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AN EFFECTIVE 3D RIGID STRING AT $\theta = \pi^*$

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Abstract

An effective sigma model describing behavior of the 3d rigid string with a θ-term at $\theta = \pi$ is proposed. It contains non-perturbative corrections resulting from summation over different genera of the 2d surfaces. The effective theory is the $SU(2)$ WZW model coupled to the Nambu-Goto action. RG analysis shows the existence of a IR fixed point at which the normal to the surface has long range correlations. A similar model can describe critical behaviour of the 3d Y-M fields or the Ising model.

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It is strongly believed that the dynamics of gauge fields can be described in terms of a string theory. The $1/N_c$ expansion [1], the lattice strong coupling expansion [2] and very recently the arguments by D.Gross concerning 2d QCD [3] strongly support this idea. String theory (random surfaces theory) is also believed to describe the critical behavior of the 3d Ising model [4, 5]. In the dual picture the 3d Ising model is the 3d Z_2 gauge theory [6]. Unfortunately no satisfactory string picture was constructed till now for any of these gauge theories [7, 8]. In [7] it was claimed that it could be a kind of NSR fermionic string, but the obtained discrete version was not suitable for the continuum limit.

It is known that a gas of self-avoiding surfaces in a special 3d lattice is in the same universality class as the Ising model [9]. Recent lattice simulations [10] show the necessity of the non-perturbative renormalization. The surfaces at small scales look like sponge due to large number of microscopic handles. Yet the summation over all genera "smoothes out" the surface on large scales. This indicates the appearance of the non-perturbative renormalization effects due to the summation over genera. One can expect that the renormalization may have drastic impact on the low energy action describing the dynamics of the surface.

Similar ideas appeared in theoretical works. J.Distler [11] analyzed the Ising model again arguing that the formula for the random surface partition function could be

$$Z[\mu] = \sum_{\{M\}} (-1)^{w_2(M)} e^{-\mu A[M]} \tag{1}$$

where $\{M\}$ is a sum over all immersions of 2d surface into 3d space R^3, $A[M]$ is the area of the manifold M and w_2 is the second Stifel-Whitney class which for the closed surfaces is the modulo two reduction of the Euler characteristic: $w_2(M) = \chi(M) \bmod 2$. The claim is that the alternating factor introduces cancellation between topologically different, but physically equivalent random surfaces. Such a factor may drastically change the low energy behavior of (1) due to enormous cancellation between various topological sectors of the model. This kind of non-perturbative renormalization takes place in the 2d Ising model whose critical behavior is described by the fermionic random walk. After integrating out the world-line fermions one obtains a random walk with an alternating factor $(-1)^n$ where n is the self-intersection number of the walk. World-sheet fermions may play the same role [12]. Moreover they also produce dependence on the extrinsic curvature of the immersed surface. Thus it is quite possible that the correct action for the 2d model should include the extrinsic curvature.

In this paper we consider a model which incorporates all the mentioned above features. Thus the proposed (short distance) action is

$$S[X] = \int d^2x \sqrt{g} \left[\mu + \frac{1}{2\alpha} g^{ab} (\partial_a \vec{n})(\partial_b \vec{n}) + \frac{i}{8} R \right], \tag{2}$$

where $\vec{n}^2 = 1$ and in addition

$$\partial_a \vec{X}\, \vec{n} = 0. \tag{3}$$

i.e. $\vec{n}$ is normal to the immersed surface. The metric appearing above is the induced metric $g_{ab} \equiv \partial_a \vec{X} \partial_b \vec{X}$. The second term in the formula above is the extrinsic curvature term (or rigidity) [8, 13] which up to curvature R is $(\Delta \vec{X})(\Delta \vec{X})$. Here we can note the importance of the constraints (3), which make (2) "naively" renormalizable.[1] A similar model with dynamical metric was proposed in [14].

The last term of the formula (2) is proportional to the Euler characteristic. It is the source of the alternating factor analogous to that of (1) if one takes into account only the orientable surfaces. This contrasts with (1) where the sum runs also over the nonorientable surfaces.

Let us note that we can express (up to sign) the Euler characteristic in terms of the normal field:

$$\chi(M) = \frac{1}{4\pi} \int_M d^2x \epsilon^{ab} \vec{n}\, (\partial_a \vec{n} \times \partial_b \vec{n}) \tag{4}$$

This is just twice the winding number of the map $S^2 \rightarrow S^2$. Thus the last two terms of (2) can be considered as the $O(3)$ sigma model with the Θ term at $\Theta = \pi$. In this way the topologically different sectors of the $O(3)$ model correspond to Riemann surfaces of even Euler characteristic.

Without the Θ term the $O(3)$ sigma model is massive [15]. For $\Theta = \pi$ it is claimed to have non trivial IR fixed point at which the theory is described by the $k = 1$ $SU(2)$ WZW model [16]. This remarkable fact was conjectured (Haldane conjecture) by many authors [17, 18, 19]. The non-perturbative contributions from different topological sectors play crucial role in this change of behaviour of the $O(3)$ model. One may expect that a similar mechanism would work in the case of the model (2). Of course here the situation is much more complicated because the constraints (3) give rise to non trivial coupling between the immersions $\vec{X}$ and the normal field $\vec{n}$.

[1] The equivalence of (2) and the rigid string was checked perturbatively up to one-loop.

Hereafter we assume that the non-perturbative effects change the $O(3)$ sigma model part of (2) to the $SU(2)$ WZW model or its modifications. The source of modifications are the constraints (3). Now we face the problem: what is the proper expression for the normal vector in terms of $O(3) \propto SU(2)$ group element ? The line of reasoning given in [18] starts from the Hubbard model [20] which describes the dynamics of electrons in (1+1) dimensions. Its nonabelian bosonization leads to the $k = 1$ $SU(2)$ WZW model. In this construction the vector $\vec{n}$ is part of the spin of the electron $\vec{S}$ which in bosonization language has the low energy limit $\vec{S} \propto \mathrm{tr}(\mathrm{h}\vec{\tau})$. Thus it is seems natural to identify $\vec{n}$ with the r.h.s. of the last formula.

$$\vec{n} \propto \mathrm{tr}(\mathrm{h}\vec{\tau}) \tag{5}$$

The model described above is strongly interacting. Its solution may be not an easy task. In what follows we are going to perform perturbative calculations taking the large k. From this point of view (5) leads to troublesome, from perturbative point of view, interactions. One may expect it because the above reasoning invoke k=1 WZW model which is strongly interacting. Thus instead of (5) we shall use a proper representation in the $k \to \infty$ limit:

$$\vec{n} = \frac{1}{2}\mathrm{tr}(\vec{\tau}\,\mathrm{h}\tau^3\mathrm{h}^{-1}), \tag{6}$$

with h being $SU(2)$ group element. Its conformal dimension is $2/(k+2)$ thus equals zero for $k \to \infty$. With (6) the rotation group is $SU_L(2)$, the correct group in this limit.

In this paper we shall discuss the simplest low energy theory. The only obvious symmetry we want to preserve is the global $O(3)$ of simultaneous rotations of the vectors $(\vec{n}, \partial_a\vec{X})$. The choice is not rich because the constraint (3) indicates the following identity:

$$n^\mu n^\nu = \delta^{\mu\nu} - g^{ab}\partial_a\vec{X}^\mu\partial_b\vec{X}^\nu \tag{7}$$

Together with (3) this eliminates powers of the normal field. We also note that the normal vector (6) does not change under the $U_R(1)$ subgroup of the $SU(2)$ thus the constraints (3) break the symmetry of the sigma model. The final form of the model we shall consider is

$$\begin{aligned} S[X] &= \int_{S^2} d^2x \left[\mu\sqrt{g} - \sqrt{g}g^{ab}\left(\frac{1}{\alpha}\mathrm{tr}[(\mathrm{h}^{-1}\mathrm{D_a h})(\mathrm{h}^{-1}\mathrm{D_b h})] + \frac{1}{2\mathrm{r}}\mathrm{tr}(\mathrm{j_a}\tau^3)\mathrm{tr}(\mathrm{j_b}\tau^3)\right)\right] \\ &+ \frac{ik}{12\pi}\int_B tr(j \wedge j \wedge j) \end{aligned} \tag{8}$$

where h is $SU(2)$ element.

$$j_a = h^{-1}\partial_a h, \quad D_a h = \partial_a h + A_a h\tau^3, \quad A_a = -\frac{1}{2}\mathrm{tr}(\mathrm{j_a}\tau^3)$$

together with the nontrivial constraint (3).

In the following we shall consider RG behavior of the coupling constants of (8). The model is a modification of the rigid string [8, 13]. The rigid string has RG behaviour which shows irrelevance of the rigidity for large distances at least for small couplings α. If so the low energy theory describing the dynamics of string is the Nambu-Goto action with its well know diseases. The rigidity is irrelevant mainly because the $O(3)$ model is massive. The forthcoming analysis shows quite opposite behaviour of (8): the coupling α has an IR fixed point signaling the long range correlations of normals. The problem is also interesting on its own as an example of coupling of a CFT to the induced metric and the Nambu-Goto action (see also ([21]) .

We are going to use the background field method. Before one proceeds with calculation the gauge freedom (2d reparameterization invariance) has to be fixed. We choose to work in so-called normal gauge [23] in which the quantum part of the $\vec{X}$ field has only the normal component i.e.[2]

$$\vec{X} \to \vec{X} + \xi\vec{n}, \quad h \to h\, e^{i\pi} \tag{9}$$

where ξ, $\pi \equiv \vec{\pi}\vec{\tau}$ are the quantum fields. In order to do perturbative calculation we need to solve the constraints first. Up to terms quadratic in quantum fields

$$\pi^a = -\frac{1}{2}(E^{ab}\partial_b\xi + E^{ab}K_b^c\xi\partial_c\xi - g^{ab}\pi^3\partial_b\xi) \tag{10}$$

where[3] $E^{ab} \equiv \epsilon^{\alpha\beta}e^a_\alpha e^b_\beta$. In fact the last two terms of (10) do not contribute: the first contains too few derivatives of the quantum fields , the second is irrelevant because there is no mixing between ξ and π^3.

Now one can expand the action (8) in powers of quantum fields and calculate the one-loop renormalization of the couplings of the model. This kind of calculations have

[2]In order not to proliferate notations all the quantities (except the quantum fields π, ξ) on the r.h.s. of the eqs (9,10) will be background quantities.

[3]We change indices from the flat $(\alpha, \beta, ...)$ to curved $(a, b, ...)$ with help of the induced background "zweibien" $e_{a\alpha} \equiv \partial_a\vec{X}\vec{t}_\alpha$ and its inverse $e^a_{\ \alpha}$. The vectors $\vec{t}_\alpha$ $(\alpha = 1, 2)$ are tangent to the surface. They can be expressed by the $SU(2)$ elements as $\vec{t}_\alpha = \frac{1}{2}\mathrm{tr}(\vec{\tau}\mathrm{h}\tau^\alpha\mathrm{h}^{-1})$.

been done many times thus we refer the reader to the literature [13, 22, 23]. We just state the final result and discuss it.

$$\frac{2}{\alpha(\mu)} = \frac{2}{\alpha(\Lambda)} + 3\left(\left(\frac{k}{8\pi}\right)^2 \alpha r - 1 + \frac{\alpha}{4r}\right) \log\frac{\Lambda}{\mu} \tag{11}$$

From the above one directly infers that the flow of the coupling α has an IR stable point given by

$$\alpha^* = \frac{1}{\left(\frac{k}{8\pi}\right)^2 r + \frac{1}{4r}} \tag{12}$$

For $\alpha^* = r$ this gives $(k\alpha^*/8\pi)^2 = 3/4$ what is very close the pure WZW result [16]. We expect that k is close to 1, thus for reasonable r, the coupling α^* is a big number far from the region of applicability of the perturbative calculations. It does not mean that we should not believe in existence of the IR fixed point (although its value may be completely different) because the very reason for its appearance is the WZ term just as it happens in the WZW model.

It is interesting to note that the couplings k and r are not renormalized. The reason for such a behavior of k is, as usual, its topological origin. The technical reason for non-renormalization of r (at one-loop) is group theoretical. The current j^3_a is an isospin zero member of $SU(2)$ triplet thus it couples only to the appropriate combination of the π fields (eq.(9)) namely $j^3_a \epsilon^{\alpha\beta} \partial_a \pi^\alpha \pi^\beta$. From eq.(9) the π fields are proportional to derivatives of ξ. This in turn makes one-loop contributions zero due to the epsilons under integrals.

The string tension is also renormalized. At one-loop the WZ term does not contribute thus the renormalization is similar to what one usually gets for the rigid string. Thus the RG flow in the (α, μ) plane looks the same as that obtained in [22, 23]. Moreover the detailed discussion of the theory at the fixed point is the same. We refer the reader to these publications. What we want to emphasize here is that at the critical point the surface is smooth. There are long range correlations between normals to the surface. One may expect that the tachyon problem disappears in such a theory.

Conclusions and comments. Our analysis left a lot of questions to be answered. The basic problem is the analysis of the model for $k = 1$. Without this one can not be sure if the action (8) has been chosen properly. It could be useful to have more direct arguments for it. The result (12) is reliable only for large enough k, although it may happen that it is close to the exact one even for small k as it is for the WZW models. This could be checked by the next-to-leading contributions. Despite this the critical indices of the low k models are expected to be completely different then those of the large k models. The next group of problems lies in the use of the induced metric. Maybe, one should use the Polyakov action instead the Nambu-Goto and work with dynamical 2d gravity [14]? These problems are the subject of current research.

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