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A MICROLOCAL VERSION OF THE EDGE OF THE WEDGE THEOREM FOR TEMPERED ULTRAHYPERFUNCTIONS

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ABSTRACT. Making use of microlocal properties of tempered ultrahyperfunctions corresponding to a convex cone, we give a simple version of the celebrated Edge of the Wedge theorem for this setting.

1. Introduction

The space of tempered ultrahyperfunctions (orginally called *tempered ultradistributions*) and its dual Fourier transform space, the space of Fourier ultrahyperfunctions (orginally called *distributions of exponential growth*), have been investigated by many authors, among others we refer the reader to [1]-[18]. Tempered ultrahyperfunctions were introduced in papers of Sebastião e Silva [1, 2] and Hasumi [3] as the strong dual of the space of test functions of rapidly decreasing entire functions in any horizontal strip. While Sebastião e Silva [1] used extension procedures for the Fourier transform combined with holomorphic representations and considered the 1-dimensional case, Hasumi [3] used duality arguments in order to extend the notion of tempered ultrahyperfunctions for the case of n dimensions (see also [2, Section 11]). In a brief tour, Marimoto [6, 7] gave some more precise informations concerning the work of Hasumi. More recently, the relation between the tempered ultrahyperfuntions and Schwartz distributions and some major results, as the kernel theorem and the Fourier-Laplace transform have been established by Brüning and Nagamachi in [16]. Earlier, some precisions on the Fourier-Laplace transform theorem for tempered ultrahyperfunctions were given by Carmichael [10] (see also [18]), by considering the theorem in its simplest form, *i.e.*, the equivalence between support properties of a distribution in a closed convex cone and the holomorphy of its Fourier-Laplace transform in a suitable tube with conical basis. In this more general setting, which includes the results of Sebastião e Silva and Hasumi as special cases, Carmichael obtained new representations of tempered ultrahyperfunctions which were not considered by Sebastião e Silva [1, 2] or Hasumi [3]. Recently, Schmidt [17] has given an insight in the operations of duality and Fourier transform on the space of test and generalized functions belonging

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to new subclasses of Fourier hyperfunctions of mixed type, satisfying polynomial growth conditions at infinity, which is very similar to the studies by Sebastião e Silva [1, 2] and Hasumi [3] about tempered ultrahyperfunctions.

The purpose of this article, which represents a sequel to [18], it is to prove the Edge of the Wedge theorem for tempered ultrahyperfunctions corresponding to a convex cone.* This classical theorem [19] deals with the question about the principle of holomorphic continuation of functions of several complex variables, which arose in physics in the study of the Wightman functions and Green functions, or in connection with the dispersion relations in quantum field theory.

2. Preliminares

We shall introduce briefly here some definitions and basic properties of the tempered ultrahyperfunction space of Sebastião e Silva [1, 2] and Hasumi [3] (we indicate the Refs. for more details).

Notations: We will use the standard multi-index notation. Let $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) be the real (resp. complex) n-space whose generic points are denoted by $x=(x_1,\ldots,x_n)$ (resp. $z=(z_1,\ldots,z_n)$), such that $x+y=(x_1+y_1,\ldots,x_n+y_n)$, $\lambda x=(\lambda x_1,\ldots,\lambda x_n)$, $x\geq 0$ means $x_1\geq 0,\ldots,x_n\geq 0$, $\langle x,y\rangle = x_1y_1+\cdots+x_ny_n$ and $|x|=|x_1|+\cdots+|x_n|$. Moreover, we define $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}_o^n$, where $\mathbb{N}_o$ is the set of non-negative integers, such that the length of α is the corresponding ℓ^1-norm $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\alpha+\beta$ denotes $(\alpha_1+\beta_1,\ldots,\alpha_n+\beta_n)$, $\alpha\geq\beta$ means $(\alpha_1\geq\beta_1,\ldots,\alpha_n\geq\beta_n)$, $\alpha!=\alpha_1!\cdots\alpha_n!$, $x^\alpha=x_1^{\alpha_1}\ldots x_n^{\alpha_n}$, and

$$D^\alpha\varphi(x)=\frac{\partial^{|\alpha|}\varphi(x_1,\ldots,x_n)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_1}\ldots\partial x_n^{\alpha_n}}\ .$$

We consider two n-dimensional spaces – x-space and ξ-space – with the Fourier transform defined

$$\widehat{f}(\xi)=\mathscr{F}[f(x)](\xi)=\int_{\mathbb{R}^n} f(x)e^{i\langle\xi,x\rangle}d^nx\ ,$$

while the Fourier inversion formula is

$$f(x)=\mathscr{F}^{-1}[\widehat{f}(\xi)](x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f}(\xi)e^{-i\langle\xi,x\rangle}d^n\xi\ .$$

The variable ξ will always be taken real while x will also be complexified – when it is complex, it will be noted $z=x+iy$. The above formulas, in which we employ the symbolic "function notation," are to be understood in the sense of distribution theory.

We shall consider the function

$$h_K(\xi)=\sup_{x\in K}\bigl|\langle\xi,x\rangle\bigr|\ ,\quad \xi\in\mathbb{R}^n\ ,$$

*Other versions of the theorem for tempered ultrahyperfunctions can be found in [8, 15].

the indicator of K, where K is a compact set in $\mathbb{R}^n$. $h_K(\xi) < \infty$ for every $\xi \in \mathbb{R}^n$ since K is bounded. For sets $K = \big[-k, k\big]^n$, $0 < k < \infty$, the indicator function $h_K(\xi)$ can be easily determined:

$$h_K(\xi) = \sup_{x \in K} \big|\langle \xi, x\rangle\big| = k|\xi| \ , \quad \xi \in \mathbb{R}^n \ , \quad |\xi| = \sum_{i=1}^{n} |\xi_i| \ .$$

Let K be a convex compact subset of $\mathbb{R}^n$, then $H_b(\mathbb{R}^n; K)$ (b stands for bounded) defines the space of all functions $\in C^\infty(\mathbb{R}^n)$ such that $e^{h_K(\xi)} D^\alpha f(\xi)$ is bounded in $\mathbb{R}^n$ for any multi-index α. One defines in $H_b(\mathbb{R}^n; K)$ seminorms

$$\|\varphi\|_{K,N} = \sup_{x \in \mathbb{R}^n; \alpha \le N} \big\{ e^{h_K(\xi)} |D^\alpha f(\xi)| \big\} < \infty \ , \quad N = 0, 1, 2, \dots \ . \tag{2.1}$$

If $K_1 \subset K_2$ are two compact convex sets, then $h_{K_1}(\xi) \le h_{K_2}(\xi)$, and thus the canonical injection $H_b(\mathbb{R}^n; K_2) \hookrightarrow H_b(\mathbb{R}^n; K_1)$ is continuous. Let O be a convex open set of $\mathbb{R}^n$. To define the topology of $H(\mathbb{R}^n; O)$ it suffices to let K range over an increasing sequence of convex compact subsets $K_1, K_2, \dots$ contained in O such that for each $i = 1, 2, \dots$, $K_i \subset K_{i+1}^\circ$ (K_{i+1}° denotes the interior of K_{i+1}) and $O = \bigcup_{i=1}^\infty K_i$. Then the space $H(\mathbb{R}^n; O)$ is the projective limit of the spaces $H_b(\mathbb{R}^n; K)$ according to restriction mappings above, *i.e.*

$$H(\mathbb{R}^n; O) = \lim_{K \subset O} \text{proj}\ H_b(\mathbb{R}^n; K) \ , \tag{2.2}$$

where K runs through the convex compact sets contained in O.

Theorem 2.1 ([3, 6, 16])**.** *The space $\mathscr{D}(\mathbb{R}^n)$ of all C^∞-functions on $\mathbb{R}^n$ with compact support is dense in $H(\mathbb{R}^n; K)$ and $H(\mathbb{R}^n; O)$. Moreover, the space $H(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $H(\mathbb{R}^n; O)$ and $H(\mathbb{R}^m; \mathbb{R}^m) \otimes H(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $H(\mathbb{R}^{m+n}; \mathbb{R}^{m+n})$.*

From Theorem 2.1 we have the following injections [6]: $H'(\mathbb{R}^n; K) \hookrightarrow H'(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n)$ and $H'(\mathbb{R}^n; O) \hookrightarrow H'(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n)$.

The dual space $H'(\mathbb{R}^n; O)$ of $H(\mathbb{R}^n; O)$ is the space of Fourier ultrahyperfunctions, V, such that

$$V = D_\xi^\gamma [e^{h_K(\xi)} g(\xi)] \ ,$$

where $g(\xi)$ is a bounded continuous function.

Now, we pass to the definition of tempered ultrahyperfunctions. In the space $\mathbb{C}^n$ of n complex variables $z_i = x_i + iy_i$, $1 \le i \le n$, we denote by $T(\Omega) = \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$ the tubular set of all points z, such that $y_i = \operatorname{Im} z_i$ belongs to the domain Ω, *i.e.*, Ω is a connected open set in $\mathbb{R}^n$ called the basis of the tube $T(\Omega)$. Let K be a convex compact subset of $\mathbb{R}^n$, then $\mathfrak{H}_b(T(K))$ defines the space of all continuous functions φ on $T(K)$ which are holomorphic in the interior $T(K^\circ)$ of $T(K)$ such that the estimate

$$|\varphi(z)| \le \mathbf{C}(1 + |z|)^{-N} \tag{2.3}$$

is valid for some constant $\mathsf{C} = \mathsf{C}_{K,N}(\varphi)$. The best possible constants in (2.3) are given by a family of seminorms in $\mathfrak{H}_b(T(K))$

$$\|\varphi\|_{K,N} = \sup_{z\in T(K)} \big\{(1+|z|)^N|\varphi(z)|\big\} < \infty \;, \quad N=0,1,2,\dots \;. \tag{2.4}$$

If $K_1 \subset K_2$ are two convex compact sets, then $\mathfrak{H}_b(T(K_2)) \hookrightarrow \mathfrak{H}_b(T(K_1)$. Given that the spaces $\mathfrak{H}_b(T(K_i))$ are Fréchet spaces, the space $\mathfrak{H}(T(O))$ is characterized as a projective limit of Fréchet spaces

$$\mathfrak{H}(T(O)) = \lim_{K\subset O}\text{proj}\; \mathfrak{H}_b(T(K)) \;, \tag{2.5}$$

where K runs through the convex compact sets contained in O and the projective limit is taken following the restriction mappings above.

Proposition 2.2 ([6]). *If $f \in H(\mathbb{R}^n;O)$, the Fourier transform of f belongs to the space $\mathfrak{H}(T(O))$, for any open convex non-empty set $O \subset \mathbb{R}^n$. By the dual Fourier transform $H'(\mathbb{R}^n;O)$ is topologically isomorphic with the space $\mathfrak{H}'(T(-O))$.*

Definition 2.3. *A* **tempered ultrahyperfunction** *is a continuous linear functional defined on the space of test functions $\mathfrak{H} = \mathfrak{H}(T(\mathbb{R}^n))$ of rapidly decreasing entire functions in any horizontal strip. The space of all tempered ultrahyperfunctions is denoted by $\mathscr{U}(\mathbb{R}^n)$.*

The space $\mathscr{U}(\mathbb{R}^n)$ is characterized in the following way [3]; let $\mathscr{H}_{\boldsymbol{\omega}}$ be the space of all functions $f(z)$ such that:

- $f(z)$ is analytic for $\{z \in \mathbb{C}^n \mid |\operatorname{Im} z_1| > p, |\operatorname{Im} z_2| > p, \dots, |\operatorname{Im} z_n| > p\}$.
- $f(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n \mid |\operatorname{Im} z_1| \geqq p, |\operatorname{Im} z_2| \geqq p, \dots, |\operatorname{Im} z_n| \geqq p\}$, where $p = 0,1,2,\dots$ depends on $f(z)$.
- $f(z)$ is bounded by a power of z: $|f(z)| \leq \mathsf{C}(1+|z|)^N$, where C and N depend on $f(z)$.

Let $\boldsymbol{\Pi}$ be the set of all z-dependent pseudo-polynomials, $z \in \mathbb{C}^n$. Then $\mathscr{U}$ is the quotient space $\mathscr{U} = \mathscr{H}_{\boldsymbol{\omega}}/\boldsymbol{\Pi}$. By a pseudo-polynomial we understand a function of z of the form $\sum_s z_j^s G(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n)$, with $G(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n) \in \mathscr{H}_{\boldsymbol{\omega}}$.

3. Tempered Ultrahyperfunctions Corresponding to a Convex Cone

Next, we consider tempered ultrahyperfunctions in a setting which includes the results of [1, 2, 3] as special cases, by considering functions analytic in tubular radial domains [10, 18]. We start by introducing some terminology and simple facts concerning cones. An open set $C \subset \mathbb{R}^n$ is called a cone if $x \in C$ implies $\lambda x \in C$ for all $\lambda > 0$. Moreover, C is an open connected cone if C is a cone and if C is an open connected set. In the sequel, it will be sufficient to assume for our purposes that the open connected cone C in $\mathbb{R}^n$ is an open convex cone with vertex at the origin. A cone C' is called compact in C – we write $C' \Subset C$ – if the projection $\mathsf{pr}\overline{C}' \overset{\text{def}}{=} \overline{C}' \cap S^{n-1} \subset \mathsf{pr}C \overset{\text{def}}{=} C \cap S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Being given a cone C in x-space, we associate with C a closed convex cone C^* in ξ-space which is the set $C^* = \big\{\xi \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq 0, \forall x \in C\big\}$. The cone C^* is called the *dual cone* of C. By $T(C)$ we will denote the set $\mathbb{R}^n + iC \subset \mathbb{C}^n$. If C is open and connected, $T(C)$ is called the tubular radial domain in $\mathbb{C}^n$, while if C is only open $T(C)$ is referred to as a tubular cone. An important example of tubular radial domain in quantum field theory is the forward light-cone

$$V_+ = \left\{z \in \mathbb{C}^n \mid \operatorname{Im} z_1 > \Big(\sum_{i=2}^{n} \operatorname{Im}^2 z_i\Big)^{\frac{1}{2}}, \operatorname{Im} z_1 > 0\right\} .$$

We will deal with tubes defined as the set of all points $z \in \mathbb{C}^n$ such that

$$T(C) = \Big\{x + iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in C, |y| < \delta\Big\} ,$$

where $\delta > 0$ is an arbitrary number.

Let C be an open convex cone and let C' be an arbitrary compact cone of C. Let $B[0;r]$ denote a closed ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Denote $T(C';r) = \mathbb{R}^n + i\big(C' \setminus \big(C' \cap B[0;r]\big)\big)$. We are going to introduce a space of holomorphic functions which satisfy certain estimate according to Carmichael [10]. We want to consider the space consisting of holomorphic functions $f(z)$ such that

$$\big|f(z)\big| \leq \mathbf{C}(C')(1 + |z|)^N e^{h_{C^*}(y)} , \quad z = x + iy \in T(C';r) , \tag{3.1}$$

where $h_{C^*}(y) = \sup_{\xi \in C^*} |\langle \xi, y\rangle|$ is the indicator of C^*, $\mathbf{C}(C')$ is a constant that depends on an arbitrary compact cone C' and N is a non-negative real number. The set of all functions $f(z)$ which are holomorphic in $T(C';r)$ and satisfy the estimate (3.1) will be denoted by $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$.

Lemma 3.1 ([10, 18])**.** *Let C be an open convex cone, and let C' be an arbitrary compact cone contained in C. Let $h(\xi) = e^{k|\xi|}g(\xi)$, $\xi \in \mathbb{R}^n$, be a function with support in C^*, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$. Let y be an arbitrary but fixed point of $C' \setminus \big(C' \cap B[0;r]\big)$. Then $e^{-\langle \xi, y\rangle}h(\xi) \in L^2$, as a function of $\xi \in \mathbb{R}^n$.*

Definition 3.2. *We denote by $H'_{C^*}(\mathbb{R}^n;O)$ the subspace of $H'(\mathbb{R}^n;O)$ of Fourier ultrahyperfunctions with support in the cone C^*:*

$$H'_{C^*}(\mathbb{R}^n;O) = \Big\{V \in H'(\mathbb{R}^n;O) \mid \operatorname{supp}(V) \subseteq C^*\Big\} . \tag{3.2}$$

Lemma 3.3 ([10, 18])**.** *Let C be an open convex cone, and let C' be an arbitrary compact cone contained in C. Let $V = D_\xi^\gamma[e^{h_K(\xi)}g(\xi)]$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = \big[-k,k\big]^n$. Let $V \in H'_{C^*}(\mathbb{R}^n;O)$. Then $f(z) = (2\pi)^{-n}(V, e^{-i\langle \xi, z\rangle})$ is an element of $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$.*

It has been shown that $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ can be recovered as the (inverse) Fourier-Laplace transform of the constructed Fourier ultrahyperfunction $V \in H'_{C^*}(\mathbb{R}^n; O)$. This result is a generalization of the Paley-Wiener-Schwartz theorem for the setting of tempered ultrahyperfunctions.

Theorem 3.4 (Paley-Wiener-Schwartz-type Theorem [18]). *Let $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$, where C is an open convex cone. Then the Fourier ultrahyperfunction $V \in H'_{C^*}(\mathbb{R}^n; O)$ has a uniquely determined inverse Fourier-Laplace transform $f(z) = (2\pi)^{-n}(V, e^{-i\langle \xi, z\rangle})$ which is holomorphic in $T(C'; r)$ and satisfies the estimate (3.1).*

In light of the results of Ref. [10, Sections 4 and 5], we define $\mathscr{U}_c = \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}/\boldsymbol{\Pi}$ as being the quotient space of $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ by set of pseudo-polynomials. Here the set $\mathscr{U}_c$ is the space of tempered ultrahyperfunctions corresponding to the open convex cone $C \subset \mathbb{R}^n$. The space $\mathscr{U}_c$ is algebraically isomorphic to the space of generalized functions $\mathfrak{H}'$. This result, which represents a generalization of Hasumi [3, Proposition 5], was obtained by Carmichael [10, Theorem 5] in the case where C is an open cone, but not necessarily connected.

4. The Edge of the Wedge Theorem

In what follows, we give Edge of the Wedge theorem for the space of the tempered ultrahyperfunctions. In order to prove the Edge of the Wedge theorem, we must develop our machinery a little further.

A useful property of tempered ultrahyperfunctions corresponding to a cone is the distributional boundary value theorem concerning analytic functions. The following proposition shows that functions in $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ have distributional boundary values in $\mathfrak{H}'$.

Proposition 4.1. *Let C be an open convex cone and let C' be an arbitrary compact cone contained in C. Let $V \in H'_{C^*}(\mathbb{R}^n; \mathbb{R}^n)$. Then there exist a function $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ such that $f(z) \to \mathscr{F}^{-1}[V] \in \mathfrak{H}'$ in the weak topology of $\mathfrak{H}'$ as $y = \operatorname{Im} z \to 0$, $y \in C' \subset C$.*

Proof. Consider the function

$$\begin{aligned} f(z) = (2\pi)^{-n}(V, e^{-i\langle \xi, z\rangle}) &= (2\pi)^{-n} \int_{C^*} D_\xi^\gamma [e^{k|\xi|} g(\xi)] e^{-i\langle \xi, z\rangle} d^n\xi \\ &= (2\pi)^{-n}(-i)^{|\gamma|} z^\gamma \int_{C^*} [e^{k|\xi|} g(\xi)] e^{-i\langle \xi, z\rangle} d^n\xi \ . \end{aligned} \tag{4.1}$$

Then the proof that $f(z)$ defined by (4.1) is an element of $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ follows from Lemma 3.3, with $O = \mathbb{R}^n$. Thus we need only show that $f(z) \to \mathscr{F}^{-1}[V]$ in the weak topology of $\mathfrak{H}'$ as $y = \operatorname{Im} z \to 0$, $y \in C' \subset C$. But, this can be proved by a minor modification of the second part of Theorem 2 in [9], by considering the convention of signs in the Fourier transform which is used here. □

Remark 1. The weak convergence in $\mathfrak{H}'$ of $f(z)$ to $\mathscr{F}^{-1}[V]$ in Proposition 4.1 can in fact be replaced by strong convergence in $\mathfrak{H}'$, since $\mathfrak{H}$ is a Montel space [20, Corollary 3.5]. According to Treves [21, Corollary 1, p.358], in the dual of a Montel space, every weakly convergent sequence is strongly convergent.

We shall consider the singularity structure of tempered ultrahyperfunctions corresponding to a convex cone. Here, we follow the results and ideas contained in the Hörmander's textbook [22] and characterize the singularities of a tempered ultrahyperfunction u via the notion of analytic wave front set, denoted by $WF_A(u)$. There are several definitions of analytic wave front set, which are equivalent to each other. Putting in simple way, the $WF_A(u)$ is composed of pairs (x, ξ) in the phase space, where x runs through the set of those points that have no neighborhoods wherein u is an analytic function, while ξ runs through the cone of those directions of a "bad" behavior of the Fourier transform of u, which are responsible for the appearance of a singularity at the point x. So we shall usually want $\xi \neq 0$. Namely, we have the following

Theorem 4.2. *If $u \in \mathscr{U}_c(\mathbb{R}^n)$ and $V \in H'_{C^*}(\mathbb{R}^n; O)$ (with $O \subseteq \mathbb{R}^n$), then $WF_A(u) \subset \mathbb{R}^n \times C^*$.*

Proof. The proof is similar to the proof of Lemma 2 of Ref. [23]. Let $\{C_j^*\}_{j\in L}$ be a finite covering of closed properly convex cones of C^*. Decompose $V \in H'_{C^*}(\mathbb{R}^n; O)$ as follows [9, Thm.4]:

$$V = \sum V_j\,, \quad \text{such that } V_j \in H'_{C_j^*}(\mathbb{R}^n; O) = \Big\{V_j \in H'(\mathbb{R}^n; O) \mid \operatorname{supp}(V_j) \subseteq C_j^*\Big\}\,. \tag{4.2}$$

Next apply the Theorem 3.4 and Proposition 4.1 for each V_j. Then the decomposition (4.2) will induce a representation of u in the form of a sum of boundary values of functions $f_j(z) \in \mathscr{H}^{\boldsymbol{o}}_{\boldsymbol{c_j}}$, such that $f_j(z) \to \mathscr{F}^{-1}[V_j] \in \mathfrak{H}'$ in the strong topology of $\mathfrak{H}'$ as $y = \operatorname{Im} z \to 0$, $y \in C'_j \subset C_j$. By Theorem 3.4 each $f_j(z)$ is holomorphic at $T(C'_j; r)$ unless $\langle \xi, Y\rangle \geq 0$ for $\xi \in C_j^*$ and $Y \in C'_j$, with $|Y| < \delta$. Since Y has an arbitrary direction in C'_j, this shows that cones of "bad" directions responsible for the singularities of the boundary values of functions $f_j(z) \in \mathscr{H}^{\boldsymbol{o}}_{\boldsymbol{c_j}}$ are contained in the dual cones of the base cones. So, we have the inclusion

$$WF_A(u) \subset \mathbb{R}^n \times \bigcup_j \Big(C_j^* = \big\{\xi \in \mathbb{R}^n \setminus \{0\} \mid \langle \xi, Y\rangle \geq 0\big\}\Big)\,. \tag{4.3}$$

Then, by making a refinement of the covering and shrinking it to C^*, we obtain the desired result. □

Proposition 4.3. *Let $u \in \mathscr{U}_c(\mathbb{R}^n)$. Then we have*

$$\Big\{x \in \mathbb{R}^n \mid (x, \xi) \in WF_A(u)\Big\} \subset \operatorname{supp}(u)\,.$$

Proof. It is obvious from definition of wave front set. □

From Theorems 3.4 and 4.2 and Proposition 4.1, we can draw the following

Corollary 4.4. *Let C be an open convex cone and let C' be an arbitrary compact cone contained in C. Let C^* be the dual cone of C. If an element $u \in \mathfrak{H}'$ is the boundary value in the distributional sense of a function $f(z)$ which is analytic in $T(C';r)$, then $WF_A(u) \subset \mathbb{R}^n \times C^*$.*

We now state the main theorem of this paper: the tempered ultrahyperfunction version of edge of the wedge theorem.

Theorem 4.5 (Edge of the Wedge Theorem). *Let $\mathscr{O}$ be an open set of $\mathbb{C}^n$ which contains a real environment, X, with X some open set of $\mathbb{R}^n$. Let C_1 and C_2 two open convex cones and $\boldsymbol{C}$ the convex hull of $C_1 \cup C_2$. If the boundary values of two holomorphic functions f_j, $(j = 1, 2)$, in $\mathscr{B}_j = T_j(C'_j; r) \cap \mathscr{O}$, $(C'_j \subset C_j)$, agree on X, i.e., $f_o = b_{C_1} f_1 = b_{C_2} f_2$, then f_o is a real analytic function which extends both f_1 and f_2.*

Remark 2. Following Hörmander [22], we use the notation $b_C f$ in order to emphazise that the boundary value of f on X is obtained taking the limit $y = \operatorname{Im} z \to 0$ from the directions of the cone C.

The Theorem 4.5 obtains a simple and elegant proof if one introduces the normal set of any closed set $X_1 \subset X$ (see [22]):

Definition 4.6. *For a closed subset $X_1 \subset X$ the exterior normal set $N_e(X_1)$ is defined to be the set of all (x, ξ) such that $x \in X_1$ and there is a real valued function $f \in C^2(X)$ with $df(x) = \xi \neq 0$ and $f(y) \leq f(x)$ for all $y \in X_1$. The normal set $N(X_1)$ is the union $N_e(X_1) \cup -N_e(X_1)$.*

Let $u \in \mathscr{U}_c(\mathbb{R}^n)$ and assume that f is a real valued real analytic function in $X \subset \mathbb{R}^n$. For a closed subset $X_1 \subset X$, there is a point $x \in \operatorname{supp}(u)$, such that $df(x) = \xi \neq 0$ and $f(y) \leq f(x)$ if $y \in X_1$. According to Theorem 8.5.6' in [22], we have that $N(\operatorname{supp}(u)) \subset WF_A(u)$. If $u \in \mathscr{U}_c(\mathbb{R}^n)$ has the property that $WF_A(u) \cap -WF_A(u) = \varnothing$, and since $N(\operatorname{supp}(u)) = -N(\operatorname{supp}(u))$, then $N(\operatorname{supp}(u)) = \varnothing$. This means that $N_e(\operatorname{supp}(u)) = \varnothing$. By the Proposition 8.5.8 also in [22], the projection of $N_e(X_1)$ in $\mathbb{R}^n$ is dense in ∂X_1. Hence, it follows that $\partial(\operatorname{supp}(u)) = \varnothing$. It now follows from connectedness of $\mathbb{R}^n$ that either $\operatorname{supp}(u) = \varnothing$ or $\operatorname{supp}(u) = \mathbb{R}^n$. The latter is excluded by $u|_X = 0$. This fact implies that u is identically zero. Thus, we prove the[†]

Theorem 4.7 (Uniqueness Theorem). *If $u \in \mathscr{U}_c(\mathbb{R}^n)$ has the property that $WF_A(u) \cap -WF_A(u) = \varnothing$, then $u|_X = 0 \Longrightarrow u = 0$ for each non-empty open set $X \subset \mathbb{R}^n$.*

[†]A similar proof has been given in Ref. [24] for $u \in \mathscr{D}'(M)$, where M is a real analytic connected manifold.

Proof of Theorem 4.5. From the assumptions, $WF_A(b_{C_1} f_1) \subset X \times C_1^*$ and $WF_A(b_{C_2} f_2) \subset X \times C_2^*$. Let $f_o = b_{C_1} f_1 = b_{C_2} f_2$, then $WF_A(f_o) \subset X \times \big(C_1^* \cap C_2^*\big)$. If $C_1^* \cap C_2^* = \varnothing$, then $\boldsymbol{C} = \mathbb{R}^n$ and it follows from Theorem 4.2 that $WF_A(f_o) = \varnothing$, which means that f_o is in fact a real analytic function. If $C_1^* \cap C_2^* \neq \varnothing$, then there exists an analytic function $\boldsymbol{f}$ at X such that $f_o = b_{\boldsymbol{C}} \boldsymbol{f}$ by Corollary 4.4. Since $f_o = b_{\boldsymbol{C}} \boldsymbol{f} = b_{C_1} f_1 = b_{C_2} f_2$, it follows that the differences $\boldsymbol{f} - f_j$, $(j = 1, 2)$, vanish on the domain of definition of each f_j, by Theorem 4.7, according to which a holomorphic function whose boundary value from the directions of a given cone vanishes on an open real domain is identically zero. This proves the theorem. □

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