THE DEPTH OF ULTRAPRODUCTS
OF BOOLEAN ALGEBRAS

SH853

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Abstract. We show that if $\mu$ is a compact cardinal then the depth of ultraproducts of less than $\mu$ many Boolean Algebras is at most $\mu$ plus the ultraproduct of the depths of those Boolean Algebras.

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Monk has looked systematically at cardinal invariants of Boolean Algebras. In particular, he has looked at the relations between $\prod_{i<\kappa} \text{inv}(B_i/D)$ and $\prod_{i<\kappa} \text{inv}(B_i)/D$, i.e., the invariant of the ultraproducts of a sequence of Boolean Algebras vis the ultraproducts of the sequence of the invariants of those Boolean Algebras for various cardinal invariants inv of Boolean Algebras. That is: is it always true that $\prod_{i<\kappa} \text{inv}(B_i/D) \leq \prod_{i<\kappa} \text{inv}(B_i)/D$? Is it consistently always true? Is it always true that $\prod_{i<\kappa} \text{inv}(B_i)/D \leq \text{inv}(\prod_{i<\kappa} B_i/D)$? Is it consistently always true? See more on this in Monk [Mo96]. Roslanowski Shelah [RoSh 534] deals with specific inv and with more on kinds of cardinal invariants and their relationship with ultraproducts. Monk [Mo90a], [Mo96], in his list of open problems raises the question for the central cardinal invariants, most of them have been solved by now; see Magidor Shelah [MgSh 433], Peterson [Pe97], Shelah [Sh 345], [Sh 462], [Sh 479], [Sh 589, §4], [Sh 620], [Sh 641], [Sh 703], Shelah and Spinas [ShSi 677].

We here throw some light on problem 12 of [Mo96], pg.287 and will be continued in [Sh:F683]. We thank the referee for many helpful comments.

0.1 Definition. For a Boolean Algebra $B$ let

(a) $\text{Depth}(B) = \sup\{\theta: \text{ in } B \text{ there is an increasing sequence of length } \theta\}$

(b) $\text{Depth}^+(B) = \sup\{\theta^+: \text{ in } B \text{ there is an increasing sequence of length } \theta\}$.

0.2 Remark. So $\text{Depth}^+(B) = \lambda^+ \Rightarrow \text{Depth}(B) = \lambda$ and if $\text{Depth}^+(B)$ is a limit cardinal then $\text{Depth}^+(B) = \text{Depth}(B)$. 
§1 Above a compact cardinal

The following claim gives severe restrictions on any try to build a ZFC example for \(\text{Depth}(\prod_{\varepsilon < \kappa} B_{\varepsilon})/D > \prod_{\varepsilon < \kappa} \text{Depth}(B_{\varepsilon})/D\) if \(V\) is near \(L\), see [Sh 652] for complementary to §1.

1.1 Claim. 1) Assume

(a) \(\kappa < \mu \leq \lambda\)
(b) \(\mu\) is a compact cardinal
(c) \(D\) is an ultrafilter on \(\kappa\)
(d) \(\lambda = \text{cf}(\lambda)\) such that \((\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)\)
(e) \(B_i (i < \kappa)\) is a Boolean Algebra with \(\text{Depth}^+(B_i) \leq \lambda\)
(f) \(B = \prod_{i<\kappa} B_i/D\).

Then \(\text{Depth}^+(B) \leq \lambda\).

2) Instead \((\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)\) it suffices that \((\forall \alpha < \lambda)(|\alpha|^\kappa/D < \lambda = \text{cf}(\lambda))\).

3) We can weaken clause (e) (for parts (1) and (2)) to

(g) \(\{i < \kappa : B_i\text{ is a Boolean Algebra with }\text{Depth}^+(B_i) \leq \lambda\} \in D\).

Proof. 1) Toward contradiction assume that \(\langle a_\alpha : \alpha < \lambda \rangle\) is an increasing sequence in \(B\). So let \(a_\alpha = \langle a_\alpha^i : i < \kappa \rangle/D\), so for \(\alpha < \beta, A_{\alpha, \beta} =: \{i < \kappa : B_i \models a_\alpha^i < a_\beta^i\} \in D\).

Let \(E\) be a \(\mu\)-complete uniform ultrafilter on \(\lambda\).

For each \(\alpha < \lambda\) let \(A_\alpha\) be such that the set \(\{\beta : \alpha < \beta < \lambda \text{ and } A_{\alpha, \beta} = A_\alpha\}\) is a member of \(E\) so an unbounded subset of \(\lambda\) (exist as \(\lambda = \text{cf}(\lambda) \geq \mu > 2^\kappa\)).

We choose \(C\) as follows

\[
C =: \{\delta < \lambda : \delta \text{ is a limit ordinal and if } u \subseteq \delta \\
\text{is bounded of cardinality } \leq \kappa \text{ then } \delta = \sup(S_u \cap \delta)\}
\]

where

\[
S_u =: \{\beta < \lambda : \beta > \sup(u) \text{ and } (\forall \alpha \in u)(A_{\alpha, \beta} = A_\alpha)\}.
\]
As \( \lambda = \text{cf}(\lambda) > 2^\kappa = |D| \), for some \( A_* \in D \) the set \( S =: \{ \alpha < \lambda : \text{cf}(\alpha) > \kappa \text{ and } A_\alpha = A_* \} \) is a stationary subset of \( \lambda \).

As we have assumed \( \lambda = \text{cf}(\lambda) \) and \((\forall \alpha < \lambda)[|\alpha|^\kappa \leq \lambda] \), clearly \( C \) is a club of \( \lambda \).

Let \( \{ \delta_\varepsilon : \varepsilon < \lambda \} \subseteq C, \delta_\varepsilon \) increases continuous with \( \varepsilon \) and \( \delta_{\varepsilon + 1} \in S \). For each \( \varepsilon < \lambda \) the family \( \mathfrak{A}_\varepsilon = \{ S_u \cap \delta_{\varepsilon + 1} \setminus \delta_\varepsilon : u \in [\delta_{\varepsilon + 1}]^{\leq \kappa} \} \) is a downward \( \kappa^+ \)-directed family of non-empty subsets of \( [\delta_\varepsilon, \delta_{\varepsilon + 1}] \) hence there is a \( \kappa^+ \)-complete filter \( E_\varepsilon \) on \( [\delta_\varepsilon, \delta_{\varepsilon + 1}] \) extending \( \mathfrak{A}_\varepsilon \).

For \( \varepsilon < \lambda \) and \( i < \kappa \) let \( W_{\varepsilon, i} =: \{ \beta : \delta_\varepsilon \leq \beta < \delta_{\varepsilon + 1} \text{ and } i \in A_{\beta, \delta_{\varepsilon + 1}} \} \) and let

\[
B_\varepsilon =: \{ i < \kappa : W_{\varepsilon, i} \in E_\varepsilon^+ \}.
\]

As \( E_\varepsilon \) is \( \kappa^+ \)-complete clearly \( W_\varepsilon =: \bigcap \{ [\delta_\varepsilon, \delta_{\varepsilon + 1}] \setminus W_{\varepsilon, i} : i \in \kappa \setminus B_\varepsilon \} \in E_\varepsilon \) hence there is \( \beta \in W_\varepsilon \); if \( i \in A_{\beta, \delta_{\varepsilon + 1}} \) then \( \{ \gamma : \delta_\varepsilon \leq \gamma < \delta_{\varepsilon + 1} \text{ and } i \in A_{\gamma, \delta_{\varepsilon + 1}} \} \in E_\varepsilon^+ \), so \( A_{\beta, \delta_{\varepsilon + 1}} \) is a subset of \( B_\varepsilon \) and belongs to \( D \) hence \( B_\varepsilon \in D \).

So for each \( \varepsilon \) for some \( i_{\delta_{\varepsilon + 1}} \in A_* \) we have

\[
\{ \beta : \delta_\varepsilon \leq \beta < \delta_{\varepsilon + 1} \text{ and } i_{\delta_{\varepsilon + 1}} \in A_{\beta, \delta_{\varepsilon + 1}} \} \in E_\varepsilon^+.
\]

We can find \( i_* \in A_* \) such that

\[
Y = \{ \varepsilon < \lambda : \varepsilon \text{ is an even ordinal and } i_{\delta_{\varepsilon + 1}} = i_* \}
\]

has cardinality \( \lambda \), and let \( Z = \{ \delta_{\varepsilon + 1} : \varepsilon \in Y \} \) so \( Z \in [\lambda]^\lambda \). Now

\[ (*)_0 \quad \varepsilon \in Y \Rightarrow A_{\delta_{\varepsilon + 1}} = A_* \quad \text{[why? as } \delta_{\varepsilon + 1} \in S] \]

\[ (*)_1 \quad i_* \in A_* \in D \quad \text{[trivial; note if } \forall \alpha < \lambda, |\alpha|^{2^\kappa} < \lambda \text{ we can have } E_\varepsilon \text{ is } (2^\kappa)^+ \text{-complete filter so we have } B_{\delta_{\varepsilon + 1}} \text{ instead of } i_{\delta_\varepsilon} \text{ so we can weaken } \text{“} D \text{ ultrafilter” to: } D \subseteq \mathcal{P}(\kappa) \text{ upward closed and the intersection of any two non-empty}] \]

\[ (*)_2 \quad \text{if } \alpha < \beta \text{ are from } Z \text{ then } i_* \in A_{\alpha, \beta} \quad \text{[why? let } \alpha = \delta_{\varepsilon + 1}, \beta = \delta_{\zeta + 1} \text{ so } \varepsilon < \zeta; \text{ let } \]

\[
\mathcal{U}_1 := \{ \gamma : \delta_\zeta < \gamma < \delta_{\zeta + 1}, A_{\alpha, \gamma} = A_\alpha (= A_{\delta_{\varepsilon + 1}}) \}
\]

so

\[
\mathcal{U}_1 = S_{(\delta_{\varepsilon + 1})} \cap [\delta_\zeta, \delta_{\zeta + 1}] \in \mathfrak{A}_\zeta \subseteq E_\zeta
\]

and let

\[
\mathcal{U}_2 := \{ \gamma : \delta_\zeta \leq \gamma < \delta_{\zeta + 1}, i_* \in A_{\gamma, \delta_{\zeta + 1}} \} \in E_\zeta^+.
\]
[Why? As this is how \( i_{\delta_{\xi+1}} \) is defined.]

So for any \( \alpha < \beta \) from \( Z \) as \( \mathcal{U}_1 \in E_\alpha \) and \( \mathcal{U}_2 \in E_\beta^+ \) clearly there is \( \gamma \in \mathcal{U}_1 \cap \mathcal{U}_2 \) hence \( (\alpha = \delta_{\xi+1} < \delta_\xi \leq \gamma < \delta_{\xi+1} = \beta \) and) for \( i = i_* \) we have \( B_i \models a_i^{\delta_{\xi+1}} < a_i^{\gamma} \) (because \( \gamma \in \mathcal{U}_1 \)) and \( B_i \models a_i^{\gamma} < a_i^{\delta_{\xi+1}} \) (because \( \gamma \in \mathcal{U}_2 \)) so together \( B_i \models a_i^{\delta_{\xi+1}} < a_i^{\delta_{\xi+1}} \) but \( \alpha = \delta_{\xi+1}, \beta = \delta_{\xi+1} \) so we have gotten \( B_i \models a_i^\alpha < a_i^\beta \) so we are done.

2) We change the choice of the club \( C \). By the assumption, for each \( \alpha < \lambda \) let \( \langle f_\alpha^\gamma/D : \gamma < \gamma_\alpha \rangle \) be a list of the members of \( \alpha^\kappa/D \) without repetitions, so \( \gamma_\alpha < \lambda \).

Let

\[
C = \{ \delta : (i) \; \delta < \lambda \text{ is a limit ordinal} \\
(ii) \text{ if } \alpha < \delta \text{ then } \gamma_\alpha < \delta \\
(iii) \text{ if } \alpha < \delta \text{ and } \gamma < \gamma_\alpha \text{ and} \\
\quad \bar{A} = \langle A_i : i < \kappa \rangle \in \kappa D \text{ and there is } \xi \in [\delta, \lambda] \text{ such that} \\
\quad i < \kappa \Rightarrow A_{f_\gamma^\alpha(i), \xi} = A_i \text{ then there is} \\
\quad \xi \in (\alpha, \delta) \text{ such that } i < \kappa \Rightarrow A_{f_\gamma^\alpha(i), \xi} = A_i \}
\]

Clearly \( C \) is a club of \( \lambda \). The only additional point in the proof is

\( (*) \) if \( \delta_1 < \delta_2 \) are from \( C \) and \( A_{\delta_2} = A_* \) then there is \( i_* \in A_* \) such that: for every \( \alpha \in S \cap \delta_1 \) there is \( \beta \in [\delta_1, \delta_2) \) satisfying \( A_{\alpha, \beta} = A_s \land i_* \in A_{\alpha, \beta} \).

[Why \( (*) \) holds? If not, then for every \( i \in A_* \) there is \( \alpha_i \in S \cap \delta_1 \) satisfying \( \beta \in [\delta_1, \delta_2) \land A_{\alpha_i, \beta} = A_* \Rightarrow i \notin A_{\beta, \delta_2} \). Let \( f \in \kappa \alpha \) be defined by: \( f(i) = \alpha_i \), if \( i \in A_* \), \( f(i) = 0 \) otherwise, so for some \( \gamma < \gamma_\delta_1 \) we have \( f = f_{\gamma_1}^\delta \) mod \( D \) hence \( A = \{ i \in A_* : f(i) = f_{\gamma_1}^\delta(i) \} \in D \). As \( \kappa < \mu \) and \( D \) is \( \mu \)-complete there is \( \xi_1 \in (\delta_2, \lambda) \) such that \( i < \kappa \Rightarrow A_{f_{\gamma_1}^\delta(i), \xi_1} = A_{f_{\gamma_1}^\delta(i)} \) hence by the choice of \( C \) there is \( \xi_2 \in (\delta_1, \delta_2) \) such that \( i < \kappa \Rightarrow A_{f_{\gamma_1}^\delta(i), \xi_2} = A_{f_{\gamma_1}^\delta(i), \xi_1} = A_{f_{\gamma_1}^\delta(i)} \). But \( i \in A \Rightarrow f_{\gamma_1}^\delta(i) = f(i) = \alpha_i \in S \Rightarrow A_{\alpha_i, \xi_2} = A_{f_{\gamma_1}^\delta(i), \xi_2} = A_{f_{\gamma_1}^\delta(i)} = A_* \) so \( i \in A \Rightarrow A_{\alpha_i, \xi_2} = A_* \). Now \( A_{\xi_2, \delta_2} \in D \) hence there is \( i_* \in A_s \cap A_{\xi_2, \delta_2} \) and for it we get contradiction.]

Of course, the set of such \( i_* \)'s belongs to \( D \).

3) Obvious.

1.2 Conclusion: Let \( \mu \) be a compact cardinal. If \( \kappa < \mu \) and \( D \) is an ultrafilter on \( \kappa \), \( B_i \) is a Boolean Algebra for \( i < \kappa \) then

\( (*) \) \( (a) \) if \( D \) is a regular ultrafilter then \( \text{Depth}(\prod_{i<\kappa} B_i/D) \leq \mu + \prod_{i<\kappa} \text{Depth}(B_i)/D \)

\( (b) \) this holds if \( \kappa = \aleph_0 \).
Proof. If this fails, let \( \lambda = (\mu + \prod_{i<\kappa} \text{Depth}(B_i)/D)^+ \), so \( \lambda \) is a regular cardinal > \( \mu \) and \((\forall \alpha<\lambda)[|\alpha^\kappa/D| < \lambda]\) - see below and \( \lambda \leq \text{Depth}(\prod_{i<\kappa} B_i/D) \), so by 1.1 we get a contradiction.

1.3 Remark. 1) Actually we prove that if \( \mu \) is a compact cardinal, \( \kappa < \mu \leq \lambda = \text{cf}(\lambda) \) and \((\forall \alpha<\lambda)[|\alpha^\kappa/D| < \lambda]\), then we can find an increasing sequence \( \langle \alpha_{\varepsilon}, \gamma \rangle \) of ordinals < \( \lambda \) and \( \text{cf}(\lambda) \) such that for every \( \varepsilon < \zeta < \lambda \) for some \( \gamma \) satisfying \( \alpha_{\varepsilon} < \gamma < \alpha_{\zeta} \) we have \( \text{c}\{\alpha_{\varepsilon}, \gamma\} = i, \text{c}\{\gamma, \alpha_{\zeta}\} = j \) (the result follows using \( \text{c}[\lambda]^2 \to D \)).

2) We use \( i_* \) rather than some \( B \in D \) in order to help clarify what we need.

3) Note that if \( D \) is a normal ultrafilter on \( \kappa > \aleph_0 \) and \( \langle \lambda_i : i < \kappa \rangle \) is increasing continuous with limit \( \lambda \), then \( \prod_{j\leq i} \lambda_j < \lambda_{i+1} \) then \( \lambda = \prod_{i<\kappa} \lambda_i/D \) but \( \lambda^\kappa/D > \lambda \). This is essentially the only reason for the undesirable extra assumption “\( D \) is regular” in 1.2.

Note

1.4 Claim. 1) In 1.1 instead “\( \mu \in (\kappa, \lambda) \) is a compact cardinal” it suffices to demand: \( \oplus_{\kappa+1,2\kappa,\lambda} \) where

\[ \oplus_{\sigma, \theta, \lambda} \text{ if } c : [\lambda]^2 \to \theta \text{ then we can find a stationary } S \subseteq \lambda \text{ and } \gamma < \theta \text{ such that for every } u \in [S]^{<\sigma} \text{ the set } S_u = \{ \beta < \lambda : (\forall \alpha \in u)[c(\alpha, \beta) = \gamma] \} \text{ is unbounded in } \lambda. \]

2) If \( \mu \) is supercompact \( \sigma < \theta = \text{cf}(\theta) < \mu < \lambda = \text{cf}(\lambda) \) and \( Q = \text{adding } \mu \text{ Cohen subsets of } \theta \text{ in } V \), \( \oplus_{\sigma, \mu, \lambda} \) holds (even \( \oplus_{\sigma, \mu, \lambda} \) if \( \mu_{<\sigma} < \lambda \) in \( V \)).

In 1.4 we cannot get such results for \( \kappa > \mu \) because for \( \mu \) supercompact Laver indestructible and regular \( \lambda > \kappa > \mu \) we can force \{\delta < \lambda : \text{cf}(\delta) > \mu\} \) to have a square preserving the supercompactness.

1.5 Claim. Assume \( \lambda = \text{cf}(\lambda) > \kappa^+ \) and \( \kappa = \text{cf}(\kappa) \), and there is a square on \( S = \{ \delta < \lambda : \text{cf}(\delta) \geq \kappa \} \) (see 1.6 below). Then

(a) there is a sequence \( \langle B_i : i < \kappa \rangle \) of Boolean Algebras such that

(\( \alpha \)) \( \text{Depth}^+(B_i) \leq \lambda \)

(\( \beta \)) for any uniform ultrafilter \( D \) on \( \kappa \), \( \text{Depth}^+(\prod_{i<\kappa} B_i/D) > \lambda \)
(b) the proof of [Sh 652, 5.1] can be carried.

Where

1.6 Definition. For \( \lambda = \text{cf}(\lambda) > \aleph_0, S \subseteq \lambda = \text{sup}(S) \) we say that \( S \) has a square when we can find \( S^+ \) and \( \langle C_\alpha : \alpha \in S^+ \rangle \) such that

(a) \( S \setminus S^+ \) is not a stationary subset of \( \lambda \)
(b) \( C_\alpha \) is a closed subset of \( \alpha \)
(c) \( \beta \in C_\alpha \Rightarrow \beta \in S \cap C_\beta = C_\alpha \cap \beta \)
(d) we stipulate \( C_\alpha = \{\emptyset\} \) for \( \alpha \notin S^+ \).

Proof of 1.5. As in [Sh 652, 5.1] using \( \bar{C} = \langle C_\alpha : \alpha \in S^+ \rangle \) from 1.6 instead \( \langle \text{acc}(C_\alpha) : \alpha < \lambda^+ \rangle \). The only change being that in the proof of [Sh 652, Fact 5.3] in case 3 we have just \( \text{cf}(\alpha) \leq \kappa \) and let \( \langle \beta_\xi : \xi < \text{cf}(\alpha) \rangle \) be increasing continuous with limit \( \alpha \). If \( \text{cf}(\alpha) < \kappa \) we can find \( \varepsilon(*) < \kappa \) such that \( \zeta_1 < \zeta_2 < \kappa \Rightarrow \beta_{\zeta_1} \in A_{\beta_{\zeta_2},\varepsilon(*)} \) and let \( A_{\alpha,\varepsilon} = \emptyset \) if \( \varepsilon < \varepsilon(*) \) and \( A_{\alpha,\varepsilon} = \cup \{ A_{\beta_\zeta,\varepsilon} : \zeta < \text{cf}(\kappa) \} \) if \( \varepsilon \in [\varepsilon(*), \kappa) \). \( \square_{1.6} \)
REFERENCES.


