Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm

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Abstract

Archimedean t-conorm and t-norm are generalizations of a lot of other t-conorms and t-norms, such as Algebraic, Einstein, Hamacher and Frank t-conorms and t-norms or others, and some of them have been applied to intuitionistic fuzzy set, which contains three functions: the membership function, the non-membership function and the hesitancy function describing uncertainty and fuzziness more objectively. Recently, Beliakov et al. constructed some operations about intuitionistic fuzzy sets based on Archimedean t-conorm and t-norm, from which an aggregation principle is proposed for intuitionistic fuzzy information. In this paper, we propose some other operations on intuitionistic fuzzy sets, study their properties and relationships, and based on which, we study the properties of the aggregation principle proposed by Beliakov et al. and give some specific intuitionistic fuzzy aggregation operators, which can be considered as the extensions of the known ones. In the end, we develop an approach for multi-criteria decision making under intuitionistic fuzzy environment, and illustrate an example to show the behavior of the proposed operators.

1. Introduction

An intuitionistic fuzzy set (IFS)[1] A on a fixed set X is defined as $A = \{ (x, \mu_x(x), \nu_x(x)) | x \in X \}$ with the condition that $0 \leq \mu_x(x) + \nu_x(x) \leq 1$, $\mu_x(x)$ and $\nu_x(x)$ $\geq$ 0. We can find that an IFS is constructed by two information functions, which not only describe the membership degree $\mu_x(x)$ of $x \in X$ in $A$, but also describe the non-membership degree $\nu_x(x)$ of $x \in X$ in $A$. Moreover, the hesitancy information of $x \in X$ in $A$ can be denoted by $v_x(x) = 1 - \mu_x(x) - \nu_x(x)$ which is called the hesitant index, and therefore IFS can describe the uncertainty and fuzziness more objectively than the usual fuzzy set (FS) [40]. For convenience, the pair $x = (\mu_x, v_x)$ is called an intuitionistic fuzzy number (IFN) [29], where $\mu_x, v_x \geq 0$ and $\mu_x + v_x \leq 1$.

Since it was introduced, IFS has attracted more and more attention from researchers and has been used to deal with many problems, especially the multi-criteria decision making problem which can be roughly described as: Let $y = (y_1, y_2, \ldots, y_m)$ be the set of alternatives, $c = (c_1, c_2, \ldots, c_n)$ be the set of criteria, the degree that alternative $y_i$ satisfies to the criterion $c_j$ can be denoted as $\mu_{ij}$, the degree that the alternative $y_i$ does not satisfy the criterion $c_j$ can be denoted as $v_{ij}$, then the performance of the alternative $y_i$ under the criteria $c_j$ can be described as an IFN $x_{ij} = (\mu_{ij}, v_{ij})$ with the condition that $0 \leq \mu_{ij}, v_{ij} \leq 1$ and $\mu_{ij} + v_{ij} \leq 1$. When all the performances of the alternatives are provided, the intuitionistic fuzzy decision matrix $D = (x_{ij})_{m \times n} = ((\mu_{ij}, v_{ij}))_{m \times n}$ is constructed. Up to now, many methods have been proposed to deal with the multi-criteria decision making under intuitionistic fuzzy environment, which fall into two groups, the first group is to calculate the relative values of the alternatives and the second group is to calculate the actual aggregation values of the alternatives. By comparing these two types of methods for obtaining the ranking of the alternatives, the second one can reflect the actual results of the alternatives more objectively, while the first one can only obtain the relative results of the alternatives to the ideal alternative or others. To calculate the relative values of the alternatives, many classical methods have been extended to intuitionistic fuzzy environment, such as the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method [5,15,19,21], the grey relational analysis (GRA) method [11,26,43], the Elimination et Choice Translating Reality (ELECTRE) method [28], the Vlse Kriterijuamska Optimizacija Kompromisno Resenje (VIKOR) method [12,23], the maximizing deviation method [25,31] and the entropy method [9,28,37,38] and so on.

To calculate the actual aggregation values of the alternatives, a lot of aggregation operators have been developed. Xu [29] proposed some operational laws for IFNs based on Algebraic t-conorm and t-norm, and developed the intuitionistic fuzzy weighed averaging operator, the intuitionistic fuzzy ordered weighted averaging operator and the intuitionistic fuzzy hybrid averaging operator.
based on which, Xu and Yager [34] gave some other aggregation operators combining the geometric mean. Based on the generalized ordered weighted aggregation operator proposed by Yager [35], Zhao et al. [44] and Li [20] introduced the generalized intuitionistic fuzzy aggregation operator, which gives a mapping to the arguments before aggregation, and an inverse mapping to the aggregated results at the end. Yang and Chen [36] proposed the quasi-arithmetic intuitionistic fuzzy ordered weighted averaging operator, the quasi-intuitionistic fuzzy aggregation operator based on the Choquet integral and the Dempster–Shafer belief structure. Chen et al. [10] adopted three families of parametric fuzzy unions and intersections of fuzzy operations, to create other aggregation operators for interval-valued intuitionistic fuzzy sets (IVIFSs) [2], which are the generalizations of ISFs allowing the membership and non-membership degrees represented by interval-valued fuzzy numbers (IVFNs) [41]. Wang and Liu [24] introduced some operations on ISFS, such as Einstein sum, Einstein product, Einstein exponentiation, etc., and developed some new geometric aggregations on IFSs, such as Einstein sum, Einstein product, Einstein t-norm if it satisfies the following four conditions:

1. Let \( g(t) = -\log(t) \), then \( h(t) = -\log(1-t) \), \( g^{-1}(t) = e^{-t} \), \( h^{-1}(t) = 1 - e^{-t} \), and Algebraic t-conorm and t-norm [4] are obtained as follows:

\[
S^*(x,y) = x + y - xy, \quad T^*(x,y) = xy
\]

2. Let \( g(t) = \log(t) \), then \( h(t) = \log(1-t) \), \( g^{-1}(t) = e^t \), \( h^{-1}(t) = 1 - e^t \), and we can get Einstein t-conorm and t-norm [4]:

\[
S^e(x,y) = \frac{x + y}{1 + xy}, \quad T^e(x,y) = \frac{xy}{1 + (1-x)(1-y)}
\]

3. Let \( g(t) = \log(\frac{1-\gamma t}{1-t}) \), \( \gamma > 0 \), then we have \( h(t) = \log(\frac{1-\gamma(1-t)}{1-t}) \), \( g^{-1}(t) = e^{\frac{t}{1-\gamma}} \), \( h^{-1}(t) = 1 - e^{\frac{t}{1-\gamma}} \), and Hamacher t-conorm and t-norm [4] are obtained as follows:

\[
S^h_\gamma(x,y) = x + y - xy - (1-\gamma)xy, \quad \gamma > 0
\]

\[
T^h_\gamma(x,y) = \frac{xy}{\gamma + (1-\gamma)(x+y-xy)}, \quad \gamma > 0
\]

Especially, if \( \gamma = 1 \), then Hamacher t-conorm and t-norm reduce to the Algebraic t-conorm and t-norm respectively; if \( \gamma \to 2 \), then Hamacher t-conorm and t-norm reduce to the Einstein t-conorm and t-norm respectively.

4. Let \( g(t) = \log(\frac{1}{1-t}) \), \( \gamma > 1 \), then \( h(t) = \log(\frac{1}{1-t}) \), \( g^{-1}(t) = \frac{-t}{\log(t)} \), \( h^{-1}(t) = 1 - g^{-1}(t) = \frac{-t}{\log(t)} \), and we have Frank t-conorm and t-norm [4] as follows:

\[
S^f_\gamma(x,y) = 1 - \log\left(1 + \frac{(\gamma^{-1}-1)(\gamma y - 1)}{\gamma y - 1}\right), \quad \gamma > 1
\]

\[
T^f_\gamma(x,y) = \log\left(1 + \frac{(\gamma^{-1}-1)(\gamma y - 1)}{\gamma y - 1}\right), \quad \gamma > 1
\]

Especially, if \( \gamma \to 1 \), then we have

\[
\lim_{\gamma \to 1} g(t) = \lim_{\gamma \to 1} \log\left(\frac{\gamma - 1}{\gamma^\gamma - 1}\right) = -\log t
\]
which indicates that \( \lim_{y \to 0} S'(x, y) = S'(x, y) \) and \( \lim_{y \to 0} T'(x, y) = T'(x, y) \).

Based on Archimedean t-norm and t-conorm [17], Beliakov et al. [3] defined the sum operation on two IFNs
\( x_1 = (\mu_{x_1},\nu_{x_1})(i=1,2) \) as \( x_1 \oplus x_2 = (\tilde{S}(\mu_{x_1},\mu_{x_2}), T(\nu_{x_1},\nu_{x_2})) \), which can be expressed by the following:

\[
\begin{align*}
x_1 \oplus x_2 &= (\tilde{S}(\mu_{x_1},\mu_{x_2}), T(\nu_{x_1}, \nu_{x_2})) \\
&= \left( h^{-1}\left(h(\mu_{x_1}) + h(\mu_{x_2})\right), \tilde{g}^{-1}\left(g(\nu_{x_1}) + g(\nu_{x_2})\right) \right).
\end{align*}
\]

Beliakov et al. [3] also mentioned that for an IFN \( x = (\mu_{x},\nu_{x}) \), let \( \lambda \) \( = (\mu_{x},\nu_{x}) \), then \( g(\nu_{x}) = g(\nu_{x}) \) and \( h(\mu_{x}) = h(\mu_{x}) \).

With the above analysis, the operations about IFNs based on Archimedean t-norm and Archimedean t-conorm [17] can be also expressed as follows:

**Definition 5**. Let \( x_1 = (\mu_{x_1},\nu_{x_1})(i=1,2) \) and \( x = (\mu_{x},\nu_{x}) \) be three IFNs, then we have

\[
\begin{align*}
(1) & x_1 \otimes x_2 = (\tilde{S}(\mu_{x_1},\mu_{x_2}), T(\nu_{x_1}, \nu_{x_2})) \\
&= \left( h^{-1}\left(h(\mu_{x_1}) + h(\mu_{x_2})\right), \tilde{g}^{-1}\left(g(\nu_{x_1}) + g(\nu_{x_2})\right) \right),
\end{align*}
\]

\( (2) \ x_1 \otimes x_2 = (\tilde{T}(\mu_{x_1},\mu_{x_2}), \tilde{S}(\nu_{x_1}, \nu_{x_2})) = \left( \tilde{g}^{-1}(g(\mu_{x_1}) + g(\mu_{x_2})), h^{-1}(\nu_{x_1} + \nu_{x_2}) \right) \),

\( (3) \ x_1 \otimes x_2 = \left( h^{-1}(1 - h(\mu_{x_1})),\tilde{g}^{-1}(1 - g(\nu_{x_1}))\right) > 0 \)

**Theorem 1**. Let \( x_1 = (\mu_{x_1},\nu_{x_1})(i=1,2) \) and \( x = (\mu_{x},\nu_{x}) \) be three IFNs, then the relations of these operational laws are given as:

\( (1) \ x_1 \otimes x_2 = x_1 \otimes x_2 \)

\( (2) \ x_1 \otimes x_2 = x_2 \otimes x_1 \)

\( (3) \ \lambda(x_1 \otimes x_2) = x_1 \otimes x_2 \lambda \)

\( (4) \ (x_1 \otimes x_2)^\gamma = x_1 \otimes x_2 \gamma \)

\( (5) \ \lambda x \otimes \lambda x = (\lambda x)^\gamma \)

\( (6) \ x^\gamma \otimes x^\gamma = (x^\gamma)^\gamma \)

**Proof.** (1) and (2) are obvious, we prove the others:

\( (3) \ \lambda(x_1 \otimes x_2) = \left( h^{-1}(h(\mu_{x_1}) + h(\mu_{x_2})), \tilde{g}^{-1}(g(\nu_{x_1}) + g(\nu_{x_2}))\right) = (h^{-1}(h(\mu_{x_1}) + h(\mu_{x_2})), \tilde{g}^{-1}(g(\nu_{x_1}) + g(\nu_{x_2}))) \),

\( (4) \ \lambda x_1 \otimes x_2 = (h^{-1}(h(\mu_{x_1})), \tilde{g}^{-1}(g(\nu_{x_1}))) \otimes (h^{-1}(h(\mu_{x_2})), \tilde{g}^{-1}(g(\nu_{x_2}))) \),

\( (5) \ \lambda x_1 \otimes x_2 = \left( h^{-1}(h(\mu_{x_1}) + h(\mu_{x_2})), \tilde{g}^{-1}(g(\nu_{x_1}) + g(\nu_{x_2}))\right) = (h^{-1}(h(\mu_{x_1}) + h(\mu_{x_2})), \tilde{g}^{-1}(g(\nu_{x_1}) + g(\nu_{x_2}))) \)

\( \tilde{g}^{-1}(g(\nu_{x_1}) + g(\nu_{x_2})) = \left( h^{-1}(h(\mu_{x_1}) + h(\mu_{x_2})), \tilde{g}^{-1}(g(\nu_{x_1}) + g(\nu_{x_2}))\right) \)

Similarly, (4) and (6) can be proven which completes the proof of the theorem. \( \square \)

**Theorem 2**. Let \( x_1 = (\mu_{x_1},\nu_{x_1})(i=1,2) \) and \( x = (\mu_{x},\nu_{x}) \) be three IFNs, and \( \gamma > 0 \), then the following are also valid:

\( (1) \ (x^\gamma)^\gamma = (x^\gamma)^\gamma \)

\( (2) \ \lambda(x\gamma)^\gamma = (\lambda x)^\gamma \)

\( (3) \ x_1 \otimes x_2 = (x_1 \otimes x_2)^\gamma \)

\( (4) \ x_1 \otimes x_2 = (x_1 \otimes x_2)^\gamma \)

where \( x^\gamma = (\nu_{x_1},\mu_{x_1}) \) denotes the complement of an IFN \( x \).

**Proof.** Based on the operations defined in **Definition 5**, we have:

\( (1) \ (x^\gamma)^\gamma = \tilde{g}^{-1}(g(\nu_{x_1}) + h^{-1}(h(\mu_{x_1}))) = x^\gamma \)

\( (2) \ \lambda(x\gamma)^\gamma = \left( h^{-1}(h(\mu_{x_1})), \tilde{g}^{-1}(g(\nu_{x_1}))\right) = x\gamma \)

\( (3) \ x_1 \otimes x_2 = \left( h^{-1}(h(\mu_{x_1}) + h(\mu_{x_2})), \tilde{g}^{-1}(g(\nu_{x_1}) + g(\nu_{x_2}))\right) = (x_1 \otimes x_2)^\gamma \).
The operational laws defined in Section 2 can be used to aggregate the intuitionistic fuzzy information, which is the focus of this section.

**Definition 6** (3). Let \( x_i = (\mu_x, \nu_x) \) for \( i = 1, 2, \ldots, n \) be a collection of IFNs, and \( w = (w_1, w_2, \ldots, w_n) \) be the weight vector of \( x_i(i = 1, 2, \ldots, n) \), where \( w_i \) indicates the importance degree of \( x_i \), satisfying \( w_i > 0 \) \( (i = 1, 2, \ldots, n) \) and \( \sum_{i=1}^{n} w_i = 1 \); if

\[
\text{ATS-IFWA}(x_1, x_2, \ldots, x_n) = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}
\]

then ATS-IFWA is called the Archimedean t-conorm and t-norm based intuitionistic fuzzy weighted averaging (ATS-IFWA) operator.

**Theorem 3** (3). Let \( x_i = (\mu_x, \nu_x) \) for \( i = 1, 2, \ldots, n \) be a collection of IFNs, and \( w = (w_1, w_2, \ldots, w_n) \) be the weight vector of \( x_i(i = 1, 2, \ldots, n) \), where \( w_i \) indicates the importance degree of \( x_i \), satisfying \( w_i > 0 \) \( (i = 1, 2, \ldots, n) \) and \( \sum_{i=1}^{n} w_i = 1 \), then the aggregated value by using the ATS-IFWA operator is also an IFN, and

\[
\text{ATS-IFWA}(x_1, x_2, \ldots, x_n)
\]

Next we give a further study:

**Proof.** By using mathematical induction on \( n \): For \( n = 2 \), we have:

\[
\text{ATS-IFWA}(x_1, x_2) = \frac{w_1 x_1 + w_2 x_2}{w_1 + w_2}
\]

Suppose Eq. (10) holds for \( n = k \), that is

\[
\text{ATS-IFWA}(x_1, x_2, \ldots, x_n) = \frac{\sum_{i=1}^{k} w_i x_i}{\sum_{i=1}^{k} w_i}
\]

then

\[
\text{ATS-IFWA}(x_1, x_2, \ldots, x_{n+1}) = \frac{\sum_{i=1}^{k} w_i x_i + w_{k+1} x_{k+1}}{\sum_{i=1}^{k} w_i + w_{k+1}}
\]

i.e., Eq. (10) holds for \( n = k + 1 \). Thus Eq. (10) holds for all \( n \).

In addition, we have known that \( h(t) = g(1 - t) \), and \( g : [0, 1] \rightarrow [0, \infty] \) is a strictly decreasing function, then \( h(t) \) is a strictly increasing function which indicates that

\[
0 < h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_{x_i}) \right) \cdot g^{-1} \left( \sum_{i=1}^{n} w_i g(\nu_{x_i}) \right) \leq 1
\]

and

\[
h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_{x_i}) \right) + g^{-1} \left( \sum_{i=1}^{n} w_i g(\nu_{x_i}) \right)
\]

\[
\leq h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_{x_i}) \right) + g^{-1} \left( \sum_{i=1}^{n} w_i g(1 - \mu_{x_i}) \right)
\]

\[
= h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_{x_i}) \right) + 1 - h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_{x_i}) \right) = 1
\]

which completes the proof of Theorem 3.

Then we can investigate some desirable properties of the ATS-IFWA operator, before doing this, a definition is given firstly:

**Definition 7** (7). Let \( s(\mu_x) = \mu_x - \nu_x \) for \( i = 1, 2 \) be the scores of \( \mu_x, \nu_x \) respectively, if \( s(\mu_x) > s(\nu_x) \), then \( \mu_x \) is larger than \( \nu_x \), denoted by \( \mu_x > \nu_x \).

**Property 1.** If all \( \mu_x(i = 1, 2, \ldots, n) \) are equal, i.e., \( \mu_x = (\mu_{x_i}, \nu_{x_i}) \), for all \( i \), then

\[
\text{ATS-IFWA}(x_1, x_2, \ldots, x_n) = \mu_x
\]

**Proof.** Let \( x_i = (\mu_{x_i}, \nu_{x_i}) \), we have

\[
\text{ATS-IFWA}(x_1, x_2, \ldots, x_n) = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} = \mu_x
\]

**Property 2.** Let \( \mu_x = (\mu_{x_i}, \nu_{x_i}) \) and \( \beta_i = (\mu_{\beta_i}, \nu_{\beta_i}) \) for \( i = 1, 2, \ldots, n \) be two collections of IFNs, if \( \mu_{x_i} \leq \mu_{\beta_i} \) and \( \nu_{x_i} \geq \nu_{\beta_i} \), for all \( i \), then

\[
s(\text{ATS-IFWA}(x_1, x_2, \ldots, x_n)) \leq s(\text{ATS-IFWA}(\beta_1, \beta_2, \ldots, \beta_n))
\]

**Proof.** We have known that \( h(t) = g(1 - t) \), and \( g : [0, 1] \rightarrow [0, \infty] \) is a strictly decreasing function, then \( h(t) \) is a strictly increasing function. Since \( \mu_{x_i} \leq \mu_{\beta_i} \) and \( \nu_{x_i} \geq \nu_{\beta_i} \), then we have

\[
h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_{x_i}) \right) \geq h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_{\beta_i}) \right)
\]

\[
g^{-1} \left( \sum_{i=1}^{n} w_i g(\nu_{x_i}) \right) \leq g^{-1} \left( \sum_{i=1}^{n} w_i g(\nu_{\beta_i}) \right)
\]

then

\[
s(\text{ATS-IFWA}(x_1, x_2, \ldots, x_n)) \leq s(\text{ATS-IFWA}(\beta_1, \beta_2, \ldots, \beta_n))
\]

which completes the proof.
Property 3. Let \( x_i = (\mu_x, v_x) (i = 1, 2, \ldots, n) \) be a collection of IFNs, and
\[
\begin{align*}
  \alpha^- &= \left( \min\{\mu_x\}, \max\{v_x\} \right), \\
  \alpha^+ &= \left( \max\{\mu_x\}, \min\{v_x\} \right)
\end{align*}
\]
then
\[
  s(\alpha^-) \leq s(\text{ATS-IFWA} (x_1, x_2, \ldots, x_n)) \leq s(\alpha^+) \tag{21}
\]

Property 4. Let \( x_i = (\mu_x, v_x) (i = 1, 2, \ldots, n) \) be a collection of IFNs, 
\( w = (w_1, w_2, \ldots, w_n)^T \) be their weight vector such that \( w_i \geq 0 \)
\( (i = 1, 2, \ldots, n) \) and \( \sum_{i=1}^{n} w_i = 1 \), if \( \beta = (\mu_0, v_0) \) is an IFN, then
\[
\text{ATS-IFWA} (x_1 \oplus \beta, x_2 \oplus \beta, \ldots, x_n \oplus \beta) = \text{ATS-IFWA} (x_1, x_2, \ldots, x_n) \oplus \beta \tag{23}
\]
Proof. Since
\[
\begin{align*}
  x_i \oplus \beta &= \left( h^{-1}(h(\mu_x) + h(\mu_0)), g^{-1}(g(v_x) + g(v_0)) \right) \\
  x_i \ominus \beta &= \left( \frac{h}{C0/C1/C16/C17} \left( \frac{1}{C0/C1} \right) \right) \\
  x_i \ominus \beta &= \left( \frac{1}{C0/C1/C16/C17} + \frac{1}{C0/C1/C16/C17} \right)
\end{align*}
\]

and
\[
\begin{align*}
  \text{ATS-IFWA} (x_1 \oplus \beta, x_2 \oplus \beta, \ldots, x_n \oplus \beta) &= \left( \frac{1}{C0/C1/C16/C17} \left( \frac{1}{C0/C1} \right) \right) \\
  \text{ATS-IFWA} (x_1, x_2, \ldots, x_n) \oplus \beta &= \left( \frac{1}{C0/C1/C16/C17} + \frac{1}{C0/C1/C16/C17} \right)
\end{align*}
\]

which completes the proof. \( \square \)

Property 5. Let \( x_i = (\mu_x, v_x) (i = 1, 2, \ldots, n) \) be a collection of IFNs, 
and \( w = (w_1, w_2, \ldots, w_n)^T \) be their weight vector such that \( w_i \geq 0 \)
\( (i = 1, 2, \ldots, n) \) and \( \sum_{i=1}^{n} w_i = 1 \), if \( \beta = (\mu_0, v_0) \) is an IFN, then
\[
\text{ATS-IFWA} (x_1, x_2, \ldots, x_n) = r \text{ ATS-IFWA} (x_1, x_2, \ldots, x_n) \tag{27}
\]
Proof. According to Definition 5, we have
\[
\begin{align*}
  r_x &= \left( h^{-1}(h(\mu_0) + h(\mu_x)), g^{-1}(g(v_0) + g(v_x)) \right)
\end{align*}
\]
then
\[
\begin{align*}
  \text{ATS-IFWA} (x_1, x_2, \ldots, x_n) &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i h \left( h^{-1} \left( h(\mu_x) \right) \right) \right), g^{-1} \left( \sum_{i=1}^{n} w_i g \left( g^{-1} \left( g(v_x) \right) \right) \right) \right) \\
  &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i \left( h(\mu_x) \right) \right) \right), g^{-1} \left( \sum_{i=1}^{n} w_i \left( g(v_x) \right) \right) \tag{29}
\end{align*}
\]
and
\[
\begin{align*}
  r \text{ ATS-IFWA} (x_1, x_2, \ldots, x_n) &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i h \left( h^{-1} \left( h(\mu_x) \right) \right) \right), g^{-1} \left( \sum_{i=1}^{n} w_i g \left( g^{-1} \left( g(v_x) \right) \right) \right) \right) \\
  &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_x) \right), g^{-1} \left( \sum_{i=1}^{n} w_i g(\mu_x) \right) \right) \tag{30}
\end{align*}
\]

According to Properties 4 and 5, we can get Property 6 easily:

Property 6. Let \( x_i = (\mu_x, v_x) (i = 1, 2, \ldots, n) \) be a collections of IFNs, 
and \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of them such that 
\( w_i \geq 0 \) \( (i = 1, 2, \ldots, n) \) \( \sum_{i=1}^{n} w_i = 1 \), if \( \beta \neq (\mu_0, v_0) \) is an IFN, then
\[
\begin{align*}
  \text{ATS-IFWA} (x_1, x_2, \ldots, x_n) &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_x) \right), g^{-1} \left( \sum_{i=1}^{n} w_i g(\mu_x) \right) \right) \\
  &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_x) \right), g^{-1} \left( \sum_{i=1}^{n} w_i \left( h(\mu_x) \right) \right) \right)
\end{align*}
\]

and
\[
\begin{align*}
  r \text{ ATS-IFWA} (x_1, x_2, \ldots, x_n) &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_x) \right), g^{-1} \left( \sum_{i=1}^{n} w_i g(\mu_x) \right) \right) \\
  &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_x) \right), g^{-1} \left( \sum_{i=1}^{n} w_i g(\mu_x) \right) \right)
\end{align*}
\]

Property 7. Let \( x_i = (\mu_x, v_x) \) and \( \beta_i = (\mu_0, v_0) (i = 1, 2, \ldots, n) \) be two collections of IFNs, 
and \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of them such that 
\( w_i \geq 0 \) \( (i = 1, 2, \ldots, n) \) \( \sum_{i=1}^{n} w_i = 1 \), then
\[
\begin{align*}
  \text{ATS-IFWA} (x_1, x_2, \ldots, x_n) &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_x) \right), g^{-1} \left( \sum_{i=1}^{n} w_i g(\mu_x) \right) \right) \\
  &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i h(\mu_x) \right), g^{-1} \left( \sum_{i=1}^{n} w_i \left( h(\mu_x) \right) \right) \right)
\end{align*}
\]

Proof. According to Definition 5, we have
\[
\begin{align*}
  x_i \oplus \beta_i &= \left( h^{-1}(h(\mu_x) + h(\mu_0)), g^{-1}(g(v_x) + g(v_0)) \right)
\end{align*}
\]
then
\[
\begin{align*}
  \text{ATS-IFWA} (x_1, x_2, \ldots, x_n) &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i \left( h(\mu_x) \right) \right), g^{-1} \left( \sum_{i=1}^{n} w_i \left( g(\mu_x) \right) \right) \right) \\
  &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i \left( h(\mu_x) \right) \right), g^{-1} \left( \sum_{i=1}^{n} w_i \left( g(\mu_x) \right) \right) \right)
\end{align*}
\]

and
\[
\begin{align*}
  r \text{ ATS-IFWA} (x_1, x_2, \ldots, x_n) &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i \left( h(\mu_x) \right) \right), g^{-1} \left( \sum_{i=1}^{n} w_i \left( g(\mu_x) \right) \right) \right) \\
  &= \left( h^{-1} \left( \sum_{i=1}^{n} w_i \left( h(\mu_x) \right) \right), g^{-1} \left( \sum_{i=1}^{n} w_i \left( g(\mu_x) \right) \right) \right)
\end{align*}
\]
If the additive generator \( g \) is assigned different forms, then some specific intuitionistic fuzzy aggregation operators can be obtained as follows:

**Case 1.** If \( g(t) = -\log(t) \), then the ATS-IFWA operator reduces to the following:

\[
\text{IFWA}(x_1, x_2, \ldots, x_n) = \left( 1 - \prod_{i=1}^{n} (1 - \mu_{x_i})^{w_i}, \prod_{i=1}^{n} \nu_{x_i}^{w_i} \right) \tag{36}
\]

which is the intuitionistic fuzzy weighted averaging (IFWA) operator defined by Xu [29].

**Case 2.** If \( g(t) = \log(\frac{1}{1-t}) \), then the ATS-IFWA operator reduces to the following:

\[
\text{EIFWA}(x_1, x_2, \ldots, x_n) = \left( \frac{\prod_{i=1}^{n} (1 + \mu_{x_i})^{w_i} - \prod_{i=1}^{n} (1 - \mu_{x_i})^{w_i}}{\prod_{i=1}^{n} (1 + \mu_{x_i})^{w_i} + \prod_{i=1}^{n} (1 - \mu_{x_i})^{w_i}} + \frac{2\prod_{i=1}^{n} \nu_{x_i}^{w_i}}{\prod_{i=1}^{n} (2 - \nu_{x_i})^{w_i} + \prod_{i=1}^{n} \nu_{x_i}^{w_i}} \right) \tag{37}
\]

which is called the Einstein intuitionistic fuzzy weighted averaging (EIFWA) operator.

**Case 3.** If \( g(t) = \log(\frac{1}{1-t}) \), \( \gamma > 0 \), then the ATS-IFWA operator reduces to the following:

\[
\text{HIFWA}(x_1, x_2, \ldots, x_n) = \left( \prod_{i=1}^{n} \left( 1 + (\gamma - 1) \mu_{x_i} \right)^{w_i} - \prod_{i=1}^{n} \left( 1 - \mu_{x_i} \right)^{w_i}, \prod_{i=1}^{n} \left( 1 + (\gamma - 1) \mu_{x_i} \right)^{w_i} + (\gamma - 1) \prod_{i=1}^{n} \left( 1 - \mu_{x_i} \right)^{w_i} \right) \tag{38}
\]

which is called the Hammer intuitionistic fuzzy weighted averaging (HIFWA) operator. Especially, if \( \gamma = 1 \), then the HIFWA operator reduces to the IFWA operator; if \( \gamma = 2 \), then the HIFWA operator reduces to the EIFWA operator.

**Case 4.** If \( g(t) = \log(\frac{1}{1-t}) \), \( \gamma > 1 \), then the ATS-IFWA operator reduces to the following:

\[
\text{FIFWA}(x_1, x_2, \ldots, x_n) = \left( 1 - \log_\gamma \left( 1 + \prod_{i=1}^{n} \left( \gamma^{\mu_{x_i} - 1} - 1 \right)^{w_i} \right), \log_\gamma \left( 1 + \prod_{i=1}^{n} \left( \gamma^{\mu_{x_i} - 1} - 1 \right)^{w_i} \right) \right) \tag{39}
\]

which is called the Frank intuitionistic fuzzy weighted averaging (FIFWA) operator. Especially, if \( \gamma \to 1 \), then the FIFWA operator reduces to the IFWA operator.

Motivated by the geometric mean, the following definition is given:

**Definition 8.** Let \( x_i = (\mu_{x_i}, \nu_{x_i}) (i = 1, 2, \ldots, n) \) be a collection of IFNs, and \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of \( x_i (i = 1, 2, \ldots, n) \), where \( w_i \) indicates the importance degree of \( x_i \), satisfying \( w_i > 0 \) (i = 1, 2, \ldots, n) and \( \sum_{i=1}^{n} w_i = 1 \), if

\[
\text{ATS-IFWG}(x_1, x_2, \ldots, x_n) = \bigwedge_{i=1}^{n} x_i^{w_i} \tag{40}
\]

then ATS-IFWG is called the Archimedean t-cornorm and t-norm based the intuitionistic fuzzy geometric (ATS-IFWG) operator.

Based on the operational laws of the IFNs given in **Definition 5**, we can derive the following theorem:

**Theorem 4.** Let \( x_i = (\mu_{x_i}, \nu_{x_i}) (i = 1, 2, \ldots, n) \) be a collection of IFNs, and \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of \( x_i (i = 1, 2, \ldots, n) \), where \( w_i \) indicates the importance degree of \( x_i \), satisfying \( w_i > 0 \) (i = 1, 2, \ldots, n) and \( \sum_{i=1}^{n} w_i = 1 \), then the aggregated value by using the ATS-IFWG operator is also an IFN, and

\[
\text{ATS-IFWG}(x_1, x_2, \ldots, x_n) = \bigwedge_{i=1}^{n} x_i^{w_i} \tag{41}
\]

Similarly, we can prove the ATS-IFWG operator also satisfies the properties that the ATS-IFWA operator has, here we will not repeat them. Moreover, if the additive generator \( g \) is assigned different forms, then the following intuitionistic fuzzy aggregation operators can be obtained:

**Case 1.** If \( g(t) = -\log(t) \), then the ATS-IFWG operator reduces to:

\[
\text{IFWG}(x_1, x_2, \ldots, x_n) = \left( \prod_{i=1}^{n} \mu_{x_i}^{w_i} - \prod_{i=1}^{n} \left( 1 - \mu_{x_i} \right)^{w_i}, \prod_{i=1}^{n} \nu_{x_i}^{w_i} \right) \tag{42}
\]

which is the intuitionistic fuzzy weighted geometric (IFWG) operator defined by Xu and Yager [34].

**Case 2.** If \( g(t) = \log(\frac{1}{1-t}) \), then the ATS-IFWG operator reduces to:

\[
\text{EIFWG}(x_1, x_2, \ldots, x_n) = \left( \prod_{i=1}^{n} \left( 1 + (\gamma - 1) \mu_{x_i} \right)^{\frac{w_i}{\gamma-1}} - \prod_{i=1}^{n} \left( 1 - \mu_{x_i} \right)^{\frac{w_i}{\gamma-1}}, \prod_{i=1}^{n} \left( 1 + (\gamma - 1) \mu_{x_i} \right)^{\frac{w_i}{\gamma-1}} + (\gamma - 1) \prod_{i=1}^{n} \left( 1 - \mu_{x_i} \right)^{\frac{w_i}{\gamma-1}} \right) \tag{43}
\]

which is called the Einstein intuitionistic fuzzy weighted geometric (EIFWG) operator defined by Wang and Liu [24].

**Case 3.** If \( g(t) = \log(\frac{1}{1-t}) \), \( \gamma > 0 \), then the ATS-IFWG operator reduces to:

\[
\text{HIFWG}(x_1, x_2, \ldots, x_n) = \left( \prod_{i=1}^{n} \left( 1 + (\gamma - 1) \mu_{x_i} \right)^{\frac{w_i}{\gamma-1}} - \prod_{i=1}^{n} \left( 1 - \mu_{x_i} \right)^{\frac{w_i}{\gamma-1}}, \prod_{i=1}^{n} \left( 1 + (\gamma - 1) \mu_{x_i} \right)^{\frac{w_i}{\gamma-1}} + (\gamma - 1) \prod_{i=1}^{n} \left( 1 - \mu_{x_i} \right)^{\frac{w_i}{\gamma-1}} \right) \tag{44}
\]

which is called the Hammer intuitionistic fuzzy weighted geometric (HIFWG) operator. Especially, if \( \gamma = 1 \), then the HIFWG operator reduces to the IFWA operator; if \( \gamma = 2 \), then the HIFWG operator reduces to the EIFWA operator.

**Case 4.** If \( g(t) = \log(\frac{1}{1-t}) \), \( \gamma > 1 \), then the ATS-IFWG operator reduces to:

\[
\text{FIFWG}(x_1, x_2, \ldots, x_n) = \left( \log_\gamma \left( 1 + \prod_{i=1}^{n} \left( \gamma^{\mu_{x_i} - 1} - 1 \right)^{w_i} \right), 1 - \log_\gamma \left( 1 + \prod_{i=1}^{n} \left( \gamma^{\mu_{x_i} - 1} - 1 \right)^{w_i} \right) \right) \tag{45}
\]

which is called the Frank intuitionistic fuzzy weighted geometric (FIFWG) operator. Especially, if \( \gamma \to 1 \), then the FIFWG operator reduces to the IFWG operator.
4. An approach to intuitionistic fuzzy multi-criteria decision making

For a multi-criteria decision making under intuitionistic fuzzy environment, let \( Y = \{y_1, y_2, \ldots, y_n\} \) be a set of alternatives to be selected, and \( C = \{c_1, c_2, \ldots, c_n\} \) be a set of criteria to be evaluated. To evaluate the performance of the alternative \( y_i \) under the criterion \( c_j \), the decision maker is required to provide not only the information that the alternative \( y_i \) satisfies the criterion \( c_j \), but also the information that the alternative \( y_i \) does not satisfy the criterion \( c_j \). These two part information can be expressed by \( \mu_{ij} \) and \( \nu_{ij} \) which denote the degrees that the alternative \( y_i \) satisfies the criterion \( c_j \) and does not satisfy the criterion \( c_j \), then the performance of the alternative \( y_i \) under the criterion \( c_j \) can be expressed by an IFN \( x_{ij} = (\mu_{ij}, \nu_{ij}) \) with the condition that \( 0 \leq \mu_{ij}, \nu_{ij} \leq 1 \) and \( \mu_{ij} + \nu_{ij} \leq 1 \). When all the performances of the alternatives are provided, the intuitionistic fuzzy decision matrix \( D = (x_{ij})_{m \times n} \) can be constructed. To obtain the ranking of the alternatives, the following steps are given:

**Step 1.** Transform the intuitionistic fuzzy decision matrix \( D = (x_{ij})_{m \times n} \) into the normalized intuitionistic fuzzy decision matrix \( B = (\beta_{ij})_{m \times n} \), where

\[
\beta_{ij} = \begin{cases} x_{ij}, & \text{for benefit attribute} \ x_i \; i = 1,2,\ldots,m, \; j = 1,2,\ldots,n \\ x_{ij}^c, & \text{for cost attribute} \ x_i \; i = 1,2,\ldots,n \end{cases}
\]  

(46)

**Step 2.** Aggregate the IFNs \( \beta_i \) of the alternative \( y_i (i = 1,2,\ldots,m) \), by the ATS-IFWA operator or the ATS-IFWG operator:

\[
\beta_i = \text{ATS-IFWA} (\beta_{i1}, \beta_{i2}, \ldots, \beta_{im}) = \bigoplus_{j=1}^{m} w_j \beta_{ij}, \quad i = 1,2,\ldots,m
\]

(47)

or

\[
\beta_i = \text{ATS-IFWG}(\beta_{i1}, \beta_{i2}, \ldots, \beta_{im}) = \bigotimes_{j=1}^{m} \beta_{ij}^w, \quad i = 1,2,\ldots,m
\]

(48)

**Step 3.** Calculate the scores \( s(\beta_i) \) of \( \beta_i \) by Definition 3, and obtain the priority of the alternatives according to the ranking of \( s(\beta_i) (i = 1,2,\ldots,m) \), the bigger the \( s(\beta_i) \), the better the alternative \( y_i \).

To illustrate the proposed method, we give an example adapted from Ref. [8] as follows:

**Example 1.** The purchasing manager in a small enterprise considers various criteria involving \( c_1 \): financial factors (e.g., economic performance, financial stability), \( c_2 \): performance (e.g., delivery, quality, price), \( c_3 \): technology (e.g., manufacturing capability, design capability, ability to cope with technology changes), and \( c_4 \): organizational culture and strategy (e.g., feeling of trust, internal and external integration of suppliers, compatibility across levels and functions of the buyer and supplier). The set of evaluative criteria is denoted by \( C = \{c_1, c_2, c_3, c_4\} \), whose weight vector is \( w = (0.34,0.23,0.22,0.21)^T \). There are six suppliers available, and the set of all alternatives is denoted by \( Y = \{y_1, y_2, \ldots, y_6\} \). The characteristics of the supplier \( y_i (i = 1,2,\ldots,6) \) in terms of the criteria in \( C \) are expressed by the following decision matrix (see Table 1):

To obtain the alternative(s), the following steps are given:

**Step 1.** Considering all the criteria \( c(j = 1,2,3,4) \) are the benefit criteria, the performance values of the alternatives \( y(i = 1,2,\ldots,6) \) do not need normalization.

![Scores for alternatives obtained by HIFWA](image-url)
Step 2. Aggregate the intuitionistic fuzzy values $a_i$ of the alternative $c_i$ by the HIFWA operator (without loss of generality, let $c = 1$):

$$a_1 = (0.4075, 0.2964), \ a_2 = (0.4005, 0.2466),$$

$$a_3 = (0.5079, 0.3163), \ a_4 = (0.4437, 0.4049),$$

$$a_5 = (0.3783, 0.2734), \ a_6 = (0.3955, 0.1689).$$

Step 3. Calculate the scores $s(a_i)$ of $a_i$ by Definition 7:

$$s(a_1) = 0.1111, \ s(a_2) = 0.1539, \ s(a_3) = 0.1915,$$

$$s(a_4) = 0.0388, \ s(a_5) = 0.1049, \ s(a_6) = 0.2266.$$

Since $s(a_6) > s(a_3) > s(a_2) > s(a_1) > s(a_5) > s(a_4)$, we can obtain the priority of the alternatives $y_i$ ($i = 1, 2, ..., 6$):

$$y_6 > y_3 > y_2 > y_1 > y_5 > y_4.$$

To investigate the variation trend of the scores and the rankings of the alternatives with the change of the value of the parameter $\gamma$, we use figures to illustrate these issues. Fig. 1 gives the scores of the alternatives obtained by the HIFWA operator as the parameter $\gamma$ is assigned different values, from which we can find that the scores of the alternatives decrease as the value of the parameter $\gamma$ increases from 0 to 10. Fig. 2 shows the scores of the alternatives obtained by the HIFWG operator, and as the value of $\gamma$ increases from 0 to 10, we can find that the scores of alternatives increase. Fig. 3 illustrates the deviation values between the scores obtained by the HIFWA operator and the ones obtained by the HIFWG operator. It is noted that the scores obtained by the HIFWA operator are bigger than the ones obtained by the HIFWG operator, and as the value of the parameter $\gamma$ increases, the deviation decreases. Moreover, if $\gamma = 1$, then the scores and rankings of the alternatives obtained in Fig. 1 are the ones obtained by the IFWA operator.
and the results obtained in Fig. 2 are just the ones obtained by the IFWG operator [30].

If we use the FIFWA or FIFWG operator instead of the HIFWA or HIFWG operator to aggregate the attribute values of alternatives, then the scores of alternatives can be found in Figs. 4 and 5, respectively. Fig. 4 gives the scores of the alternatives obtained by the FIFWA operator as the parameter $c$ is assigned different values, from which we can find that the scores of the alternatives decrease as the value of the parameter $c$ increases from 1 to 100. Fig. 5 shows the scores of the alternatives obtained by the FIFWG operator, and as the value of the parameter $c$ increase from 1 to 100, we can find that the scores of the alternatives increase. Fig. 6 illustrates the deviation values between the scores obtained by FIFWA operator and the ones obtained by the FIFWG operator, it is noted that the scores obtained by the HIFWA operator are bigger than the ones obtained by the HIFWG operator, and as the value of $\gamma$ increases, the deviation decreases.

From the above analysis, we can find that the parameter $c$ can be considered as a reflection of the decision makers' preferences, as the parameter $c$ is assigned different values, the scores of the alternatives are different, and the rankings of the alternatives are also different. Therefore, the proposed aggregation operators with parameters can provide the decision makers more choices and thus the proposed method is more flexible than the existing ones.
because we can choose different values of the parameter according to the different situations, which is an interesting topic and is worthy to be further studied in the future.

5. Concluding remarks

In this paper, we have given a further study about the application of Archimedean t-conorm and t-norm under intuitionistic fuzzy environment, and given some new operational laws for IFNs, studied their properties and correlations, based on which, some properties of the aggregation principle given by Beliakov et al. [3] have been further discussed, and some important conclusions have been obtained. Some specific intuitionistic fuzzy aggregation operators have been developed including the Hammer intuitionistic fuzzy aggregation operator and the Frank intuitionistic fuzzy aggregation operator. As the parameter changes in the proposed aggregation operators, some existing ones can be obtained, moreover, the proposed aggregation operators also satisfy all the properties that the existing ones have. An approach for multi-criteria decision making has been developed based on the proposed aggregation operators, and a detailed discussion has been given about the variation trend of the scores and rankings of the alternatives as the parameter changes in the aggregation operators.

Acknowledgments

The authors are very grateful to the anonymous reviewers for their insightful and constructive comments and suggestions that have led to an improved version of this paper. The work was partly supported by the National Natural Science Foundation of China (No. 71071161), the National Science Fund for Distinguished Young Scholars of China (No. 70625005), the Ministry of Education Foundation of Humanities and Social Sciences (No. 10YJC630269) and the Pre-Research Foundation of PLA University of Science and Technology (No. 20110511).

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