Report On Cobweb Posets' Tiling Problem

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Summary
Responding to Kwaśniewski's cobweb posets' problems posed in [3, 4] we present here some results on one of them - namely on Cobweb Tiling Problem. Kwaśniewski cobweb posets with tiling property are designated-coded by their correspondent tiling sequences. We show that the family of all cobweb tiling sequences includes Natural numbers, Fibonacci numbers, Gaussian integers and show that there are more other cobweb tiling sequences. We show also that cobweb tiling problem is a particular case of clique problem in a certain graph. To this end we illustrate the area of our reconnaissance by means of the Venn type map of various cobweb sequences families according to definitions from [3, 4].

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1 Introduction

1.1 Upside down notation

Let us recall basic information about Kwaśniewski's cobweb posets as a quote from his papers [3, 4, 7]:

> $F-nomial$ **coefficients**. The source papers are [11, 12, 13, 14, 15] from which indispensable definitions and notation are taken for granted including Kwaśniewski [11, 12] upside - down notation $n_F \equiv F_n$ being used for mnemonic reasons - as in the case of Gaussian numbers in finite geometries and the so called "quantum groups").
>
> Given any sequence $\{F_n\}_{n\geq 0}$ of nonzero reals ($F_0 = 0$ being sometimes acceptable as $0! = F_0! = 1$.) one defines its cor-

responding binomial-like $F - nomial$ coefficients as in Ward's Calculus of sequences [16] as follows.

Definition 1 ([3, 4]) .$(n_F \equiv F_n \neq 0, \quad n > 0)$

$$\binom{n}{k}_F = \frac{F_n!}{F_k!F_{n-k}!} \equiv \frac{n_F^{\underline{k}}}{k_F!}$$

$$n_F! \equiv n_F(n-1)_F(n-2)_F(n-3)_F \dots 2_F 1_F;$$

$$0_F! = 1; \quad n_F^{\underline{k}} = n_F(n-1)_F \dots (n-k+1)_F.$$

We have made above an analogy driven identifications in the spirit of Ward's Calculus of sequences [16]. Identification $n_F \equiv F_n$ is the notation used in extended Fibonomial Calculus case [11, 12, 13, 14, 15, 10] being also there inspiring as n_F mimics n_q established notation for Gaussian integers exploited in much elaborated family of various applications including quantum physics (see [11, 12, 15] and references therein). (end of quote)

1.2 Cobweb poset's definition [3, 4]

Nevertheless let us at first recall that cobweb poset in its original form [3, 4] is defined as a partially ordered graded infinite poset $\Pi = \langle P, \leq \rangle$, designated uniquely by any sequence of nonnegative integers $F = \{n_F\}_{n \geq 0}$ and it is represented as a directed acyclic graph (DAG) in the graphical display of its Hasse diagram. P in $\langle P, \leq \rangle$ stays for set of vertices while $\leq$ denotes partially ordered relation.

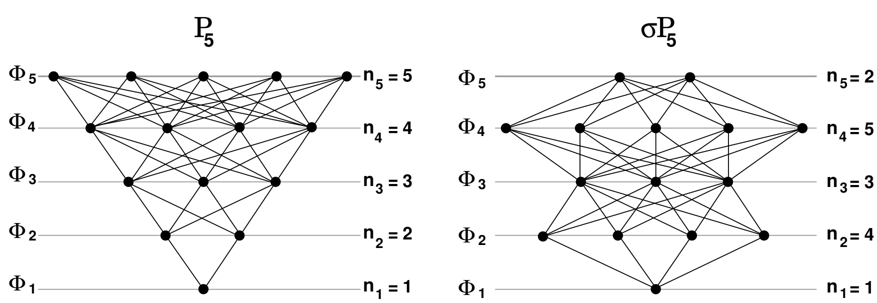

Figure 1: Display of Natural numbers' Cobweb sub-posets P_5 and σP_5

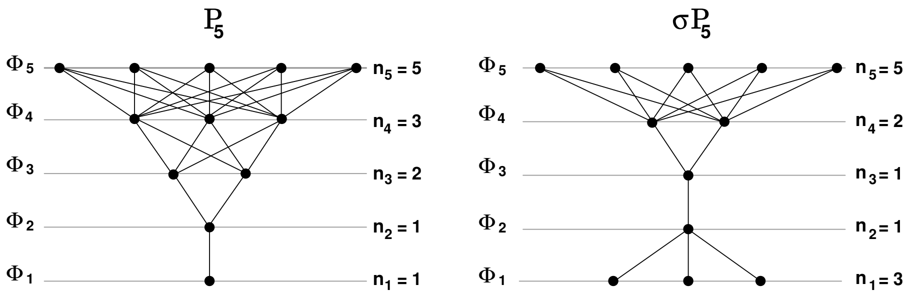

Figure 2: Display of Fibonacci numbers' Cobweb sub-posets P_5 and σP_5

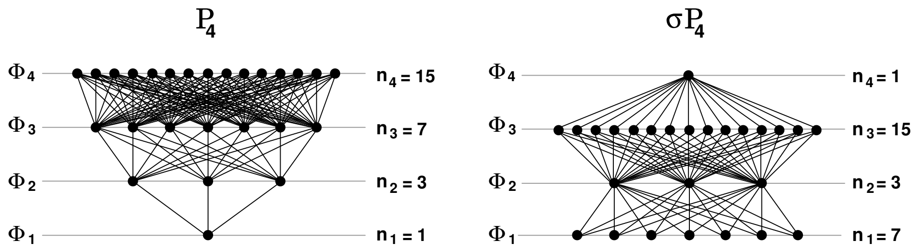

Figure 3: Display of Gaussian numbers' $(q = 2)$ Cobweb sub-posets P_4 and σP_4

1.3 Cobweb posets and chains of binary relations [3] (see also [1, 2])

The Kwaśniewski cobweb posets under consideration once represented by their Hasse di-graphs are examples of oderable directed acyclic graphs (oDAG) which we call [1] from now in brief: KoDAGs. As pointed out in [3] these digraphs are structures of universal importance for the whole of mathematics - in particular for discrete "mathemagics" (*http://ii.uwb.edu.pl/akk/*) and computer sciences in general (quotation from [3]):

> For any given natural numbers valued sequence the graded (layered) cobweb posets' DAGs are equivalently representations of a chain of binary relations. Every relation of the cobweb poset chain is biunivocally represented by the uniquely designated **complete** bipartite digraph-a digraph which is a di-biclique designated by the very given sequence. The cobweb poset is then to be identified with a chain of di-bicliques i.e. by definition - a chain of complete bipartite one direction digraphs. Any chain of relations is therefore obtainable from the cobweb poset chainof complete relations via deleting arcs (arrows) in di-bicliques.
> Let us underline it again : *any chain of relations is obtainable*

from the cobweb poset chain of complete relations via deleting arcs in di-bicliques of the complete relations chain. For that to see note that any relation R_k as a subset of $A_k \times A_{k+1}$ is represented by a one-direction bipartite digraph D_k. A "complete relation" C_k by definition is identified with its one direction di-biclique graph $d{-}B_k$. Any R_k is a subset of C_k. Correspondingly one direction digraph D_k is a subgraph of an one direction digraph of $d - B_k$.
The one direction digraph of $d - B_k$ is called since now on **the di-biclique** i.e. by definition - a complete bipartite one direction digraph. Another words: cobweb poset defining di-bicliques are links of a complete relations' chain. (end of quote)

2 Preliminaries

2.1 The graded layer structure of Kwaśniewski cobweb posets [3]

In Kwaśniewski's cobweb posets' tiling problem one considers finite cobweb sub-posets for which we have finite number of levels in layer $\langle \Phi_k \to \Phi_n \rangle$ [3, 4], where $k \leq n$, $k, n \in \mathbb{N} \cup \{0\}$ with exactly k_j vertices on Φ_j level $k \leq j \leq n$. For $k = 0$ the sub-posets $\langle \Phi_0 \to \Phi_n \rangle$ are named *prime cobweb posets* and these are those to be used - up to permutation of levels equivalence - as a block to partition finite cobweb sub-poset.

2.2 The cobweb tiling problem [3, 4]

Suppose now that F is a cobweb admissible sequence. Under which conditions any layer $\langle \Phi_n \to \Phi_k \rangle$ may be partitioned with help of max-disjoint blocks of established type σP_m? Find effective characterizations and/or find an algorithm to produce these partitions.

3 Family of cobweb tiling sequences

Definition 2 (Cobweb Tiling sequence) *The cobweb admissible sequences that designate cobweb posets with tiling are called cobweb tiling sequences.*

Notation. *Let $\mathcal{T}$ denotes the family of all cobweb tiling sequences.*

Now we define subfamily $\mathcal{T}_\lambda \subset \mathcal{T}$ of cobweb tiling sequences with some combinatorial identities and formulas for the number of different tilings of any layer. This family contains sequences like Natural numbers, Fibonacci numbers, Gaussian coefficients and many others.

Definition 3 *Let $\mathcal{T}_\lambda$ denotes the family of natural number's valued sequences $F \equiv \{n_F\}_{n\geq 0}$ such that for any $m, k \in \mathbb{N}$ its terms satisfy*

$$n_F = (m+k)_F = \lambda_m \cdot m_F \ + \ \lambda_k \cdot k_F \tag{1}$$

for certain coefficients $\lambda_m \equiv \lambda_m(m,k)$ and $\lambda_k \equiv \lambda_k(m,k)$ as a functions $\lambda_m : \mathbb{N}^2 \to \mathbb{N}$, $\lambda_k : \mathbb{N}^2 \to \mathbb{N}$.

Theorem 1 (Cobweb tiling family) *Any sequence from $\mathcal{T}_\lambda$ family is cobweb tiling.*

Proof

Consider now a cobweb poset Π designated by sequence F from $\mathcal{T}_\lambda$ family. Take any layer $\langle\Phi_{k+1} \to \Phi_n\rangle$. Consider Φ_n level, with n_F vertices and note that the number of vertices in this level is the sum of $\lambda_m \cdot m_F \ + \ \lambda_k \cdot k_F$. Therefore we now separate them by cutting into two disjoint subsets as illustrated by Fig. 4 and cope at first $\lambda_m \cdot m_F$ vertices in Step 1. Then we shall cope rest $\lambda_k \cdot k_F$ ones in Step 2.

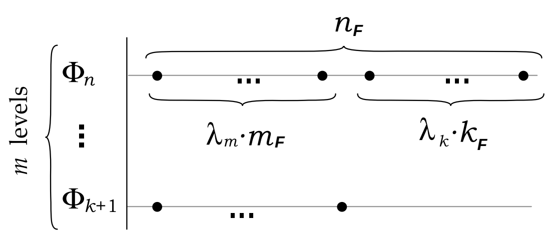

Figure 4: Picture of Theorem 1 proof's idea.

Step 1. Temporarily we have $\lambda_m \cdot m_F$ fixed vertices on Φ_n level to consider. Let us cover them λ_m times by m-th level of block σP_m, which has exactly m_F vertices-leafs. If $\lambda_m = 0$ we skip this step. What was left is the layer $\langle\Phi_{k+1} \to \Phi_{n-1}\rangle$ and we might eventually partition it with smaller max-disjoint blocks σP_{m-1} in next induction step.

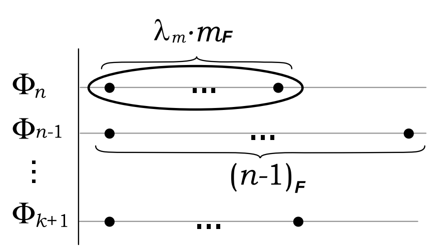

Figure 5: Picture of Theorem 1 proof's Step 1.

Step 2. Consider now the second complementary situation, where we have $\lambda_k \cdot k_F$ vertices on Φ_n level being fixed. Observe that is we move this level lower than Φ_{k+1} level, we obtain exactly λ_k the same $\langle \Phi_k \to \Phi_{n-1} \rangle$ layers to be partitioned with max-disjoint blocks of the form σP_m. This "*move*" operation is just permutation of levels' order.

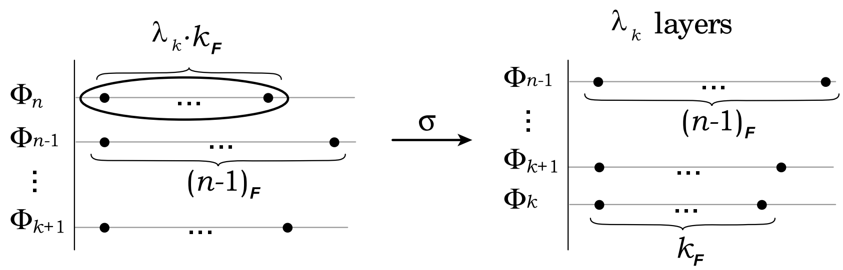

Figure 6: Picture of Theorem 1 proof's Step 2.

Recapitulation:
The layer $\langle \Phi_{k+1} \to \Phi_n \rangle$ may be partitioned with σP_m blocks if $\langle \Phi_{k+1} \to \Phi_{n-1} \rangle$ may be partitioned with σP_{m-1} blocks and $\langle \Phi_k \to \Phi_{n-1} \rangle$ by σP_m again. Continuing these steps by induction, we are left to prove that $\langle \Phi_k \to \Phi_k \rangle$ may be partitioned by σP_0 blocks and $\langle \Phi_1 \to \Phi_m \rangle$ by σP_m ones, which is obvious ■

Observation 1 (Number of blocks) *Given any F cobweb tiling sequence from the family $\mathcal{T}_\lambda$. Then the number of blocks σP_m of the layer $\langle \Phi_{k+1} \to \Phi_n \rangle$ satisfies the recurrence*

$$\binom{n}{k}_F = \lambda_m \cdot \binom{n-1}{k}_F + \lambda_k \cdot \binom{n-1}{k-1}_F \tag{2}$$

with initial values $\binom{n}{n}_F = 1$, $\binom{n}{0}_F = 1$.

Proof
In the proof of Theorem 1, set of points on n-th level we separate into two disjoint subsets. At first we tile λ_m times the layer $\langle \Phi_{k+1} \to \Phi_{n-1} \rangle$ and next λ_k times the layer $\langle \Phi_k \to \Phi_{n-1} \rangle$. From observation 3 in [3] the number of blocks in $\langle \Phi_{k+1} \to \Phi_n \rangle$ is equal to $\binom{n}{k}_F$ what finishes the proof ■

Observation 2 (Number of tilings) *Given any F cobweb tiling sequence from the family $\mathcal{T}_\lambda$. Then the number $\left\{ n \atop k \right\}_F^1$ of different tilings of layer $\langle \Phi_k \to \Phi_n \rangle$ where $n, k \in \mathbb{N}$, $n, k \geq 1$ is equal to:*

$$\left\{ n \atop k \right\}_F^1 = \frac{n_F!}{(m_F!)^{\lambda_m} \cdot ((k-1)_F!)^{\lambda_k}} \cdot \left\{ n-1 \atop k \right\}_F^1 \cdot \left\{ n-1 \atop k-1 \right\}_F^1 \tag{3}$$

where $\left\{ {n \atop n} \right\}_F^1 = 1$ and $\left\{ {n \atop 1} \right\}_F^1 = 1$.

Proof

According to steps of the proof of Theorem 1 we may choose on n-th level m_F vertices λ_m times and next $(k-1)_F$ vertices λ_k times out of n_F ones in $\left\{ {n \atop k} \right\}_F^1 = \frac{n_F!}{(m_F!)^{\lambda_m} \cdot ((k-1)_F!)^{\lambda_k}}$ ways. Next recurrent steps of the proof of Theorem 1 result in formula (3) via product rule of counting ■

Note. $\left\{ {n \atop k} \right\}_F^1$ is not the number of all different tilings of the layer $\langle \Phi_k \to \Phi_n \rangle$ i.e. $\left\{ {n \atop k} \right\}_F^1 \leq \left\{ {n \atop k} \right\}_F$ as computer experiments [9] show. There are much more other tilings with blocks σP_m.

Note. There are more cobweb tiling sequences than family $\mathcal{T}_\lambda$ has. For example easy periodic sequence $B_{c,M}$ [8] is tiling however it does not satisfy (1).

4 Examples of cobweb tiling sequences

Throughout this section we shall consequently use $n = m + k$.

Example 1 (Natural numbers and binomial coefficients)

For $\lambda_m = 1$ and $\lambda_k = 1$ we obviously have the recurrence $n_F = m_F + k_F$ which results in a sequence $n_F = 1_F \cdot n$. When $1_F = 1$ we obtain Natural numbers' sequence and well-known formula of Newton's symbol recurrence:

$$\binom{n}{k}_F = \binom{n-1}{k}_F + \binom{n-1}{k-1}_F$$

Example 2 (Fibonacci numbers and Fibonomial coefficients)

Coefficients $\lambda_m = (k+1)_F$ and $\lambda_k = (m-1)_F$ generate a recurrence $n_F = (k+1)_F \cdot m_F + (m-1)_F \cdot k_F$ and then explicit formula $1_F = 1$ and $n_F = \frac{1}{\sqrt{\Delta}} (\phi_1^n - \phi_2^n)$ for $n \geq 2$ where $\phi_{1,2} = \frac{2_F \pm \sqrt{\Delta}}{2}$, $\Delta = 2_F^2 + 4$. For initial values $2_F = 1$ we obtain Fibonacci numers' sequence and analogous to Newton symbol, recursive formula

$$\binom{n}{k}_F = (k+1)_F \binom{n-1}{k}_F + (m-1)_F \binom{n-1}{k-1}_F$$

Example 3 (Gaussian integers and and Gaussian coefficients)

Coefficients $\lambda_m = q^k$ and $\lambda_k = 1$ generate a recurrence $n_F = q^k \cdot m_F + k_F$ with explicit formula $n_F = 1_F \cdot \frac{q^n - 1}{q-1}$ where $1_F \in \mathbb{N}$, $q \in \mathbb{N}$, what gives us

Gaussian coefficients' sequence when $1_F = 1$ and q is a primary number. Recurrence (2) for such sequences will be

$$\binom{n}{k}_F = q^k \binom{n-1}{k}_F + \binom{n-1}{k-1}_F$$

Example 4 (Modified Gaussian coefficients)
For coefficients $\lambda_m = q^k$ and $\lambda_k = q^m$ we obtain a recurrence $n_F = q^k \cdot m_F + q^m \cdot k_F$ and explicit formula $n_F = 1_F \cdot n \cdot q^{n-1}$ where $1_F \in \mathbb{N}$, $q \in \mathbb{N}$ and recurrence (2) :

$$\binom{n}{k}_F = q^k \binom{n-1}{k}_F + q^m \binom{n-1}{k-1}_F$$

5 Cobweb poset tiling problem as a particular case of clique problem

Definition 4 *Denote $B\left(\langle \Phi_k \to \Phi_n \rangle\right)$ as a family of all blocks of the form σP_m, where $m = n - k + 1$, in $\langle \Phi_k \to \Phi_n \rangle$ layer and Φ_k^n as the cardinality of that family i.e. $\Phi_k^n = \left|B\left(\langle \Phi_k \to \Phi_n \rangle\right)\right|$.*

Observation 3 *Given any cobweb admissible sequence F. The number Φ_k^n of blocks of the form σP_m in the layer $\langle \Phi_k \to \Phi_n \rangle$ is equal to*

$$\Phi_k^n = \sum_{\sigma_i \in S_m} \prod_{s=1}^{m} \binom{(k+s-1)_F}{(\sigma_i \cdot s)_F}$$

where $m = n - k + 1$ and S_m is a set of levels' permutation of the layer $\langle \Phi_k \to \Phi_n \rangle$.

Proof
Given any layer $\langle \Phi_k \to \Phi_n \rangle$. Let any permutation $\sigma_i \in S_m$ of m levels of the block σP_m. For such order of levels, cope $(\sigma_i \cdot s)_F$ vertices by s-th element of the block σP_m from all of vertices i.e. $(k+s-1)_F$ of the $(k+s)$-th level in the layer $\langle \Phi_k \to \Phi_n \rangle$. On the end sum after all of permutation σ_i ■

Theorem 2 (Tiling - clique problem) *Cobweb tiling problem is a particular case of the clique problem.*

Proof
Let any cobweb poset layer $\langle \Phi_k \to \Phi_n \rangle$ with $\eta = n_F^{\underline{m}}$ vertices, $\lambda = m_F!$ vertices in each of the blocks σP_m and $\kappa = \eta / \lambda$ blocks. There exists a not directed graph G without loops such that V stays a set of all the blocks σP_m

in considered layer i.e. $V = B(\langle \Phi_k \to \Phi_n \rangle)$ and E as a set of edges between such blocks which are disjoint with each other i.e.

$$\langle u, v \rangle \in E \leftrightarrow u \cap v = \emptyset$$

where $u, v \in V$ and $u \cup v = \emptyset$ means that two blocks are disjoint.

Recapitulation:
Any layer $\langle \Phi_k \to \Phi_n \rangle$ can be partitioned with help of the blocks σP_m if, and only if, graph G has a clique of size κ. The number of all different tilings of the layer $\langle \Phi_k \to \Phi_n \rangle$ is equal to the number of all different cliques of size κ in graph G. There are no cliques larger than κ in graph G ■

6 Cobweb poset's sequences map

Here down we present a Venn type diagram map. Note that the boundary of the whole family of Cobweb Tilling sequences is still not known (Tiling problem 2 from [3, 4].

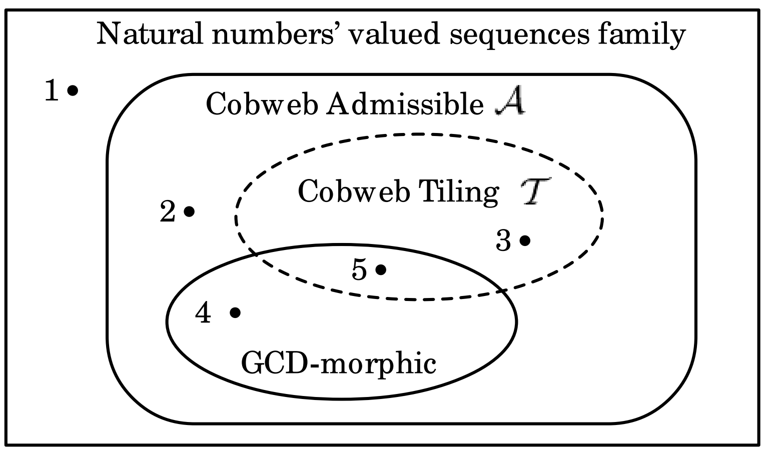

Figure 7: Venn type map of various cobweb sequences families.

Cobweb Admissible sequences family $\mathcal{A}$ is defined in [6].
GCD-morphic sequences family is defined in [5].
The whole family $\mathcal{T}$ of **Cobweb Tiling** sequences is still not known (open problem).

1. $A = (1, 3, 5, 7, 9, ...)$
2. $B = (1, 2, 2, 2, 1, 4, 1, 2, ...) = B_{2,2} \cdot B_{2,3}$
3. $C = (1, 2, 2, 2, 2, ...) = A_{2,2}$

4. $D = (1, 2, 3, 2, 1, 6, 1, ...) = B_{2,2} \cdot B_{3,3}$

5. Natural numbers, Fibonacci numbers

Where the sequence $B_{c,M}$, $A_{c,t}$ are defined in [8].

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