetilde{P}_2(\alpha)$ are both compact subsets of $\mathbb{R}^2$, they are bounded. Any z in $\mathbb{A}(\alpha)$ can be obtained by taking a combination of same points that belong to $\widetilde{P}_1(\alpha)$ and $\widetilde{P}_2(\alpha)$. Thus, $\mathbb{A}(\alpha)$ is bounded trivially.
Now we prove that $\mathbb{A}(\alpha)$ is closed. If the set of all limit points of $\mathbb{A}(\alpha)$ is empty, then this part is obviously true. If the set is not empty, then let z_0 be a limit point of $\mathbb{A}(\alpha)$. If possible, let $z_0 \notin \mathbb{A}(\alpha)$. Let $\mu(z_0|\widetilde{P}_1 + \widetilde{P}_2) = \beta$. Thus, $\beta < \alpha$. Now $z_0 \notin \{z : \mu(z|\widetilde{P}_1 + \widetilde{P}_2) \geq \alpha\} \subset \{z : \mu(z|\widetilde{P}_1 + \widetilde{P}_2) \geq \beta\}$. Let ε be the distance between z_0 and $\mathbb{A}(\alpha)$. It is easily perceived that $\varepsilon > 0$. Now $\mathbb{A}(\alpha)$ and the open ball $B(z_0, \varepsilon)$ have empty intersection and hence z_0 cannot be a limit point of $\mathbb{A}(\alpha)$, which is a contradiction. Thus, $z_0 \in \mathbb{A}(\alpha)$. Since z_0 is arbitrarily taken, $\mathbb{A}(\alpha)$ is closed.
Obviously, $\forall t \in \mathbb{R}$ the set $\{z : \mu(z\widetilde{P}_1 + \widetilde{P}_2) \geq t\}$ is closed. Thus, the membership function $\mu(z|\widetilde{P}_1 \oplus \widetilde{P}_2)$ is upper semi-continuous.
Since $\mathbb{A}(\alpha)$ is closed and bounded, $\mathbb{A}(\alpha)$ is a compact subset of $\mathbb{R}^2$.
Let $\widetilde{P}_1, \widetilde{P}_2$ be fuzzy points at the points (a, b) and (c, d), respectively. Then $\mu((a+c, b+d)|\widetilde{P}_1 + \widetilde{P}_2) = 1$. Thus, $\widetilde{P}_1 + \widetilde{P}_2$ is a fuzzy point.
(3) This part is an application of the previous two parts, and the proof is omitted. □

Definition 3.6 (*Inverse points with respect to continuous fuzzy points*). Let (x_1, y_1) and (x_2, y_2) be two points belonging to the supports of two different continuous fuzzy points $\widetilde{P}(a, b)$ and $\widetilde{P}(c, d)$, respectively. The points (x_1, y_1), (x_2, y_2) are said to be inverse points with respect to $\widetilde{P}(a, b)$ and $\widetilde{P}(c, d)$ if (x_1, y_1), $(-x_2, -y_2)$ are same points with respect to $\widetilde{P}(a, b)$ and $-\widetilde{P}(c, d)$, where $-\widetilde{P}(c, d)$ is $\lambda\widetilde{P}(c, d)$, with $\lambda = -1$.

Fig. 2 explains same and inverse points for two continuous fuzzy points $\widetilde{P}(Q_1)$ and $\widetilde{P}(Q_2)$. The interior and boundary of outer circle centered at Q_1 and the outer ellipse centered at Q_2 are their respective supports. The inner circle is the α-cut of $\widetilde{P}(Q_1)$ and the inner ellipse is the α-cut of $\widetilde{P}(Q_2)$. ML is a line joining Q_1 and Q_2. A_1B_1 and A_2B_2 are lines passing through Q_1 and Q_2, respectively. Both A_1B_1 and A_2B_2 make the same angle with ML, $\angle A_1Q_1L = \angle A_2Q_2L = \theta$ (say). The points A_1, A_2; P_1, P_2; R_1, R_2; ... are pairs of same points and P_1, R_2; A_1, B_2; ... are pairs of inverse points.

Example 3.6.1. Consider the fuzzy points $\widetilde{P}(2, 2)$, $\widetilde{P}(5, 6)$ in Example 3.3.1. The points $P_1(1.5, 2.5)$, $R_2(5.5, 5.5)$ are inverse points with respect to $\widetilde{P}(2, 2)$ and $\widetilde{P}(5, 6)$.

Note 5. Note that subtraction of two fuzzy points/numbers, $\widetilde{P}_1$ and $\widetilde{P}_2$ say, can be done by taking the supremum over the combination of inverse points in the sup-min composition of the fuzzy sets, because it can be proved that $\widetilde{P}_1 \ominus \widetilde{P}_2 = \widetilde{P}_1 \oplus (-\widetilde{P}_2)$. The same observation can be applied in evaluating a fuzzy distance.

Next we discuss a reference frame in fuzzy plane geometry for use in research using the concepts of same and inverse points.

4. Basic concepts of fuzzy plane geometry

In research on fuzzy plane geometry, two reference frames may be visualized to define a fuzzy geometrical plane. In the first reference frame, the axes are real and the membership functions of fuzzy points on the plane $\mathbb{R}^2$ are realized as surfaces in $\mathbb{R}^3$. In the second reference frame, the axes are also fuzzy (i.e., fuzzy numbers are situated on the axes). However, this leads to a restricted environment, because if two fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$ lie on fuzzy axes, then this reference frame cannot offer a location for the fuzzy number $\widetilde{a} \oplus \widetilde{b}$ on the axes (as the extended addition $\widetilde{a} \oplus \widetilde{b}$ is not equal to $\widetilde{a+b}$ in general). In this paper, definitions are given using the first reference frame, as used by Buckley and Eslami [3,4].

4.1. Fuzzy distance

Definition 4.1 (*Fuzzy distance between two fuzzy points*). The fuzzy distance $\widetilde{D}$ between two continuous fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ may be defined by its membership function: $\mu(d|\widetilde{D}) = \sup\{\alpha$: where $d = d(u, v)$, $u \in \widetilde{P}_1(0)$ and $v \in \widetilde{P}_2(0)$ are inverse points, $\mu(u|\widetilde{P}_1) = \mu(v|\widetilde{P}_2) = \alpha\}$. Here, $d(,)$ is the usual Euclidean distance metric.

Theorem 4.1. *For two continuous fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$,*

(1) $\widetilde{D}(\alpha) = \{d : d = d(u, v)$, *where* $u \in \widetilde{P}_1(\alpha)$, $v \in \widetilde{P}_2(\alpha)$ *are inverse points*$\}$ $\forall \alpha \in [0, 1]$.
(2) $\widetilde{D}$ *is a fuzzy number in* $\mathbb{R}$.

Proof.

(1) Let $\mathbb{A}(\alpha) = \{d : d = d(u, v)$, where $u \in \widetilde{P}_1(\alpha)$ and $v \in \widetilde{P}_2(\alpha)$ are inverse points$\}$. We prove that $\mathbb{A}(\alpha) = \widetilde{D}(\alpha)$ for $0 < \alpha \leq 1$. If this result is true for $0 < \alpha \leq 1$, then obviously $\widetilde{D}(0) = \mathbb{A}(0)$, since support of a fuzzy number is the union of all of its α-cuts.
To prove that $\widetilde{D}(\alpha)$ is a subset of $\mathbb{A}(\alpha)$ for any $\alpha \in (0, 1]$, let $d \in \widetilde{D}(\alpha)$ and $\mu(d|\widetilde{D}) = \beta$, say. Then $\beta \geq \alpha$.
If $\beta > \alpha$, then there exists $\gamma \in \mathbb{R}$ with $\alpha < \gamma \leq \beta$ such that $d \in \mathbb{A}(\gamma)$. As $\mathbb{A}(\gamma) \subseteq \mathbb{A}(\alpha)$, so $d \in \mathbb{A}(\alpha)$. Hence, in this case $\widetilde{D}(\alpha)$ is a subset of $\mathbb{A}(\alpha)$.
For the case when $\beta = \alpha$, observe that $\mu(d|\widetilde{D}) = \sup\{t : d = d(u, v)$, where $u \in \widetilde{P}_1(0)$ and $v \in \widetilde{P}_2(0)$ are inverse points, $\mu(u|\widetilde{P}_1) = \mu(v|\widetilde{P}_2) = t\} = \beta = \alpha$. Obviously, there exist sequences of inverse points $\{u_n\}, \{v_n\}$ with $\mu(u_n|\widetilde{P}_1) = \mu(v_n|\widetilde{P}_2) = \delta_n$ and $d = d(u_n, v_n)$ such that $\{\delta_n\}$ is a nondecreasing sequence that converges to α. Therefore, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $\beta - \varepsilon < \delta_n$ for all $n \geq K$. Here, $d \in \mathbb{A}(\delta_n)$ for any n and $\mathbb{A}(\delta_n) \subseteq \mathbb{A}(\beta - \varepsilon)$ for all $n \geq K$. This implies that $d \in \mathbb{A}(\beta)$ because $\varepsilon > 0$ is arbitrarily taken. Therefore, in this case $\widetilde{D}(\alpha)$ is also a subset of $\mathbb{A}(\alpha)$.
Thus, $\widetilde{D}(\alpha)$ is a subset of $\mathbb{A}(\alpha)$ for any $\alpha \in (0, 1]$.
Consider $d \in \mathbb{A}(\alpha)$, where $\alpha \in (0, 1]$. From the definition of $\mathbb{A}(\alpha)$ and $\mu(d|\widetilde{D})$, we obtain $\mu(d|\widetilde{D}) \geq \alpha$. Thus, d belongs to $\widetilde{D}(\alpha)$ and therefore $\mathbb{A}(\alpha)$ is a subset of $\widetilde{D}(\alpha)$.
Thus, $\widetilde{D}(\alpha) = \mathbb{A}(\alpha) \forall \alpha \in (0, 1]$, and hence $\forall \alpha \in [0, 1]$.

(2) Since $\widetilde{P}_1(\alpha)$ and $\widetilde{P}_2(\alpha)$ are closed and bounded subsets of $\mathbb{R}^2$, $\mathbb{A}(\alpha)$ is a closed and bounded interval of $\mathbb{R}$ for all $\alpha \in [0, 1]$, and therefore so is $\widetilde{D}(\alpha)$. Let $\widetilde{D}(\alpha) = [a(\alpha), c(\alpha)]$ and $\widetilde{D}(0) = [a, c]$. Thus, $\mu(d|\widetilde{D}) = 0$ for all d not in $[a, c]$.
It is obvious from the definition of $\widetilde{D}(\alpha)$ that for $0 \leq \alpha \leq \beta \leq 1$, $[a(\beta), c(\beta)] = \widetilde{D}(\beta) \subseteq \widetilde{D}(\alpha) = [a(\alpha), c(\alpha)]$. Therefore, as α increases, $a(\alpha)$ increases and $c(\alpha)$ decreases.
Now, for all $t \in \mathbb{R}$, the set $\{d : \mu(d|\widetilde{D}) \geq t\}$ is closed and bounded. Therefore, the membership function of $\widetilde{D}$ is upper semi-continuous.
Let $\widetilde{P}_1(1) = (a, b)$ and $\widetilde{P}_2(1) = (p, q)$. Now, $\widetilde{D}(1) = \mathbb{A}(1) = d((a, b), (p, q)) = a(1) = c(1)$.
Hence, $\widetilde{D}$ is a fuzzy number. □

Example 4.1.1. Let $\widetilde{P}_1$ and $\widetilde{P}_2$ be two fuzzy points at (1, 0) and (2, 0), respectively.
The shape of $\widetilde{P}_1$ is a right circular cone with base $\widetilde{P}_1(0) = \{(x, y) : (x-1)^2 + y^2 \leq \frac{1}{4}\}$ and vertex (1, 0).
The shape of $\widetilde{P}_2$ is a right circular cone with base $\widetilde{P}_2(0) = \{(x, y) : (x-2)^2 + y^2 \leq \frac{1}{4}\}$ and vertex (2, 0).
For each $\alpha \in [0, 1]$, the inverse points with respect to $\widetilde{P}_1$ and $\widetilde{P}_2$ with membership value α are $P : (1 + \frac{1}{2}(1-\alpha)\cos\theta, \frac{1}{2}(1-\alpha)\sin\theta)$ and $Q : (2 - \frac{1}{2}(1-\alpha)\cos\theta, -\frac{1}{2}(1-\alpha)\sin\theta)$, respectively ($\theta \in [0, 2\pi]$).
The distance between P and Q is $d(P, Q) = \sqrt{\{1 + (1-\alpha)^2 - 2(1-\alpha)\cos\theta\}}$.
Now, $\inf_{\theta\in[0,2\pi]} d(P, Q) = \sqrt{\{1+(1-\alpha)^2-2(1-\alpha)\}} = \alpha$, and $\sup_{\theta\in[0,2\pi]} d(P, Q) = \sqrt{\{1+(1-\alpha)^2+2(1-\alpha)\}} = 2-\alpha$.

Clearly, if the distance between $\widetilde{P}_1$ and $\widetilde{P}_2$ is $\widetilde{D}$, say, then for any $d \in [\alpha, 2-\alpha]$, $\mu(d|\widetilde{D}) \geq \alpha$, and more precisely $\widetilde{D}(\alpha) = [\alpha, 2-\alpha]$.

Thus, $\widetilde{D}$ is the fuzzy number defined by the following membership function:

$$\mu(d|\widetilde{D}) = \begin{cases} d & \text{if } 0 \leq d \leq 1, \\ 2-d & \text{if } 1 \leq d \leq 2, \\ 0 & \text{elsewhere.} \end{cases}$$

Definition 4.2 (*Coincidence of two fuzzy points*). The degree of fuzzy coincidence (κ) of two fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ may be defined as

$$\kappa = \begin{cases} 0 & \text{if } \widetilde{P}_1(1) \neq \widetilde{P}_2(1), \\ 1 & \text{if } \widetilde{P}_1 = \widetilde{P}_2, \\ 1 - \sup\limits_{(x,y)\in\mathbb{R}^2} |\mu((x,y)|\widetilde{P}_1) - \mu((x,y)|\widetilde{P}_2)| & \text{if } \widetilde{P}_1(1) = \widetilde{P}_2(1) \text{ but } \widetilde{P}_1 \neq \widetilde{P}_2. \end{cases}$$

Note 6. If two fuzzy points coincide fuzzily (i.e., $\kappa > 0$), then their fuzzy distance $\widetilde{D}$ is a fuzzy number with $\mu(0|\widetilde{D}) = 1$, i.e., $\widetilde{D}$ is a fuzzy number $\widetilde{0}$.

Example 4.2.1. Let $\widetilde{P}_1$ and $\widetilde{P}_2$ be two fuzzy points whose membership functions are right circular cones with bases $\widetilde{P}_1(0) = \{(x,y) : x^2 + y^2 \leq 1\}$ and $\widetilde{P}_2(0) = \{(x,y) : (x-1)^2 + y^2 \leq 1\}$ and vertices (0, 0) and (1, 0), respectively. The degree of coincidence of $\widetilde{P}_1$ and $\widetilde{P}_2$ is zero because $\widetilde{P}_1(1) = (0,0) \neq (1,0) = \widetilde{P}_2(1)$.

Example 4.2.2. Let $\widetilde{P}_1$ and $\widetilde{P}_2$ be two fuzzy points whose membership functions are right circular cones with bases $\widetilde{P}_1(0) = \{(x,y) : x^2 + y^2 \leq 1\}$ and $\widetilde{P}_2(0) = \{(x,y) : x^2 + y^2 \leq 2\}$, both of which have vertex (0, 0). In this example, the degree of coincidence of $\widetilde{P}_1$ and $\widetilde{P}_2$ is $1 - \sup_{(x,y)\in\mathbb{R}^2} |\mu((x,y)|\widetilde{P}_1) - \mu((x,y)|\widetilde{P}_2)| = \frac{1}{\sqrt{2}}$.

4.2. Fuzzy line segments

Theorem 2.2 implies that for two fuzzy points $\widetilde{P}$ and $\widetilde{Q}$, only the combinations of the points $(x_1^*, y_1^*) \in \widetilde{P}(0)$ and $(x_2^*, y_2^*) \in \widetilde{Q}(0)$ are sufficient to evaluate $\widetilde{P} + \widetilde{Q}$. Similarly, it can be proved that those combinations are also sufficient to evaluate the convex combination $\lambda\widetilde{P} + (1-\lambda)\widetilde{Q}$ for any $\lambda \in [0,1]$. In proving so, the binary composition $\varphi(a,b) = \lambda a + (1-\lambda)b$ is taken as the continuous and increasing operator instead of $\varphi(a,b) = a+b$ in Theorem 2.2. We propose that the fuzzy line segment joining two fuzzy points $\widetilde{P}$ and $\widetilde{Q}$ is the union of all possible convex combinations of $\widetilde{P}$ and $\widetilde{Q}$, i.e., $\bigcup_{\lambda\in[0,1]}(\lambda\widetilde{P} + (1-\lambda)\widetilde{Q})$). Therefore, only the points $(x_1^*, y_1^*) \in \widetilde{P}(0)$ and $(x_2^*, y_2^*) \in \widetilde{Q}(0)$ are joined to construct the fuzzy line segment. Thus, a fuzzy line segment may be defined as follows.

Definition 4.3 (*Fuzzy line segment joining two fuzzy points*). The fuzzy line segment $\widetilde{\widetilde{L}}_{P_1P_2}$ joining the fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ may be defined by its membership function as

$$\mu((x,y)|\widetilde{\widetilde{L}}_{P_1P_2}) = \sup\{\alpha : \text{ where } (x,y) \text{ lies on the line joining same points } u \in \widetilde{P}_1(0) \text{ and } v \in \widetilde{P}_2(0) \text{ and } \mu(u|\widetilde{P}_1) = \mu(v|\widetilde{P}_2) = \alpha\}.$$

The fuzzy point internally dividing the fuzzy line segment in a given ratio $m : n$ is the fuzzy point $(n/(m+n))\widetilde{P}_1 + (m/(m+n))\widetilde{P}_2$. The midpoint of the two fuzzy points can be obtained by taking $m = 1,\ n = 1$.

Example 4.3.1. We consider the fuzzy points taken in Example 3.3.1. The fuzzy point that internally divides the line segment joining those two fuzzy points in the ratio 2:3 is $\widetilde{P}(3.2, 3.6)$, whose membership function is the right circular cone with base $\{(x,y) : (x-3.2)^2/1.36^2 + (y-3.6)^2/1.48^2 \leq 1\}$ and vertex (3.2, 3.6).

We now obtain an equation form of the fuzzy line segment $\widetilde{\widetilde{L}}_{P_1P_2}$ joining $\widetilde{P}_1(a,b)$ and $\widetilde{P}_2(c,d)$. Let $p = \min\{x : (x,y) \in \widetilde{P}_1 \text{ or } (x,y) \in \widetilde{P}_2\}$, $q = \max\{x : (x,y) \in \widetilde{P}_1 \text{ or } (x,y) \in \widetilde{P}_2\}$, $r = \min\{y : (x,y) \in \widetilde{P}_1 \text{ or } (x,y) \in \widetilde{P}_2\}$ and $s = \max\{y : (x,y) \in \widetilde{P}_1 \text{ or } (x,y) \in \widetilde{P}_2\}$.

It is worth noting that for the fuzzy line segment $\widetilde{\widetilde{L}}_{P_1P_2}$, there always exist two curves $f(x,y)=0$ and $g(x,y)=0$ in $[p,q]\times[r,s]$ that are the boundaries of $\widetilde{\widetilde{L}}_{P_1P_2}(0)$ on either side of $\widetilde{\widetilde{L}}_{P_1P_2}(1)$. If we consider a line, l say, perpendicular to $\widetilde{\widetilde{L}}_{P_1P_2}(1)$, then the cross-section of $\widetilde{\widetilde{L}}_{P_1P_2}(0)$ on the vertical plane passing through l must be an *LR*-type fuzzy number along l. Considering different l, we obtain different fuzzy numbers along l whose reference functions L and R are identical. Thus, we can write the equation of the fuzzy line segment as $(f(x,y)/(y-b)-((d-b)/(c-a))(x-a)/g(x,y))_{LR}=0$, where the membership function $\mu(\cdot|\widetilde{\widetilde{L}}_{P_1P_2})$ gradually increases from 0 to 1 on either side of $(y-b)-((d-b)/(c-a))(x-a)=0$; L and R are suitable reference functions. The equation $(f(x,y)/(y-b)-\frac{d-b}{c-a}(x-a)/g(x,y))_{LR}=0,|$ means that along any line perpendicular to $(y-b)-((d-b)/(c-a))(x-a)=0$ there exists an *LR*-type fuzzy number. In addition, this equation does not mean that $\exists (x,y)\in\mathbb{R}^2$ for which $f(x,y)$, $(y-b)-((d-b)/(c-a))(x-a)$, $g(x,y)$ vanish together.

Definition 4.4 (*Containment of a fuzzy point on a fuzzy line segment* $\widetilde{\widetilde{L}}$). Let $\widetilde{P}$ be a fuzzy point and let $\widetilde{\widetilde{L}} \equiv (f(x,y)/(y-b)-((d-b)/(c-a))(x-a)/g(x,y))_{LR}=0\ \forall (x,y)\in[p,q]\times[r,s]$. If $\widetilde{P}(1)\in\widetilde{\widetilde{L}}(1)$, then the fuzzy point $\widetilde{P}$ must be fuzzily contained in $\widetilde{\widetilde{L}}$ with some membership value, β say. This β may be obtained as

$$\beta=\begin{cases} 1 & \text{if } \widetilde{P}\le\widetilde{\widetilde{L}} \text{ or } \widetilde{P}(0)\subset\widetilde{\widetilde{L}}(0),\\ \beta_1 & \text{if } \widetilde{P}(0) \text{ exceeds } \widetilde{\widetilde{L}}(0) \text{ on the side of } f,\\ \beta_2 & \text{if } \widetilde{P}(0) \text{ exceeds } \widetilde{\widetilde{L}}(0) \text{ on the side of } g,\\ \min\{\beta_1,\beta_2\} & \text{if } \widetilde{P}(0) \text{ exceeds } \widetilde{\widetilde{L}}(0) \text{ on the either sides of } \widetilde{\widetilde{L}}(1),\end{cases}$$

where $\beta_1=\sup_{(x,y):f(x,y)=0}\mu((x,y)|\widetilde{P})$ and $\beta_2=\sup_{(x,y):g(x,y)=0}\mu((x,y)|\widetilde{P})$.

If $\widetilde{P}(1)\notin\widetilde{\widetilde{L}}(1)$, then $\widetilde{P}$ cannot be fuzzily contained in $\widetilde{\widetilde{L}}$ and we define $\beta=0$ in this situation.

Note 7. If a fuzzy point $\widetilde{P}$ is contained in $\widetilde{\widetilde{L}}_{P_1P_2}$, then for any point $(x,y)\in\widetilde{P}(0)\cap\widetilde{\widetilde{L}}(0)$, $\mu((x,y)|\widetilde{P})\ge\beta$.

The following question arises: How can the proposed method be extended to obtain the fuzzy line segment determined by two fuzzy points when another fuzzy point with a core collinear to those of the other two points? More precisely, if same points of the three fuzzy points are not collinear, how should the points be used? The answer to this question is as follows.

Suppose that $\widetilde{\widetilde{L}}_{P_1P_2}$ is the line segment joining $\widetilde{P}_1$ and $\widetilde{P}_2$. Let $\widetilde{P}_3$ be an additional fuzzy point to be added to $\widetilde{\widetilde{L}}_{P_1P_2}$ to extend it. It is given that $\widetilde{P}_1(1)$, $\widetilde{P}_2(1)$ and $\widetilde{P}_3(1)$ are collinear. According to our suggested method, one of the following can be done. Let $\widetilde{\widetilde{L}}$ represent the required extended form of $\widetilde{\widetilde{L}}_{P_1P_2}$ and let $\widetilde{P}_1$ be situated on the left-hand side of $\widetilde{P}_2$ in $\widetilde{\widetilde{L}}_{P_1P_2}$. Now $\widetilde{\widetilde{L}}$ can be obtained as follows.

(i) If $\widetilde{P}_3$ lies in between $\widetilde{P}_1$ and $\widetilde{P}_2$, then $\widetilde{\widetilde{L}}=\widetilde{\widetilde{L}}_{P_1P_3}\cup\widetilde{\widetilde{L}}_{P_3P_2}$.

(ii) If $\widetilde{P}_3$ lies on the left-hand side of $\widetilde{P}_1$, then $\widetilde{\widetilde{L}}=\widetilde{\widetilde{L}}_{P_3P_1}\cup\widetilde{\widetilde{L}}_{P_1P_2}$.

(iii) If $\widetilde{P}_3$ lies on the right-hand side of $\widetilde{P}_2$, then $\widetilde{\widetilde{L}}=\widetilde{\widetilde{L}}_{P_1P_2}\cup\widetilde{\widetilde{L}}_{P_2P_3}$.

Definition 4.5 (*Angle between two fuzzy line segments*). Let $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ be three continuous fuzzy points and let $\widetilde{\widetilde{L}}_{P_1P_2}$, $\widetilde{\widetilde{L}}_{P_2P_3}$ be fuzzy line segments joining $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_2$, $\widetilde{P}_3$ respectively. The angle between $\widetilde{\widetilde{L}}_{P_1P_2}$ and $\widetilde{\widetilde{L}}_{P_2P_3}$ is denoted by $\widetilde{\Theta}$ and is defined by

$$\mu(\theta|\widetilde{\Theta})=\sup\{\alpha:\theta \text{ is the angle between the line segments } \overline{L}_{uv} \text{ and } \overline{L}_{vw}, \text{ where } u, v \text{ and } v, w \text{ are same points with membership value } \alpha;\ u\in\widetilde{P}_1(0),\ v\in\widetilde{P}_2(0),\ w\in\widetilde{P}_3(0)\}.$$

Theorem 4.2. *For three continuous fuzzy points* $\widetilde{P}_1$, $\widetilde{P}_2$ *and* $\widetilde{P}_3$,

(1) $\widetilde{\Theta}(\alpha) = \{\theta : \theta$ *is the angle between line segments* $\overline{L}_{uv}$ *and* $\overline{L}_{vw}$, *where* $u \in \widetilde{P}_1(\alpha)$, $v \in \widetilde{P}_2(\alpha)$ *and* $w \in \widetilde{P}_3(\alpha)$, *and* u, v *and* v, w *are same points*$\}$ $\forall \alpha \in [0, 1]$.
(2) $\widetilde{\Theta}$ *is a fuzzy number in* $\mathbb{R}$.

Proof. The proof is similar to that for Theorem 4.1 and is omitted. □

Example 4.5.1. Let $\widetilde{P}(7, 6)$, $\widetilde{P}(1, 2)$ and $\widetilde{P}(14, 15)$ be three fuzzy points.

The membership function of $\widetilde{P}(1, 2)$ is the right circular cone with base $\{(x, y) : (x-1)^2 + (y-2)^2 \le 1\}$ and vertex $(1, 2)$.

The shape of the membership function of $\widetilde{P}(7, 6)$ is the right elliptical cone with base $\{(x, y) : (x-7)^2/4 + (y-6)^2/9 \le 1\}$ and vertex $(7, 6)$.

The shape of the membership function of $\widetilde{P}(14, 15)$ is the right elliptical cone with base $\{(x, y) : (x-14)^2 + (y-15)^2/4 \le 1\}$ and vertex $(14, 15)$.

Here the angle between the fuzzy line segments joining the first two and last two fuzzy points is a fuzzy number $\widetilde{\Theta}$ with support $[\pi/4 - \tan^{-1}\frac{131}{134}, \pi/4 - \tan^{-1}\frac{47}{126}]$; $\mu(\pi/4 - \tan^{-1}\frac{131}{134}|\widetilde{\Theta}) = 0 = \mu(\pi/4 - \tan^{-1}\frac{131}{134}|\widetilde{\Theta})$ and the core of $\widetilde{\Theta}$ is $\{\pi/4 - \tan^{-1}\frac{2}{3}\}$.

5. Discussion

In the Introduction, we mentioned that methods and definitions prior to the work of Buckley and Eslami [3,4] either lack inner conformity when reduced to the usual definitions for crisp sets, or different measures of fuzzy objects are crisp numbers. Thus, comparisons are made only to results reported by Buckley and Eslami [3,4] only.

• *Fuzzy point*: We use the fuzzy point definition of Buckley and Eslami here [3].

• *Fuzzy distance*: Buckley and Eslami define the fuzzy distance between two fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ as $\bigvee\{d : d = d(u, v), u \in \widetilde{P}_1(0)$ and $v \in \widetilde{P}_2(0)\}$ [3]. If S_1^α and S_2^α are the boundaries of $\widetilde{P}_1(\alpha)$ and $\widetilde{P}_2(\alpha)$, respectively, then this definition determines the fuzzy distance as $\bigvee_{\alpha\in[0,1]}[d_{min}^\alpha, d_{max}^\alpha]$, where $d_{min}^\alpha := \min_{X_1\in S_1^\alpha, X_2\in S_2^\alpha} d(X_1, X_2)$ and $d_{max}^\alpha := \max_{X_1\in S_1^\alpha, X_2\in S_2^\alpha} d(X_1, X_2)$.

By contrast, Definition 4.1 evaluates fuzzy distance as $\bigvee_{\alpha\in[0,1]}[d_{min}^\alpha, d_{max}^\alpha]$, where

$$d_{min}^\alpha := \min_{\substack{X_1\in S_1^\alpha, X_2\in S_2^\alpha \\ X_1, X_2:\text{ inverse points}}} d(X_1, X_2) \quad \text{and} \quad d_{max}^\alpha := \max_{\substack{X_1\in S_1^\alpha, X_2\in S_2^\alpha \\ X_1, X_2:\text{ inverse points}}} d(X_1, X_2).$$

Note that the above two methods essentially depend on two nonlinear constrained optimization problems and they differ in the constraint set. The constraint set for the proposed method is a subset of the constraint set for the method in [3]. Thus, the support of the proposed fuzzy distance must always be a subset of the fuzzy distance in [3] and their cores are identical. Therefore, the proposed fuzzy distance is less imprecise than that in [3].

• *Fuzzy line segment*: The fuzzy line segment $\widetilde{\overline{L}}_{P_1P_2}$, according to the definition in [3], implies that $\widetilde{\overline{L}}_{P_1P_2} = \bigvee\{\bar{l} : \bar{l}$ is a line segment joining a point in $\widetilde{P}_1(0)$ to a point in $\widetilde{P}_2(0)\}$. Therefore, evaluation of the membership value $\mu((x_0, y_0)|\widetilde{\overline{L}}_{P_1P_2})$ of a particular point (x_0, y_0) is obtained by taking the supremum over the minimum of the membership values of the two extremities of all the line segments on which (x_0, y_0) lies.

By contrast, Definition 4.3 is equivalent to stating that $\widetilde{\overline{L}}_{P_1P_2} = \bigvee\{\bar{l} : \bar{l}$ is a line segment joining same points in $\widetilde{P}_1(0)$ and $\widetilde{P}_2(0)\}$, and thus $\mu((x_0, y_0)|\widetilde{\overline{L}}_{P_1P_2}) = \sup\{\alpha : (y_0 - y_1)/(y_2 - y_1) = (x_0 - x_1)/(x_2 - x_1)$, where $(x_1, y_1) \in \widetilde{P}_1(0)$ and $(x_2, y_2) \in \widetilde{P}_2(0)$ are same points with membership value $\alpha\}$, which is the supremum of the membership value of same points that are two extremities of the line segments on which (x_0, y_0) lies.

Note that Definition 4.3 takes the union of the line segments joining only same points to form a fuzzy line segment, whereas the fuzzy line segment in [3] combines all possible line segments joining points in the supports of the fuzzy points. Therefore, the fuzzy line segment according to the proposed method has less spread than that in [3]. Moreover, the following discussion shows that quite often the fuzzy line segment in [3] does not utilize all the information provided when we try to extend it, whereas our method fully uses all the information given.

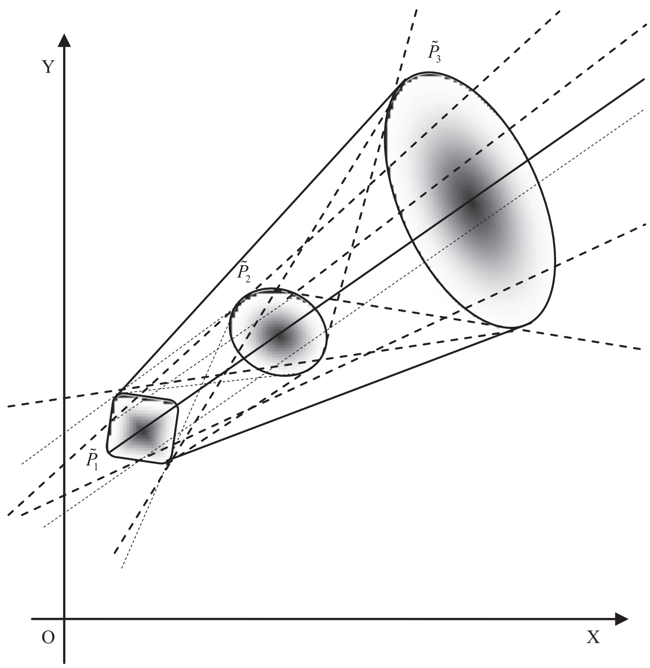

Fig. 3. Fuzzy line segment joining $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ according to [3].

Figs. 3 and 4 depict the extended forms of the fuzzy line segment $\widetilde{\widetilde{L}}_{P_1P_2}$ when one additional fuzzy point $\widetilde{P}_3$ is added according to [3] and according to the proposed method, respectively. In Fig. 3, information for the fuzzy point $\widetilde{P}_2$ is lost and $\widetilde{P}_2$ does not have any influence on the extended fuzzy line segment when $\widetilde{P}_3$, with larger support, is added. By contrast, according to our method, no such situation, except when $\widetilde{P}_3$ is a subset of $\widetilde{P}_1$ or $\widetilde{P}_2$, can arise whereby any information is lost or has no influence. In addition, the proposed methodology considers all information given about $\widetilde{P}_2$ and utilizes it fully in constructing a fuzzy line segment. It is also worth mentioning that even though the boundary of the support of an extended fuzzy line segment is not a straight line, the core of the fuzzy line is always a straight line in the proposed method.

• *Coincidence of two fuzzy points*: Buckley and Eslami measured the coincidence of two fuzzy points as the height of their intersection [3]. Therefore, according to [3], the coincidence between two fuzzy points can be measured even if their cores are not identical. By contrast, we propose that if the cores of two fuzzy points do not coincide, then they are not at all fuzzily coincident, and if their cores coincide then they are fuzzily coincident to some degree. Definition 4.2 provides a measurement of fuzzy coincidence.

In Example 4.2.1, the cores of the fuzzy points $\widetilde{P}_1(1)$ and $\widetilde{P}_2(1)$ are not identical. Even so, according to [3] their degree of coincidence is $\frac{1}{2}$, so they are half-coincident. However, according to the proposed method they are not coincident and their degree of coincidence is zero.

In Example 4.2.2, the cores of the fuzzy points $\widetilde{P}_1(1)$ and $\widetilde{P}_2(1)$ are identical. Thus, the height of $\widetilde{P}_1 \cap \widetilde{P}_2$ is 1 and their degree of coincidence is 1, so they are fully coincident according to [4]. However, according to the proposed method they are not fully coincident and have a positive degree of coincidence $\frac{1}{\sqrt{2}}$.

• *Containment of a fuzzy point on a fuzzy line segment*: According to [3], a fuzzy point $\widetilde{Q}$ is contained on a fuzzy line segment $\widetilde{\widetilde{L}}$ if $\mu((x, y)|\widetilde{Q}) \le \mu((x, y)|\widetilde{\widetilde{L}})$ for all $(x, y) \in \mathbb{R}^2$. This definition indicates that if $\widetilde{Q}(0) \not\subset \widetilde{\widetilde{L}}(0)$, then $\widetilde{Q}$ is not fuzzily contained on $\widetilde{\widetilde{L}}$ even if $\widetilde{Q}(1) \in \widetilde{\widetilde{L}}(1)$.

By contrast, we propose that the condition $\widetilde{Q} \le \widetilde{\widetilde{L}}$ alone is not sufficient for containment of a fuzzy point in a fuzzy environment; instead, measurement of the degree to which $\widetilde{Q}$ belongs in $\widetilde{\widetilde{L}}$ only when $\widetilde{Q}(1) \in \widetilde{\widetilde{L}}(1)$ would be more reasonable. Definition 4.4 proposes a method for this measurement.

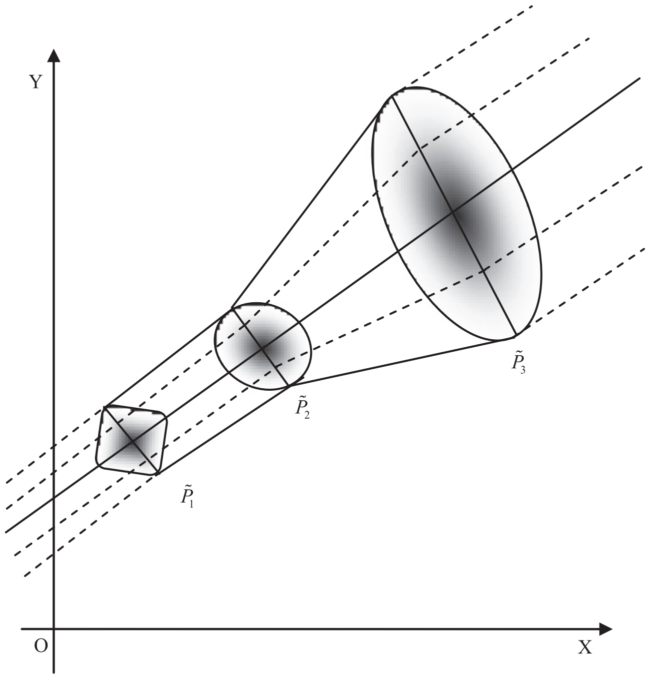

Fig. 4. Fuzzy line segment joining $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ according to the proposed method.

• *Fuzzy angle between two fuzzy line segments*: Buckley end Eslami defined the fuzzy angle between two fuzzy line segments by direct use of the sup-min composition of fuzzy sets [4], whereas we defined this angle using the same points concept. Let $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ be three fuzzy points. For both methods, the fuzzy angle between $\widetilde{\widetilde{L}}_{P_1P_2}$ and $\widetilde{\widetilde{L}}_{P_2P_3}$ is determined as $\bigvee_{\alpha\in[0,1]}[\theta^\alpha_{min}, \theta^\alpha_{max}]$, where, according to [4],

$$\theta^\alpha_{min} = \min_{u\in\widetilde{P}_1(\alpha), v\in\widetilde{P}_2(\alpha), w\in\widetilde{P}_3(\alpha)} \angle(\overline{uv}, \overline{vw}) \quad \text{and} \quad \theta^\alpha_{max} = \max_{u\in\widetilde{P}_1(\alpha), v\in\widetilde{P}_2(\alpha), w\in\widetilde{P}_3(\alpha)} \angle(\overline{uv}, \overline{vw}).$$

By contrast, for Definition 4.5,

$$\theta^\alpha_{min} = \min_{\substack{u\in\widetilde{P}_1(\alpha), v\in\widetilde{P}_2(\alpha), w\in\widetilde{P}_3(\alpha) \\ u,v:\ \text{same points} \\ v,w:\ \text{same points}}} \angle(\overline{uv}, \overline{vw}) \quad \text{and} \quad \theta^\alpha_{max} = \max_{\substack{u\in\widetilde{P}_1(\alpha), v\in\widetilde{P}_2(\alpha), w\in\widetilde{P}_3(\alpha) \\ u,v:\ \text{same points} \\ v,w:\ \text{same points}}} \angle(\overline{uv}, \overline{vw}).$$

In the proposed method, the constraint set for the optimization problems is a subset of the constraint set of Buckley and Eslami for the corresponding optimization problem. Thus, support of the proposed fuzzy angle must always be a subset of the fuzzy angle in [4] and their core angles are identical. Therefore, the proposed fuzzy angle is less imprecise than that of Buckley and Eslami [4].

6. Conclusions

This paper discussed a few basic fuzzy geometrical concepts. The sup-min composition of fuzzy sets and the newly defined concepts of same and inverse points were used in this discussion. We studied fuzzy reference frames, fuzzy point, linear combinations of fuzzy points, the fuzzy distance between fuzzy points, the fuzzy coincidence of two fuzzy points, fuzzy line segment, a fuzzy point on a fuzzy line segment and the fuzzy angle between two fuzzy line segments. According to the methodologies and definitions proposed, measurement of the fuzzy distance and fuzzy angle yields a fuzzy number and is less imprecise than existing methods. The proposed concepts were investigated in a two-dimensional plane. All the ideas can easily be extended to an n-dimensional plane, $n \geq 3$. Future research can focus on this extension.

Our future research on fuzzy plane geometry will include detailed studies of fuzzy lines and fuzzy circles. Other fuzzy geometrical concepts such as fuzzy distances and fuzzy trigonometric properties may also be investigated.

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