Graph colourings and partitions

V. Yegnanarayanan

Prof and Head, Department of Science and Humanities, Arulmigu Meenakshi Amman College of Engineering, Vadavancudal-604410, Near Kanchipuram, India

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Abstract

In this paper we have investigated mainly the three colouring parameters of a graph $G$, viz., the chromatic number, the achromatic number and the pseudoachromatic number. The importance of their study in connection with the computational complexity, partitions, algebra, projective plane geometry and analysis were briefly surveyed. Some new results were found along these directions. We have redefined the concept of perfect graphs in terms of these parameters and obtained a few results. Some open problems are raised. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we discuss mostly finite undirected simple graphs. For any graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. Given a graph $G$ with $n$ vertices, one can observe several variations and interesting aspects as far as the problem of colouring of $G$ is concerned. It is well-known that graph colouring is a difficult (i.e., NP-complete) problem. By and large, we have chosen to address in this paper on vertex colourings and propose to address on other colourings in the subsequent parts. Although there does exist several colouring parameters in literature, its potential for application to the other allied areas are so vast that new parameters are being defined with a specific purpose in mind. We focus our attention in this paper mostly on the three important colouring parameters, namely, the chromatic number ($\chi(G)$) the achromatic number ($\alpha(G)$) and the pseudoachromatic number ($\Psi(G)$). Many heuristic algorithms exist for finding good colourings. Almost all these algorithms are known to have extremely bad worst case behaviour, but empirical tests have shown that some of them do quite well in colouring most graphs on 100 or fewer vertices. Behaviour
of graphs with much more than 100 vertices is largely unexplored, because of the
difficulty in determining the chromatic number of very large graphs.

The chromatic number, of course, is a well-studied parameter, whose history dates
back to the famous four colour conjecture and the early work of Kempe in 1879 [24]
and Heawood in 1890 [21]. The achromatic number was first studied as a parameter
by Harary et al. in 1967 [20] and later named and studied by Harary and Hedetniemi
in 1970 [19]. The definition of pseudoachromatic number of a graph appeared first in
a paper by Gupta in 1969 [16] and later referred by Chartrand and Mitchem in 1971
[5]. Bhave [1] investigated this parameter for certain classes of graphs and obtained a
few results. The author in [37–43] has further probed this parameter.

For a simple graph $G$ with chromatic number $\chi(G)$, Nordhaus–Gaddum inequalities
[27] give upper and lower bounds for $\chi(G)\chi(G^c)$ and $\chi(G) + \chi(G^c)$. Based on a
coloration by Fink [12] of the extremal graphs $G$ attaining the lower bounds for
the product and sum, the author in [37,38] has characterized the extremal graphs $G$
for which $A(G)B(G^c)$ is minimum, where $A$ and $B$ are each of chromatic number,
achromatic number and pseudoachromatic number. Characterizations are also provided
for several cases in which $A(G)+B(G^c)$ is minimum. The problem of investigating the
existence of graphs with prescribed coloring parameters has been studied by many in
the past. The author in [37,43] has obtained a few such existence results and the most
remarkable one is as follows: “For any three positive integers $a, b, c$ with $2 \leq a \leq b \leq c$,
there exists a graph $G$ such that $\chi(G) = a$, $\chi(G^c) = b$ and $\Psi(G) = c$”.

A $k$-colouring of a graph $G$ is a map $f : V(G) \rightarrow \{1, \ldots, k\}$. Consider the following
two conditions: For every edge $uv \in E(G)$, $f(u) \neq f(v)$, and for every $i, j \in \{1, \ldots, k\},
i \neq j$, there exists an edge $uv \in E(G)$ such that $f(u) = i$ and $f(v) = j$. A $k$-colouring is
called a proper $k$-colouring if it satisfies (i), and the same is called a pseudocomplete
$k$-colouring if it satisfies (ii). By a complete $k$-colouring of a graph $G$ we mean a
$k$-colouring which is both proper and pseudocomplete. The chromatic number $\chi(G)$ and
the achromatic number $\chi(G)$ of $G$ are respectively, the smallest and the largest integer
$k$ such that $G$ has a $k$-colouring satisfying both (i) and (ii). The pseudoachromatic number $\Psi(G)$ of $G$ is the largest integer $k$ such that $G$ has a $k$-colouring satisfying
only condition (ii).

2. $\chi$, $\alpha$, $\Psi$ and its computational complexity

**Chromatic Number Problem.**

**Instance:** Graph $G = (V, E)$, positive integer $k \leq |V|$.

**Question:** Is $G$ $k$-colourable, i.e., does there exist a function $f : V \rightarrow \{1, \ldots, k\}$
such that $f(u) \neq f(v)$ whenever $(u, v) \in E$?

**Achromatic Number Problem.**

**Instance:** Graph $G = (V, E)$, positive integer $k \leq |V|$.

**Question:** Does $G$ have achromatic number $k$ or greater, i.e., is there a partition
of $V$ into disjoint sets $V_1, \ldots, V_n$, $k_0 \geq k$, such that each $V_i$ is an independent set for
G and such that for each pair of distinct sets \( V_i, V_j \), \( V_i \cup V_j \) is not an independent set for \( G \)?

**Psuedoachromatic Number Problem.**

**Instance:** Graph \( G = (V, E) \), positive integer \( k \leq |V| \).

**Question:** Does \( G \) have pseudoachromatic number \( k \) or greater, i.e., is there a partition of \( V \) into disjoint sets \( V_1, \ldots, V_k \), \( k \geq k \), such that each \( V_i \) may or may not be an independent set for \( G \) and such that for each pair of distinct sets \( V_i, V_j \), \( V_i \cup V_j \) is not an independent set for \( G \) ?

There is an increasing interest in studying the computational complexity of certain fundamental graph problems because such investigations extend the knowledge about the ‘borderline’ (of course assuming that one such exists) between NP-completeness and solvability in polynomial time.

Obviously, the chromatic number of \( G \) is 1 if and only if \( G \) is \( K_1 \) or \( nK_1 \), where \( n \) is the number of vertices of \( G \) and \( \chi(G) = 2 \) if and only in \( G \) is bipartite. No efficient and convenient procedure is known for finding the chromatic number of any arbitrary graph. According to Cook [6] and Karp [23] there exists a family of problems called NP-complete no member of which is known to have a polynomial time algorithm, but if any of them does have one, then all those have. For proving that a problem \( P \) is NP-complete it is enough to prove that \( P \in NP \) and to show that a known NP-complete problem is reducible to \( P \) in polynomial time. Garey and Johnson [14] and others have already proved that the chromatic number computation is NP-complete. But there are some well-known bounds for this number in the general case and some formulas in a few special cases involving the degrees of \( G \). The maximum and minimum among the degrees of a graph \( G \) are denoted by \( \Delta(G) \) and \( \delta(G) \), respectively.

A set \( A \) of vertices in a graph \( G \) of order \( n \) is an independent set in \( G \) if no two vertices in \( A \) are adjacent in \( G \) and the cardinality of a largest independent set (in \( G \)) is the vertex independence number denoted by \( \beta_0(G) \). A set \( Z \) of vertices in \( G \) is called a clique in \( G \) if there is an edge (in \( G \)) between every pair of vertices in \( Z \) and the cardinality of a largest clique in \( G \) is the clique number of \( G \), denoted by \( \omega(G) \). Obviously, both \( \omega(G) \) and \( n/\beta_0(G) \) are the lower bounds for \( \chi(G) \). Clearly \( 1 \leq \omega \leq \chi \leq n \). Zykov [44] has shown that there is no further relation possible between \( \omega \) and \( \chi \) by constructing for each graph \( G \) a graph \( H \) of order greater than that of \( G \) for which \( \omega(H) = \omega(G) \) and \( \chi(H) = \chi(G) + 1 \). By iteration of that construction, one can obtain a graph for which \( \chi - \omega \) is arbitrarily large.

An obvious colouring procedure to obtain an upper bound for \( \chi(G) \) is by adopting a greedy method known as sequential colouring algorithm: For any ordering \( v_1, \ldots, v_n \) of \( n \) vertices of a graph and any sequence \( c_1, \ldots, c_n \) of \( n \) colours, the colour to be assigned to \( v_i \) is the smallest indexed colour not already assigned to one of its lower-indexed neighbouring vertices. Since each vertex has at most \( \Delta(G) \) neighbouring vertices, we do not need more than \( \Delta(G) + 1 \) colours to colour the vertices of \( G \).

In intuitive terms, we are trying to colour the vertices of \( G \) with as many colours as possible so that no two distinct colour classes can be coalesced into one. We can
therefore consider $\chi(G)$ as a reasonable upper bound on how badly a greedy colouring algorithm can colour the vertices of $G$. Clearly $\chi(G) = 1$ if and only if $G$ is either $K_1$ or $nK_1$, where $n$ is the order of $G$. $\chi(G) = 2$ if and only if each component of $G$ is complete bipartite. For, if each component of $G$ is complete bipartite then $\chi(G) = 2$. For the converse, assume that $\chi(G) = 2$. Then $\chi(G) \leq 2$ and hence $G$ must be bipartite. Further each component of $G$ cannot contain $P_4$ as an induced subgraph since $\chi(P_4) = 3$. Thus, each component of $G$ must be complete bipartite. There is a close relationship between independent sets and edge dominating sets in total graphs. Using this relationship, Yannakakis and Gavril [36] have proved that the achromatic number problem is NP-complete. The idea is briefly outlined as follows:

Given a graph $G$ we can construct the independence graph $S$ of $G$ as follows: We represent each independent set of $G$ by a vertex in $S$ and two vertices of $S$ are joined by an edge if and only if the corresponding independent sets $I_1, I_2$ of $G$ are disjoint and there are two adjacent vertices $v_1, v_2$ of $G$ such that $v_1 \in I_1$ and $v_2 \in I_2$. Now it is easy to see that every achromatic colouring of $G$ corresponds to a maximally connected set of $S$ and conversely. Therefore the smaller are the independent sets of $G$, the easier would be the problem. Assume that $G$ is the complement of a bipartite graph $G_1$. Then, an independent set of $G$ is either a vertex or a pair of vertices. By this observation, it is easy to see that an independence graph $S$ of $G$ is exactly the complement of the total graph $T(G_1)$ of $G$. Thus, the achromatic colourings of $G$ correspond to the maximal independent sets of $T(G_1)$.

Theorem 1 (Yannakakis and Gavril [36]). The independent set problem for the total graphs of bipartite graphs is NP-complete and hence,

Corollary 1.1 (Yannakakis and Gavril [36]). The achromatic number problem is NP-complete even for complement of bipartite graphs.

The psuedoachromatic number $\psi(G)$ of a graph $G$ is 1 if and only if $G$ is either $K_1$ or $nK_1$, where $n$ is the order of $G$. $\psi(G) = 2$ if and only if $G$ is one of the following: $K_2, 2K_2, K_2 \cup (n-2)K_1, 2K_2 \cup (n-4)K_1, P_3, P_3 \cup (n-3)K_1$, where $n$ is the order of $G$. To demonstrate the NP-completeness of the psuedoachromatic number problem we proceed as follows. First we state the partition problem:

3. Partition Problem

Instance: Positive integers $k, n, a_1, \ldots, a_n$ such that $k$ divides $\sum_{i=1}^n a_i$.

Question: Is there a partition $S_1, S_2, \ldots, S_k$ of $\{1, \ldots, n\}$ such that $\sum_{i \in S_j} a_i = \sum_{i \in S_m} a_i$ for each $j, m \in \{1, \ldots, k\}$?

Farber et al. [11] have shown that the exact bipartite achromatic number problem is NP-complete by providing a polynomial transformation between this problem and the partition problem which is already known to be strongly NP-complete [29]. This
observation together with the following lemma establishes that the pseudoachromatic number problem is NP-complete.

**Lemma 1.** If $G$ is a bipartite graph with $n(n-1)/2$ edges, then $\chi(G)=n$ if and only if $\Psi(G)=n$.

**Proof.** Suppose that $\chi(G)=n$. By definition a complete colouring is also a pseudo-complete colouring and hence $\Psi(G)=n$. The reverse implication is immediate since $\Psi(G)=n$. For the converse, let $\mathcal{C}$ be any optimal pseudo-complete colouring with $\Psi(G)=n$. Then for every $i,j \in \{1,\ldots,n\}$ with $i \neq j$, there exists an edge $uv \in E(G)$ such that $\mathcal{C}(u)=i$ and $\mathcal{C}(v)=j$. If $\mathcal{C}$ is not proper, then there exists an edge joining two distinct vertices in the same colour class. But as $|E(G)|=n(n-1)/2$, it follows that $\Psi(G)<n$, a contradiction. Hence $\mathcal{C}$ is proper. Now as $\mathcal{C}$ is both proper and pseudo-complete, it is complete and $\chi(G)=n$. □

4. Partitions

A partition of a nonnegative integer $n$ is a sequence of nonnegative integers $n_1,\ldots,n_k$ where $n_1 \geq n_2 \geq \cdots \geq n_k$ and $n_1+\cdots+n_k=n$. Such a partition is said to be of length $r$. $[n_1^{i_1},\ldots,n_1^{i_j},\ldots,n_k^{i_k}]$ denotes a partition in which $n_j$ occurs $i_j$ times.

The degree sequence of a $(p,q)$ graph, after a possible reordering, forms a partition of $2q$. This partition is of length $p$, and is called the partition of a graph. Only some partitions of an integer $2q$ are partitions of graphs and those which are, usually called graphical. Hakimi [17] has given the necessary and sufficient conditions for a given sequence of positive integers to be graphical.

**The basic theorem.** The necessary and sufficient conditions for positive integers $n_1,\ldots,n_k$ to be graphical are (i) $\sum_{i=1}^{k} n_i = 2q$, where $q$ is an integer, (ii) $\sum_{i=1}^{k-1} n_i \geq n_k$.

In [17], he also considered other related problems, such as “when a set of positive integers is graphical as a connected graph, connected graph without “parallel” elements, separable graph, non-separable graph”, etc.

Associated with the partition $[n_1^{i_1},\ldots,n_1^{i_j},\ldots,n_k^{i_k}]$ is the sequence $i_1,\ldots,i_k$. Properly ordered, this is a partition of the length of the original partition. This new partition may be called the frequency partition of the original partition. This process, applied to a graphical partition, yields a partition of $p$ called the frequency partition of the graph.

**Theorem 2.** A graph and its complement both have the same frequency partition.

**Proof.** Assume that $G$ has frequency partition $i_1,\ldots,i_k$. Because $G$ has exactly $i_j$ vertices of degree $d_j$, its complement has exactly $i_j$ vertices of degree $p-1-d_j$ and thus has the same frequency partition. Since every connected graph must have at least
two vertices of different degrees, it is clear that there is no \( p \)-graph with \( [1^p] \). \( \square \)

**Theorem 3.** Given any partition of an integer \( p \geq 2 \), other than \( [1^p] \) there is at least one connected \( p \)-graph having this partition as its frequency partition.

**Proof.** If \( p = 2 \), the only connected graph of order 2 has two vertices of the same degree.

Assume that the theorem holds for any partition of all non-negative integers not greater than \( p \) and consider a partition of \( p + 1 \) given by \( n_1, \ldots, n_k \) where \( k \geq 2 \), and \( n_1 > 1 \). The integers \( n_1, \ldots, n_{k-1} \) form a partition of \( p + 1 - n_k \) with \( n_k > 1 \) and \( p + 1 - n_k \leq p \). By induction hypothesis, there is a connected graph with this as frequency partition. Add \( n_k \) isolated vertices to this graph. The new graph has the proper frequency partition but it is not connected. Its complement has the same frequency partition (by Theorem 2) and is connected. If \( k = 1 \), the complete \( (p = 1) \)-graph has the frequency partition \( [(p = 1)^1] \). \( \square \)

The achromatic (chromatic) partition cover \( \pi_1 = \pi_1(G) \) \( (\pi_2 = \pi_2(G)) \) is the maximum (minimum) number of vertex disjoint complete graphs which cover the vertices of \( G \).

**Theorem 4.** \( \pi_1 = \chi^C \) and \( \pi_1^C = \chi \).

**Proof.** Consider a proper \( \chi \)-colouring of the vertices of a graph \( G \) having achromatic number \( \chi \). The \( \chi \)-colour classes are sets of independent vertices, so in \( G^C \) there arise \( \chi \) disjoint complete graphs covering the vertices of \( G^C \) and \( \pi_1^C \leq \chi \), by the maximum property of achromatic partition cover. Next consider a decomposition of the vertex set of \( G^C \) into a maximum number \( \pi_1^C \) of disjoint complete graphs. Each of these \( \pi_1^C \) sets of vertices is an independent set in \( G \). If we use these sets as colour classes, we have a \( \pi_1^C \) colouring of \( G \) and \( \chi \leq \pi_1^C \) by the maximum property of the achromatic number. Hence \( \pi_1^C = \chi \) and similarly \( \pi_1 = \chi^C \). \( \square \)

**Corollary 4.1.** \( \pi_2 = \chi^C \) and \( \pi_2^C = \chi \).

**Proof.** On similar lines as in Theorem 4. \( \square \)

5. \( \chi, \alpha, \Psi \) morphisms and partitions

Let \( G \) and \( H \) be graphs with \( |V(H)| \leq |V(G)| \). A homomorphism \( f : G \rightarrow H \) is a map from \( V(G) \) into \( V(H) \) with the property that \( uv \in E(G) \) implies \( f(u)f(v) \in E(H) \) and \( f(u)f(v) \in E(H) \) implies the existence of \( r \in f^{-1}(f(u)) \) and \( s \in f^{-1}(f(v)) \) such that \( rs \in E(G) \). Since loops are not allowed, adjacent vertices of \( G \) cannot have the same image under \( f \). Hence the inverse image \( f^{-1}(u) \) for any \( u \in V(H) \), is an independent
set in $G$ and we get an induced partition of $V(G)$, $V(G) = \bigcup_{u \in V(H)} f^{-1}(u)$. Now for any $v, w \in V(H)$, there exist $r \in f^{-1}(v)$ and $s \in f^{-1}(w)$ such that $rs \in E(G)$ if and only if $vw \in E(H)$. In case $H = K_n$ write the induced partition as $V(G) = \bigcup_{i=1}^{n} V_i$ and we call $V_i$ the classes of the particular partition being fixed by the context. If $f : G \to H$ is a homomorphism and if $f(G) = H$, then $f$ is called an epimorphism. An isomorphism is a bijective homomorphism whose inverse is also a homomorphism. An $n$-colouring of $G$ corresponds to a homomorphism of $G$ to $K_n$, since the preimage of each vertex in $K_n$ is an independent set in $G$. Also, a complete $n$-colouring of $G$ corresponds to an epimorphism of $G$ to $K_n$. Thus $\chi(G) = \min \{ n : f : G \to K_n \text{ is a homomorphism} \} = \min \{ n : f : G \to K_n \text{ is an epimorphism} \}$ and $\chi(G) = \max \{ n : f : G \to K_n \text{ is an epimorphism} \}$. $f : G \to H$ is a homomorphism implies $\chi(G) \leq \chi(H)$. Since for $g : H \to K_n$ is a homomorphism, we have $g \circ f : G \to K_n$ is a homomorphism. Similarly, $f : G \to H$ is an epimorphism implies $\chi(G) \geq \chi(H)$. An edge-monomorphism of $G$ to $H$ is a homomorphism of $G$ to $H$ which is injective on edges, i.e., if $vw \neq u'v'$, then $f(u)f(v) \neq f(u')f(v')$. If $G$ has exactly $n(n-1)/2$ edges then a complete $n$-colouring of $G$ corresponds to an edge-monomorphism of $G$ to $K_n$. So a graph $G$ is eulerian if and only if it is a homomorphic image of a cycle under some edge-monomorphism.

**Theorem 5.** If $G$ is an Eulerian graph with $n(n-1)/2$ edges, where $n$ is even then $\chi(G) < n$.

**Proof.** Suppose $G$ has $n(n-1)/2$ edges and $\chi(G) = n$. Then $K_n$ is a homomorphic image of $G$ under some edge-monomorphism. Clearly $K_n$ is Eulerian. But we know that $K_n$ is Eulerian if and only if $n$ is odd. So we conclude that either $n$ is odd or $G$ is not Eulerian. $\square$

A congruence on $G$ is an equivalence relation on $V(G)$ whose classes are independent sets. Every homomorphism $f : G \to H$ induces a congruence on $G$, namely the equivalence relation with classes $f^{-1}(u)$ for $u \in V(H)$. Conversely, for every congruence $\sim$ on $G$ we can define the quotient graph $G/\sim$, whose vertices correspond to the classes of $\sim$, two vertices being adjacent if $G$ has at least one edge joining the corresponding classes. That is, there is a canonical epimorphism $f : G \to G/\sim$ taking each vertex to the class that contains it. An epimorphism $e : G \to H$ is called an elementary homomorphism if it induces a congruence which has one class containing two elements and the remaining classes each contain one vertex. So we can view the elementary homomorphism as the identification of two non-adjacent vertices. Clearly, any epimorphism is a composition of elementary homomorphisms.

The idea of partitions in terms of set-theory terminology is as follows. A collection $\mathcal{P} = \{ V_1, \ldots, V_n \}$ of non-empty subsets of a non-empty set $V$ is a partition of $V$ if (i) $V = \bigcup_{i=1}^{n} V_i$ and (ii) $V_i \cap V_j = \emptyset$ for $i \neq j$.

Let $\mathcal{P}$ be a partition of $V(G)$ of a graph $G$. The partition graph $\mathcal{P}(G)$ of $G$ is the graph with vertex set $\mathcal{P}$ where $V_i$ and $V_j$ are adjacent if and only if there exist $v_i \in V_i$ and $v_j \in V_j$ such that $v_iv_j$ is an edge in $G$. A partition $\mathcal{P}$ of $V(G)$ is complete
if $\mathcal{P}(G)$ is a complete graph. Corresponding to a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of $V(G)$ of $G$, consider the partition graph $\mathcal{P}(G)$. If $\{V_1, \ldots, V_r\}$ is the collection of all $V_i \in \mathcal{P}$ with $|V_i| > 1$, then $r$ is called the order of the partition $\mathcal{P}$ and the norm of $\mathcal{P}$, denoted $\|\mathcal{P}\| := \sum_{i=1}^{n} |V_i|$. If $r = 1$, then $\mathcal{P}$ is an elementary partition. A partition graph $\mathcal{P}(G)$ is a homomorphic image or a contraction of $G$ according as every set in $\mathcal{P}$ is an independent set in $G$ or induces a complete subgraph of $G$, respectively. When this is the case then we call $\mathcal{P}$ itself a homomorphism or a contraction accordingly. It is easy to see that an elementary homomorphism of $G$ is an elementary partition of norm two. Note that for any partition $\mathcal{P}$ of $V(G)$, $\Psi(\mathcal{P}(G)) \leq \Psi(G)$. In particular, if $\mathcal{P}$ is any homomorphism or a contraction, then $\alpha(\mathcal{P}(G)) \leq \alpha(G)$. □

For a graph $G$, the edge independence number $\beta_1(G)$ is the largest number of edges in an independent set. A vertex and an edge cover each other if they are incident. The vertex covering number $\alpha_0(G)$ is the smallest number of vertices which cover all the edges of $G$. The edge covering number $\alpha_1(G)$ is the smallest number of edges which cover all the vertices of $G$.

**Theorem 6.** For any graph $G$ with $n$ vertices $\Psi(G) \leq n - \beta_0 + 1$.

**Proof.** Let $\Psi = r$. Clearly there exists a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of $V(G)$ such that $\mathcal{P}(G) = K_r$. Let $S$ be a set of $\beta_0$ independent vertices of $G$. Since any two $V_i, V_j$, $i \neq j$ are adjacent in $\mathcal{P}(G)$, it can be seen that $V_i \cup V_j$ is not contained in $S$ of all $i \neq j$. That is at least $(r - 1)$ of the sets in $\mathcal{P}$ intersect $V(G) - S$. Thus $r - 1 \leq |V(G) - S| = n - \beta_0$. Hence $\Psi \leq n - \beta_0 + 1$. □

**Corollary 6.1.** $\chi(G) \leq \alpha(G) \leq n - \beta_0 + 1$.

**Proof.** It follows from Theorem 6, since $\chi(G) \leq \alpha(G) \leq \Psi(G)$. □

**Corollary 6.2.** $\Psi(G) \leq \alpha_0 + 1$.

**Proof.** This follows easily from Theorem 6, once we recall the known equality $\alpha_0 + \beta_0 = \alpha_1 + \beta_1 = n$ as given by Gallai [13]. □

**Theorem 7.** For any graph $G$, $\Psi(G) \leq 2\beta_1(G) + 1$ and $\Psi(G) \leq 2\alpha_1(G)$.

**Proof.** If $n = \Psi$, then there exists a partition $\mathcal{P}$ of $V(G)$ such that $\mathcal{P}(G) = K_n$. Now $\beta_1(K_n) = n/2$ or $(n - 1)/2$ according as $n$ is even or odd. Hence $n = 2\beta_1(K_n) + 1 = 2\beta_1(\mathcal{P}(G)) + 1 \leq 2\beta_1(G) + 1$ since $\beta_1(\mathcal{P}(G)) \leq \beta_1(G)$. Similarly as $\alpha_1(K_n) = n/2$ or $(n + 1)/2$ according as $n$ is even or odd, we have $n = 2\alpha_1(K_n) = 2\alpha_1(\mathcal{P}(G)) \leq 2\alpha_1(G)$, since $\alpha_1(\mathcal{P}(G)) \leq \alpha_1(G)$. □

**Corollary 7.1.** For any graph $G$, $\chi(G) \leq \alpha(G) \leq 2\beta_1(G) + 1$ and $\chi(G) \leq \alpha(G) \leq 2\alpha_1(G)$.  

Theorems 6 and 7 and its corollaries were obtained by Harary and Hedetniemi [19] for achromatic number $\chi(G)$ using the concept of homomorphism. We caution the readers that they have denoted the achromatic number of a graph $G$ by $FVT(G)$ and the pseudoachromatic number of a graph are not widely known to them.

6. $\chi$ and projective plane geometry

A projective plane geometry $\pi$ is a mathematical system composed of undefined elements called points and undefined sets of points called lines (at least two in number) subject to the following three postulates. 

P$_1$: Two distinct points are contained in a unique line.

P$_2$: Two distinct lines contain at least three points.

P$_3$: Each line contains at least three points.

The projective plane $\pi$ is finite if it consists of a finite number of points. If $\pi$ is finite, then there exists a positive integer $N$ such that each line of $\pi$ contains exactly $N + 1$ distinct points, and each point is contained in exactly $N + 1$ distinct lines. Also $\pi$ has exactly $N^2 + N + 1$ distinct points and $N^2 + N + 1$ distinct lines (see [18,25,35]).

In all known finite geometries the integer $N$ is a power of a prime, that is, for every prime $p$ and for every positive integer $n$ finite geometries with $N = p^n$ have been constructed by means of the Galois fields $GF(p^n)$ [34]. It is still an unsettled question whether or not $N$ must be a power of a prime.

The non-existence of finite projective geometries for some typical $N$’s is seen from the following.

**Theorem 8** (Bruck and Ryser [4]). If $N \equiv 1$ or $2 \ (mod\ 4)$ and if the square-free part of $N$ contains at least one prime factor of the form $4k + 3$, then there does not exist a finite projective plane geometry with $N + 1$ points on a line.

A Latin square of order $N$ may be defined as an arrangement of $N$ symbols, say, 1,2,...,N in a $N \times N$ square such that each symbol occurs exactly once in every row and once in every column. Two Latin squares are said to be orthogonal if, when they are superposed, each symbol of the first square occurs just once with each symbol of the second square. A set of mutually orthogonal Latin squares is a set of Latin squares any two of which are orthogonal.

According to Bose [2] and Mann [26] from a given complete set of mutually orthogonal Latin squares of order $N \geq 3$, a finite plane with $(N + 1)$ points on a line can always be constructed, and hence for those $N$ of Theorem 8, there does not exist a complete set of orthogonal Latin squares of order $N$.

Further in 1782 itself, Euler has conjectured that a pair of orthogonal Latin squares of order $N$ cannot exist if $N$ has the form $4k + 2$. So the truth of Euler’s conjecture would ensure the non-existence of finite projective planes with $N \equiv 2\ (mod\ 4)$. 
The importance of the study of the achronatic number of a graph is seen from the following remarkable observation which has a direct bearing on the existence of finite projective planes of a particular order. This aspect makes the study of the achronatic number both interesting and difficult.

**Theorem 9 (Bouchet [3]).** Suppose \( q \) is odd and \( n = q^2 + q + 1 \), then \( \chi(L(K_n)) = qn \) if and only if a finite projective plane of orders \( q \) exists, where \( L(K_n) \) denotes the line-graph of \( K_n \).

If \( \Gamma \) is any colour class in the achronatic edge coloring of \( K_n \), the vertices covered by the edges in \( \Gamma \) is called the support of \( \Gamma \). By the Theorem 9, if \( \chi(L(K_n)) = qn \), then the supports of the colour classes in any optimal coloring, form the lines of the projective plane with the vertices of \( K_n \) as points. For more on this topic, an excellent reference is Jamison [22].

### 7. \( \chi \), \( \alpha \), \( \Psi \) and perfect graphs

For any graph \( G \), it is clear that \( \omega(G) \leq \chi(G) \leq \alpha(G) \leq \Psi(G) \). For \( \alpha \) and \( \beta \) distinct members of \( \{ \omega, \chi, \alpha, \Psi \} \), a graph is called \( \alpha \beta \)-perfect if for each induced subgraph \( H \) of \( G \), \( \alpha(H) = \beta(H) \). A \( \omega \)-perfect graph is often called a perfect graph. The smallest cycle for which the above four parameters are not equal is the graph \( C_{11} \). It is easy to see that \( \omega(C_{11}) = 2 \), \( \chi(C_{11}) = 3 \), \( \alpha(C_{11}) = 4 \) and \( \Psi(C_{11}) = 5 \).

By definition, there is no complete \( n \)-colouring of a graph \( G \) for \( n < \chi(G) \) or \( n > \alpha(G) \). Harary, Hedetniemi and Prins [20] proved that for each \( G \) and each \( n \) with \( \chi(G) \leq n \leq \alpha(G) \), there is a complete \( n \)-colouring of \( G \).

**Theorem 10.** For any finite graph \( G \) the following are equivalent. (1) \( G \) is \( \Psi \omega \)-perfect, (2) \( G \) is \( \Psi \chi \)-perfect, (3) \( G \) is \( \Psi \alpha \)-perfect, (4) \( G \) does not contain an induced subgraph isomorphic to \( C_4 \).

**Proof.** (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (3) are trivial, because for each \( G \), \( \omega(G) \leq \chi(G) \leq \alpha(G) \leq \omega(G) \). (3) \( \Rightarrow \) (4) \( \alpha(C_4) = 2 \) but \( \Psi(C_4) = 3 \). Hence, if \( G \) is \( \Psi \alpha \)-perfect it does not contain an induced subgraph isomorphic to \( C_4 \). (4) \( \Rightarrow \) (1) Suppose that \( G \) does not contain an induced subgraph isomorphic to \( C_4 \). Let \( \mathcal{E} \) be any optimal pseudocomplete \( \Psi \)-colouring of \( G \). Since \( G \) is finite, the maximum number of colours in \( \mathcal{E} \) exists and obviously it is finite. If \( \max |\mathcal{E}| = 1 \), then there is nothing to prove and \( G \) is \( \Psi \omega \)-perfect. Suppose that for some \( (m-1) \) with \( m \in \mathbb{Z}^+ \), \( G \) contains a complete subgraph on \( (m-1) \) vertices \( v_1, \ldots, v_{m-1} \) with \( (m-1) \) distinct colours. If \( \max |\mathcal{E}| = \Psi = m-1 \), then we are through. Otherwise we construct a complete subgraph of size more than \( (m-1) \) as follows. Let \( S_i \) denote the set of vertices coloured \( C^* \notin \{ C_1, \ldots, C_{m-1} \} \) adjacent to \( v_i \), \( 1 \leq i \leq m-1 \). We claim that there exists no \( i, j \) with \( 1 \leq i, j \leq m-1 \) and \( i \neq j \) such that the cardinality of the set of edges between \( S_i \) and \( S_j \) is non-empty. Otherwise, for some \( u_i \in S_i \) and
which is adjacent to in $G$ are adjacent to $G$. Suppose the component which induces be the vertex of $FSO$.

Proof. Let $u,v$ be distinct vertices of $G$. For each pair of distinct vertices $u,v$ in $K_{\omega}$, let $H_{uv}$ consist of those vertices $x$ such that $x$, is adjacent to neither $u$ nor $v$. It is easy to see that $H_{uv}$ is independent, otherwise, for $u,v \in K_{\omega}$ with $u \neq v$ an edge in $H_{uv}$ together with the edge $uv$, would produce a $2K_2$ in $G$, a

**Corollary 10.1.** Every $\Psi \chi$-perfect graph is $\chi \chi$-perfect and every $\chi \chi$-perfect graph is $\chi_0 \omega$-perfect.

**8. Some bounds for $\chi$ and $\Psi$**

We give here an upper bound for $\chi$ and $\Psi$ in terms of $\omega(G)$, the size of the largest complete subgraph of $G$, where $G$ is marked by the property that it contains no $2K_2$ as an induced subgraph. First we give the following lemma.

**Lemma 2.** If $G$ is connected and simple with $\delta(G) \geq 2$, then there exists adjacent vertices $u,v \in V(G)$ such that $G - u, G - v, G - \{u,v\}$ are all connected.

**Proof.** Let $P$ be a longest path in $G$, and let $u$ be any end vertex of $P$. Further let $v$ be the vertex of $P$ adjacent to $u$ in $P$. The maximality of $|V(P)|$ implies that $G - u$ is connected. Suppose $G - \{u,v\}$ is disconnected, then it has a component $G^*$ other than the component which induces $P - \{u,v\}$. Since $G$ is connected, there is a vertex $w$ of $G^*$ which is adjacent to either $u$ or $v$. The maximality of $|V(P)|$ ensures that $w$ is not adjacent to $u$ and so $w$ is adjacent to $v$. Now all the vertices of $G$ other than $v$ which are adjacent to $w$ lie in $G^*$ and since $\delta(G) \geq 2$, there is a vertex $x \in V(G^*) - \{w\}$ which is adjacent to $v$. It follows that $\langle V(P) - \{u\} \cup \{w,x\} \rangle$ is the vertex set of a path in $G$ which is longer than $P$, a contradiction. Hence $G - \{u,v\}$ is connected. Since $\delta(G) \geq 2$ it follows that $u$ is adjacent to at least one vertex of $G - \{u,v\}$ and so $G - v$ is also connected. □

We make use of this Lemma 2 invariably in the following theorem.

**Theorem 11.** If a graph $G$ contains no $2K_2$ as an induced subgraph, then $\chi(G) \leq (\omega(G)(\omega(G) - 1)/2) + f(\omega)$, where $f(\omega)$ is some linear function of $\omega$ with $f(\omega) \geq \omega$.

**Proof.** Let $\omega = \omega(G)$ and let $K_\omega$ be the complete subgraph of $G$. For each pair of distinct vertices $u,v$ in $K_\omega$, let $H_{uv}$ consist of those vertices $x$ such that $x$, is adjacent to neither $u$ nor $v$. It is easy to see that $H_{uv}$ is independent, otherwise, for $u,v \in K_\omega$ with $u \neq v$ an edge in $H_{uv}$ together with the edge $uv$, would produce a $2K_2$ in $G$, a
contradiction. Now let \( H = \bigcup H_w \). Then as there may or may not be edges between \( K_o \) and \( H \), we conclude that \( \chi(H) \leq \omega(\omega - 1)/2 \). Consider \( K_o \cup H \). We claim that a vertex \( x \) not in \( K_o \cup H \) is adjacent to all but one vertex of \( K_o \). For, if \( x \) is not adjacent to two or more vertices in \( K_o \) then \( x \) is in \( H \) and again \( x \) adjacent to all the vertices in \( K_o \) would mean that \( K_o \cup \{x\} \) is a complete subgraph of size \( \omega + 1 \). We observe finally that for each vertex \( u \in K_o \), let \( I_u \) consist of those vertices not in \( H \) which are not adjacent to \( u \). Then \( \{u\} \cup I_u \) is an independent set. This is so, because if \( v, r \) are adjacent vertices in \( I_u \), then \( \{v, r\} \cup K_o - \{u\} \) is a complete subgraph of size \( \omega + 1 \). These facts amounts to the fact that depending upon the number of edges between \( K_o \) and those set of vertices not in \( H \) and between \( K_o \) and \( H \). The vertices not in \( K_o \cup H \) may be coloured with \( f(\omega) \) colours, where \( f(\omega) \geq \omega \) is some linear function of \( \omega \). Thus it follows that \( \chi(G) \leq \omega(\omega - 1)/2 + f(\omega) \). \( \square \)

**Corollary 11.1.** If a graph \( G \) contains no \( 2K_2 \) as induced subgraph, then \( \Psi(G) \leq \omega (\omega - 1)/2 + f(\omega) \), where \( f(\omega) \) is some linear function of \( \omega \) with \( f(\omega) \geq \omega \).

**Observation.** If \( G \) is a graph with \( q \) edges, then \( \chi(G) \leq \Psi(G) \leq r \), where \( r \) is the maximum integer with \( r(r - 1)/2 \leq q \). For, let \( P(G) = K_n \), where \( n = \Psi(G) \). Then it is easy to see that \( G \) has at least \( n(n - 1)/2 \) edges, i.e., \( n(n - 1)/2 \leq q \). If \( G = qK_2 \), where \( q = r(r - 1)/2 \) then \( \Psi(G) = r \). This shows that the bound is attained.

### 9. Representation of graphs by integers

Trying to describe graphs by integers we may ask: Is there a set \( A \) of integers with the following property? The vertices of each finite graph can be labelled by different numbers such that two arbitrary vertices are adjacent if and only if the difference of their labels belongs to \( A \)? Schnabel and Kiel [31] have shown that such a set \( A \) exist by using the theory of group graphs.

Let \( H \) be a group and let \( A \) be a subset of \( H \) such that \( 1 \notin A, A^{-1} \subseteq A \). The relation \( \rho_A \) defined by \( x \rho_A y, \) if \( xy^{-1} \in A \) for all \( x, y \in H \). Clearly \( \rho \) is an irreflexive symmetric relation and \( (H, \rho_A) \) is called a group graph. We know that every graph can be extended to a complete graph and since every complete graph is a group graph, it follows that every graph is embeddable in a group graph.

**Theorem 12** (Schnabel [31]). Every finite graph is isomorphic to a group subgraph on the group \( Z \) of integers.

**Theorem 13** (Schnabel [31]). There is a group graph on \( Z \) which contains all finite graphs as induced subgraphs up to isomorphism.

Let \( N_n \subseteq Z^+ \), be any set of positive integers denoted elaborately by \( N_n = \{1, 2, \ldots, n\} \) for \( n \in Z^+ \). Let \( D_m \subseteq Z^+ \) be any other set with \( D_m = \{1, 2, \ldots, m\} \) and \( |N_n| \leq |D_m| \). Form
the graph $G(N_n,D_m)$, whose vertex set is $N_n$ and the edge set $E(G) = \{(x,y) \mid |x-y| \in D_m \ \forall x,y \in N_n\}$. The set $D_m$ is called the distance set of the graph. Here we consider the problem of determining the achromatic number and pseudoachromatic number of $G(N_n,D_m)$ and compute them in this paper when $m=1$ and propose to take the rest in the subsequent parts. It is easy to observe that $G(N_n,D_1) \cong P_n$.

**Theorem 14** (Geller and Kronk [15]). Let $P_n$ be the path on $n$ vertices. Find the largest $k$ such that $n \geq (k(k-1)/2) + 1$ if $k$ is odd, or $n \geq (k^2/2)$ if $k$ is even. Then $\chi(P_n) = k$.

The pseudoachromatic number of $G(N_n,D_1)$ is computed as follows:

**Theorem 15.** Let $P_n$ be the path on $n$ vertices. Then $\Psi(P_n) = 2k - 1$ or $2k$ or $2k + 1$ as $(2k - 1)k + 1 \leq n \leq k(2k - 1) + (k - 1)$ where $k > 1$; $k(2k - 1) + k \leq n \leq k(2k + 1)$ where $k \geq 1$; $k(2k + 1) + 1 \leq n \leq (k + 1)(2k + 1)$ where $k \geq 1$.

**Proof.** Let $\Psi = \Psi(P_n)$. As $\Psi$ is a pseudocomplete colouring, $(\Psi(\Psi - 1)/2) \leq n - 1$ and as for any $n$, there exists a unique $k$ with $k(2k - 1) + 1 \leq n \leq k(2k + 1)$, it can be easily be seen that $\Psi(P_n) \leq 2k$ if $k(2k - 1) + 1 \leq n \leq k(2k + 1)$ and $\Psi(P_n) \leq 2k + 1$ if $k(2k + 1) + 1 \leq n \leq (k + 1)(2k + 1)$.

Case 1: $k(2k - 1) + 1 \leq n \leq k(2k + 1)$. Clearly $\Psi(P_n) \leq 2k$. Suppose that $\Psi(P_n) = 2k$ and $C$ is an optimal pseudocomplete colouring of $P_n$, with colour classes $V_1, V_2, \ldots, V_{2k}$, where for $1 \leq i \leq 2k$, $V_i$ is the set of all vertices receiving the colour $c_i$. Obtain a new graph $G^*$ with vertex set $\{V_1, \ldots, V_{2k}\}$. The edge set $E(G^*)$ is obtained by introducing $s_{ij}$ edges joining $V_i$ and $V_j$, where $s_{ij} = \|V_i, V_j\|$, the number of edges of $P_n$ having one end in $V_i$ and the other end in $V_j$. As $\Psi$ is pseudocomplete, $s_{ij} \geq 1$ for each $i$ and $j$, $i \neq j$. Further, introduce $t$ loops at $v_i$ if and only if if the subgraph induced by $V_i$ contains $t$ edges. Clearly $G^*$ contains an Eulerian trail. An Eulerian trail of $G^*$ can be obtained by replacing each vertex of $V_i$ in $P_n$ by $v_i$ and reading the vertices of $G^*$ in the order of $P_n$. Here the underlying simple graph of $G^*$ is $K_{2k}$, and to obtain a spanning supergraph with Eulerian trail from $K_{2k}$, one has to add at least $(k - 1)$ new edges. Hence $(n - 1) \geq k(2k - 1) + (k - 1)$, a contradiction. Thus $\Psi(P_n) \leq 2k - 1$. To obtain equality we have to exhibit a pseudocomplete $(2k - 1)$-colouring for $P_n$. For this, consider $K_{2(k-1)} + e$ where $e$ is any edge of $K_{2(k-1)}$, and label the vertices of it by $v_1, \ldots, v_{2(k-1)}$. Let $T$ be any Eulerian trail of $K_{2(k-1)} + e$. Now colour the $i$th vertex of $P_{(k-1)(2k-1) + 2}$ by the $i$th vertex of $T$. This yields a pseudocomplete colouring for $P_{(k-1)(2k-1) + 2}$ with $(2k - 1)$ colours. Now this colouring of $P_{(k-1)(2k-1)+2}$ can be extended to a pseudocomplete $(2k - 1)$-colouring of $P_n$ by subdividing an edge $e = (u,v)$ of $P_{(k-1)(2k-1)+1}$ $n - (k - 1)(2k - 1)$ times and assigning to each new vertex either the colour of $u$ or the colour of $v$. Hence $\Psi(P_n) = 2k - 1$.

Case 2: $k(2k - 1) + k \leq n \leq (k + 1)(2k + 1)$. Here, once again $\Psi(P_n) \leq 2k$. To establish equality, consider a perfect matching $F$ of $K_{2k}$ and obtain a new graph $K_{2k}^*$ from $K_{2k}$...
by duplicating the edges of $F$. Clearly $K_{2k}^*$ is $2k$-regular and hence it is Eulerian. Let $T$ be any Euler tour of $K_{2k}^*$. Then $K_{2k}^* - e$ contains an Eulerian trail for an edge $e \in F$, a perfect matching of $K_{2k}$. Now proceed as in case 1 with $G^* = K_{2k}^* - e$.

Case 3: $k(2k + 1) + 1 \leq n \leq (k + 1)(2k + 1)$. In this case, consider the graph $K_{2k+1}$ and proceed as in case 2 with the assumption that the origin and terminus of $P_n$ share the same colour.

We conclude this part with some open problems:

What is the least number of colours which can be used to colour all the points of the euclidean plane so that no two points which are unit distance apart have the same colour? This problem has resisted all attempts at solution. The corresponding problem for the real line is easy: There are various ways in which two colours can be used to colour the real line so that points which are unit distance apart have different colours. But how many colours are needed to avoid assigning the same colour to points whose distance apart lies between $1 - \varepsilon$ and $1 + \varepsilon$ for some $0 < \varepsilon < 1$? Eggleton et al. [7–9] have discussed this and other related problems. More specifically. Given any set $D$ of positive real numbers, let $G(R, D)$ denote the graph whose vertices are all the points of the real line $R^1$, such that any two points $x, y$ are adjacent if and only if $|x - y| \in D$. They have treated the problem of determining the chromatic number of $G(R, D)$ when (1) $D$ is a closed interval of distances, (2) $D$ is an open interval of distances, (3) $D$ is the union of infinitely many closed intervals, (4) $D$ is a finite set of distances and (5) $D$ is a set of prime distances. For a variety of problems related to distances in graphs one can refer to [10, 30, 32, 33].

Let $G_0$ denote the graph on $Z \times Z$ which can be obtained by joining two integer vertices with an edge if and only if their Euclidean distance is 1. This intuition is due to the observation that if a straight line $pq$ is almost parallel to the $x$-axis or to the $y$-axis, then the graph distance in $G_0$ and the Euclidean distance are very close to each other, even though $\sup((d_G(p, q))/(d(p, q))) = \sqrt{2}$. Pach et al. [28] have proved that for every $\varepsilon > 0$ there is a graph $G = G_{\varepsilon}$ and a constant $d = d_\varepsilon$ such that $|d_G(p, q) - d(p, q)| < \varepsilon d(p, q)$ for every pair $p, q \in V$ with $d(p, q) \geq d$.

Problem. For every $\varepsilon > 0$, does there exist a graph $G$ and a suitable constant $k$ which satisfies $|d_G(p, q) - d(p, q)| < k$ for all $p, q \in Z \times Z$.

References


[42] V. Yegnanarayanan, On some extremal graph problems of the Nordhaus–Gaddum class, submitted for publication.
