The Method of Fundamental Solutions
– A Meshless Method

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Dynamic Publishers, Inc.
Atlanta
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©Dynamic Publishers, Inc. 2008
P.O.Box 48654
Atlanta, GA 30362, USA.

ISBN 1890888-04-4

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Preface

The first international workshop devoted entirely to the Method of Fundamental Solutions (MFS) was held in the resort town of Ayia Napa, Cyprus from 11-13 June 2007. The ideas behind the MFS have been around for many years and were developed primarily by V. D. Kupradze and M. A. Alexidze in the late 1950s and early 1960s. However, the method was proposed as a computational technique much later by R. Mathon and R. L. Johnston in the late 1970s. A number of papers by Mathon, Johnston and Graeme Fairweather with applications followed, and slowly but surely the MFS became a useful tool for the solution of a large variety of physical and engineering problems. A major obstacle was overcome when, in the 1990s, M. A. Golberg and C. S. Chen extended the method to deal with inhomogeneous equations and time-dependent problems. In addition, recent developments indicate that the MFS can also be used to solve partial differential equations with variable coefficients. The MFS has proved particularly effective for certain classes of problems such as inverse problems and free-boundary problems. The method today is applied to disciplines as varied as computer-aided design and biomedical engineering. The MFS is known by many different names in the literature. Among these are the charge simulation method, the superposition method, the desingularized method, the indirect boundary element method and the virtual boundary element method, to name a few.

The great advantage of the MFS over other methods is the ease with which it can be implemented for problems in three-dimensional and irregular domains. This factor is very important to people who actually compute, but unfortunately it is often overlooked. The MFS has become more widely used in the last decade because of the popularity gained by meshless methods in general. Clearly, there are still many limitations in the range of applications of the method as well as in its theoretical foundations and development. We felt, however, that the MFS had sufficiently matured as a numerical technique to deserve a meeting devoted entirely to
The purpose of this book is to present some of the recent significant developments in the MFS which were presented at the Ayia Napa meeting.

The international workshop, entitled MFS 2007, was attended by thirty-two participants from thirteen countries from as far away as the USA and Taiwan. Among all participants, there was an even mix of mathematicians and engineers. Thus, the MFS 2007 provided an excellent platform to bridge the gap between theory and practice.

The programme included three plenary talks which were delivered by Graeme Fairweather (Colorado School of Mines, USA), Alex H.-D. Cheng (University of Mississippi, USA), and R. Schaback (Universität Göttingen, Germany). We would like to take this opportunity to thank all participants for their contributions. We would also like to thank our sponsors, namely the University of Cyprus, the University of Southern Mississippi, the Electricity Authority of Cyprus, and the Cyprus Tourism Organization for their generous support.

After the end of the workshop, we invited all participants to contribute papers to be considered for publication as chapters in this book. After a careful refereeing process, fourteen papers were selected. In this selection, we tried to maintain a balance among theoretical analysis, numerical algorithms, and engineering applications of the MFS.

The first part of this book is focused on the theoretical analysis and the development of new numerical algorithms. In the MFS, one of the outstanding problems is how to choose the locations of the source points. In Chapter 1 Schaback contributes to this question by developing a greedy adaptive technique for the automatic choice of suitable source points. In Chapter 2 Li provides an improved error bound for solving the Laplace equation in a bounded simply connected domain. In Chapter 3 Alves and Martin discuss the application of the MFS to an inverse potential problem that consists of detecting inclusions or cavities using a single boundary measurement on an external boundary. They also develop an iterative MFS approach to inverse problems. In Chapter 4 Chen, Fan, and Monroe extend the MFS to solving general elliptic PDEs with variable coefficients. Despite its effectiveness, the MFS has so far been restricted to a small class of PDEs with constant coefficients. The new developments presented in this chapter are expected to generate more interest for solving a large class of science and engineering problems using the MFS. In Chapter 5 Karageorghis, Mogilevskaya, and Stolarski develop an efficient complex MFS.
algorithm for the solution of 2D potential and linear elasticity problems. In Chapter 6 Karageorghis and Lesnic propose an MFS approach for the solution of steady-state nonlinear heat conduction problems. In Chapter 7 Gáspár presents a regularization technique for the MFS. The computational problems caused by the singularity of the fundamental solution can be avoided by introducing the fundamental solution of a higher order auxiliary problem. As a result, the use of a large, dense, and ill-conditioned matrix can be avoided.

The second part of this book provides a collection of papers on engineering applications which include fluid dynamics, solid mechanics, and wave propagation. In Chapter 8 Uscilowska uses the MFS for solving axisymmetric isothermal gas flow in a porous medium. Picard iteration is used to treat the nonlinearity. In Chapter 9 Young, Chen, and Fan propose to combine the MFS and an Eulerian-Lagrangian method to solve the unsteady nonlinear Navier-Stokes equations for low Reynolds number flows with a moving rigid body. In Chapter 10 Kolodziej and Klekiel apply the MFS and radial basis functions for solving the fully developed laminar flow of non-Newtonian fluids inside ducts with constant but arbitrary cross-sections. Picard iteration is also used to tackle the non-linear governing equation. In Chapter 11 Wen, Chen, and Liu investigate the application of the MFS in the shear deformable plate problem under either static or dynamic loads. Durbin's inversion method for Laplace transform is employed for dynamic loads. In Chapter 12 Fujisaki focuses on the accuracy of the MFS by calculating the maximum stress of a circle and an elliptical hole under internal pressure. To verify the effectiveness of the MFS, the effect of a point load and its shape are investigated. In Chapter 13 Mohareb, Rashed, and Akl propose a new MFS approach using the concept of dipoles for Reissner's plate theory. In Chapter 14 Gudinho, Tadeu, and Mendes study the applicability of the MFS for the computation of the rotational motion generated by a line load in an elastic medium. In Chapter 15 Särler extends the desingularised modified method of fundamental solutions to solution of potential flow problems. In Chapter 16 Rodriguez et al proposed to use the MFS to analyze micro strip patch antennas of arbitrary shape.

This first MFS workshop was dedicated to Professor Graeme Fairweather, in honor of his 65th birthday and in recognition of his pioneering work in the development of the method.
In 1965 Graeme Fairweather was appointed Lecturer of Applied Mathematics at the University of St Andrews where he remained until 1969. Then he joined the faculty of the Department of Mathematics at Rice University in Houston as an Assistant Professor. In 1971 he moved to the University of Kentucky where he remained until 1994. While at Kentucky he also served as Acting Director and Associate Director at the Center for Computational Sciences of the University. Since 1994 he has been the head of the Department of Mathematical and Computer Sciences of the Colorado School of Mines. Graeme has always been a keen traveler and has held visiting appointments at a number of places. He is the author of one book and over 100 journal articles on numerical analysis and scientific computing. He is also the recipient of numerous NSF grants.

Graeme’s earlier contributions were primarily in the numerical solution of partial differential equations by finite differences. Later he also worked on finite element Galerkin methods for the same class of problems. This interest resulted in the publication of his book Finite Element Galerkin Methods for Differential Equations published by Marcel Dekker in 1978. From his work on finite difference and finite element methods resulted interesting contributions in other areas of numerical analysis such as numerical integration and numerical linear algebra. While at Kentucky, Graeme also collaborated with Frank Rizzo and David Shippy on the boundary integral equation method. While on sabbatical in Toronto, Graeme collaborated with Laurie Johnston on a new idea which was to use approximations with linear combinations of fundamental solutions. Laurie had worked on it previously with R. Mathon and the idea appeared promising. Their collaboration led to Graeme’s first two papers on the MFS in the early 80’s. Graeme remained interested in the method and has since been instrumental in its development. In recent years, he has also worked on the numerical solution of partial differential equations with more conventional methods, notably on spline collocation methods with B. Bialecki and A. Karageorghis.

On behalf of the MFS community, we thank Graeme Fairweather for the inspiration he has offered to us all and are looking forward to more of his contributions in all areas of numerical mathematics.

The production of this book required the help of a number of individuals whose efforts we would like to acknowledge. We first thank all the authors whose work has made this volume possible. We would also like to
thank all the referees who contributed their time to review all these papers. Finally, we are grateful to Dr. John Dudley at the University of Southern Mississippi who provided services for proofreading some of the submitted manuscripts.

While this book was in the last stage of preparation, we received the sad news that Dr. Michael Golberg had passed away after a long illness. We deeply regret that we lost a colleague and friend whose work has inspired much of our research on the MFS.

C. S. Chen
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Y.S. Smyrlis
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CHAPTER 1

Adaptive Numerical Solution of MFS Systems

R. Schaback

Abstract. The linear systems arising from MFS calculations share certain numerical effects with other systems involving radial basis functions. These effects concern approximation error and stability, which are closely related, and they can already be studied for simple interpolation problems without PDEs. In MFS calculations, they crucially depend on the position and density of the source points and the collocation points. In turn, the choice of these points must depend on the smoothness and possible singularities of the solution. This contribution provides an adaptive method which chooses good source points automatically. A series of examples shows that the adaptive choice of source points follows the theoretical predictions quite well.

1.1 Introduction

The Method of Fundamental Solutions (MFS) solves a homogeneous boundary value problem via approximation of the boundary data by traces of fundamental solutions centered at source points outside the domain in question. The method has been used extensively in recent years, and there are excellent surveys [6, 8, 5]. However, this contribution focuses on the linear systems arising in MFS calculations and ignores applications in engineering and science. Since our observations will easily generalize to other

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In this case, we keep the presentation and the examples simple by restricting ourselves to the homogeneous Poisson problem

\[ \begin{align*}
\Delta u &= 0 \quad \text{in } \Omega \subset \mathbb{R}^2 \\
u &= \varphi \quad \text{on } \Gamma := \partial \Omega
\end{align*} \]

(1.1)

with the Laplace operator. In this case, the fundamental solution (up to a multiplicative constant) is the singular radial kernel function

\[ \Phi(x,y) := \log \|x-y\|_2, \quad x,y \in \mathbb{R}^2. \]

The source points will be taken from a curve \( \Sigma \) outside \( \overline{\Omega} \) which is often called the “fictitious” boundary. In particular, users normally choose \( N \) points \( y_1, \ldots, y_N \in \Sigma \) and take linear combinations

\[ s(x) := \sum_{j=1}^{N} \alpha_j \log \|x-y_j\|^2_2, \quad x \in \Omega \]

(1.2)

of fundamental solutions as trial functions being homogeneous solutions of the Laplacian, i.e. harmonic functions. Of course, other homogeneous solutions can also enrich the trial space, and there are plenty of such possibilities, including harmonic polynomials. Methods like this date back to Trefftz [16] in much more general form, and are currently revived under the name of boundary knot methods [4].

1.2 Error Bounds

Whatever homogeneous solutions the trial functions \( s \) are composed of, the maximum principle will under mild assumptions on the regularity of the domain and the boundary data [11] imply that the true solution \( u \) and the trial approximation \( s \) satisfy the error bound

\[ \|u - s\|_{\infty, \Pi} \leq \|u - s\|_{\infty, \partial \Omega}. \]

This means that users only have to worry about the \( L_\infty \) approximation error on the boundary. If a fixed space of general linear combinations

\[ s(x) := \sum_{j=1}^{N} \alpha_j s_j(x), \quad x \in \Omega \]
of smooth homogeneous solutions $s_j$ are admitted, the natural numerical approach induced by the Maximum Principle would be to minimize the $L_\infty$ norm of the error on the boundary. This is a semi–infinite linear optimization problem

$$\text{Minimize } \eta \quad -\eta \leq \varphi(x) - \sum_{j=1}^{N} \alpha_j s_j(x) \leq \eta, \quad x \in \Gamma$$

(1.3)

with $N + 1$ variables $\eta, \alpha_1, \ldots, \alpha_n$ and infinitely many affine–linear constraints. The literature on optimization deals with such problems [10, 9], but in many cases it suffices to come up with a cheap but suboptimal approximation. We shall focus on this situation and give examples later.

### 1.3 Linear Systems

In particular, users often try to get away with picking $N$ collocation points $x_1, \ldots, x_N$ on the boundary $\Gamma$ and setting up an $N \times N$ linear system

$$\sum_{j=1}^{N} \alpha_j s_j(x_k) = \varphi(x_k), \quad 1 \leq k \leq N$$

(1.4)

for interpolation at these points. This works well in many cases, but the main theoretical problem with such systems is that the coefficient matrix with entries $s_j(x_k)$ may be singular. This clearly occurs for $N > 1$ and the MFS, because the determinant of the $N \times N$ system with matrix entries

$$s_j(x_k) = \log \|y_j - x_k\|_2^2, \quad 1 \leq j, k \leq N$$

will be a smooth function of the source points $y_j$, and swapping two source points will change the sign of the determinant. Thus there are plenty of configurations of source and test points where the system is necessarily singular. Confining source points to curves may help in 2D cases, but not in 3D if source points are restricted to surfaces.

Consequently, it does not make any sense to head for theorems proving nonsingularity of the above systems. The same holds for other unsymmetric collocation–type techniques like the one introduced by E. Kansa [12, 13] for general PDE problems in strong form, or the meshless local Petrov–Galerkin method of S.N. Atluri and collaborators [1, 2].
Instead, systems like (1.4) should not be expected to be solvable exactly. In view of the maximum principle and the semi–infinite optimization problem (1.3) one can take many more collocation points than source points and solve the overdetermined linear system

\[
\sum_{j=1}^{N} a_j s_j(x_k) = \phi(x_k), \quad 1 \leq k \leq M \geq N
\]

approximately, e.g. by a standard least–squares solver. We shall focus on such systems from now on.

1.4 Choice of Test and Collocation Points

If a good linear combination \( s \) of the form (1.2) is found by any method whatsoever, users will check the maximum boundary error \( \| \phi - s \|_{\infty, \Gamma} \) by evaluating the error in sufficiently many test points on the boundary. Though this test also needs a thorough mathematical analysis in order to be safe, we ignore it here. We just remark that users will need very many test points in case of steep gradients of the trial functions, and this inevitably occurs if the MFS is used with source points close to the boundary. Adding more test points still is computationally cheap if \( N \) is not too large, and most users will be satisfied with a simple plot of the boundary errors evaluated at test points guaranteeing graphic accuracy, i.e. at most 1000 points per plot. We shall use this rule–of–thumb in later examples.

Choosing \( M \) collocation points for setting up the system (1.4) is somewhat more difficult, but it will always stabilize the system if more points are taken. Independent of the choice of trial functions, users can repeat the calculation with more or other collocation points, if they are not satisfied with the first result. This is a simple way of introducing adaptivity into the numerical solution strategy:

Adaptivity of Testing:

If the evaluation of the boundary error on certain test points yields values that are intolerably large, take these test points as collocation points and repeat the calculation.

As long as the trial space \( S \) is not changed, this can improve the results, but if the trial space is poorly chosen, the final boundary error cannot be
Adaptive Numerical Solution of MFS Systems

less than
\[
\inf_{s \in \mathcal{S}} \| \mathbf{s} \|_{\infty, \Gamma}
\]

no matter how collocation and testing is done and how many points are used.

But there is another argument that needs consideration. If the linear optimization problem (1.3) is solved for a large but finite subset \( \Gamma_0 \) of the boundary instead of the full boundary, the Karush–Kuhn–Tucker conditions applied to the dual reformulation [3] of a linear minimax problem will imply that there is a subset \( \Gamma_1 \) of \( \Gamma_0 \) consisting of at most \( N + 1 \) points such that

\[
\inf_{s \in \mathcal{S}} \| \mathbf{s} \|_{\infty, \Gamma_0} = \inf_{s \in \mathcal{S}} \| \mathbf{s} \|_{\infty, \Gamma_1}.
\]

This is related to the notion of support vectors in support vector machines, and it has the following implication:

Reducibility of collocation points:
If a system (1.4) with \( M \gg N \) has a good approximate solution, it even has a good approximate solution determined already by a subset of at most \( N + 1 \) collocation points.

Unfortunately, these collocation points are not known beforehand, but users should be aware of the fact that a large system with a good approximate solution will have a much smaller subsystem with an equally good solution. This fact will reappear later, and we shall provide examples.

1.5 Choice of Trial Space
The lower bound (1.6) for the achievable boundary error reveals that the main design problem consists in picking good trial functions, or, in case of the MFS, in picking good source points.

Let us postpone the MFS for a while. Users can take all homogeneous solutions as trial functions, and this will work well in certain examples we shall look at later. For the Laplace operator in 2D, the real part of any differentiable function of a complex variable will be harmonic and can serve as a possible trial function. The standard fundamental solution just is a special case of a real part of a complex function with a singularity, but there are many others without singularities, e.g. harmonic polynomials or entire functions like \( f(x, y) := \exp(y) \cos(x) \).
How to choose? We shall later let an algorithm decide adaptively, but there is a general though trivial rule:

Take trial functions with similar analytic properties as the expected solution. In particular, be careful when the solution or one of its derivatives will necessarily have singularities somewhere.

1.6 Harmonic Polynomials

We explain this first for the case of using harmonic polynomials. If the solution $u$ of the given Poisson problem is itself a real part of a function of a complex variable without singularities anywhere, it can be well approximated by harmonic polynomials on any curve, namely by the real part of its partial sums of its power series. The shape of the domain does not matter at all, and the background PDE problem is completely irrelevant because we only have to recover a partial power series. A full power series of an analytic function is determined by values on any countable set with an accumulation point, and thus recovery of globally harmonic functions from point evaluation data will work almost anywhere.

By analogy to certain theorems on polynomial approximations to analytic functions [7], the rate of approximation can be expected to be spectral, i.e. the error should behave like $C\lambda^n \to \infty$ as a function of the degree $n$ of the harmonic polynomials used, and $\lambda > 0$ can be arbitrarily small. Then the choice of harmonic polynomials should be superior to all choices of fundamental solutions. In many engineering applications where MFS users report that source points of the MFS taken far away from the domain work best, the special examples usually have solutions without singularities anywhere, but users tend to ignore that harmonic polynomials will do even better in such situations.

If the solution, when viewed as a global function, is still harmonic but has a singularity at a positive distance to the boundary $\Gamma$, the rate of approximation will again be like $C\lambda^n \to \infty$, but with $\lambda < 1$ now being bounded below, and related to the distance of the singularity to the boundary, with $\lambda \to 1$ if the singularity moves towards the boundary. Again, the shape and smoothness of the boundary is irrelevant. The crucial quantity is the distance of the closest singularity of the solution from the boundary, when the solution is extended harmonically as far as possible. Again, this
case is hard to beat by the MFS, if the singularity is sufficiently far away from the domain.

The situation gets serious if the solution or one of its derivatives has a singularity directly on the boundary \( \Gamma \). Note that this case occurs whenever the boundary data, however smooth, are given by a function which is not itself harmonic. In such a case, the rate of approximation of boundary values by harmonic polynomials can be very poor, depending on the smoothness of the solution when restricted to the boundary. The upshot of this discussion of harmonic polynomials is that the MFS makes sense only if the boundary data come from a non–harmonic function or if there is no harmonic extension of the solution without singularities close to the boundary. Users working in application areas do not seem to be aware of this fact.

1.7 Rescaling Fundamental Solutions

Before we go over to the problem of choosing good source points for the MFS, let us consider the case of far–away source points \( y \in \mathbb{R}^2 \) while the evaluation of a fundamental solution \( \log \left( x - y \right) \) is at \( x \in \mathbb{R}^2 \) with a relatively small value of \( \| x \|_2 \). In such cases, the functions \( \log \| x - y \|_2 \) will not differ much if \( y \) varies, and consequently the resulting matrix gets a bad condition. But we can rewrite the function for large \( \| y \|_2 \neq 0 \) as

\[
\log \| x - y \|_2^2 = \log \left( \| x \|_2^2 - 2(x, y)_2 + \| y \|_2^2 \right) \\
= \log \left( \| y \|_2^2 \left( \frac{\| x \|_2^2}{\| y \|_2^2} - 2 \left( \frac{y}{\| y \|_2} , \frac{x}{\| y \|_2} \right)_2 + 1 \right) \right) \\
= \log \| y \|_2^2 + \log \left( 1 + \left( \frac{\| x \|_2^2}{\| y \|_2^2} - 2 \left( \frac{x}{\| y \|_2} , \frac{y}{\| y \|_2} \right)_2 \right) \right).
\]

If we use the expansion

\[
\log(1 + z) = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{z^j}{j}
\]
for \(|z| < 1\), we get for sufficiently large \(||y||_2\) the expansion

\[
\log ||x-y||_2^2 - \log ||y||_2^2 = \log \left( 1 + \left( ||x||_2^2 - \frac{2}{||y||_2^2} x, y \right) \right)
\]

\[
= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \left( \frac{||x||_2^2}{||y||_2^2} - \frac{2}{||y||_2^2} x, y \right)_j^j
\]

\[
= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \sum_{m=0}^{j} \binom{j}{m} \left( \frac{||x||_2^2}{||y||_2^2} \right)^{j-m} \left( -2 \left( x, \frac{y}{||y||_2^2} \right) \right)^m
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{||y||_2^2} \sum_{k/2 \leq j \leq k} \frac{(-1)^{j-1}}{j} \left[ \sum_{m=0}^{j} \binom{j}{2j-k} \left( \frac{||x||_2^2}{||y||_2^2} \right)^{2j-k} \left( -2 \left( x, \frac{y}{||y||_2^2} \right) \right)^{2j-k} \right]
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{||y||_2^k} P_k(x, y)
\]

of the fundamental solution at \(y\) into harmonic polynomials

\[
P_k(x, y) := \sum_{k/2 \leq j \leq k} \frac{(-1)^{j-1}}{j} \sum_{m=0}^{j} \binom{j}{2j-k} \left( \frac{||x||_2^2}{||y||_2^2} \right)^{2j-k} \left( -2 \left( x, \frac{y}{||y||_2^2} \right) \right)^{2j-k}
\]

with respect to \(x\) of degree \(k\). If we push the source point \(y\) to infinity by writing it as \(y = rz\) for large \(r > 0\) and fixed \(z \in \mathbb{R}^2\) with \(||z||_2 = 1\), we get

\[
\log ||x-rz||_2^2 = 2\log r + \sum_{k=1}^{\infty} \frac{1}{r^k} P_k(x, z)
\]

and this is something like a “far field expansion” of the fundamental solution. Note that \(z\) and \(r\) are considered to be fixed, and thus users are strongly advised to include constants into the space of trial functions in order to cope with the \(2\log r\) term.

Now let us look at the span of fundamental solutions based on points \(y_j = rz_j\) on a circle of radius \(r\) for large \(r\). We want to find functions which are in the span when taking the limit \(r \to \infty\), and we call this the
“asymptotic span”. The linear combinations are
\[
\begin{align*}
\quad s_r(x) &= \sum_{j=1}^{N} \alpha_j(r) \left( 2 \log r + \sum_{k=1}^{\infty} \frac{1}{r^k} p_k(x, z_j) \right) \\
&= 2 \log r \sum_{j=1}^{N} \alpha_j(r) + \sum_{k=1}^{\infty} \frac{1}{r^k} \sum_{j=1}^{N} \alpha_j(r) p_k(x, z_j)
\end{align*}
\]
and thus have specific expansions in terms of harmonic polynomials. If constants are not added to the span, and if the MFS works at all for large \( r \) in a specific case, the sum of the coefficients \( \alpha_j(r) \) will tend to zero for \( r \to \infty \) while the coefficients themselves cannot stay all bounded. In all “pure MFS” examples with far-away source points, the sum of coefficients will always be close to zero while the sum of the absolute values will be extremely large.

To avoid computational crimes, we now add the constant 1 to the span of trial functions and use a coefficient \( \alpha_0 \) for it. Then we have a span of
\[
\begin{align*}
\quad s_r(x) &= 1 \left( \alpha_0(r) + 2 \log r \sum_{j=1}^{N} \alpha_j(r) \right) + \sum_{k=1}^{\infty} \frac{1}{r^k} \sum_{j=1}^{N} \alpha_j(r) p_k(x, z_j)
\end{align*}
\]
which we can analyze somewhat easier. We have the constants in the span, of course, but for arbitrary \( \alpha_1(r), \ldots, \alpha_N(r) \) we can always set
\[
\alpha_0(r) := -2 \log r \sum_{j=1}^{N} \alpha_j(r)
\]
to cancel the first term. Now \( rs_r(x) \) must be in the span, and this asymptotically is in the span of the \( p_1(x, z_j) \), \( 1 \leq j \leq N \), which necessarily is a subspace \( V_1 \) of the linear polynomials. To proceed inductively, we now look at the subspace \( A_1 \) of coefficient vectors \( \alpha \in \mathbb{R}^N \) with
\[
\sum_{j=1}^{N} \alpha_j p_1(x, z_j) = 0.
\]
If we take a vector \( \alpha \in A_1 \) and form the functions \( r^2 s_r(x) \), we find that the asymptotic span of the fundamental solutions contains the polynomial space
\[
V_2 := \left\{ \sum_{j=1}^{N} \alpha_j p_2(x, z_j) : \alpha \in A_1 \right\}
\]
of maximally second-degree polynomials. Inductively we can define $A_0 := \mathbb{R}^N$ and

$$A_m := \left\{ \alpha \in \mathbb{R}^N : \sum_{j=1}^{N} \alpha_j p_i(x, z_j) = 0, \ 1 \leq i \leq m \right\}$$

for all $m \geq 1$ and use it for defining a space

$$V_m := \left\{ \sum_{j=1}^{N} \alpha_j p_m(x, z_j) : \alpha \in A_{m-1} \right\}$$

of polynomials of degree at most $m$. The spaces $A_m$ form an inclusion chain

$$\mathbb{R}^N = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

and if we take an appropriate orthogonal basis for that chain, we get

**Theorem 1.** The asymptotic span for $r \to \infty$ of fundamental solutions with source points of the form $y_j = rz_j$ for fixed points $z_j$ on the unit circle is a space of harmonic polynomials spanned by constants and the union of all $V_m$.

Unfortunately, it seems to be difficult to calculate the dimension of that space, because it will depend on the number and the geometry of the points $z_j$.

The upshot of all of this is that the MFS for far-away source points, if it works at all, is asymptotically nothing else than a fit of the boundary data by specific harmonic polynomials. Thus the MFS should not be used at all for far-away source points, but rather be replaced by use of harmonic polynomials. For this reason, we do not elaborate the above argument any further, though it would result in a way of preconditioning MFS matrices for far-away points. It does not make sense to precondition a matrix one should not use.

However, a rather primitive but still somewhat useful change of basis induced by the above argument is to add constants to the MFS span and replace the fundamental solution at $y \neq 0$ by

$$(\log ||x - y||_2^2 - \log ||y||_2^2) \ ||y||_2$$

behaving like a linear polynomial in $x$ when $y$ is far away from $x$. A full preconditioning will use such basis changes plus coefficient vectors from
an orthogonal basis of $\mathbb{R}^N$ which is compatible with the chain of the $A_m$ spaces. Details can be worked out similarly to [19]. As an aside, we remark that it is no problem to replace the standard fundamental solutions by rational trial functions arising when taking derivatives of $\log ||x - ry||^2_2$ with respect to $r$.

Finally, we present an example supporting the results of this and the previous section. In Fig. 1.1 we show the $L_\infty$ error $\varepsilon_\infty(r)$ on the full circle when we recover the harmonic function $f(x, y) = e^x \cos(y)$ from boundary values only on a half circle. We collocate at 100 test points on the right half unit circle, using 20 source points on the right half circle of radius $r$. We stopped the calculation when the numerical rank of the $100 \times 20$ collocation matrix, as given by MATLAB$^\circledR$ was less than 20, and this occurred for $r \approx 5.5$ already. The error decreases nicely with increasing $r$, because the setting converges towards harmonic polynomials for $r \to \infty$, as was shown in this section, and since the discussion in the previous section showed that recovery by harmonic polynomials should work on any arc.

Figure 1.1: $L_\infty$ error as function of $r$
1.8 Choice of Source Points

It should be clear by now that a good placement of source points will crucially depend on the distance of the closest singularity arising when extending the solution harmonically outside the domain. In many cases, the user normally does not have this information, but there are a few guidelines.

We start by an upside–down argument. If the MFS works for sufficiently many source points on a fixed curve $\Sigma$, and if the results are getting better when taking more source points, the solution will have a harmonic extension up to $\Sigma$, because the MFS constructs it. But if there necessarily is a singularity inside $\Sigma$ for some reason or other, the MFS cannot work satisfactorily on $\Sigma$.

We now have to find a–priori indicators for singularities close to the boundary. The first and simplest case arises when the known boundary data are such that there is no $C^\infty$ extension locally into $\mathbb{R}^2$. This always happens if the boundary data are not $C^\infty$ on smooth parts of the boundary. If users know where the “boundary points of data nonsmoothness” are, source points should be placed close to those. Unfortunately, there currently is no general way to guess the type of singularity beforehand, even if the position is known. Thus this case usually must be handled experimentally.

A second and partially independent case arises for incoming corners of the domain. Even if the boundary data have a $C^\infty$ extension to $\mathbb{R}^2$, e.g. if they are non–harmonic polynomials, users must expect a singularity at the boundary, but the type of singularity is known, depending on the boundary angle. Again, users should either add the correct type of singularity or place source points close to corners in such cases. But the situation is different if the data come from an extendable harmonic function, even if corners are present. Then the MFS can ignore the corners. Note that MFS examples on domains with corners are useless as long as they consider specific boundary data which are values of functions with a harmonic extension.

Finally, the convergence rate of the MFS when adding more and more source points will be strongly influenced by the smoothness of both the data function and the MFS trial functions on the boundary. Approximation theory proves in many situations that convergence rates are completely
controlled by the minimal smoothness of the data function and the trial functions. Thus smooth boundary data on smooth boundaries will lead to good convergence rates improving with the smoothness properties. If source points can be kept at a fixed positive minimal distance from the boundary (this requires the solution to have a harmonic extension), then the trial functions are analytic and the convergence rate will be completely determined by the smoothness of the data on the boundary. But then the approximation by harmonic polynomials on the boundary will also have a good convergence rate depending on the smoothness of the boundary data, and it is not easy to predict superiority of the MFS over approximation by harmonic polynomials.

If singularities of derivatives are on the boundary or if there are incoming corners, the convergence rate of approximation of the boundary data by harmonic polynomials will deteriorate seriously, and the MFS can be competitive by placing source points closer and closer to the singularities. A general rule is not known, but there are certain adaptive techniques [18, 14, 15, 6] to handle this case. We shall provide a simple adaptive method in the next section.

1.9 Greedy Adaptive Techniques

Overdetermined systems like (1.5) can be approximately solved by a stepwise adaptive techniques even if they are huge. We applied the method of [17] to MFS problems, but it turned out to be less stable than the algorithm we describe now, because the previous one did not keep all collocation points under control.

The basic idea can be formulated independent of the MFS in terms of solving a linear unsymmetric over- or underdetermined $m \times n$ system of the form $Ax = b$. The goal is to pick useful columns of the $m \times n$ matrix $A$ in a data-dependent way without cutting the number of rows down. This is also done by any reasonable solution algorithm, e.g. by the backslash operator in MATLAB®, and we shall present examples later. However, standard $QR$ routines do not account for the right-hand side $b$, and they do not stop early when only a few columns of the matrix suffice to reproduce the right-hand side with small error. To maintain stability, we use orthogonal transformations like in any $QR$ decomposition, but we make the choice of columns dependent on the right-hand side.
In short, our adaptive algorithm for selecting good columns works as follows:

1. Pick the column of $A$ whose multiples approximate $b$ best.

2. Then transform the problem to the space orthogonal to that column and repeat.

If the algorithm has selected a number of columns this way, take this column selection for a trial space and use your algorithm of choice for solving the given problem on that trial space. For instance, in MFS applications one can use $L_\infty$ minimization of boundary errors after the selection process has provided a small set of useful source points.

The actual implementation of the algorithm needs some further explanation. Approximation of $b$ by multiples of a single nonzero vector $a$ is optimal in $L_2$, if the error vector has the form $b - a \cdot \frac{b^T a}{a^T a}$, and its squared norm then takes the minimal possible value

$$||b||^2 - \frac{(b^T a)^2}{||a||^2}.$$ 

If we denote the columns of $A$ by $a^1, \ldots, a^n$, we thus can implement Step 1 by taking the maximum of

$$\frac{(b^T a_j)^2}{||a^j||^2}, \ 1 \leq j \leq n, \ ||a^j||_2 \neq 0$$

to pick the best column for approximation of $b$. If we denote this column by $u$, we form the normalized vector $v := u/||u||_2$ and transform both $A$ and $b$ into

$$A_1 := A - v(v^T A)$$
$$b_1 := b - v(v^T b)$$

to let both $b_1$ and the columns of $A_1$ be orthogonal to $u$ and $v$. The new matrix has a zero column where once was $u$. To avoid roundoff problems, we insert exact zeros there, but we do not delete the zero column in order to avoid unnecessary storage transformations. Instead, we store the column index of $u$ for later use and proceed. The following steps will always automatically ignore the columns we already picked. Note that we do not care for the approximate solution of the system and about accumulation of
roundoff during the transformations. The $L_2$ norms of the vectors $b, b_1, \ldots$ will necessarily decrease, and this can be used for stopping the algorithm. Going for a strong error reduction will finally use the full matrix, while users can get away with just a few columns and a very simple numerical solution if they admit larger errors. A MATLAB\textsuperscript{®} package is available from the author.

Application of this technique to MFS calculations is particularly appealing in cases where the user lets the algorithm decide which trial function to choose. One can offer harmonic polynomials up to a fixed degree and plenty of fundamental solutions at different distances to the boundary, and the algorithm will pick suitable ones without knowing background mathematics like harmonic extendibility of the solution. Running the algorithm several times will provide the user with information about hazardous places at the boundary, and the user can offer refined choices of source points close to these when preparing the next run. The actual calculation of the solution is done after column selection in order to keep the accumulated errors small.

1.10 Examples

To illustrate the mathematical issues of the previous sections, we now provide a series of examples, but we have to explain the notation in the tables and figures first. Following (1.4), the number of collocation points will be $M$, and $N$ will denote the number of source points offered to the algorithm. If only a smaller subset of source points is actually used for the calculation, we use $n \leq N$. Similarly, $K$ denotes the number of harmonic polynomials included into the trial space, and $k$ stands for the number actually arising in the solution. For approximate evaluation of the $L_\infty$ norm $\varepsilon_\infty$ of the error on the boundary we use $\max(1000, 5 \ast N)$ points for graphic accuracy. The number $M$ of collocation points is always defined as $1 + \max(200, 2N + 2K)$. The approximate solution of the system will be done by algorithms labeled as

- $L2$: the MATLAB\textsuperscript{®} backslash operator, i.e. a standard least–squares solver with internal selection of columns,

- $A2$: the same as above, but applied to the reduced matrix after adaptive column selection along the lines of the previous section,
Note that the solution algorithm will have quite some influence on the final error and the number of nonzero solution coefficients. The $L^2$ solver always exploits maximal machine accuracy, while our adaptive solvers $A^2$ and $A^\infty$ are tuned for a compromise between error and complexity.

We denote the $L^\infty$ error on our test points as $\varepsilon^\infty$. It is debatable to use the RMSE error measure at all, because the Maximum Principle is guiding the error behavior, but we also include it as $\varepsilon^2$. The curves we use for boundaries or source points consist of circles $C_r$ with radius $r$ around zero, or they are obtained from a polar coordinate representation of the standard boundary $B_0$ by adding a distance $h$ to get a source boundary $B_h$ at polar distance $h$.

The first example in Table 1.1 concerns a more or less trivial case where the data come from a globally harmonic function $f(x, y) = \exp(x) \cos(y)$ which is the real part of the entire complex function $\exp(z) = \exp(x + iy)$. The real part of the power series of $\exp(z)$ yields a sequence of perfect approximations by harmonic polynomials on each domain whatsoever, and the recovery by collocation via harmonic polynomials even works on open arcs anywhere. As expected, the MFS cannot outperform harmonic polynomials, independent of where the sources are. The domain is a lemniscate with an incoming corner as in Fig. 1.2, while the source points are on the circle of radius 4 around the origin.

<table>
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<tr>
<th>$N$</th>
<th>$M$</th>
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Table 1.1: Recovery of harmonic function
The table should be interpreted as follows. The first three lines used no source points at all ($N = 0$), but $M = 201$ collocation points and allowed $K = 25$ harmonic polynomials. Only up to $k = 13$ nonzero polynomial coefficients were calculated due to the symmetry of the data function, and the recovery quality via the first terms of the power series is around $1.0e-10$, independent of the algorithm used. This happened for all domains tested. But the adaptive algorithms in lines 2 and 3 solve only a subproblem with 12 degrees of freedom after picking the most useful harmonic polynomials.

The next three lines are a pure MFS offering 200 source points on the circle. With the $L_2$ solver, the results are even better than for harmonic polynomials, and surprisingly the MATLAB backslash solver yields only 36 nonzero coefficients, i.e. only 36 source points were necessary. The adaptive solvers are satisfied with less accuracy, but also use a simpler approximation by 25 or 28 source points.
The final three lines offered the same 200 source points, but allowed also 25 harmonic polynomials. All algorithms prefer harmonic polynomials over fundamental solutions. This is to be expected, because the solution has no finite singularities. The 25 marked source points in Fig. 1.2 belong to the situation of the fifth line of Table 1.1. The adaptive $L_2$ algorithm picks these 25 source points with no connection to the domain corner, as is to be expected. However, the error is worse than for harmonic polynomials in this case.

The $L_\infty$ norm of the error using the $A\infty$ solver in line 3 is only slightly better than the one from the $A2$ solver in line 2. It cannot be worse because they use the same trial space. However, in some of the later cases, the $A\infty$ algorithm, after starting from the same trial space as the $A2$ algorithm, is less stable and often ends prematurely with a larger $L_\infty$ error than the $A2$ solver. Future work should add a more sophisticated $L_\infty$ solver.

Table 1.2 we present the same situation, but with boundary data given by the function $x^2y^3$. This is still smooth, but the domain has an incoming corner causing problems.

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Table 1.2: Smooth boundary data on lemniscate, source points on circle

With source points on the circle, none of the methods outperforms harmonic polynomials seriously.

To demonstrate that the incoming corner is the culprit, we replace the lemniscate now by the unit circle and get Table 1.3. Again, harmonic polynomials do best. We now go back to the lemniscate, but place the source points at a polar radial distance of 0.2 outside the boundary (i.e. we
Adaptive Numerical Solution of MFS Systems

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<td>3</td>
<td>7.31e-011</td>
<td>4.79e-011</td>
</tr>
</tbody>
</table>

Table 1.3: Smooth boundary data on circle

calculate them by adding 0.2 to the radius of boundary points in polar coordinates). This gives Table 1.4 and should be compared to Table 1.2. The

<table>
<thead>
<tr>
<th>N</th>
<th>M</th>
<th>K</th>
<th>Alg</th>
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<th>m</th>
<th>k</th>
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<td>$A_2$</td>
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<td>3.09e-004</td>
<td>4.75e-005</td>
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<td>$A_\infty$</td>
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<td>119</td>
<td>13</td>
<td>4.82e-004</td>
<td>1.28e-004</td>
</tr>
</tbody>
</table>

Table 1.4: Smooth boundary data on lemniscate, source points at 0.2

results are not better than for sources on the circle, but maybe the sources are not close enough. Thus we go for a distance of 0.02 in Table 1.5, and Fig. 1.3 shows the source point distribution (circles, while the offered but unused source points are small dots) for the $A_2$ technique in the eighth line of Table 1.5. Note that the source points are automatically picked close to the singularity.
Going even closer does not pay off, unless we enhance the collocation resolution.

Now we admit source points at different distances, starting at 0.02 and ending at 10.24 after repeated multiplications by 2. At each distance, we choose random source points such that their total is roughly 201 as in the previous cases. Now the algorithms can pick source points at very different distances, and they do.

Finally, we allowed roughly 700 source points and varying distances from 0.002 to 2.048 to get Table 1.7. The distances were prescribed by multiplying 0.002 by powers of 2 until 2.048 was reached, while the points for fixed distance were uniformly sampled with respect to the parametrization of the boundary. Zooming in on the adaptively selected source point placements gives Fig. 1.4 for the first line of Table 1.7, while Figs 1.5 and 1.6 are close-ups for line 2 with the A2 algorithm. Note that the small dots are the offered source points, while circles indicate the selected source
Users may suspect that things are better if there is no incoming corner of the domain. Thus let us take the unit circle and prescribe the continuous boundary values $\phi(x,y) := \max(0,y)$. This leads to two derivative singularities of the harmonic solution at $(1,0)$ and $(-1,0)$. Numerical results are quite similar to the case on the lemniscate, and thus we confine our-

### Table 1.5: Smooth boundary data on lemniscate, source points at 0.02

<table>
<thead>
<tr>
<th>N</th>
<th>M</th>
<th>K</th>
<th>Alg</th>
<th>n</th>
<th>m</th>
<th>k</th>
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<td>3.63e-004</td>
<td>5.75e-005</td>
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<td>447</td>
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<td>6.12e-004</td>
<td>1.45e-004</td>
</tr>
</tbody>
</table>

### Table 1.6: Smooth boundary data on lemniscate, about 200 source points at varying distances

<table>
<thead>
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<th>K</th>
<th>Alg</th>
<th>n</th>
<th>m</th>
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<td>4.80e-003</td>
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</table>
Adaptive Numerical Solution of MFS Systems

Figure 1.4: Lemniscate with source points, $L_2$ algorithm

selves to offering about 700 source points on circles with distances 0.002 to 2.048 in Table 1.8. The $L_2$ solver does not care about the right-hand side of the system, and thus it does not realize the two singularities. This is shown in a close-up in Fig. 1.7, while the $A_2$ algorithm (see Fig. 1.8) selects source points close to the problematic boundary locations. Note that this case is offered 729 degrees of freedom and uses maximally 490 of these.

<table>
<thead>
<tr>
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Table 1.8: MFS results for $\max(0, y)$
If we drop the MFS completely and offer 751 harmonic polynomials on the circle instead, we get Table 1.9. Note that this performs slightly better than the MFS and uses 377 of the possible degrees of freedom.

<table>
<thead>
<tr>
<th>N</th>
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<th>Alg</th>
<th>n</th>
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Table 1.9: Results for $\max(0,y)$, harmonic polynomials only

Finally, Table 1.10 shows how the sum of coefficients and the $L_1$ norm of coefficients vary with the radius $r$ of the distance of the source points to the boundary. To avoid symmetries, we took the boundary data function $f(x,y) := \max(0,|y|)$ on the unit circle, offered 200 source points on a
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Figure 1.6: Lemniscate with source points, $A^2$ algorithm

circle of radius $r$ and used 401 collocation points. The solution method was the standard backslash $L_2$ solver from MATLAB. Note that the increase of condition is counteracted by the solver in a very nice way, using fewer and fewer source points. This effect is even more significant when using the $A^2$ or $A^\infty$ methods.

<table>
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<td>3.89e-002</td>
<td>4.57e-002</td>
<td>1.62e+012</td>
</tr>
</tbody>
</table>

Table 1.10: Sums of coefficients as functions of $r$
Figure 1.7: Source points selected by $L2$ algorithm

References


Figure 1.8: Source points selected by A2 algorithm


CHAPTER 2

The Method of Fundamental Solutions for the Laplace Equation with Mixed Boundary Problems

Zi-Cai Li

Abstract. The method of fundamental solutions (MFS) was first used in Kupradze [16] in 1963. Since then, there have appeared numerous reports of MFS for computation, but there exist only a few for analysis. For the Dirichlet problem of Laplace’s equation on the bounded simply-domain S, when the source points of fundamental solutions (FS) are located uniformly on an outside circle of S, the polynomial convergence rates were proved in Bogomolny [2], and the exponential convergence rates were provided in Katsurada and Okumoto [14]. This paper is devoted to Laplace’s equation with mixed boundary problems, and bounds of both errors and condition numbers are first provided for bounded simply-connected domains. Numerical experiments are carried out for the benchmark of singularity problems, Motz’s problem. Moreover, the analysis of MFS is briefly viewed, to display the nature of MFS.

2.1 Introduction

The method of fundamental solutions (MFS) was first used in Kupradze [16] in 1963. Since then, there have appeared numerous reports of MFS for computation, see the review of MFS in Fairweather and Karageorghis [9], and Golberg and Chen [10], and a systemic introduction on MFS is given in Chen et al. [3]. To celebrate the progress of MFS, there held the first
The MFS for Laplace Equation

Inter. Workshop on the Method of Fundamental Solutions (MFS2007), Ayia Napa, Cyprus, June 11-13, 2007. However, there exist only a dozen of papers for analysis. Even for the analysis, most of publications are confined to the disk domains (see Kitagawa [15], Christiansen [5, 6], Smyrnis and Karageorghis [32, 33], and Li [17, 18]), except the work of Bogomolny [2] and Katsurada [11, 12, 13]. So far, there exist no analysis of MFS for the mixed boundary problems. Hence this paper is devoted for MFS for Laplace’s equation with mixed boundary conditions, and bounds of both errors and condition numbers are provided for bounded simply-connected domains. Numerical experiments are carried out for Motz’s problem, by adding singular solutions, or local refinements of collocation nodes. The values of Cond are huge, and those of Cond eff are moderately large. Moreover, the expansion coefficients obtained by MFS are oscillatingly large, to cause another kind of instability: subtraction cancelation in the final harmonic solutions. Hence for practical applications, the errors and the ill-conditioning must be balanced to each other.

This paper is organized as follows. In the next section, the algorithms of MFS are described, and stability is measured by Cond and Cond eff. In Section 2.3, error analysis is briefly proved. In Section 2.4, numerical experiments are carried out for Motz’s problem by MFS, and in the last section, a few remarks for the analysis on MFS are made.

2.2 Algorithms and Stability

2.2.1 Algorithms

Consider Laplace’s equation with the mixed boundary problems of the Dirichlet and the Robin boundary conditions,

\[ \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ in \ S, \]  \( \text{(2.1)} \)

\[ u = f \ on \ \Gamma_D, \]  \( \text{(2.2)} \)

\[ \frac{\partial u}{\partial v} + \alpha u = g \ on \ \Gamma_R, \]  \( \text{(2.3)} \)

where \( \alpha \) is a non-negative constant, \( S \) is a bounded and simply-connected domain with the boundary \( \partial S = \Gamma_D \cup \Gamma_R \), and \( v \) is the exterior normal of \( \Gamma \).
Denote in Figure 2.1,

\[ r_{\text{max}} = \max_S r, \quad r_{\text{min}} = \max_{S_{\text{in}}(S \subseteq S)} r, \]  

(2.4)

where \( S_{\text{in}} \) is the maximal disk inside of \( S \). Let the source (charge) points \( Q \) be located outside of \( S \), the fundamental solutions

\[ \phi(r, \theta) = \ln|PQ_j|, \quad P \in S \cup \partial S \]  

(2.5)

are harmonic, where

\[ P = \{(x, y) \mid x = r\cos \theta, \ y = r\sin \theta \}. \]  

(2.6)
A circle surrounding $S$ is given by
\[ \ell_R = \{ (r, \theta) \mid r = R, \quad 0 \leq \theta \leq 2\pi \}, \quad R > r_{\text{max}}. \] (2.7)

Based on Bogomolny [2], the source points $Q_i$ may be simply located uniformly on $\ell_R$
\[ Q_i = \{ (x, y) \mid x = R \cos ih, \ y = R \sin ih \}, \] (2.8)
where $R > r_{\text{max}}$ and $h = \frac{2\pi}{N}$. We obtain the fundamental solutions
\[ \phi_i(P) = \ln |PQ_i|, \quad i = 1, 2, \ldots, N, \] (2.9)
and the numerical solution is given by a linear combination
\[ u_N = \sum_{i=1}^{N} c_i \phi_i(P), \] (2.10)
where $c_i$ are the unknown coefficients to be sought. Since $u_N$ satisfies Laplace’s equation in $S$ already, the coefficients $c_i$ can be sought by enforcing the boundary conditions (2.2) and (2.3) only. We will follow the Trefftz method (TM) [34] in [22, 27, 29], to seek $u_N$ (i.e., $c_i$). Denote the energy
\[ I(u) = \int_{\Gamma_D} (u - f)^2 + w^2 \int_{\Gamma_R} \left( \frac{\partial u}{\partial \nu} + \alpha u - g \right)^2, \] (2.11)
where $w$ is a positive weight. We choose $w = \frac{1}{N}$ in our computations (see [22]). Also denote by $V_N$ the space of (2.10). Then the numerical solution $u_N$ can be obtained by
\[ I(u_N) = \min_{v \in V_N} I(v). \] (2.12)

When the integrals in (2.11) involve approximation, denote
\[ \hat{I}(v) = \hat{\int}_{\Gamma_D} (v - f)^2 + w^2 \hat{\int}_{\Gamma_R} \left( \frac{\partial v}{\partial \nu} + \alpha v - g \right)^2, \] (2.13)
where $\hat{\int}_{\Gamma_D}$ and $\hat{\int}_{\Gamma_R}$ are the numerical approximations of $\int_{\Gamma_D}$ and $\int_{\Gamma_R}$ by some quadrature rules, such as the central or the Gaussian rule. Hence, the numerical solution $\tilde{u}_N \in V_N$ is obtained by
\[ \hat{I}(\tilde{u}_N) = \min_{v \in V_N} \hat{I}(v). \] (2.14)
We may establish the collocation equations directly from (2.2) and (2.3), to yield

\begin{equation}
\sum_{i=1}^{N} c_i \phi_i(P_j) = f(P_j), \quad P_j \in \Gamma_D, \tag{2.15}
\end{equation}

\begin{equation}
\sum_{i=1}^{N} c_i \left[ \frac{\partial}{\partial \nu} \phi_i(P_j) + \alpha \phi_i(P_j) \right] = g(P_j), \quad P_j \in \Gamma_R. \tag{2.16}
\end{equation}

First, let \( \Gamma_D \) and \( \Gamma_R \) be divided into small \( \Gamma_D^j \) and \( \Gamma_R^j \) with the mesh spacings \( \Delta h_j \), i.e.,

\begin{equation}
\Gamma_D = \bigcup_{j=1}^{M_1} \Gamma_D^j, \quad \Gamma_R = \bigcup_{j=1}^{M_2} \Gamma_R^j. \tag{2.17}
\end{equation}

We obtain from (2.15) and (2.16) by multiplying different weights,

\begin{equation}
\sqrt{\Delta h_j} \sum_{i=1}^{N} c_i \phi_i(P_j) = \sqrt{\Delta h_j} f(P_j), \quad P_j \in \Gamma_D^j, \quad j = 1, 2, \cdots, M_1, \tag{2.18}
\end{equation}

\begin{equation}
w \sqrt{\Delta h_j} \sum_{i=1}^{N} c_i \left\{ \frac{\partial}{\partial \nu} \phi_i(P_j) + \alpha \phi_i(P_j) \right\} = w \sqrt{\Delta h_j} g(P_j), \quad P_j \in \Gamma_R^j, \quad j = M_1 + 1, \cdots, M_1 + M_2, \tag{2.19}
\end{equation}

where for simplicity, \( P_j \) are the midpoints of \( \Gamma_D^j \) and \( \Gamma_R^j \). Following Lu et al. [29], Eqs (2.18) and (2.19) are just equivalent to (2.14), where the central rule is chosen for \( \int_{\Gamma_D} \) and \( \int_{\Gamma_R} \). In computation, we may choose the number of collocation points to be equal or larger than that of source points, i.e.,

\begin{equation}
M = M_1 + M_2 \geq N. \tag{2.20}
\end{equation}

When the Gaussian rule is chosen, the following collocation equations are obtained.

\begin{equation}
\beta_j \sum_{i=1}^{N} c_i \phi_i(P_j) = \beta_j f(P_j), \quad P_j \in \Gamma_D, \tag{2.21}
\end{equation}

\begin{equation}
\beta_j w \sum_{i=1}^{N} c_i \left\{ \frac{\partial}{\partial \nu} \phi_i(P_j) + \alpha \phi_i(P_j) \right\} = \beta_j w g(P_j), \quad P_j \in \Gamma_R. \tag{2.22}
\end{equation}
where \( P_j \) are the Gaussian nodes, the weights \( \beta_j = O(\sqrt{\Delta h}) \), and \( \Delta h = \max_j \Delta h_j \). Eqs. (2.21) and (2.22) (i.e., (2.14)) are called the collocation Trefftz method (CTM) in [27].

### 2.2.2 Stability

Consider

\[
Fx = b, \tag{2.23}
\]

where \( F \in \mathbb{R}^{m \times n} \) \( (m \geq n) \), \( x \in \mathbb{R}^n \) and \( b \in \mathbb{R}^m \). Assume that \( \text{rank}(F) = n \).

The condition number for (2.23) is defined by

\[
\text{Cond} = \frac{\sigma_1}{\sigma_n}, \tag{2.24}
\]

where \( \sigma_1 \) and \( \sigma_n \) are the maximal and the minimal singular values of matrix \( F \), respectively. From [24], we may define the effective condition number

\[
\text{Cond}_{\text{eff}} = \frac{\|b\|}{\sigma_n\|x\|}. \tag{2.25}
\]

The bound of Cond of MFS for bounded simply-connected domains is given in Li, et al. [25] by

**Theorem 2.2.1.** Let \( \mu \in [1, 2] \) and choose \( w = O(\frac{1}{N}) \). For (2.1) – (2.3) by the MFS, there exists the bound

\[
\text{Cond}(F) = \frac{\sigma_{\text{max}}(F)}{\sigma_{\text{min}}(F)} \leq CN^{(2+\tilde{\delta})} \left( \frac{R}{r_{\text{min}}} \right)^{\frac{N}{2}}, \tag{2.26}
\]

where \( \tilde{\delta} = \max\{0, \frac{\mu-1}{2}\} \).

### 2.3 Error Estimates

In this section, the error analysis is made for Laplace’s equation by the method of fundamental method (MFS), based on [22, 27, 2, 8]. The error bounds are derived for the mixed boundary problems in bounded simply-connected domains. Since the MFS can be classified into the Trefftz method (TM) using the FS, we may follow the analysis of TM in [22, 27], and pay an attention of the errors between harmonic polynomials and the solutions
of MFS in [2]. By our analysis, when the Laplace’s solutions are infinitely smooth, the exponential convergence rates can also be achieved as those of the TM in [27] (see Remark 3.1). However, when \( u \in H^p(S) (p > \frac{3}{2}) \), only the polynomial convergence rates are obtained. In this section, we have extended our analysis of the TM to that of the MFS. Then more interesting results of the analysis of MFS may follow [22, 27]. The MFS is exactly the TM using FS.

Choose

\[
\begin{align*}
  u_N &= \sum_{i=1}^{N} c_i \phi_i(r, \theta), \quad (r, \theta) \in S, \\
  \phi_i(r, \theta) &= \ln \sqrt{R^2 + r^2 - 2Rr \cos(\theta - \xi_i)}, \\
  \psi_i(r, \theta) &= \frac{\partial}{\partial \nu} \phi_i(r, \theta) = -\frac{a \cos(\theta - \xi_i) - 1}{\rho(a^2 + 1 - 2a \cos(\theta - \xi_i))},
\end{align*}
\]

(2.27)

with \( \xi_i = i h, h = \frac{2\pi}{N} \) and \( a = \frac{R}{\rho} > 1 \). In this section, only the error bounds of \( u_N \) by the TM are derived, and those by the CTM may follow [27, 29].

Let \( V_N \) denote the set of the admissible functions in (2.27). Denote the boundary norm

\[
\|v\|_B = \left\{ \|v\|_{0, \Gamma_D}^2 + w^2 \left\| \frac{\partial v}{\partial \nu} \right\|_{0, \Gamma_R}^2 + \alpha u \right\}^{\frac{1}{2}},
\]

(2.30)

where \( \|v\|_{0, \Gamma_D} \) is the Sobolev norm. The solution by the TM, (2.12), also satisfies

\[
\|u - u_N\|_B = \min_{v \in V_N} \|u - v\|_B.
\]

(2.31)

In Figure 2.1, denote

\[
S_{\min} \subset S \subset S_{\max},
\]

(2.32)

where

\[
\begin{align*}
  S_{\min} &= \{(r, \theta) | r \leq r_{\min}, \ 0 \leq \theta \leq 2\pi\}, \\
  S_{\max} &= \{(r, \theta) | r \leq r_{\max}, \ 0 \leq \theta \leq 2\pi\},
\end{align*}
\]

(2.33)
The MFS for Laplace Equation

where \( r_{\text{max}} \) and \( r_{\text{min}} \) are given in (2.4). Denote the fundamental solutions

\[
v_N = \sum_{i=1}^{N} c_i \ln |PQ_i|,
\]

(2.34)

where \( P \in (S \cup \partial S) \), and \( Q_i \) are given in (2.8)

\[
Q_i = (R \cos i\theta, R \sin i\theta),
\]

(2.35)

We obtain the following lemma.

**Lemma 2.3.1.** Let (2.32) and \( u \in H^p(S)(p > \frac{3}{2}) \) hold. Choose \( w = \frac{1}{N} \).

Suppose that \( N \) satisfies

\[
2^{2q+1} \left( \frac{R}{r_{\text{max}}} \right)^{-2N} < 1
\]

(2.36)

with \( q = 1 \) and

\[
\left( \frac{R}{r_{\text{max}}} \right)^{2n-N} \left( \frac{r_{\text{max}}}{r_{\text{min}}} \right)^n \leq n^{-p},
\]

(2.37)

where \( C \) is a constant independent of \( N \). Then the solution \( u_N \) by the MFS has the error bound,

\[
\| u - u_N \|_B = O \left( \frac{1}{N^{p-\frac{1}{2}}} \right).
\]

(2.38)

**Proof:** We have

\[
u(r, \theta) = P_n(r, \theta) + R_n(r, \theta),
\]

(2.39)

where \( P_n(r, \theta) \) is the harmonic polynomials of order \( n \),

\[
P_n(r, \theta) = \frac{\alpha_0}{2} + \sum_{i=1}^{n} r^i (\alpha_i \cos i\theta + \beta_i \sin i\theta),
\]

(2.40)

\( \alpha_i \) and \( \beta_i \) are the Fourier coefficients, and \( R_n = R_n(r, \theta) \) is the the residual. From \( u \in H^p(S) \), there exists the bound of \( R_n \),

\[
\| R_n(r, \theta) \|_{q,S} = \| u(r, \theta) - P_n(r, \theta) \|_{q,S}
\]

\[
= O \left( \frac{1}{N^{p-q}} \| u \|_{p,S} \right), \quad q < k.
\]

(2.41)
For the given $P_n(r, \theta)$, there exists the special solution of FS denoted as $u_N^* = \sum f N(r, \theta) \in V_N$ given in Bogomolny [2]. Then we have from (2.31)

$$\|u - u_N\|_B \leq \|u - u_N^*\|_B.$$  \hspace{1cm} (2.42)

Also from the embedding theorem [7], there exist the bounds,

$$\|v\|_0,\Gamma \leq C\|v\|_{\frac{1}{2},S}, \quad \|\partial v / \partial v\|_0,\Gamma \leq C\|v\|_{\frac{3}{2},S},$$  \hspace{1cm} (2.43)

where $C$ is a constant independent\(^2\) of $N$. Then we have from (2.30), (2.43) and $w = \frac{1}{N}$

$$\|v\|_B \leq \|v\|_0,\Gamma_D + w\|\partial v / \partial v\|_0,\Gamma_R + w\alpha\|v\|_0,\Gamma_R$$  \hspace{1cm} (2.44)

$$\leq (1 + w\alpha)\|v\|_{\frac{1}{2},S} + w\|v\|_{\frac{3}{2},S}$$

$$\leq C\left\{\|v\|_{\frac{1}{2},S} + w\|v\|_{\frac{3}{2},S}\right\}.$$  \hspace{1cm} 

Hence, we obtain from (2.42), (2.39) and the triangle inequality

$$\|u - u_N\|_B \leq C\left\{\|u - u_N\|_{\frac{1}{2},S} + w\|u - u_N^*\|_{\frac{3}{2},S}\right\}$$  \hspace{1cm} (2.45)

$$\leq C\left\{\|R_N(r, \theta)\|_{\frac{1}{2},S} + w\|R_N(r, \theta)\|_{\frac{3}{2},S}ight\}$$

$$+ \left(\|P_n(r, \theta) - u_N^*\|_{\frac{1}{2},S} + w\|P_n(r, \theta) - u_N^*\|_{\frac{3}{2},S}\right)$$

$$:= C(T_1 + T_2).$$

Since there exist the bounds from (2.41)

$$\|R_N\|_{\frac{1}{2},S} = O\left(\frac{1}{n^{p-k}}\right)\|u\|_{p,S}, \quad k = \frac{1}{2}, \frac{3}{2},$$  \hspace{1cm} (2.46)

we have from $w = \frac{1}{N}$

$$T_1 = \|R_N\|_{\frac{1}{2},S} + w\|R_N\|_{\frac{3}{2},S} = O\left(\frac{1}{n^{p-k}}\right).$$  \hspace{1cm} (2.47)

\(^2\)In the following, $C$ is always a constant independent of $N$, but the values of $C$ may be different in different texts.
Next, for $\Delta v = 0$, we have from Oden and Reddy [30], p. 195,

$$||v||_{k,S} \leq C ||v||_{k-\frac{1}{2}, \partial S^c}. \tag{2.48}$$

Then we have from (2.32), (2.48) and (2.43)

$$T_2 = \|P_n(r, \theta) - u_N^*\|_{\frac{1}{2}, S} + w\|P_n(r, \theta) - u_N^*\|_{\frac{1}{2}, S} \tag{2.49}$$

$$\leq \|P_n(r, \theta) - u_N^*\|_{S_{\max}} + w\|P_n(r, \theta) - u_N^*\|_{S_{\max}}$$

$$\leq C \{\|P_n(r, \theta) - u_N^*\|_{0, \partial S_{\max}} + w\|P_n(r, \theta) - u_N^*\|_{1, \partial S_{\max}}\}.$$

Under (2.36), there exists the special solution $u_N^* \in V_N$ of FS such that (see Bogomolny [2])

$$\|P_n(r, \theta) - u_N^*\|_{q, \partial S_{\max}} \leq C N^q \left(\frac{R}{r_{\max}}\right)^{2n-N} \left(\frac{r_{\max}}{r_{\min}}\right)^n \sqrt{n} \|P_n(r, \theta)\|_{0, \Gamma}. \tag{2.50}$$

When (2.36) with $q = 1$ is satisfied, we obtain from (2.49) and (2.50),

$$T_2 \leq C \left(\frac{R}{r_{\max}}\right)^{2n-N} \left(\frac{r_{\max}}{r_{\min}}\right)^n \sqrt{n} \|P_n(r, \theta)\|_{0, \Gamma}. \tag{2.51}$$

From $u \in H^p(S)(p > \frac{3}{2})$, we conclude that $u \in C(S)$ and $P_n(r, \theta) \approx u$ such that $\|P_n(r, \theta)\|_{0, \Gamma} = O(1)$. Eq. (2.51) leads to

$$T_2 \leq C \left(\frac{R}{r_{\max}}\right)^{2n-N} \left(\frac{r_{\max}}{r_{\min}}\right)^n \sqrt{n}. \tag{2.52}$$

Moreover, when Eq. (2.37) holds, i.e.,

$$N = 2n + \frac{n \ln(r_{\max}^n)}{\ln(R/r_{\min})} \leq C n, \tag{2.53}$$

We have

$$T_2 \leq C \frac{1}{n^{p-\frac{1}{2}}} \tag{2.54}$$

Combining (2.45), (2.47) and (2.54) gives the desired result (2.38) by noting $\frac{1}{n} \leq C \frac{1}{N}$. This completes the proof of Lemma 2.3.1.

---

3In the recent study in [20], the factor $\sqrt{n}$ in (2.50) can be removed.
Lemma 2.3.2. Suppose that there exists a positive constant $\mu$ independent of $N$ such that

$$|v|_{1,\Gamma_D} \leq CN^\mu \|v\|_{0,\Gamma_D}, \ v \in V_N, \quad (2.55)$$

where $V_N$ is the set of (2.10). For $\Delta v = 0$, there exists the bound,

$$\|v\|_{1, S} \leq C \left( N^\frac{\mu}{2} + \frac{1}{w} \right) \|v\|_B. \quad (2.56)$$

Proof: For the harmonic functions $v \in V_N$ in $S$, we have from Oden and Reddy [30]

$$\|v\|_{1, S} \leq C \left\{ \|v\|_{\frac{1}{2}, \Gamma_D} + \left\| \frac{\partial v}{\partial v} + \alpha_v \right\|_{-\frac{1}{2}, \Gamma_R} \right\}. \quad (2.57)$$

Since (2.55) we have

$$\|v\|_{\frac{1}{2}, \Gamma_D} \leq C \left\{ \|v\|_{1, \Gamma_D} \|v\|_{0, \Gamma_D} \right\} \leq CN^\frac{\mu}{2} \|v\|_{0, \Gamma_D} \quad (2.58)$$

and

$$\left\| \frac{\partial v}{\partial v} + \alpha_v \right\|_{-\frac{1}{2}, \Gamma_R} \leq C \left\| \frac{\partial v}{\partial v} + \alpha_v \right\|_{0, \Gamma_R}. \quad (2.59)$$

Then we have from (2.57) – (2.59) for $\Delta v = 0$,

$$\|v\|_{1, S} \leq C \left\{ \|v\|_{\frac{1}{2}, \Gamma_D} + \left\| \frac{\partial v}{\partial v} + \alpha_v \right\|_{-\frac{1}{2}, \Gamma_R} \right\} \quad (2.60)$$

$$\leq C \left\{ N^\frac{\mu}{2} \|v\|_{0, \Gamma_D} + \left\| \frac{\partial v}{\partial v} + \alpha_v \right\|_{0, \Gamma_R} \right\}$$

$$\leq C \left( N^\frac{\mu}{2} + \frac{1}{w} \right) \|v\|_B. \quad (2.60)$$

This completes the proof of Lemma 2.3.2.

From Lemmas 2.3.1 and 2.3.2, we have the following theorem.

Theorem 2.3.1. Let the conditions in Lemmas 2.3.1 and 2.3.2 hold. Then the numerical solutions by the MFS (i.e., the TM using FS) have the error bound,

$$\|u - u_N\|_{1, S} = O \left( \frac{1}{N^{p-t}} \right), \quad (2.61)$$

where $t = \frac{1}{2} + \max\{1, \frac{\mu}{2}\}$. 

To obtain the inverse inequality (2.55), we may also use a similar bound in Katsurada and Okamoto [14], Theorem A.1. In fact, a detailed analysis of (2.55) is given in [25], to show \( \mu \in [1, 2] \) for some simple cases.

**Remark 3.1.** For the infinite smooth function \( u \), we have \( u = P_n(r, \theta) + R_n(r, \theta) \), where the residual \( R_n \) is exponential convergence,

\[
\| R_n(r, \theta) \|_{1, S} = O(\sigma_1^p), \quad 0 < \sigma_1 \leq \sigma < 1. \tag{2.62}
\]

Hence we may choose the \( N \) such that

\[
n^q \left( \frac{R}{r_{\text{max}}} \right)^{2n-N} \left( \frac{r_{\text{max}}}{r_{\text{min}}} \right)^n \sqrt{n} \leq \sigma_2^n, \quad 0 < \sigma_2 < \sigma < 1, \tag{2.63}
\]

i.e.,

\[
N = 2n + \frac{n \ln \left( \frac{r_{\text{max}}}{r_{\text{min}}} \right) + \ln \frac{1}{\sigma_2} + \left( q + \frac{1}{2} \right) \ln n}{\ln (\frac{R}{r_{\text{max}}})} \leq Cn. \tag{2.64}
\]

Hence, by following the proof of Theorem 2.3.1, we have the exponential convergence,

\[
\| u - u_N \|_B \leq \| u - u_N^* \|_B \leq \| u_N^* - P_n(r, \theta) \|_{1, S} + \| R_n(r, \theta) \|_{1, S} \leq C \left( \sigma_1^n + \sigma_2^n \left( \frac{N}{n} \right)^q \right) = O(\sigma_1^n + \sigma_2^n). \tag{2.65}
\]

### 2.4 Numerical Experiments

Consider Motz’s problem (see Figure 2.2)

\[
\Delta u = 0, \tag{2.66}
\]

\[
u = 500 \quad \text{on} \ AB, \quad u = 0 \quad \text{on} \ DO, \tag{2.67}
\]

\[
\frac{\partial u}{\partial v} = 0 \quad \text{on} \ BC \cup CD \cup OA. \tag{2.68}
\]

There exists a singularity at \( O \) due to the intersection of the Dirichlet and
the Neumann boundary conditions. Since the fundamental functions (2.9) are smooth on $\partial S$, we first add the singular functions into (2.10), to obtain

$$u_N = \sum_{i=0}^{L} D_i r^{i+\frac{1}{2}} \cos \left( i + \frac{1}{2} \right) \theta + \sum_{i=1}^{N} c_i \phi_i(P),$$

(2.69)

where the singular particular solutions

$$\sum_{i=1}^{L} D_i r^{i+\frac{1}{2}} \cos \left( i + \frac{1}{2} \right) \theta$$

(2.70)

are given in [22]. To deal with the singularity, the admissible functions as in (2.69) are also in [1, 3] for an enrich MFS domain decomposition technique. Let $M$ denote the number of collocation points along $\overline{AB}$. Then the total number of collocation points on $\partial S$ in Figure 2.2 is $m = 6M$. The number of unknown coefficients in (2.69) is $L+N+1$. In our computation we choose

$$m = 6M \geq L+N+1 = n.$$ 

(2.71)
The MFS for Laplace Equation

From [29], using the Gaussian rule is beneficial to the accuracy of the singular coefficient $D_i$ in (2.70). Then we choose the Gaussian rules.

Define the boundary errors

$$k_{\|} = \left\{ \int_{\partial B \times \partial \Omega} \varepsilon^2 + \int_{\partial B \times \partial \Omega} \left( \frac{\partial \varepsilon}{\partial \nu} \right)^2 \right\}^{\frac{1}{2}},$$

where $\varepsilon = u - \bar{u}_N$. We use the SVD to solve (2.18) and (2.19). Choose $R = \sqrt{3}$ and $L = 3$ in (2.69), the errors and condition numbers are listed in Table 2.1. From Table 2.1, we can see the empirical rates for $N \leq 70$

$$\|\varepsilon\|_B = O(0.8^N), \quad \frac{\Delta D_0}{D_0} = O(0.8^N), \quad (2.72)$$

and

$$\text{Cond} = O(1.8^N), \quad \text{Cond}_{\text{eff}} = O(1.2^N). \quad (2.73)$$

From (2.72) and (2.73), both errors and condition numbers are exponential, with respect to $N$. Since $R = \sqrt{3}$ and $r_{\min} = 0.5$, we have from Theorem 2.2.1,

$$\text{Cond}(\mathbf{F}) \leq C N^{(2+\delta)} \left( \frac{\sqrt{3}}{0.5} \right)^{\frac{N}{2}} = C N^{(2+\delta)}(1.86)^N, \quad (2.74)$$

where $\delta = \max\{0, \frac{\mu-1}{2}\}$ with $\mu \in [1, 2]$. Evidently, Eqs. (2.73) and (2.74) are consistent with each other. Note that the growth rates of Cond_{eff} are lower than those of Cond, to indicate that Cond_{eff} is a better estimate on the true stability of rounding errors, also see [24].

From Table 2.1 we can see that the norm $\|\mathbf{x}\|$ is large, which is caused by small $\sigma_{\min}$. After $D_i$ and $c_i$ have been obtained, the final harmonic solutions are computed by (2.69). Note that since $R - r_{\max} \geq c_0 > 0$, $|\phi_i(P)| = \ln|\overrightarrow{PQ}|$ is bounded. Also the harmonic solutions of (2.66) – (2.68) satisfy that $|\mathbf{u}_N| \leq 1$. Then there must occur the severe subtraction cancelation in (2.69), which is another kind of instability. On the other hand, the Cond and the Cond_{eff} are used for the stability analysis for $D_i$ and $c_i$ by MFS, to evaluate the ill-conditioning of their rounding errors. Hence the final
The MFS for Laplace Equation

Table 2.1: The errors and condition numbers for Motz’s problem by the MFS with (2.69) for $R = \sqrt{3}$ and $L = 3$, where $M$ is the number of Gaussian nodes on $\overline{AB}$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>28</th>
<th>56</th>
<th>70</th>
<th>84</th>
<th>112</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>20</td>
<td>40</td>
<td>50</td>
<td>60</td>
<td>80</td>
</tr>
<tr>
<td>$|e|_R$</td>
<td>5.364(-5)</td>
<td>5.811(-8)</td>
<td>1.982(-8)</td>
<td>6.371(-9)</td>
<td>1.319(-9)</td>
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<tr>
<td>$D_0$</td>
<td>401.159855</td>
<td>401.162417</td>
<td>401.162434</td>
<td>401.162446</td>
<td>401.162451</td>
</tr>
<tr>
<td>$\frac{\partial\mathbf{u}}{\partial r}$</td>
<td>6.477(-6)</td>
<td>9.082(-8)</td>
<td>4.802(-8)</td>
<td>1.864(-8)</td>
<td>6.011(-9)</td>
</tr>
<tr>
<td>$D_1$</td>
<td>87.567622</td>
<td>87.6556399</td>
<td>87.6554638</td>
<td>87.6558065</td>
<td>87.6558657</td>
</tr>
<tr>
<td>$\frac{\partial\mathbf{u}}{\partial r}$</td>
<td>1.006(-3)</td>
<td>3.299(-6)</td>
<td>5.206(-6)</td>
<td>1.296(-8)</td>
<td>6.011(-9)</td>
</tr>
<tr>
<td>$D_2$</td>
<td>16.9236032</td>
<td>17.1955434</td>
<td>17.2084924</td>
<td>17.2189297</td>
<td>17.2271946</td>
</tr>
<tr>
<td>$\frac{\partial\mathbf{u}}{\partial r}$</td>
<td>2.786(-1)</td>
<td>2.458(-3)</td>
<td>1.707(-3)</td>
<td>1.101(-3)</td>
<td>6.219(-4)</td>
</tr>
<tr>
<td>$D_3$</td>
<td>87.6556399</td>
<td>87.6554638</td>
<td>87.6558065</td>
<td>87.6558657</td>
<td>87.6558657</td>
</tr>
<tr>
<td>$\frac{\partial\mathbf{u}}{\partial r}$</td>
<td>2.786(-1)</td>
<td>2.458(-3)</td>
<td>1.707(-3)</td>
<td>1.101(-3)</td>
<td>6.219(-4)</td>
</tr>
<tr>
<td>$\beta_k$</td>
<td>1.572(-5)</td>
<td>7.370(-8)</td>
<td>-9.758(-9)</td>
<td>-4.665(-9)</td>
<td>-4.640(-10)</td>
</tr>
<tr>
<td>$|\mathbf{x}|$</td>
<td>4.346(2)</td>
<td>1.297(4)</td>
<td>7.901(5)</td>
<td>1.660(8)</td>
<td>4.445(12)</td>
</tr>
<tr>
<td>$|\mathbf{b}|$</td>
<td>1.118(2)</td>
<td>7.906(1)</td>
<td>7.071(1)</td>
<td>6.455(1)</td>
<td>5.591(1)</td>
</tr>
<tr>
<td>$\sigma_{\text{max}}$</td>
<td>9.967(-1)</td>
<td>9.787(-1)</td>
<td>9.752(-1)</td>
<td>9.729(-1)</td>
<td>9.700(-1)</td>
</tr>
<tr>
<td>$\sigma_{\text{min}}$</td>
<td>6.106(-7)</td>
<td>5.851(-12)</td>
<td>1.308(-14)</td>
<td>3.727(-17)</td>
<td>2.381(-22)</td>
</tr>
<tr>
<td>Cond</td>
<td>1.632(6)</td>
<td>1.672(11)</td>
<td>7.458(13)</td>
<td>2.610(16)</td>
<td>4.074(21)</td>
</tr>
<tr>
<td>Cond$_{\text{eff}}$</td>
<td>4.213(5)</td>
<td>1.042(9)</td>
<td>6.844(9)</td>
<td>1.043(10)</td>
<td>5.282(10)</td>
</tr>
<tr>
<td>Cond$_{\text{EE}}$</td>
<td>7.112(6)</td>
<td>1.073(9)</td>
<td>7.269(9)</td>
<td>1.384(10)</td>
<td>1.205(11)</td>
</tr>
</tbody>
</table>

solutions (2.69) suffer in severe instability from both rounding errors and subtraction cancelation.

For the singularity at $O$ in Figure 2.2, the second technique is to adopt the local refinements of $P_j$ near $O$ (see Figure 2.2),

$$|\overline{OP_j}| = (jh)^q, \quad j = 1, 2, \cdots \quad (2.75)$$

where $q \geq 1$. Choose (2.10) with the FS only, where the coefficients are obtained from MFS. Afterwards, the leading coefficients in (2.70) can be evaluated by

$$D_\ell = \frac{2}{\pi r^{\ell+\frac{1}{2}}} \int_0^\pi u_N(r, \theta) \cos \left( \ell + \frac{1}{2} \right) \theta d\theta, \quad \ell = 0, 1, 2, \cdots \quad (2.76)$$

The leading coefficient $D_0$ indicates the intensity factor, which is important in both theory and application.
The uniform collocation points with $h = \frac{1}{M}$ are used on $\partial S$, except those on $\overline{OE} \cup \overline{OF}$ in Figure 2.2, where the collocation points $P_j$ are given by (2.75). In computation, we choose $|\overline{OE}| = |\overline{OD}| = \frac{1}{M}$ and retain the invariant $M$ of collocation points on $\overline{OA}$ and $\overline{OD}$. The radius $R = \sqrt{3}$ in (2.7) and $q = 4$ in (2.75) are chosen, based on our numerical experiments. For Motz’s problem, the errors and condition numbers are listed in Table 2.2, and the coefficients $c_i$ and $D_j$ in Table 2.3.

<table>
<thead>
<tr>
<th>$N$</th>
<th>28</th>
<th>42</th>
<th>56</th>
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<th>84</th>
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<tbody>
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<td>$M$</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
</tr>
<tr>
<td>$|\varepsilon|_B$</td>
<td>0.877(-2)</td>
<td>0.214(-3)</td>
<td>0.864(-5)</td>
<td>0.274(-6)</td>
<td>0.130(-7)</td>
</tr>
<tr>
<td>$D_0$</td>
<td>402.091842</td>
<td>405.634884</td>
<td>401.645653</td>
<td>403.9010585</td>
<td>401.5045143</td>
</tr>
<tr>
<td>$\frac{\Delta D}{D_0}$</td>
<td>0.232(-2)</td>
<td>0.112(-1)</td>
<td>0.728(-3)</td>
<td>0.443(-4)</td>
<td>0.728(-3)</td>
</tr>
<tr>
<td>$D_1$</td>
<td>85.4862012</td>
<td>83.1566289</td>
<td>86.5255205</td>
<td>84.9189393</td>
<td>86.8840334</td>
</tr>
<tr>
<td>$\frac{\Delta D}{D_1}$</td>
<td>0.248(-1)</td>
<td>0.513(-1)</td>
<td>0.312(-1)</td>
<td>0.683(-1)</td>
<td>0.347(-1)</td>
</tr>
<tr>
<td>$\frac{\Delta D}{D_2}$</td>
<td>0.193(-2)</td>
<td>0.518(-1)</td>
<td>0.261(-1)</td>
<td>0.312(-1)</td>
<td>0.881(-2)</td>
</tr>
<tr>
<td>$D_3$</td>
<td>-7.7106008</td>
<td>-8.1404903</td>
<td>-7.80776930</td>
<td>-8.0527179</td>
<td>-7.84821114</td>
</tr>
<tr>
<td>$\frac{\Delta D}{D_3}$</td>
<td>0.447(-1)</td>
<td>0.858(-2)</td>
<td>0.326(-1)</td>
<td>0.533(-1)</td>
<td>0.347(-1)</td>
</tr>
<tr>
<td>$\beta_n$</td>
<td>0.502(5)</td>
<td>0.214(7)</td>
<td>0.935(8)</td>
<td>0.471(10)</td>
<td>0.220(12)</td>
</tr>
<tr>
<td>$|\beta|_B$</td>
<td>0.232(-2)</td>
<td>0.112(-1)</td>
<td>0.728(-3)</td>
<td>0.443(-4)</td>
<td>0.728(-3)</td>
</tr>
<tr>
<td>$\sigma_{\max}$</td>
<td>0.865</td>
<td>0.865</td>
<td>0.865</td>
<td>0.865</td>
<td>0.865</td>
</tr>
<tr>
<td>$\sigma_{\min}$</td>
<td>0.293(-6)</td>
<td>0.257(-9)</td>
<td>0.153(-11)</td>
<td>0.150(-15)</td>
<td>0.983(-19)</td>
</tr>
<tr>
<td>Cond</td>
<td>0.295(7)</td>
<td>0.337(10)</td>
<td>0.566(13)</td>
<td>0.576(16)</td>
<td>0.880(19)</td>
</tr>
<tr>
<td>Cond$_{\text{eff}}$</td>
<td>0.758(4)</td>
<td>0.166(6)</td>
<td>0.553(7)</td>
<td>0.999(8)</td>
<td>0.298(10)</td>
</tr>
<tr>
<td>Cond$_{\text{EE}}$</td>
<td>0.794(4)</td>
<td>0.171(6)</td>
<td>0.566(7)</td>
<td>0.101(9)</td>
<td>0.303(10)</td>
</tr>
</tbody>
</table>

Table 2.2: The error norms and condition numbers for Motz’s problem by the MFS with local refinements for $R = \sqrt{3}$ and $q = 4$.

From Table 2.2, the following empirical rates can be observed:

$$\|\varepsilon\|_B = O(0.74^N), \quad \Delta D_0 \quad \frac{\Delta D_0}{D_0} = O(0.97^N),$$

$$\sigma_{\max} = O(1), \quad \sigma_{\min} = O(0.487^N),$$

$$\text{Cond} = O(2.05^N), \quad \text{Cond}_{\text{eff}} = O(1.37^N),$$

$$\|\beta\|_B = O(1.47^N).$$

Evidently, the Cond$_{\text{eff}}$ is much smaller than Cond. The norm $\|\beta\|$ is large. Note that some coefficients $c_i$ in Table 2.3 are huge and highly oscillating.
The MFS for Laplace Equation

<table>
<thead>
<tr>
<th>$D_0$</th>
<th>401.504514344611</th>
<th>$D_1$</th>
<th>86.8840334693767</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_2$</td>
<td>16.6395941822623</td>
<td>$D_3$</td>
<td>-7.8482111479329</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$c_{17}$</td>
<td>.512088955005D+07</td>
<td>$c_{59}$</td>
<td>.165586759379D+11</td>
</tr>
<tr>
<td>$c_{18}$</td>
<td>-.17686899498D+07</td>
<td>$c_{60}$</td>
<td>-.35761445585D+11</td>
</tr>
<tr>
<td>$c_{19}$</td>
<td>.895598609608D+07</td>
<td>$c_{61}$</td>
<td>.634402739441D+11</td>
</tr>
<tr>
<td>$c_{20}$</td>
<td>-.99648035454D+07</td>
<td>$c_{62}$</td>
<td>-.92121325213D+11</td>
</tr>
<tr>
<td>$c_{21}$</td>
<td>.986061783032D+07</td>
<td>$c_{63}$</td>
<td>.109312719485D+12</td>
</tr>
<tr>
<td>$c_{22}$</td>
<td>-.86704303939D+07</td>
<td>$c_{64}$</td>
<td>-.10599564313D+12</td>
</tr>
<tr>
<td>$c_{23}$</td>
<td>.677226463301D+07</td>
<td>$c_{65}$</td>
<td>.841202271679D+11</td>
</tr>
<tr>
<td>$c_{24}$</td>
<td>-.46990352789D+07</td>
<td>$c_{66}$</td>
<td>-.54819461868D+11</td>
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<tr>
<td>$c_{25}$</td>
<td>.289846804493D+07</td>
<td>$c_{67}$</td>
<td>.294891303375D+11</td>
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<tr>
<td>$c_{26}$</td>
<td>-.15916310560D+07</td>
<td>$c_{68}$</td>
<td>-.13193813281D+11</td>
</tr>
<tr>
<td>$c_{27}$</td>
<td>.78015841711D+06</td>
<td>$c_{69}$</td>
<td>.496146614829D+10</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Table 2.3: All coefficients for Motz’s problem by the MFS with local refinements with $R = \sqrt{3}$, $q = 4$, $M = 60$ and $N = 84$, where the order of $c_i$ is shown as that of $Q_i$ in Figure 2.2.

Hence, the subtraction cancelation occurs in the final harmonic solution (2.69). From (2.79) we can see

\[ \text{Cond} \approx \text{Cond}_{\text{eff}} \| x \|. \]  \hspace{1cm} (2.80)

### 2.5 Concluding Remarks

For numerical PDEs, once a method is designed by numerical algorithms with some preliminary computation, three important issues of analysis need to be studied: (1) Convergence with error bounds, (2) Stability with bounds of condition numbers, (3) Combined algorithms with other numerical methods, since no method is perfect. A numerical method, such as the finite element method (FEM), can stand as a competitive and popular method if its efficiency has been proven numerically, and if the theoretical analysis of the above three issues has been established. The book [22] focuses the third issue: combinations of different methods, accompanied with error and stability analysis. One important method in [22] is called the boundary approximation method (BAM), which is the very Trefftz method (TM).

Below we discuss the MFS. Since there have appeared numerous (even
I. Error Analysis. Since MFS is a special one of TM using fundamental solutions (FS) as basis functions (or admissible functions), the analytical framework of TM in [28, 21, 22] and the recent book [27] is also valid for MFS. Take Laplace’s equation for example. When the harmonic polynomials are chosen, the error bounds of TM are studied in [26]. Since the extra errors between FS and harmonic polynomials can be found from (2.50) in Bogomolny [2], the error bounds of the solutions by MFS are obtained. By following [2], the analysis of MFS for annular domains is given in [20], to retain the same polynomial convergence rates as in [2]. For Laplace’s equation, the error bounds of MFS are derived for the Dirichlet, the Neumann [23] and the Robin problems, as well as their mixed boundary problems in this paper. The error bounds can also be derived for the problems with interior boundary conditions, so that different FS with different source circles can be applied simultaneously, for complicated geometrical domains.

II. Stability Analysis. For MFS, the instability is the most severe issue of both computation and analysis, since both Cond and Cond_eff are huge, and since they grow exponentially as the number of fundamental solutions increases. From (2.72) & (2.73) and (2.77) & (2.79), we can see

\[
\text{Cond} = O \left( \frac{1}{\| \epsilon \|_B} \right)^\alpha, \quad \alpha > 1.
\]

(2.81)

Eq. (2.81) indicates that the rate of Cond grows faster than that of the error diminishing. Usually, the Cond and Cond_eff are so huge that the computation of MFS for problems of PDE is difficult under double precision. Of course, we may solicit Mathematics using unlimited working digits, to produce useful numerical solutions. However, the more the working digits are used, the more the CPU time and the computer storage are needed. From our experiments in Chen [4] to seek the extremely \( D_0 \) in (2.69) with more than 10000 significant digits, there exists the empirical relation,

\[
\text{CPU Time} = O(p^{2.4}),
\]

(2.82)

where \( p \) is the number of working digits.
Moreover, the ill-conditioning of MFS results from the infinitesimal $\sigma_{\text{min}}$ of the discrete matrix, which also causes the huge, oscillating coefficients $c_i$ in (2.69). Hence, for the final harmonic solution $u_N$ from (2.69), a new instability of substraction cancelation occurs, to lose $\ln_{10}(||x||)$ significant digits. Hence, the MFS is a monster in instability, who needs to be controlled. To reduce the ill-conditioning, the truncated singular value decomposition (TSVD) and the Tikhonov regularization should be employed, in order to control the ill-conditioning of MFS. For stability analysis of MFS, bounds of condition numbers are derived for Dirichlet, Neumann and Robin problems, as well as their mixed boundary problems not only for circular domains but also for bounded, simply-connected domains. Theorem 2.1 is for the mixed boundary problems in bounded simply-connected domains; details are reported in [25, 23]

III. Combined Methods of MFS. From Section 2.3, for Laplace’s equation, the errors of MFS will not be smaller than those of TM using harmonic polynomials. It is also proven in Schaback[31] that when $R \to \infty$, the FS goes to the harmonic polynomials. However, the instability of MFS (even with small $R$) is worse than that of TM using harmonic polynomials [27]. Hence, using TM with harmonic polynomials is superior to the MFS. Moreover, the MFS are not efficient for the corner singularity, either, except by some techniques, such as adding singular functions or local refinements in Section 4. On the other hand, the MFS is simple and easy for some domains with curved boundary. A nature strategy is to use the singular solutions to deal the corner singularity, and the FS for the rest of S. Hence, the TM using both FS and the singular solutions should first be adapted in the TM. By following [22], the algorithms and error analysis of combinations of MFS can be established. For the non-homogeneous PDE, or those with variable coefficients, the FEM is efficient. Hence for the complicated problems, we may combine FMS with FEM. Of course, the MFS can also be combined with other methods, such as the finite difference method (FDM), the finite volume method (FVM), etc. Details are reported in [19].

References


References


CHAPTER 3

Reconstruction of Inclusions or Cavities in Potential Problems Using the MFS

Carlos J.S. Alves and Nuno F.M. Martins

Abstract. We discuss the application of the Method of Fundamental Solutions (MFS) to an inverse potential problem that consists of detecting inclusions or cavities using a single boundary measurement on an external boundary. The application of the method of fundamental solutions presented here is closely related to a method introduced by Kirsch and Kress [12], in another context – obstacle detection in an exterior acoustic inverse scattering problem. We briefly address the identifiability questions on the shape reconstruction, presenting a counterexample for the case of Robin boundary conditions. Using fundamental solutions on auxiliary boundary curves we prove density results for separated data on the whole of the boundary or for Cauchy data on the accessible part of the boundary. This justifies the reconstruction of the solution from the Cauchy data using the MFS. Moreover, this connection relates the linear part of the Kirsch-Kress method to the direct MFS, using boundary operator matrices. Numerical examples are presented, showing good reconstructions for the shape and location with this MFS version of the Kirsch-Kress method. Although the method proved to be robust for noisy data, results can be improved using an iterative Quasi-Newton scheme here presented.

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3.1 Introduction
The Method of Fundamental Solutions has been mostly considered as a meshfree method for direct problems, since the first papers by Kupradze and Alekside [13], Oliveira [8] or Mathon and Johnston [16]. An excellent account of the MFS as a numerical method for direct problems can be found in the review paper by Fairweather and Karageorghis [9].

On the other hand the MFS is also being considered as a tool for inverse problems, mainly for Cauchy data reconstruction. An example of this application can be found in the early work by Kirsch and Kress [12], twenty years ago, where it was used to fit the boundary data in an external problem, in the context of obstacle detection for exterior acoustic inverse scattering problems. Recently, it has gained some popularity as a method to solve linear inverse problems, that involve Cauchy reconstruction (eg. [14]).

In this work, we aim to reconstruct the shape and location of the object using a single boundary Cauchy measurement on an accessible part of the external boundary. The buried object is either a rigid inclusion (defined by a Dirichlet boundary condition), a cavity (defined by a Neumann type boundary condition) or a more general inclusion (defined by a Robin type boundary condition).

The inverse problem that consists of the identification of inclusions or cavities in potential problems using single external boundary measurements, is a problem in nondestructive testing. It has been addressed in the literature, with applications to thermal imaging and corrosion detection (see for instance [4], [10], [11]) and more recently in [5] where unknown Robin boundary conditions were also considered. The identifiability questions are discussed in Section 2. In Section 3 we consider the numerical resolution of the inverse problem using the MFS. Following the original ideas of the Kirsch-Kress Method (KKM) as in [2], the numerical scheme can be divided in two parts:

(i) linear part - uses the MFS as a solver for the Cauchy problem,
(ii) nonlinear part - recover the level set.

This connection is made clear by analysing the operator matrices with boundary potentials for both the MFS and the KKM.

Density results play an important role for the justification of the MFS approximation (e.g. [3] for boundary approximations, [1] for domain ap-
Reconstruction of inclusions or cavities using the MFS

proximations). Unlike the classical case, the density results presented here allow for the approximation of both function and normal derivative, therefore justifying the Cauchy reconstruction.

In Section 3 we also present a method based on a reciprocity gap functional, introduced by Caldéron [6], to approximate the barycenter of a rigid inclusion.

Finally, in Section 4, several numerical examples show the potential of the method. It proved to be a fast numerical scheme that enables a good approximation of the location and shape of the unknown inclusion even with noisy data. To improve the approximations we use a Levenberg–Marquardt method, already presented in [15] (in the context of a Stokes problem), which lead to even more accurate results.

3.2 Direct and Inverse Problem

Consider a medium \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) with inclusions or cavities represented by \( \omega \). We assume that \( \Omega \) is an open bounded simply connected set and \( \omega \) is open, bounded and the disjoint union of simply connected sets, with regular \( C^1 \) boundaries \( \Gamma = \partial \Omega \) and \( \gamma = \partial \omega \) such that \( \overline{\omega} \subset \Omega \). We define the domain of propagation by

\[
\Omega_c := \Omega \setminus \overline{\omega}.
\]

Note that \( \partial \Omega_c = \Gamma \cup \gamma \).

Direct problem:
Given \( g \in H^{1/2}(\Gamma) \), determine \( \partial_n u \) on \( \Gamma \), such that \( u \) that satisfies

\[
(\mathcal{P}_{\Omega_c}) \left\{ \begin{array}{ll}
\Delta u = 0 & \text{in } \Omega_c \\
u = g & \text{on } \partial \Omega = \Gamma \\
\mathcal{B}u = 0 & \text{on } \partial \omega = \gamma
\end{array} \right.
\]

\( \mathcal{B} \) stands for one of the classic boundary trace operators: \( \mathcal{B}u = \tau_n u \) (Dirichlet, \( \omega \) is a rigid inclusion) or \( \mathcal{B}u = \tau_n^\omega u \) (Neumann, \( \omega \) is a cavity) or \( \mathcal{B}u = \tau_n^\omega u + Z \tau_n^\omega u \) (Robin, \( \omega \) is an inclusion). This direct problem is well–posed with solution \( u \in H^1(\Omega_c) \) (for the Robin condition, we must have \( Z \in L^\infty(\gamma) \) and \( Z \geq 0 \)).

Inverse problem:
Reconstruction of inclusions or cavities using the MFS

From the known (input function) \( g \) and the measured (output function) \( \partial_n u \) on a part \( \Sigma \) of the external boundary \( \Gamma = \partial \Omega \), we aim to identify the location and shape of the internal boundary \( \gamma = \partial \omega \) (and therefore the inclusion or the cavity \( \omega \)). It is well known that this is an ill posed and non-linear inverse problem.

**Remark 1.** The direct problem will also be addressed in the more general formulation

\[
(\bar{P}_{\Omega_c}) \left\{ \begin{array}{l}
\Delta u = 0 \quad \text{in } \Omega_c \\
\bar{B} u = \bar{g} \quad \text{on } \partial \Omega_c \end{array} \right. 
\]

(3.1)

where \( \bar{B} \) is the boundary operator defined by

\[
\bar{B} = a \tau_{\partial \Omega_c}^n + \tilde{Z}_a \tau_{\partial \Omega_c}
\]

(3.2)

with \( a \in \{0, 1\} \) continuous on \( \partial \Omega_c \) and on the set \( \sigma = \sigma^{-1}(0) \) we have \( \tilde{Z}_a |\sigma = 1 \). This latter restriction on \( \tilde{Z}_a \) gives the Dirichlet boundary condition on \( \Gamma \) or \( \gamma \). For a pure Neumann problem, i.e. \( a = 1 \) and \( \tilde{Z}_a = 0 \) both on \( \partial \Omega_c \), it is well-known that, in order to obtain uniqueness, a compatibility condition is required.

### 3.2.1 Uniqueness

As shown in [5], a single boundary measurement may not suffice to identify a part of the boundary with a Robin condition. For the particular case of identification of inclusions \( \omega \), we present the following example:

Consider the harmonic function in \( \mathbb{R}^2 \setminus \{0\} \) defined by

\[
u(x,y) = x - \frac{x}{x^2 + y^2}
\]

(3.3)

and the annular domain \( \Omega_c = \Omega \setminus \overline{\omega} \) where \( \Omega = B(0,P) = \{ x \in \mathbb{R}^2 : |x| < P \} \), \( \omega = B(0, \rho) \) and \( 0 < \rho < P \). On \( \gamma = \partial \omega \), we have

\[
\partial_n u|_\gamma = \frac{1 + \rho^2}{\rho^2} n \cdot e_1 \wedge u|_\gamma = \frac{\rho^2 - 1}{\rho} n \cdot e_1.
\]

Hence, \( u \) solves the direct problem

\[
\left\{ \begin{array}{l}
\Delta u = 0 \quad \text{in } \Omega_c \\
u = g \quad \text{on } \Gamma \\
\partial_n u + Z_\rho u = 0 \quad \text{on } \gamma
\end{array} \right.
\]
where $g$ is the restriction of $u$ to $\Gamma = \partial \Omega$ and

$$Z_p \equiv \frac{1 + \rho^2}{\rho(1 - \rho^2)}.$$

The function $\rho \to \frac{1 + \rho^2}{\rho(1 - \rho^2)}$ is not injective for $0 < \rho < 1$ and we can conclude that at least two circular inclusions generate the same Cauchy data on $\Gamma$.

The previous example also shows that the identification, from a single boundary measurement, is not possible for non homogeneous boundary conditions on the inclusion/cavity.

We now address the identification of inclusions or cavities defined by homogeneous Dirichlet or Neumann conditions.

**Theorem 2.** Assume that the boundary condition on the inclusion or cavity $\gamma$ is defined by a homogeneous Dirichlet or Neumann condition. If $\Sigma \subset \Gamma$ is an open set in the topology of $\Gamma$ and $g$ is not constant (null for the case of inclusions), the Cauchy data on $\Sigma$ determines $\gamma$ uniquely.

**Proof.** Suppose that $\Omega^1_\varepsilon$ and $\Omega^2_\varepsilon$ are different non-disjoint propagation domains with boundaries

$$\partial \Omega^1_\varepsilon = \Gamma \cup \gamma_1, \partial \Omega^2_\varepsilon = \Gamma \cup \gamma_2,$$

where $\gamma_j$ refer to the boundary of the inclusion/cavity $\omega_j$. Each $\omega_j (j = 1, 2)$ can be the disjoint union of several simply connected components $\omega_j = \cup \omega_{j,k}$ and therefore $\gamma_j = \cup \gamma_{j,k}$.

Denote by $u_1$ and $u_2$ the solutions of problems $(\mathcal{P}_{\Omega^1_\varepsilon})$ and $(\mathcal{P}_{\Omega^2_\varepsilon})$ respectively. We show that, if

$$\tau_{\Sigma}u_1 = \tau_{\Sigma}u_2, \tau_{\Sigma}^b u_1 = \tau_{\Sigma}^b u_2,$$

then $u_1$ is constant (null, for the case of inclusions) in $\Omega^\sim_\varepsilon$, where $\Omega^\sim_\varepsilon$ denotes the connected component of $\Omega^1_\varepsilon \cap \Omega^2_\varepsilon$ that contains $\Gamma$. By Holmgren’s theorem, the same Cauchy data on $\Sigma$ implies

$$u_1 = u_2 \text{ in } \Omega^\sim_\varepsilon.$$
Now, $\partial \Omega_\subset = \Gamma \cup \gamma_j \cup \gamma_j^c$ with $\gamma_j \subset \gamma_j^c$ and $\gamma_j^c \cap \gamma_j^c = \emptyset$. Without loss of generality assume that $\gamma_j^c$ is not empty and let $\Omega_{\gamma^c}$ be a simply connected component such that $\gamma_j^c \cap \gamma^c$ is not empty.

If $\Omega_1 \cap \Omega_2$ is connected, i.e., $\Omega_1 \cap \Omega_2 = \Omega_\subset$, then take $\sigma = \Omega_{\gamma^c} \setminus \overline{\Omega}_1 \subset \Omega_\subset$ which is an open set with boundary $\partial \sigma \subset \gamma_j^c \cup \gamma_1$. It is clear that $\Delta u_1 = 0$ in $\sigma$ and on $\gamma_1$ we have null Dirichlet/Neumann data. By analytic continuation, $u_1$ has also null Dirichlet/Neumann data on $\gamma^c_j$.

If $\Omega_1 \cap \Omega_2$ is not connected, take $\sigma$ as the connected component of $\Omega_1 \setminus \Omega_\subset$ that intersects $\Omega_{\gamma^c}$. Again, $\partial \sigma \subset \gamma^c_j \cup \gamma_1$ and in both cases we have

$$\begin{cases}
\Delta u_1 = 0 & \text{in } \sigma, \\
\mathcal{B} u_1 = 0 & \text{on } \partial \sigma,
\end{cases}$$

(3.4)

where $\mathcal{B}$ is the trace or the normal trace operator. Thus, $u_1$ is constant on $\sigma$ (null, for the Dirichlet boundary condition) and we conclude, by analytic continuation, that $u_1$ is constant/null in $\Omega_\subset$. This implies $g = u_1|\Gamma$ is constant/null and the contradiction with the hypothesis follows.

3.3 The MFS and its Connection to Kirsch-Kress Method

3.3.1 Multi-connected Domain - Approximation by the MFS

Suppose that $\Omega_\subset \subset \mathbb{R}^d$ ($d = 2,3$) is bounded, open and multi-connected. The complementary set $\mathbb{R}^d \setminus \Omega_\subset$ has several connected components, one exterior $\Omega^c = \mathbb{R}^d \setminus \overline{\Omega}$ and $N$ interior connected components (inclusions or cavities) $\omega_1, \ldots, \omega_N \subset \Omega$ with boundaries $\gamma_1 = \partial \omega_1$. Again, we will denote by $\omega$ the union of the components $\omega_i$ and $\gamma$ is $\partial \omega$, the union of their boundaries.

To apply the Method of Fundamental Solutions, we will consider artificial sets that will define the location of the point-sources. These admissible sets are $\tilde{\gamma} = \partial \tilde{\omega}$ internal regular boundaries of $\tilde{\omega}$, simply connected open sets such that $\overline{\tilde{\omega}} \subset \omega$. Again, we will denote by $\tilde{\omega}$ the union of the components $\tilde{\omega}_i$ and $\tilde{\gamma} = \partial \tilde{\omega}$, the union of the artificial inner boundaries. Finally, we define an external boundary $\tilde{\Gamma} = \partial \tilde{\Omega}$ with $\tilde{\Omega}$ an open unbounded set $\tilde{\Omega} \subset \Omega^c$ with a boundary that encloses the domain, $\tilde{\Omega} \subset \mathbb{R}^d \setminus \overline{\Omega}$. (The case of a bounded $\Omega$ is also possible, and theoretically simpler, but it gives worst numerical results when $\tilde{\Gamma}$ does not enclose the domain $\Omega$.)
Recall that the fundamental solution for the Laplacian is given by
\[
\Phi(x) = \begin{cases} 
-\frac{1}{2\pi} \log |x|, & \text{2D case} \\
\frac{1}{4\pi} |x|, & \text{3D case}
\end{cases}
\]
and define the source function \( \Phi_y(x) := \Phi(x-y) \).

Consider the single and double layer potential given in the integral form on a boundary \( B \):
\[
L_B \phi(x) = \int_B \Phi_x(y) \phi(y) dS_y, \quad x \in \mathbb{R}^d \setminus B,
\]
\[
M_B \psi(x) = \int_B \partial_n \Phi_x(y) \psi(y) dS_y, \quad x \in \mathbb{R}^d \setminus B,
\]
with \( \phi \in H^{-1/2}(B), \psi \in H^{1/2}(B) \). The boundary operator \( \mathcal{B}_B = \alpha \tau_B + \tau_B \) is defined by the trace \( \tau \) and normal trace \( \tau^a = \partial_n \) on \( B \) as in (3.2).

Consider the solution of the direct problem \( (\mathcal{P}_{\Omega}) \) represented in terms of the single layer potential
\[
u = L_B \chi + \sum_{k=1}^N L_B \Phi_k.
\]
(3.5)

On the boundary \( \Gamma \cup \gamma \), we separate the integral equations
\[
\mathcal{B}_\Gamma u = g, \quad \mathcal{B}_\gamma u = g_i \text{ where } g = \widetilde{g}|\Gamma, g_i = \widetilde{g}|\gamma,
\]
(3.6)
with \( g \in H^{(-1)^{p_0}/2}(\Gamma) \) and \( g_i \in H^{(-1)^{p_1}/2}(\gamma) \). This leads to the matrix operator system
\[
\begin{pmatrix}
\mathcal{B}_\Gamma L_{\Gamma} & \mathcal{B}_\Gamma L_{\gamma} & \cdots & \mathcal{B}_\Gamma L_{\gamma N} \\
\mathcal{B}_\gamma L_{\Gamma} & \mathcal{B}_\gamma L_{\gamma} & \cdots & \mathcal{B}_\gamma L_{\gamma N} \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{B}_\gamma L_{\Gamma} & \mathcal{B}_\gamma L_{\gamma} & \cdots & \mathcal{B}_\gamma L_{\gamma N}
\end{pmatrix}
\begin{pmatrix}
\chi \\
\phi_1 \\
\vdots \\
\phi_N
\end{pmatrix}
= 
\begin{pmatrix}
g \\
g_1 \\
\vdots \\
g_N
\end{pmatrix}.
\]
(3.7)

**Theorem 3.** The set \( S = \text{span}\left\{(\mathcal{B}_{\Gamma} \Phi_y, \ldots, \mathcal{B}_{\gamma} \Phi_y, \ldots, \mathcal{B}_{\gamma N} \Phi_y) : y \in \widehat{\Gamma} \cup \widehat{\gamma}\right\} \) is dense in \( H^{(-1)^{p_0}/2}(\Gamma)/\mathbb{R}^{p_0} \times \cdots \times H^{(-1)^{p_N}/2}(\Gamma)/\mathbb{R}^{p_N} \) (or \( H^{(-1)^{p_0}/2}(\Gamma) \times \cdots \times H^{(-1)^{p_N}/2}(\Gamma) \))
... \times H^{(-1)\partial \Gamma / 2}(\Gamma) in 3D), where

\[
p_k = \begin{cases} 
0 & \text{if } \int_{\Gamma_k} Z_a(x) dS_x = 0, \\
1 & \text{otherwise}
\end{cases}
\]

**Proof.** We prove that \(\ker(\mathcal{M}(\Gamma \cup \gamma)^*)\) is null, and this implies the density of the range of \(\mathcal{M}(\Gamma \cup \gamma)\). Discretizing the smooth single layer potentials, the density of the linear span follows.

Note that \((B_{\gamma} L_{y})^* = \tau_{\gamma}(aM_N + L_Z)\) because

\[
\langle B_{\gamma} L_{y} \phi, \psi \rangle_{\gamma} = \langle (a\tau_{\gamma} L_{y} + Z_{a} \tau_{\gamma} L_{y}) \phi, \psi \rangle_{\gamma} \\
= \int_{\gamma} (a\partial_n + Z_{a}(x)) \int_{\gamma} \Phi_{y}(x) \phi(y) dS_y \psi(x) dS_x \\
= \int_{\gamma} \int_{\gamma} (a\partial_n + Z_{a}(x)) \Phi_{y}(x) \psi(x) dS_x \phi(y) dS_y \\
= \langle aM_N \psi + L_{y} Z_{a} \psi, \phi \rangle_{\gamma}.
\]

Hence,

\[
\mathcal{M}(\Gamma, \gamma)^* = \begin{bmatrix} 
\tau_{\Gamma}(aM_{\Gamma} + L_{\Gamma} Z_{a}) & \tau_{\Gamma}(aM_{\gamma} + L_{\gamma} Z_{a}) & \ldots & \tau_{\Gamma}(aM_{N\gamma} + L_{N\gamma} Z_{a}) \\
\tau_{\gamma}(aM_{\Gamma} + L_{\Gamma} Z_{a}) & \tau_{\gamma}(aM_{\gamma} + L_{\gamma} Z_{a}) & \ldots & \tau_{\gamma}(aM_{N\gamma} + L_{N\gamma} Z_{a}) \\
\vdots & \vdots & \ddots & \vdots \\
\tau_{N\gamma}(aM_{\Gamma} + L_{\Gamma} Z_{a}) & \tau_{N\gamma}(aM_{\gamma} + L_{\gamma} Z_{a}) & \ldots & \tau_{N\gamma}(aM_{N\gamma} + L_{N\gamma} Z_{a})
\end{bmatrix}.
\]

In 2D we consider

\[
\mathcal{H}_{Z_{a}}(\gamma) = \left\{ \phi \in H^{-1/2}(\gamma) : \int_{\gamma} Z_{a}(x) \phi(x) dS_x = 0 \right\},
\]

which can be identified to \(H^{-1/2}(\gamma) / \mathbb{R}^{p_k}\).

Now, let \(\psi \in \mathcal{H}_{Z_{a}}(\Gamma)\) and \(\phi_k \in \mathcal{H}_{Z_{a}}(\gamma)\) (2D case) or \(\psi \in H^{-1/2}(\Gamma)\) and \(\phi_k \in H^{-1/2}(\gamma)\) (3D case) and define the harmonic function

\[
u(y) = (aM_{\Gamma} \psi + L_{\Gamma} Z_{a} \psi + \sum_{k=1}^{N} (aM_{N\gamma} \phi_k + L_{N\gamma} Z_{a} \phi_k))(y), \quad (y \in \mathbb{R}^d \setminus (\Gamma \cup \gamma)).
\]
Assume that $\varphi = (\psi, \phi_1, \ldots, \phi_k, \ldots, \phi_N) \in \text{Ker}(\mathcal{M}(\Gamma, \gamma)^*)$. Then, $\tau_{\Gamma} u = \mathcal{M}(\Gamma, \gamma)^* \varphi \cdot e_1 = 0$ and we can conclude that $u$ is the solution of the well–posed exterior problem (cf. [7])

$$
\begin{align*}
\Delta u &= 0 & \text{in } \hat{\Omega} \\
u &= 0 & \text{on } \hat{\Gamma} \\
u(y) &= \begin{cases} 
AZ_a \log|y| + \mathcal{O}(1) & \text{if } d = 2 \\
\mathcal{O}(|y|^{-1}) & \text{if } d = 3 
\end{cases} & |y| \to \infty
\end{align*}
$$

and uniqueness implies $u = 0$ in $\hat{\Omega}$. In the 2D case, uniqueness follows by taking the asymptotic behavior with constant

$$
AZ_a = -\frac{1}{2\pi} \left( \int_{\Gamma} Z_a(x) \psi(x) dS_x + \sum_{k=1}^{N} \int_{\gamma_k} Z_a(x) \phi_k(x) dS_x \right),
$$

hence $AZ_a = 0$ because $\psi \in \mathcal{H}(Z_a(\Gamma)), \phi_k \in \mathcal{H}(Z_a(\gamma_k))$. In 3D the asymptotic behavior is immediately verified by the fundamental solution.

By analytic continuation, $u = 0$ in $\hat{\Omega}$ implies $u$ is null in $\Omega^C$ and has null exterior traces $u|_{\Gamma}^{\pm}, \partial_n u|_{\Gamma}^{\pm}$. On the other hand, since

$$
u(y) = \int_{\Gamma} \partial_n \Phi_{y}(x) \psi(x) dS_x + \int_{\Gamma} Z_a(x) \Phi_{y}(x) \psi(x) dS_x \\
+ \sum_{k=1}^{N} \left( \int_{\gamma_k} (a \partial_n \Phi_{y}(x) + Z_a(x) \Phi_{y}(x)) \phi_k(x) dS_x \right)
$$

the boundary jumps across $\Gamma$ are given by $[u]|_\Gamma = a \psi \wedge [\partial_n u]|_\Gamma = -Z_a \psi$ and we conclude that

$$
u|_{\Gamma}^{\pm} = -a \psi \wedge \partial_n u|_{\Gamma}^{\pm} = Z_a \psi.
$$

Also, $u$ is the solution of the well–posed interior problems for each $j = 1, \ldots, N$,

$$
\begin{align*}
\Delta u &= 0 & \text{in } \hat{\omega}_j, \\
u &= 0 & \text{on } \hat{\gamma}_j = \partial \hat{\omega}_j,
\end{align*}
$$

hence, $u = 0$ in $\hat{\omega}_j$. Again, by analytic continuation this implies that $u = 0$ in each $\omega_j$ and that the exterior traces $u|_{\gamma_j}^{\pm}, \partial_n u|_{\gamma_j}^{\pm}$ are null, thus

$$
u|_{\gamma_j}^{\pm} = -a \phi_j \wedge \partial_n u|_{\gamma_j}^{\pm} = Z_a \phi_j.
$$
Therefore, \( u \) is also a solution of the interior problem
\[
\begin{align*}
\Delta u^- &= 0 & \text{in } \Omega_c \\
\alpha \partial_n u^- + \tilde{Z}_u^- u^- &= 0 & \text{on } \partial \Omega_c = \Gamma \cup \gamma
\end{align*}
\]
and by the well-posedness of this problem we have \( u^- = 0 \) in \( \Omega_c \). This implies that all traces \( u^-|_{\Gamma \cup \gamma} \) and \( \partial_n u^-|_{\Gamma \cup \gamma} \) are null and
\[
a \psi = [u]_{\Gamma} = 0, \quad a \phi_j = [u]_{\gamma_j} = 0, \quad Z_a \psi = -[\partial_n u]_{\Gamma} = 0 \wedge Z_a \phi_j = -[\partial_n u]_{\gamma_j} = 0.
\]
From the restrictions on \( a \) and \( \tilde{Z}_u \), we can conclude that \( \psi = \phi_j = 0 \).

In the MFS, the single layer representation is discretized, i.e.,
\[
u(x) \approx \sum_{i=1}^{n} \alpha_i \Phi_{y_i}(x)
\]
for some source points \( y_1, \ldots, y_n \in \Gamma \cup \gamma \). The coefficients \( \alpha_i \in \mathbb{R} \) are computed by solving the discretized version of the previous matrix operator system which leads to the linear system
\[
\begin{bmatrix}
B_{\Gamma} \Phi_{y_1}(x_1) & \cdots & B_{\Gamma} \Phi_{y_n}(x_1) \\
\vdots & \ddots & \vdots \\
B_{\gamma} \Phi_{y_1}(x_{l-1}+1) & \cdots & B_{\gamma} \Phi_{y_n}(x_{l-1}+1) \\
\vdots & \ddots & \vdots \\
B_{\gamma} \Phi_{y_1}(x_L) & \cdots & B_{\gamma} \Phi_{y_n}(x_L)
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{bmatrix}
= \begin{bmatrix}
g(x_1) \\
\vdots \\
g(x_L)
\end{bmatrix}
\]
at some collocation points \( x_1, \ldots, x_L \in \Gamma \) and \( x_{l-1}+1, \ldots, x_L \in \gamma \). The previous system can be solved in a least squares sense, for \( n < m \), or by interpolation using Gaussian elimination, for \( n = m \).
3.3.2 Kirsch-Kress Method using MFS

We now consider the original framework \((\mathcal{P}_\Omega)\) (recall that on \(\Gamma\) we are considering a Dirichlet boundary condition) and due to the identifiability results, we assume that \(\mathcal{B}_\gamma = \tau_\gamma\) or \(\mathcal{B}_\gamma = \tau_\gamma^n\). The inverse problem now consists in obtaining the shape of the boundary \(\gamma\) from the Cauchy data on \(\Gamma\).

In the Kirsch-Kress Method, initially presented for acoustic scattering, one assumes some knowledge on \(\gamma\), such that we can prescribe an artificial boundary \(\hat{\gamma}\) inside \(\gamma\) and write the solution in terms of the inner boundary layer representation. In the acoustic scattering problem the unknown density for the artificial inner boundary layer was recovered fitting its far field pattern. In the bounded domain we need to fit the Cauchy data and it is clear that the inner boundary will not be sufficient to adjust both Dirichlet and Neumann data. An extra external boundary layer must be considered.

The adaptation of the KKM for the inverse bounded problem presented here uses the MFS and involves two steps:

(a) **linear part**: solving the system of integral equations

\[
\begin{bmatrix}
\tau_\Gamma L_{\hat{\Gamma}} & \tau_\Gamma L_{\hat{\gamma}} \\
\tau_\Gamma^n L_{\Gamma} & \tau_\Gamma^n L_{\hat{\gamma}}
\end{bmatrix}
\begin{bmatrix}
\chi \\
\varphi
\end{bmatrix} =
\begin{bmatrix}
\frac{g}{\partial_n u}
\end{bmatrix}
\]

where \(\tau_\Gamma, \tau_\Gamma^n\) are, respectively, the trace and the normal trace on \(\Gamma\). Note that \(\mathcal{K}(\Gamma, \Gamma)\) is \(\mathcal{M}(\Gamma, \Gamma)\) with \(\mathcal{B}_\Gamma = \tau_\Gamma\) and \(\mathcal{B}_\gamma = \tau_\gamma^n\), when \(\gamma = \Gamma\) (see Fig. 3.1).

(b) **nonlinear part**: the boundary \(\gamma\) will be given by the level set \(u^{-1}(0) = \{x \in \Omega : u(x) = 0\}\), for the Dirichlet problem (for the Neumann problem, an optimization scheme similar to the one presented in section 4.1 can be used).

The linear part of the KKM to solve the Cauchy problem is therefore connected to the MFS. It may use the same boundary layers on \(\hat{\gamma}\) and on \(\hat{\Gamma}\) to give the solution of both direct problem from the boundary conditions on \(\gamma\) and \(\Gamma\), and the reconstruction of the solution from the Cauchy data on \(\Gamma\). In fact, the first line of the system (3.9) would be the same – known Dirichlet data. The same densities will verify both systems (3.7) and (3.9).
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Figure 3.1: Limit situation where the operator matrix arising from the direct MFS (on the left), \( M(\Gamma, \gamma) \), formally tends to \( K(\Gamma, \Gamma) \), the matrix arising from the KKM, used as a Cauchy solver in a different region of interest (on the right).

We now prove that a pair of Cauchy data can be retrieved using this MFS version of the KKM method.

**Theorem 4.** The set

\[ S_n = \text{span} \left\{ (\tau_\gamma \Phi_y, \psi^\Gamma_n \Phi_y) : y \in \hat{\Gamma} \cup \hat{\gamma} \right\} \]

is dense in \( H^{1/2}(\Gamma) / \mathbb{R} \times H^{-1/2}(\Gamma) \) (or \( H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \) in 3D).

**Proof.** We follow the proof of Theorem 3. The adjoint of the matrix operator \( K(\Gamma, \Gamma) \) defined in (3.9) is given by

\[
K(\Gamma, \Gamma)^* = \begin{bmatrix}
\tau_\gamma L_\Gamma & \tau_\gamma L_{\hat{\gamma}} \\
\psi^\Gamma_n L_\Gamma & \psi^\Gamma_n L_{\hat{\gamma}}
\end{bmatrix}^* = \begin{bmatrix}
\tau_\gamma L_\Gamma & \tau_\gamma M_\Gamma \\
\psi^\Gamma_n L_\Gamma & \psi^\Gamma_n M_\Gamma
\end{bmatrix}.
\]

In the 2D case, again the space \( \mathcal{H}_1(\Gamma) \) is identified as \( H^{-1/2}(\Gamma) / \mathbb{R} \). Let \( \psi \in \mathcal{H}_1(\Gamma), \phi \in H^{1/2}(\Gamma) \) (2D case) or \( \psi \in H^{-1/2}(\Gamma), \phi \in H^{1/2}(\Gamma) \) (3D case) and define the function

\[ u(y) = (L_\Gamma \psi + M_\Gamma \phi)(y), \]
a combination of single and double layer potentials defined on $\Gamma$. Following the same arguments as in Theorem 3, we prove that if $u = 0$ on $\hat{\Gamma}$ then by analytic continuation of the unique null solution of the interior and exterior problems (now $A_1 = -\frac{1}{2\pi} \int_{\Gamma} \psi(x) dS_x = 0$, because $\psi \in \mathcal{H}_1(\Gamma)$), we obtain $u = 0$ in $\mathbb{R}^d \setminus \Gamma$. Then $\psi = -[\partial_n u]|_{\Gamma} = 0$, $\phi = [u]|_{\Gamma} = 0$ and the result follows.

### 3.3.3 Retrieving the Barycenter

In this paragraph, we present a method based on a reciprocity gap functional to obtain an approximation for the barycenter, for $\omega$ simply connected. Recall that, the reciprocity gap functional is defined by

$$\mathcal{R}(v) = \int_{\Gamma} (v \partial_n u - u \partial_n v), \quad (3.10)$$

and that, by Green’s formula

$$\mathcal{R}(v) = \int_{\Omega_c} (v \Delta u - u \Delta v) - \int_{\gamma} (v \partial_n u - u \partial_n v). \quad (3.11)$$

Suppose that $u$ solves the direct problem ($\mathcal{P}_{\Omega_c}$) with null Dirichlet boundary condition on $\gamma$ and that $v$ is harmonic. Then,

$$\mathcal{R}(v) = \int_{\gamma} v \partial_n u, \quad (3.12)$$

assuming that the normal direction on $\gamma = \partial \omega$ points outwards with respect to the inclusion $\omega$.

Prescribing positive data on $\Gamma$ we have

$$W := \partial_n u > 0 \text{ on } \gamma \quad (3.13)$$

by the maximum principle (still assuming that the normal on $\gamma$ points outward with respect to $\omega$).

Taking $v_i = x_i$ we define

$$c^*_i = \frac{\mathcal{R}(x_i)}{\mathcal{R}(1)} = \frac{\int_{\gamma} x_i W}{\int_{\gamma} W}, \quad i = 1, 2.$$ 

Since $W > 0$ then these values act as weights in the previous sums and $c^* = (c^*_1, c^*_2)$ belongs to the convex hull of $\overline{\omega}$. We use the point $c^*$ as an approximation for the barycenter of the inclusion (Note that for $W$ constant we obtain the barycenter).
3.4 Numerical Simulations

We present three numerical simulations of the MFS–KKM applied to the reconstruction of a single (rigid) inclusion in 2D. The accessible part of the boundary, \( \Gamma \), is \( \partial B(0,3.5) \) and the boundary of the inclusion is given by the parametrization

\[
\gamma(t) = c_i + j_i(t)(\cos t, \sin t), \quad 0 \leq t \leq 2\pi
\]

with

\[
c_1 = (1, -1), \quad j_1(t) = 1.0 + 0.3\sin(4t),
\]

\[
c_2 = (0, 0), \quad j_2(t) = 0.8 + 1.8(\sin t \cos t/2),
\]

and

\[
\gamma_3(t) = (-0.2 + 0.9\cos t + 0.4\cos(2t))e_1 + 1.1\sin(t)e_2 - 1.2e_1 + 0.5e_2
\]

(see Fig. 3.2). As input function, we considered

\[
g \equiv 1.
\]

The Neumann data on \( \Gamma \) were generated synthetically by solving numerically the direct problem with the MFS. Here, we used the artificial boundaries \( \tilde{\Gamma} = \partial B(0,4.0) \) and \( \tilde{\gamma}(t) = 0.9(\gamma(t) - c_i) + c_i, 0 \leq t \leq 2\pi \) and considered 400 points on \( \tilde{\Gamma} \cup \tilde{\gamma} \) (source points) and the same amount on \( \Gamma \cup \gamma \) (collocation points). For the presented simulations, artificial random noise was added to the data (up to 5\%) and measured it at 80 uniformly distributed observation points, for all cases.

For the linear part of the MFS–KKM we started with the computation of the barycenter: In the first simulation we obtained \( c_1^* = (1.15, -1.14) \) as an approximation of the barycenter \( c_1 = (1, -1) \). The results for the other simulations are presented in Fig. 3.2.

Using the computed approximation of the barycenter \( c_1^* \), we considered as an artificial inner curve \( \tilde{\gamma} = \partial B(c_1^*, 0.3) \). This choice of artificial inner curve is based on the assumption that no a priori information regarding the dimension or shape of the inclusion is known. The other artificial curve was \( \tilde{\Gamma} = \partial B(0,5.5) \) and we used 160 source points and 80 collocation points. In order to avoid the so called inverse crimes the considered artificial boundaries for both direct MFS and KKM–MFS were different. Thus, no information regarding \( M(\Gamma, \gamma) \) was used to construct the matrix for the KKM–MFS, \( \mathcal{K}(\Gamma, \Gamma) \). The resulting system of linear equations was
solved using the Tikhonov regularization procedure (the choice of the parameter was based on the L–curve analysis). Fig. 3.3 shows a comparison, for the third simulation, between the initial (noisy) Cauchy data on \( \Gamma \) and the corresponding ones obtained with the MFS–KKM.

Figure 3.2: Geometry of the domains. Left– first test, middle– second test and right– third test. Red dots– computed barycenter, dashed curves– artificial inner curves \( \hat{\gamma}_i \).

For the reconstruction of the inclusion (non linear part of the MFS–KKM) we computed the level curve \( u^{-1}(0) \) based on the maximum principle and a radial search: The first ensures that a harmonic function in the domain \( \Omega_c \) with the Dirichlet data under consideration (recall that on \( \Gamma \), we have \( g = 1 \) and on \( \gamma \) zero) cannot be zero in that domain. Based on this property, we performed a search along several radial segments, joining \( \Gamma \) to \( \hat{\gamma}_i \), and choose, on each, a point with image near zero (see Fig. 3.4). The
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Figure 3.3: On the left– absolute error of the inverse problem’s solution on \( \Gamma \), on the right– comparison between the noisy Cauchy data on \( \Gamma \) (red dots) and the computed data (black line) for the third simulation.

set of these points is the approximation of \( \gamma \).

The numerical simulations obtained with this procedure showed that this fast method produces accurate reconstructions for both the barycenter (Fig. 3.2) and the shape (Fig. 3.5). Note that the previous procedures based on the maximum principle to solve the non linear part of the MFS–KKM and to compute an approximation of the barycenter do not have a trivial extension to problems where this principle does not apply. Still, we were able to retrieve some information about the inclusion’s geometry in a linear elastoeastic framework (cf. [2]).
3.4.1 Iterative MFS Approach to the Inverse Problem

Minimization based methods for the reconstruction of boundaries (or part of them) usually rely on computing the "best approximation" of the boundary in some space. By best approximation, we mean the curve that produces the closest Cauchy data to the available ones. These type of iterative methods requires, at each step, several solutions of direct problems. This makes them slower than the MFS–KKM. They are, however, more accurate. In this paragraph, we present the minimization procedure, already tested with success in [15].

Figure 3.4: Left– Segment defined by two points on $\Gamma$ and $\hat{\gamma}_1$ (black dots). Right– Solution of the inverse problem along the line of the left plot. The red dot is the computed approximation.
Consider, as space of representation the space of star like curves

\[ \Upsilon_c = \{ r(t)(\cos t, \sin t) + c : r(t) > 0, t \in [0, 2\pi] \} \]

where \( r \) is the distance to the point \( c \). The distance function \( r \) is represented in the subspace spanned by the trigonometric polynomials

\[ A_p = \text{span} \{ 1, \cos t, \sin t, \ldots, \cos (pt), \sin (pt) \}. \]

The goal is to search, iteratively, for a positive function \( r \in A_p \) such that the Cauchy data generated by the corresponding curve in \( \Upsilon_c \) is the best fit
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for the given data. In the discrete version we aim to solve the nonlinear optimization problem

$$\arg \min_{\alpha \in \mathbb{R}^{2p+1}} F(\alpha)$$

where

$$F(\alpha) := \left( \partial_n u, \partial_n u^{(k)} \right)_{\mathcal{L}_2(\Gamma)} = \sum_{i=1}^{k} \left( \partial_n u(x_i) - \partial_n u^{(k)}(x_i) \right)^2, \quad x_i \in \Gamma.$$ 

The vector $[\partial_n u(x_i)]$ represents the measured (eventually noisy) data, $u^{(k)}$ solves the problem

$$\begin{cases} 
\Delta u^{(k)} = 0 \quad &\text{in } \Omega_c^{(k)} \\
u^{(k)} = g \quad &\text{on } \Gamma \\
\partial_t u^{(k)} = 0 \quad &\text{on } \gamma^{(k)}
\end{cases}$$

and $\gamma^{(k)} = \gamma^{(k-1)}(t) + \delta \gamma^{(k-1)}(t)$ where the correcting term $\delta \gamma^{(k-1)}$ is defined by the radius function

$$\gamma^{(k-1)}(t) = \alpha \cdot (1, \cos t, \sin t, \ldots, \cos (pt), \sin (pt)),$$

given $\gamma^{(0)} \in \mathcal{Y}_c$.

We use the Levenberg–Marquardt method (eg. [17]) to solve the optimization problem (the gradient is approximated by finite differences) and the MFS for the numerical solution of the direct problems. We present the results for the first two simulations, with the same noisy data used in the KKM. As a starting curve, we took the approximation computed in the MFS–KKM simulation and introduced the barycenter as an optimization variable. The iterative procedure stops at the first $k$ such that

$$\frac{|F_k - F_{k-1}|}{F_{k-1}} \leq 0.01$$

where $F_{k-1}$ and $F_k$ are the values of the objective function $F$ at $\gamma^{(k-1)}$ and $\gamma^{(k)}$, respectively.

Fig. 3.6 shows the approximations in $A_4$. For both cases, few iterations were sufficient to obtain very accurate reconstructions. We tested the procedure in $A_8$, starting from the last curve computed in $A_4$ and no valid iteration was obtained (no descent direction was encountered). However, it
is expected that numerical instabilities arise with the increase of the number of degrees of freedom. As in [15], this phased approximation over several $A_p$, starting with a small $p$ and using the last curve as a starting curve for bigger $p$, leads to stable reconstructions and helps to avoid the convergence to some local extremum.

Figure 3.6: Approximation of $\gamma_i$ in $A_4$. Full line starting curve, dashed lines intermediate curves, dotted line last curve, full bold line $\gamma_i$. 
3.5 Conclusions

In this work we discussed the question of the identification of inclusions/cavities for the Laplace equation, using a single boundary measurement. We proved the adequacy of the MFS to solve not only the direct problem, but also the inverse (Cauchy) problem with a MFS version of the Kirsch-Kress Method. We proposed two reconstruction methods: A method based on the maximum principle for the MFS–KKM method and a Quasi Newton method. The first revealed to be a fast method and gave good results. The Quasi Newton based method, relying on the computation of several direct problems, produced better results but required bigger computational effort. This effort was substantially reduced by combining the MFS-KKM with the iterative method. The use of the MFS for the solution of the direct problems arising in the iterative method provided fast and accurate solutions, making the extension of this work to the 3D case computational affordable.

Acknowledgment

The authors acknowledge the financial support of Fundação para a Ciência e Tecnologia (FCT), through projects POCI MAT/61792/2004, MAT/60863/2004 and ECM/58940/2004. N. Martins is also partially supported by FCT through the scholarship SFRH/BD/27914/2006. The authors also thank the comments made by the referees.

References


CHAPTER 4

The Method of Fundamental Solutions for Solving Elliptic Partial Differential Equations with Variable Coefficients

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Abstract. In this paper we extend the method of fundamental solutions to solve general elliptic partial differential equations with variable coefficients through a new formulation that resembles both the dual reciprocity method (DRM) and Kansa’s method. As in the case of the DRM, the current approach keeps the Laplacian as the major differential operator. Unlike Kansa’s method using MQ as the sole basis function, we employed two basis functions which contain a radial basis function and the fundamental solution of Laplacian. In contrast to the traditional formulation which uses a two-stage process to find the particular solution and the homogeneous solution separately, the current approach only requires one matrix system. The particular solution and homogeneous solution can thus be obtained simultaneously.

Numerical results show the current approach is more stable and accurate using truncated singular value decomposition than LU decomposition. In particular, the results of the approximate derivatives are also excellent.

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4.1 Introduction

During the past decade, various types of meshless methods have emerged as effective numerical methods for solving problems in science and engineering which involve partial differential equations. The major attraction of meshless methods is to alleviate the difficulty of meshing the domain and boundary, particularly in a 3D case. Among all meshless methods, the method of fundamental solutions (MFS) has been proven to be a highly effective boundary meshless method when the fundamental solutions of the governing equations are available. The MFS was first introduced by Kupradze and Aleksidze [18] in 1964 as the method of generalized Fourier series. In 1977, Mathon and Johnston [20] further developed the MFS as a numerical technique. In the 1980’s, Fairweather and Karageorghis [9] made extensive contributions of the MFS through solving a large class of science and engineering problems. However, during this period of time, the MFS was essentially restricted to solving homogeneous partial differential equations. It was not until 1996 that Golberg et. al. [12] first proposed to extend the MFS to nonhomogeneous partial differential equations through the use of radial basis functions (RBFs). Many RBFs, such as the higher order conical functions, spline-types, multiquadric, compactly supported RBFs, etc., were introduced. Subsequently, the method has further extended to time-dependent problems, nonlinear problems, etc. Detailed reviews of the MFS can be found in several survey papers [8, 9, 10, 11]. The success of extending the MFS to various types of nonhomogeneous problems is largely due to the efficient evaluation of the particular solutions. Since the particular solution is not unique, which allows a wide variety of choices for its computation, many numerical techniques based on polynomial, trigonometric, and radial basis functions have been developed and are available in the literature [8, 11]. In 2001, Balakrishman and Ramachandran [3] proposed to combine the two parts of the solutions, the homogeneous and inhomogeneous solutions, for solving nonlinear Poisson problems of the form $\Delta u = f(x, u)$. Following a similar technique, Wang and Qin [25] extended the method to solving Poisson-type problems $\Delta u = f(x, u, u_x, u_y, u_{xx}, u_{yy})$. In the above mentioned approaches, one can only handle Poisson-type equations with constant coefficients. Recently, Alves and Valtchev [1] developed a Kansa-type scheme [13] using the MFS to solve a certain class of elliptic partial differential equations with variable coefficients. This is the first paper of the MFS to go beyond solving nonhomogeneous equations with constant coefficients. However,
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It is not clear if the convection equations with variable coefficients can be efficiently treated.

The derivation of the closed form particular solutions for a general differential operator is a non-trivial task. Before 1998, the derivation of a particular solution was restricted to the cases of Laplacian or biharmonic operators due to the difficulty of deriving the closed form particular solution using RBFs. It was not until the work of Chen and Rashed [5] that the construction of a particular solution of Helmholtz-type operators could be efficiently treated. The ability of deriving a particular solution for nonhomogeneous Helmholtz-type equations has made significant progress for effectively solving time-dependent problems via integral transformations or time-differencing [8, 11].

Once the particular solution of the given linear partial differential equation is available, the nonhomogeneous equation can be reduced to a homogeneous equation through the use of the method of particular solution [11]. A boundary meshless method such as the MFS can then be used to obtain the homogeneous solution. The original solution can be recovered by adding the particular solution and the homogeneous solution. This is a two-stage procedure that is commonly called the method of particular solutions (MPS) or splitting method. In this approach, we try to keep the nonhomogeneous term as simple as possible so that it can be approximated easily and accurately. It is known that the accuracy of the particular solution is directly related to the accuracy of interpolating the nonhomogeneous term of a given partial differential equation. On the other hand, by keeping the nonhomogeneous term simple, the differential operator will be more involved and the closed form particular solution will be more difficult to obtain. There are limited closed form particular solutions available beyond Helmholtz-type operators.

It is the purpose of this paper to develop a one-stage method similar to Balakrishnan and Ramachandran [3] and Wang and Qin [25] by combining the particular solution and homogeneous solution together in the solution process for solving partial differential equations with variable coefficients. Similar to the DRBEM [22, 23], the Laplacian is kept as the differential operator and all the other terms of the original differential operator are moved to the right hand side to become a part of the forcing term. As we shall see later, such a formulation requires two basis functions. The major advantage of such a formulation is that the closed form particular solutions for Laplacian are available for various RBFs and the
fundamental solution for the Laplacian is simple. Unlike the formulation of Helmholtz-type operators where the basis functions are restricted to polyharmonic functions, the current approach allows us to use a wide variety of basis functions, particularly multiquadric (MQ) or inverse MQ for superior convergence in the evaluation of the particular solution. Furthermore, our proposed method resembles Kansa’s method [13] which is very popular in the meshless community. In [19], the authors investigated the performance of Kansa’s method and the combined two-stage MFS-MPS. They concluded that there is no difference between these two methods in terms of numerical accuracy. As in the case of Kansa’s method, our new approach can solve practically all types of linear partial differential equations with variable coefficients. The major difference between the two methods is that Kansa’s method employs MQ as the basis function and we adopt the RBF and the fundamental solution as the basis functions. It is clear that our approach with a two basis system has a more sound mathematical foundation. In our numerical tests, we will demonstrate that the current approach outperforms Kansa’s method in both stability and accuracy, particularly in the evaluation of partial derivatives.

The structure of the paper is as follows. In Section 4.2, we briefly describe the MPS, the MFS, and Kansa’s method. In Section 4.3, we add the idea of the formulation of the DRM and combine the methods mentioned in Section 2 to establish an effective numerical scheme that will allow us to solve elliptical PDEs with variable coefficients. In Section 4.4, we test four examples to demonstrate the convergence, stability, and high accuracy of current method. In Section 4.5, we give a list of computational advantages of our numerical technique and discuss possible directions of future research.

4.2 Standard Formulation

In this section, we briefly review the methods that will be used later for the formulation of our new proposed method.

4.2.1 The Method of Particular Solutions

Let $\mathcal{L}$ be a linear second order elliptic partial differential operator with constant coefficients, $\mathcal{B}$ be a boundary operator, and let a fundamental solution of $\mathcal{L}$ be known. We consider the following model boundary value problem...
(BVP)

\[ \mathcal{L}u(x) = f(x), \quad x \in \Omega, \quad \text{(4.1)} \]
\[ \mathcal{B}u(x) = g(x), \quad x \in \partial \Omega, \quad \text{(4.2)} \]

where \( \Omega \subset \mathbb{R}^d, \ d = 2, 3 \), is a bounded domain with a sufficiently regular boundary \( \partial \Omega \). We assume that the boundary value problem (4.1)--(4.2) has a unique solution \( u \) for any given continuous nonhomogeneous term \( f \) and the boundary data \( g \).

In the MPS we split the solution, \( u \), of (4.1)--(4.2) into a particular solution and a homogeneous solution [2]. Let \( u = u_p + u_h \) where \( u_p \) is a particular solution satisfying the nonhomogeneous equation

\[ \mathcal{L}u_p(x) = f(x), \quad \text{(4.3)} \]

but does not necessarily satisfy the boundary condition in (4.2). Then the homogeneous solution, \( u_h \), satisfies

\[ \mathcal{L}u_h(x) = 0, \quad x \in \Omega, \quad \text{(4.4)} \]
\[ \mathcal{B}u_h(x) = g(x) - \mathcal{B}u_p(x), \quad x \in \partial \Omega. \quad \text{(4.5)} \]

Since a fundamental solution of \( \mathcal{L} \) is known, various boundary methods can be easily applied to solve (4.4)--(4.5). In order to exploit all the advantages of a mesh free boundary method, we employ the MFS for the solution of (4.4)--(4.5). We will elaborate on the MFS later.

The key issue here is how to compute an approximation to \( u_p \) for a general forcing term, \( f \), in (4.3). The classical way to evaluate \( u_p \) is based on the Newtonian potential

\[ u_p(x) = \int_\Omega G(x, y)f(y)dV(y), \quad \text{(4.6)} \]

where \( G(x, y) \) is the fundamental solution of \( \mathcal{L} \).

To overcome the difficulty of domain integration in evaluating \( u_p \), Nardini and Brebbia [22] introduced the DRM. The success of the DRM depends on how the right-hand side is approximated. Typically, this is done by approximating \( f \) with a finite series of basis functions

\[ f(x) \simeq \hat{f}(x) = \sum_{j=1}^n a_j \phi(r_j), \quad \text{(4.7)} \]
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where \( r_j = \|x - x_j\| \) and \( \{x_j\}_{j=1}^n \) are called the centers or trial points and \( \phi : \mathbb{R}_+ \to \mathbb{R} \) is a univariate function. The coefficients \( \{a_j\}_{j=1}^n \) are usually obtained by a collocation method; i.e., by solving

\[
\sum_{j=1}^{n} a_j \phi(r_{kj}) = f(x_k), \quad 1 \leq k \leq n, \tag{4.8}
\]

where \( r_{kj} = \|x_k - x_j\|, x_k \in \mathbb{R}^d, d = 2, 3 \). An approximate particular solution, \( \hat{u}_p \), to (4.3) is given by

\[
\hat{u}_p(x) = \sum_{j=1}^{n} a_j \Phi(r_j), \tag{4.9}
\]

where \( \Phi \) is obtained by analytically solving

\[
\mathcal{L} \Phi = \phi. \tag{4.10}
\]

An accurate approximation of \( u_p \) depends on how well \( f \) is approximated. Consequently, the appropriate choice of basis function, \( \phi \), is of considerable importance. On the other hand, \( \mathcal{L} \) also affects how \( \phi \) should be chosen. For instance, when \( \mathcal{L} = \Delta \) or \( \Delta^2 \), there are a wide range of \( \phi \) that can be chosen so that the closed form \( \Phi \) can be obtained. Among them, MQ, inverse MQ, and polyharmonic splines are the most popular choices. When \( \mathcal{L} = \Delta \pm \lambda^2 \), it is not a trivial task to obtain \( \Phi \) in (4.10) [21]. Up to date, it is known that \( \Phi \) is available only if we choose \( \phi \) to be the polyharmonic splines (\( r^{2k} \ln r \) in 2D or \( r^{2k-1} \) in 3D, \( k \geq 1 \)) or compactly supported radial basis functions in 2D [5, 21]. Little is known for differential operators other than Helmholtz-type operators. The mathematical analysis of deriving \( \Phi \) is very demanding. This is the major difficulty in the application of the MPS.

4.2.2 The Method of Fundamental Solutions

After the particular solutions have been evaluated, we solve the homogeneous equation in (4.4) and (4.5) by boundary methods. The MFS is an efficient and flexible boundary method. We will briefly describe the MFS which will serve as a foundation for our proposed new unified method described in the next section.

The basic idea of the MFS is to approximate the homogeneous solution, \( u_h \), by \( \hat{u}_h \), which can be expressed as a linear combination of fundamental
The MFS for PDEs with variable coefficients

\[
\hat{u}_h(x) = \sum_{j=1}^{m} b_j G(\rho_j), \quad x \in \Omega \cup \partial \Omega, \tag{4.11}
\]

where \( \rho_j = \|x - y_j\| \) is the distance between \( x \) and the source point, \( y_j \). Note that \( G(\rho) \) is the known fundamental solution of the elliptic linear differential operator \( \mathcal{L} \) in (4.4). The source points in the MFS may be considered as lying on a fictitious boundary \( \Gamma = \partial \hat{\Omega} \) of a region \( \hat{\Omega} \) containing \( \Omega \). It is an important issue to determine the optimal location of the fictitious boundary.

In general, there are two approaches used in choosing \( \{y_j\}_1^m \), fixed and adaptive [9]. In this paper, we focus on the fixed method where \( \{y_j\}_1^m \) are chosen \textit{a priori}. Much of the work in this direction has relied on the approximation results of Bogomolny [4] and Cheng’s convergence results [6] for the Dirichlet problem for Laplace’s equation when \( \Omega \) and \( \hat{\Omega} \) are concentric circles. In their work, it was shown that the accuracy of the approximation improves as \( \Gamma \) is moved farther away from \( \partial \Omega \). Cheng’s result was generalized by Katsurada and Okamoto [16, 17, 15, 14], who showed that if \( \partial \Omega \) is a closed Jordan curve in the plane and data are analytic, then

\[
\|u - u_m\|_\infty \leq c(r/R)^m
\]

where \( r \) and \( R \) are the diameters of \( \Omega \) and \( \hat{\Omega} \) respectively.

Once the source points have been chosen, the \( \{b_j\}_{j=1}^m \) in (4.11) are generally obtained by collocation. That is, \( m \) points, \( \{x_i\}_{i=1}^m \), are chosen on \( \partial \Omega \) and then \( \{b_j\}_{j=1}^m \) satisfies (4.5); i.e.,

\[
\sum_{j=1}^{m} b_j B G(\rho_{ij}) = g(x_i) - B u_p(x_i), \quad 1 \leq i \leq m, \tag{4.12}
\]

where \( \rho_{ij} = \|x_i - y_j\| \).

Despite the ill-conditioning of the matrix \( [B G(\rho_{ij})] \), the accuracy of the numerical solution is largely unaffected if the boundary conditions are smooth and exact for the homogeneous equation [7, 11]. For nonhomogeneous problems, the error produced by the particular solution will have an impact on the homogeneous solution. In (4.12), the error in the evaluation of \( B u_p \) has the same effect as noise added to the boundary. As a result, the ill-conditioning of the MFS will magnify the error introduced in the boundary. As such, instead of using Gaussian elimination, the truncated singular
value decomposition (TSVD) is recommended in the solution process to reduce the effect of ill-conditioning in the MFS. For more details, we refer readers to Reference [7]. We will illustrate this issue through examples in the later section of numerical results.

4.2.3 Kansa’s Method

We consider the following elliptic partial differential equation in 2D

\[ \Delta u + \alpha(x,y) \frac{\partial u}{\partial x} + \beta(x,y) \frac{\partial u}{\partial y} + \gamma(x,y) u = f(x,y), \quad (x,y) \in \Omega, \quad (4.13) \]

\[ \mathcal{B}u = g(x,y), \quad (x,y) \in \partial\Omega, \quad (4.14) \]

where \( \alpha(x,y), \beta(x,y), \gamma(x,y), f(x,y), \) and \( g(x,y) \) are given functions. \( \mathcal{B} \) is a boundary differential operator.

Let \( \{(x_j,y_j)\}_{j=1}^n \) be \( n \) distinct collocation points in \( \hat{\Omega} \) of which \( \{(x_j,y_j)\}_{i=1}^n \) are interior points and \( \{(x_j,y_j)\}_{i=n+1}^n \) are boundary points. MQ is one of the most widely adopted RBFs in Kansa’s method. Though other RBFs can be used, we consider only the MQ basis function:

\[ \phi_j(x,y) = \sqrt{r_j^2 + c^2}, \quad r_j = \sqrt{(x-x_j)^2 + (y-y_j)^2}, \]

where \( c \) is a free parameter and often called the shape parameter of MQ.

By direct differentiation, we have

\[ \frac{\partial \phi_j}{\partial x} = \frac{x-x_j}{\sqrt{r_j^2 + c^2}}, \quad \frac{\partial \phi_j}{\partial y} = \frac{y-y_j}{\sqrt{r_j^2 + c^2}}, \quad (4.15) \]

\[ \frac{\partial^2 \phi_j}{\partial x^2} = \frac{(y-y_j)^2 + c^2}{(r_j^2 + c^2)^{3/2}}, \quad \frac{\partial^2 \phi_j}{\partial y^2} = \frac{(x-x_j)^2 + c^2}{(r_j^2 + c^2)^{3/2}}. \quad (4.16) \]

For the elliptic problem (4.13)–(4.14), the main idea of Kansa’s method is to approximate the solution, \( u \), by \( \phi \); i.e.,

\[ \hat{u}(x,y) = \sum_{j=1}^n c_j \phi_j(x,y), \quad (4.17) \]

where \( \{c_j\}_{j=1}^n \) are coefficients to be determined. Then, from (4.13) – (4.14), we have
\[
\sum_{j=1}^{n} c_j \left( \Delta \phi_j + \alpha \frac{\partial \phi_j}{\partial x} + \beta \frac{\partial \phi_j}{\partial y} + \gamma \phi_j \right)(x_i,y_i) = f(x_i,y_i), \quad i = 1, 2, \cdots, n_i, \tag{4.18}
\]

\[
\sum_{j=1}^{n} c_j \beta \phi_j(x_i,y_i) = g(x_i,y_i), \quad i = n_i + 1, \cdots, n, \tag{4.19}
\]

which is an \( n \times n \) linear system for the unknowns \( \{c_j\}_{j=1}^{n} \). More specifically, from (4.15) – (4.16), (4.18) becomes

\[
\sum_{j=1}^{n} c_j \left( \frac{r_j^2 + 2c^2}{(r_j^2 + c^2)^{3/2}} + \frac{\alpha (x - x_j) + \beta (y - y_j)}{\sqrt{r_j^2 + c^2}} + \gamma \sqrt{r_j^2 + c^2} \right) = f(x_i,y_i),
\]

for \( i = 1, 2, \cdots, n_i. \) Once \( \{c_j\}_{j=1}^{n} \) are known through solving (4.19) – (4.20), the approximation of \( u \) and its derivatives can be obtained. In general, Kansa’s method is very effective for the approximation of \( u \), but the approximation of the derivatives of \( u \) are normally not as accurate as we might expect.

### 4.3 Methodology

There are disadvantages to the above mentioned two-stage formulation in Section 4.2.1. First, closed form particular solutions, \( \Phi \), are available only for simple differential operators, \( \mathcal{L} \). Second, for Helmholtz-type differential operators or more complicated differential operators, the availability of RBFs are often restricted to polyharmonic splines [5, 21]. As such, the accuracy of the solution of the partial differential equation is limited. Third, such approach can only be used to solve partial differential equations with constant coefficients. In this section, we introduce a new formulation, without the need for the two-stage approach, to alleviate the difficulties mentioned above. For simplicity, we only consider the formulation in 2D cases. For 3D cases, a similar procedure can be applied.

We consider solving the elliptic partial differential equation (4.13) – (4.14). Since the general differential operators were variable coefficients, the fundamental solution is not available. Similar to the formulation of the DRBEM [22, 23], the Laplacian is kept on the left hand side and all the
other terms with differential operators containing reaction and convection terms are moved to the right hand side and are treated as the forcing term. In this way, (4.13) is rearranged into Poisson’s equation; i.e.,

$$
\Delta u(x, y) = F \left( x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right), \quad (x, y) \in \Omega,
$$

(4.21)

where

$$
F \left( x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) = -\alpha(x, y) \frac{\partial u}{\partial x} - \beta(x, y) \frac{\partial u}{\partial y} - \gamma(x, y) u + f(x, y).
$$

(4.22)

First, we assume the solution can be directly approximated by the sum of the particular solution and homogeneous solution [3, 25]

$$
u(x, y) \approx \hat{u}(x, y) = \sum_{j=1}^{n} a_j \Phi(r_j) + \sum_{j=1}^{m} b_j G(\rho_j),
$$

(4.23)

where $r_j$ and $\rho_j$ are defined in the same way as in (4.7) and (4.11) respectively. Due to the reformulation of (4.13) into (4.21), we choose the fundamental solution in (4.23) as $G(\rho) = \ln \rho$. We will soon illustrate how we choose $\Phi(r)$ as a basis function. Figure 4.1 shows the distribution of interior collocation points in $\Omega$, boundary collocation points on $\partial \Omega$, and source points on the fictitious boundary on $\Gamma$. Furthermore,

$$
\frac{\partial u}{\partial x} \approx \frac{\partial \hat{u}}{\partial x} = \sum_{j=1}^{n} a_j \frac{\partial \Phi}{\partial x}(r_j) + \sum_{j=1}^{m} b_j \frac{\partial G}{\partial x}(\rho_j),
$$

(4.24)

$$
\frac{\partial u}{\partial y} \approx \frac{\partial \hat{u}}{\partial y} = \sum_{j=1}^{n} a_j \frac{\partial \Phi}{\partial y}(r_j) + \sum_{j=1}^{m} b_j \frac{\partial G}{\partial y}(\rho_j).
$$

(4.25)

In (4.23), we apply two different basis functions with two different distance functions to approximate the solution of (4.13)–(4.14). Instead of finding the particular solution and homogeneous solution separately, we intend to obtain the undetermined coefficients $\{a_j\}_{j=1}^{n}$ and $\{b_j\}_{j=1}^{m}$ simultaneously. In (4.21), we reformulated the given differential equation (4.13) into Poisson’s equation where the particular solutions for various basis functions are well-known. Using RBFs, $\Phi$, to approximate $F$, we have

$$
F \left( x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) = \sum_{j=1}^{n} a_j \phi(r_j).
$$

(4.26)
Furthermore, we note that $\Delta G(\rho_j) = 0$ for $(x, y) \in \Omega$ and $(x_j, y_j) \in \Gamma$. As a result, from (4.23), we have

$$\Delta u \simeq \Delta \hat{u} = \sum_{j=1}^{n} a_j \Delta \Phi(r_j) + \sum_{j=1}^{m} b_j \Delta G(\rho_j)$$

$$= \sum_{j=1}^{n} a_j \phi(r_j) \quad \text{in} \quad \Omega,$$

(4.27)

where $\Delta \Phi(r) = \phi(r)$. Replacing the Laplacian in (4.21) using (4.27), we obtain

$$\sum_{j=1}^{n} a_j \phi(r_j) = -\alpha(x,y) \frac{\partial \hat{u}}{\partial x} - \beta(x,y) \frac{\partial \hat{u}}{\partial y} - \gamma(x,y) \hat{u} + f(x,y), \quad \text{for} (x,y) \in \Omega.$$

(4.28)

Using (4.23) – (4.25), the above expression (4.28) can be rearranged as follows
\[ \sum_{j=1}^{n} a_j \Psi(r_j) + \sum_{j=1}^{m} b_j \Theta(\rho_j) = f(x,y), \quad \text{for } (x,y) \in \Omega, \quad (4.29) \]

where

\[ \Psi(r_j) = \phi(r_j) + \alpha(x,y) \frac{\partial \Phi}{\partial x}(r_j) + \beta(x,y) \frac{\partial \Phi}{\partial y}(r_j) + \gamma(x,y) \Phi(r_j), \quad (4.30) \]

\[ \Theta(\rho_j) = \alpha(x,y) \frac{\partial G}{\partial x}(\rho_j) + \beta(x,y) \frac{\partial G}{\partial y}(\rho_j) + \gamma(x,y) G(\rho_j). \quad (4.31) \]

The boundary condition in (4.14) becomes

\[ \sum_{j=1}^{n} a_j \mathcal{B} \Phi(r_j) + \sum_{j=1}^{m} b_j \mathcal{B} G(\rho_j) = g(x,y), \quad (x,y) \in \partial \Omega. \quad (4.32) \]

It is clear that \( \Psi(r_j), \Theta(\rho_j), \) and \( \Phi(r_j) \) in (4.29) and (4.32) are completely known functions. In particular, \( \Psi(r_j) \) contains the radial basis function \( \phi \), the particular solution \( \Phi \), and the derivatives \( \frac{\partial \Phi}{\partial x} \) and \( \frac{\partial \Phi}{\partial y} \). Since the differential operator in (4.21) is the Laplacian, we have

\[ G(\rho) = \ln \rho, \quad \frac{\partial G(\rho)}{\partial x} = \frac{x}{\rho^2}, \quad \frac{\partial G(\rho)}{\partial y} = \frac{y}{\rho^2}. \]

If we choose \( \phi = \sqrt{r^2 + c^2} \) (MQ), it is known that [11]

\[ \Phi(r) = \frac{1}{9} (4c^2 + r^2) \sqrt{r^2 + c^2} - \frac{c^3}{3} \ln \left(c + \sqrt{r^2 + c^2}\right), \quad (4.33) \]

\[ \frac{\partial \Phi(r)}{\partial x} = \frac{x \left(c \sqrt{r^2 + c^2} + 2c^2 + r^2\right)}{3 \left(c + \sqrt{r^2 + c^2}\right)}, \quad \frac{\partial \Phi(r)}{\partial y} = \frac{y \left(c \sqrt{r^2 + c^2} + 2c^2 + r^2\right)}{3 \left(c + \sqrt{r^2 + c^2}\right)} \quad (4.34) \]

For the implementation, we choose \( n \) interior points in \( \Omega \), \( m \) boundary points on \( \partial \Omega \), and \( m \) source points on the fictitious boundary \( \Gamma \) (see Figure 4.1). By collocation, we have thus formulated a system of equations of order \( (m + n) \times (m + n) \); i.e.,

\[ \begin{bmatrix} \Psi(r_{ij}) & \Theta(\rho_{ij}) \\ \mathcal{B} \Phi(r_{ij}) & \mathcal{B} G(\rho_{ij}) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \quad (4.35) \]
where $\mathbf{a} = [a_1 \ a_2 \ \cdots \ a_n]^T$, $\mathbf{b} = [b_1 \ b_2 \ \cdots \ b_m]^T$, $\mathbf{f} = [f(x_1,y_1) \ f(x_2,y_2) \ \cdots \ f(x_n,y_n)]^T$, $\mathbf{g} = [g(x_1,y_1) \ g(x_2,y_2) \ \cdots \ g(x_m,y_m)]^T$. Once $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=1}^m$ are determined by solving (4.35), the approximate solution of $u$ can be obtained from (4.23).

The major improvement of this proposed method is that we can extend the MFS to not only nonhomogeneous differential equations, but also to a large class of elliptic partial differential equations with variable coefficients. Contradicting the conventional wisdom that sophisticated particular solutions and fundamental solutions are required for the solution of partial differential equations, in this case all are expressed in simple terms. Furthermore, as we know that MQ converges exponentially for smooth functions, we have reason to expect rapid convergence in this solution process. As we shall see in the next section, high accuracy and rapid convergence were observed in our numerical tests. Another attractive feature of the current approach is its simplicity in implementation.

Note that $\Delta u$ in (4.27) can be approximated without taking second derivatives of $u$ with respect to $x$ and $y$. As such, the numerical accuracy and stability are preserved since taking the derivative numerically is not a stable process. This gives a great advantage over Kansa’s method [13]. On the other hand, the convection terms $\partial u / \partial x$ and $\partial u / \partial y$ are integrated naturally into the matrix system in (4.35) without additional treatment which is a significant advantage over the DRBEM [22, 23]. In other words, while the current approach resembles both Kansa’s method and the DRBEM, it is also different from them through the above attractive features.

Unlike the two-stage process of the MPS described in Section 4.2.1, which requires solving two smaller matrix systems for the particular solution and the homogeneous solution individually, the current approach requires only one system of equations to be solved. The selection of the shape parameter of MQ and fictitious boundary of the MFS are basically the same as the two-stage case. These computational issues will be further illustrated in the next section.

### 4.4 Numerical Results

To demonstrate the effectiveness of the current approach, four numerical examples are presented in this section. The first three examples involve the modified Helmholtz equations with constant and variable coefficients. In the second example, we experimentally examine the convergence rate of the current approach. The fourth example is the convection-diffusion
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equation with variable coefficients, which is difficult using other standard numerical methods. In the first example, we examine the performance of different matrix solvers for the proposed method. All the numerical results are compared with both the analytical solution and the Kansa’s method. In addition, we propose a useful guide for determining the optimal shape parameter of MQ, \( \epsilon \), by tracking the residual with the known boundary conditions. In the following examples, we provide a detailed comparison and the performance for the current method.

Through all the numerical tests in this section, we use the following formula to choose the location of the source points in the MFS:

\[
x^s = x^b + \sigma \left( x^b - x^c \right),
\]

where \( x^s, x^b, \) and \( x^c \) denote the source, boundary, and central nodes. The parameter \( \sigma \) determines how far away the source points from the boundary. For the evaluation of the numerical accuracy, we use the root mean square error (RMSE) and the root mean square error of the derivative with respect to \( x \) (RMSEx), which are defined as follows:

\[
RMSE = \sqrt{\frac{1}{nt} \sum_{j=1}^{nt} (\hat{u}_j - u_j)^2},
\]

\[
RMSE_x = \sqrt{\frac{1}{nt} \sum_{j=1}^{nt} \left( \frac{\partial \hat{u}_j}{\partial x} - \frac{\partial u_j}{\partial x} \right)^2},
\]

where \( nt \) is the number of testing nodes chosen randomly in the domain. \( \hat{u}_j \) denotes the approximate solution at the \( j^{th} \) node. We shall not present the numerical results for RMSEy since they are very similar to RMSEx.

How to identify the optimal shape parameter of MQ is a challenge. For this purpose, we propose using the given boundary conditions to identify it since the boundary conditions are known and can be considered part of the analytical solution. Comparing the boundary conditions is similar to comparing the analytical solution. The definition of the residual of Dirichlet
and Neumann boundary conditions are defined as follows:

\[
\text{Residual (Dirichlet)} = \frac{1}{nr} \sum_{j=1}^{nr} |\tilde{u}_j - g_j|, \\
\text{Residual (Neumann)} = \frac{1}{nr} \sum_{j=1}^{nr} \left| \frac{\partial \tilde{u}_j}{\partial n} - g_j \right|,
\]

where \(nr\) is the number of test nodes on the boundary, \(\partial \Omega\), and \(g_j\) is the given boundary condition at the \(j^{th}\) node.

**Example 4.4.1.** We consider the following modified Helmholtz equation

\[
(\Delta - \lambda^2)u = f(x,y), \quad (x,y) \in \Omega,
\]

\[
u = y\sin(\pi x) + x\cos(\pi y), \quad (x,y) \in \partial \Omega,
\]

where \(\lambda\) is a constant. The computational domain is a Cassini curve defined by the parametric equation

\[
\Omega = \{(x,y) : x = \rho \cos \theta, y = \rho \sin \theta, 0 \leq \theta \leq 2\pi\},
\]

where

\[
\rho = \left( \cos (3\theta) + \sqrt{2 - \sin^2 (3\theta)} \right)^{\frac{1}{3}}.
\]

The analytical solution is given by

\[
u(x,y) = y\sin(\pi x) + x\cos(\pi y).
\]
Figure 4.2: Cassinian curve (left) and profile of solution in the extended domain (right).

Figure 4.3: Comparison of RMSE (left) and RMSEx (right) using different matrix solvers.

results is depicted in Figure 4.4. We observed the behaviors of the RMSE and RMSEx, both shown in this figure, with respect to different shape parameters. Note that Kansa’s method quickly becomes unstable while the current method remains stable for a wide range of shape parameters. It should be noticed that the best performance of Kansa’s method seems to be occurring at random peaks. Hence, it is very difficult to identify the optimal shape parameter using Kansa’s method. On the other hand, our method remains stable and accurate for a wide range of shape parameters. We believe that the current method is clearly more effective and more sta-
ble than Kansa’s method in this case.

![Numerical results of RMSE and RMSEx](image1)

Figure 4.4: Numerical results of RMSE (left) and RMSEx (right) by Kansa’s method and the MFS.

![Profiles of RMSE and residual](image2)

Figure 4.5: Profiles of RMSE and residual with respect to different shape parameters.

The overall profiles of the residual and RMSE are shown in Figure 4.5. The profiles of RMSE and boundary residual clearly behave in the same way. This means that the acceptable range of shape parameter can be tracked down by calculating the residual along the boundary. We tried different combinations for the number of nodes and location of source points in the MFS. The results were quite similar. In Table 4.1, we list the best RMSE and RMSEx for both Kansa’s method and our method. Upon close
observation of the RMSE, the best performances of each method seem to yield the same level of accuracy. However, the ranges of acceptable shape parameters for each method are quite different as we showed previously in Figure 4.4. Additionally, we noticed that there was a difference between the RMSEx of each method. The current method outperforms Kansa’s method in evaluating partial derivatives, which is critical for many physical problems. In Table 4.1, we choose \( \lambda \) as 1, 10, and 100. The results are consistent for all these \( \lambda \). The case for large \( \lambda \) is important to solutions of diffusion and wave equations.

**Example 4.4.2.** The purpose of this example is to experimentally observe the convergent rate of the current approach. In this example, we consider the same equation as in the previous example with \( \lambda = 1 \). The domain is the unit square. We choose uniform grid points on the unit square to carry out the computation. In addition, we choose 200 random nodes for calculating RMSE and RMSEx. As in the last example, MQ have been used as basis functions. As seen in Figure 4.6, \( \delta h = 1/n \) where \( n \) is the number of mesh on each coordinate. The optimal shape parameter \( c \) is selected for each \( \delta h \). In this example we observe rapid convergence rate of the proposed method.

![Figure 4.6: Profile of convergent rate in Example 4.4.2.](image)

**Example 4.4.3.** In this example, we consider the nonhomogeneous modi-
Table 4.1: Comparison of best RMSE, RMSEx and corresponding shape parameter for Kansa’s method and the MFS.

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>RMSE Kansa’s method</th>
<th>RMSE MFS</th>
<th>RMSEx Kansa’s method</th>
<th>RMSEx MFS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 1$</td>
<td>111</td>
<td>80</td>
<td>$2.54 \times 10^{-5}$</td>
<td>$1.37 \times 10^{-6}$</td>
<td>$1.10 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(c = 0.12)$</td>
<td>$(c = 2.51)$</td>
<td>$(c = 0.12)$</td>
</tr>
<tr>
<td></td>
<td>208</td>
<td>80</td>
<td>$2.05 \times 10^{-6}$</td>
<td>$3.20 \times 10^{-7}$</td>
<td>$6.37 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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<td>$(c = 1.80)$</td>
<td>$(c = 0.47)$</td>
</tr>
<tr>
<td></td>
<td>310</td>
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<td>$3.97 \times 10^{-6}$</td>
<td>$1.76 \times 10^{-7}$</td>
<td>$5.96 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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<td>$4.56 \times 10^{-6}$</td>
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<td>$1.29 \times 10^{-3}$</td>
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<tr>
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<td>$(c = 1.78)$</td>
<td>$(c = 0.92)$</td>
</tr>
<tr>
<td></td>
<td>208</td>
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<td>$1.82 \times 10^{-3}$</td>
</tr>
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<td>$(c = 1.26)$</td>
<td>$(c = 0.89)$</td>
</tr>
<tr>
<td></td>
<td>310</td>
<td>80</td>
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<td>$4.91 \times 10^{-7}$</td>
<td>$5.14 \times 10^{-4}$</td>
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<td>$(c = 0.41)$</td>
<td>$(c = 1.06)$</td>
<td>$(c = 0.41)$</td>
</tr>
<tr>
<td>$\lambda = 100$</td>
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<td>80</td>
<td>$2.90 \times 10^{-6}$</td>
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<td>$7.04 \times 10^{-3}$</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>$(c = 0.10)$</td>
<td>$(c = 1.26)$</td>
<td>$(c = 0.10)$</td>
</tr>
<tr>
<td></td>
<td>208</td>
<td>80</td>
<td>$3.74 \times 10^{-7}$</td>
<td>$2.81 \times 10^{-7}$</td>
<td>$1.89 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(c = 1.25)$</td>
<td>$(c = 1.73)$</td>
<td>$(c = 1.25)$</td>
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<tr>
<td></td>
<td>310</td>
<td>80</td>
<td>$1.26 \times 10^{-6}$</td>
<td>$1.08 \times 10^{-7}$</td>
<td>$5.72 \times 10^{-4}$</td>
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<td></td>
<td></td>
<td></td>
<td>$(c = 0.34)$</td>
<td>$(c = 1.49)$</td>
<td>$(c = 0.34)$</td>
</tr>
</tbody>
</table>

Fied Helmholtz equation with variable coefficient as follows:

\[ \Delta u - (\lambda^2 y) u = f(x, y), \quad (x, y) \in \Omega, \quad (4.46) \]

\[ u = \sin (y^2 + x) - \cos (y - x^2), \quad (x, y) \in \partial \Omega^D, \quad (4.47) \]

\[ \frac{\partial u}{\partial n} = [\nabla (\sin (y^2 + x) - \cos (y - x^2))] \cdot \mathbf{n}, \quad (x, y) \in \partial \Omega^N, \quad (4.48) \]
where \( \mathbf{n} \) is the unit normal vector and \( f(x,y) \) is a given function based on the analytical solution

\[
    u(x,y) = \sin(y^2 + x) - \cos(y - x^2).
\]  

(4.49)

The computational domain is \( \Omega = \Omega_1 \cup \Omega_2 \), which is formed by combining a square and circular domain

\[
\Omega_1 = \{ (x,y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2 \},
\]

(4.50)

\[
\Omega_2 = \{ (x,y) \mid x = \cos \theta + 2, y = \sin \theta + 2, -\frac{\pi}{2} \leq \theta \leq \pi \}.
\]

(4.51)

Let us denote by \( \partial \Omega^D \) and \( \partial \Omega^N \) the boundaries on which Dirichlet and Neumann boundary conditions are given respectively. Let \( \partial \Omega = \partial \Omega^D \cup \partial \Omega^N \) and

\[
\partial \Omega^N = \{(x,y) \mid 0 \leq x \leq 2, y = 0, \text{ and } 0 \leq x \leq 1, y = 2 \},
\]

(4.52)

\[
\partial \Omega^D = \partial \Omega \setminus \partial \Omega^N.
\]

(4.53)

Figure 4.7: Computational domain (left) and variation of solution (right).

The Neumann and Dirichlet boundaries are shown in Figure 4.7 (left). The profile of the solution in the extended domain is shown in Figure 4.7 (right). To obtain the results in Figures 4.8 and 4.9, we chose \( m = 144 \) equally distributed points on the boundary, \( n = 300 \) quasi-random interior points, \( \sigma = 5, nr = 250 \), and the cut off singular value for TSVD as \( 10^{-11} \). Due to the complicated equation and mixed boundary conditions, Kansa’s method is very erratic with regard to the shape parameter of MQ. As shown
in Figure 4.8, the best performance of Kansa’s method only appears in peaks and it is difficult to locate the optimal shape parameter. In contrast, the numerical results of the MFS are very stable and accurate for various shape parameters. Also, the range of suitable shape parameters is very wide for the MFS. Though the best RMSEx for Kansa’s method is better than the MFS, the instability of Kansa’s method with respect to the shape parameter of MQ could limit its application for more complicated problems.

Figure 4.8: Numerical results of RMSE (left) and RMSEx (right) by Kansa’s and the MFS.

Figure 4.9: Profiles of RMSE and residual with respect to different shape parameters.

In Figure 4.9, the same technique as the one used in the previous ex-
ample was used to determine the optimal shape parameter. Since there are two types of boundary conditions, we calculate the residual for each boundary condition, separately. 21 and 63 nodes are used for calculating the residuals on boundaries with the Dirichlet and Neumann conditions.

We noticed that the overall profiles of the RMSE, the residual for the Dirichlet condition, and the residual for the Neumann condition are very similar. Therefore, it is clear that the criterion for choosing the optimal shape parameter is valid. Both or one of the two boundary conditions can be used to determine the optimal shape parameter.

In Table 4.2, we listed the best results for Kansa’s method and the current method using 100, 200 and 300 interior nodes with 96, 120 and 144 boundary nodes. Both methods produce excellent results for RMSE and RMSEx using only a small number of nodes. Obtaining accurate approximate derivatives is an important issue. Although Kansa’s method is as accurate as the current method, we observed that an acceptable solution with Kansa’s method is only available within a small range of shape parameters. In contrast, the current method appears to be more robust and stable. The \( c \) values shown in Table 4.2 are the optimal shape parameters.

Example 4.4.4. In this example, we consider the convection-diffusion equation:

\[
\begin{align*}
\Delta u + (x^2 + y^2) u + y \cos(y) \frac{\partial u}{\partial x} + \sinh(x) \frac{\partial u}{\partial y} &= f(x, y), \quad (x, y) \in \Omega, \\
\Delta u + (x^2 + y^2) u + y \cos(y) \frac{\partial u}{\partial x} + \sinh(x) \frac{\partial u}{\partial y} &= f(x, y), \quad (x, y) \in \Omega,
\end{align*}
\]  

where \( f(x, y) \) depends upon the analytical solution, which is given as:

\[
\begin{align*}
u(x, y) = \sin(\pi x) \cosh(y) - \cos(\pi x) \sinh(y),
\end{align*}
\]

The domain is defined by the following star shape parametric equation:

\[
\Omega = \{(x, y) \mid x = (1 + \cos^2(4\theta)) \cos \theta, y = (1 + \cos^2(4\theta)) \sin \theta, 0 \leq \theta < 2\pi\}.
\]
Table 4.2: Comparison of best RMSE with corresponding optimal shape parameters and RMSEx for Kansa’s method and the MFS.

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>RMSE</th>
<th>RMSEx</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Kansa’s method</td>
<td>MFS</td>
</tr>
<tr>
<td>100</td>
<td>96</td>
<td>$2.50 \times 10^{-3}$</td>
<td>$1.50 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($c = 1.03$)</td>
<td>($c = 1.50$)</td>
</tr>
<tr>
<td>200</td>
<td>96</td>
<td>$2.82 \times 10^{-5}$</td>
<td>$8.68 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($c = 0.43$)</td>
<td>($c = 1.44$)</td>
</tr>
<tr>
<td>300</td>
<td>96</td>
<td>$1.71 \times 10^{-5}$</td>
<td>$3.04 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($c = 0.68$)</td>
<td>($c = 1.23$)</td>
</tr>
<tr>
<td>100</td>
<td>120</td>
<td>$3.10 \times 10^{-4}$</td>
<td>$1.50 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($c = 1.46$)</td>
<td>($c = 1.50$)</td>
</tr>
<tr>
<td>200</td>
<td>120</td>
<td>$4.07 \times 10^{-6}$</td>
<td>$8.79 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($c = 0.74$)</td>
<td>($c = 1.42$)</td>
</tr>
<tr>
<td>300</td>
<td>120</td>
<td>$1.48 \times 10^{-5}$</td>
<td>$2.82 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($c = 1.13$)</td>
<td>($c = 1.24$)</td>
</tr>
<tr>
<td>100</td>
<td>144</td>
<td>$1.19 \times 10^{-4}$</td>
<td>$1.49 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>($c = 1.49$)</td>
</tr>
<tr>
<td>200</td>
<td>144</td>
<td>$1.25 \times 10^{-5}$</td>
<td>$8.30 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($c = 0.80$)</td>
<td>($c = 1.44$)</td>
</tr>
<tr>
<td>300</td>
<td>144</td>
<td>$6.12 \times 10^{-7}$</td>
<td>$2.86 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($c = 0.56$)</td>
<td>($c = 1.24$)</td>
</tr>
</tbody>
</table>

method and the MFS are shown in Figure 4.11. As in the previous examples, the optimal shape parameter for Kansa’s method is not easy to identify. In contrast, the current method remains stable and converges rapidly as the shape parameter increases. In Figure 4.11, the range of acceptable shape parameters for the current method is quite large even though the governing equation is more complicated.

Similarly, we used the residual on the boundary to determine the optimal shape parameter. The results are depicted in Figure 4.12. As was the case with the results in the previous examples, the RMSE and residual behave in similar way. In spite of different governing equations in the current and previous examples, the proposed criterion for choosing the optimal
Figure 4.10: Computational domain (left) and profile of solution in the extended domain (right).

Figure 4.11: Numerical results of RMSE (left) and RMSEx (right) by Kansa’s method and the MFS.

shape parameter is still valid.

In Table 4.3, we observed a similar pattern for RMSE and RMSEx as the previous two examples. In particular, the current method performs better than Kansa’s method for RMSEx. Again, we produce excellent solutions as well as derivatives using a small number of nodes, which indicates that the proposed method converges rapidly.

To alleviate the difficulty of finding the optimal shape parameter of MQ, we can always choose other radial basis functions such as $r^{2k+1}$ where $k$ is a nonnegative integer. In Tables 4.4 and 4.5, we compare the results
using MQ and $r^{2k+1}, k = 3, 4, 5$. We observe that the accuracy obtained with $r^{2k+1}$ is consistently one order lower than MQ. Without the difficulty of tracking the optimal shape parameter of MQ, it may be worth considering these types of basis functions in real applications. Other RBFs such as polyharmonic splines are also available. The closed form particular solution for these RBFs for the Laplacian are easy to derive. This will be the subject of a future investigation.

4.5 Concluding Remarks

In this paper, we proposed a one-stage approach to incorporate the MPS, the MFS, the DRBEM, and Kansa’s method into one matrix system for solving general elliptic partial differential equations with variable coefficients. The numerical tests reveal that we can obtain excellent results even when using a small number of interpolation points and boundary points. This indicates that our proposed method converges rapidly relative to the number of interior interpolation points and boundary points. The advantages of the proposed method are summarized as follows:

1. The method can be used to solve general elliptic PDEs with variable coefficients.

2. $\Delta u$ can be efficiently evaluated without direct numerical differentiation.
Table 4.3: Comparison of best RMSE, RMSEx and corresponding shape parameter for Kansa’s method and MFS.

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>RMSE MFS</th>
<th>RMSE Kansa’s method</th>
<th>RMSEx MFS</th>
<th>RMSEx Kansa’s method</th>
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<td>2.44 × 10^{-2}</td>
<td>1.96 × 10^{-3}</td>
</tr>
<tr>
<td></td>
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<td>(c = 0.03)</td>
<td>(c = 2.85)</td>
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<tr>
<td>212</td>
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<td></td>
<td>(c = 0.69)</td>
<td>(c = 2.35)</td>
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<td>317</td>
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<td></td>
<td>(c = 0.93)</td>
<td>(c = 1.73)</td>
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<tr>
<td>113</td>
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<tr>
<td>212</td>
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<td>2.88 × 10^{-3}</td>
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<tr>
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<td>(c = 0.33)</td>
<td>(c = 1.54)</td>
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</table>

3. The method is truly meshless and thus very easy to implement.
4. The numerical results are highly accurate and converge rapidly.
5. No complicated fundamental solution or particular solutions are required.
Table 4.4: Comparison of RMSE by the proposed method using different RBFs.

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
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<th>9</th>
<th>11</th>
<th>$\sqrt{r^2 + c^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>113</td>
<td>50</td>
<td>$6.76 \times 10^{-3}$</td>
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As was the case in the two-stage method, finding the optimal shape parameter, $c$, of MQ and the location of the fictitious boundary still pose challenges in implementation. In this paper, we identify an effective way to track the optimal shape parameter. The unified approach has great potential and is expected to make significant contributions to the future development of the MFS. The method is especially attractive for solving equations with nonconstant convection terms, which are very challenging when using other standard methods. Moreover, our proposed method has excellent results in evaluating derivatives, which is crucial for solving many engineering problems.

We believe the current numerical technique has great potential for further application to science and engineering problems. Meanwhile, there are still outstanding theoretical issues to be resolved. The current numerical scheme can be directly extended to 3D, time-dependent, and nonlinear problems. These subjects are currently under investigation.
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Acknowledgement

The first author acknowledges the support of NATO Collaborative Linkage Grant under reference ESP.CLG.982891. The authors would like to thank the referee for the constructive suggestions for improving the paper.

References


References


the Faculty of Science of Tokyo University, Section 1A, 37:635–657, 1990.


CHAPTER 5

On Efficient MFS Algorithms Using Complex Representations

A. Karageorghis\textsuperscript{1}, S. G. Mogilevskaya\textsuperscript{2}, and H. Stolarski\textsuperscript{3}

Abstract. We propose an efficient complex Method of Fundamental Solutions (MFS) algorithm for the solution of certain two–dimensional potential and linear elasticity problems. We consider the solution of such problems in rectangular domains subject to Dirichlet boundary conditions. Numerical experiments for both types of problems are presented.

5.1 Introduction
The Method of Fundamental Solutions (MFS) is a meshless boundary method which is applicable to certain elliptic boundary value problems. In the last three decades the MFS has been successfully applied to a large variety of physical problems, see for example the survey papers \cite{2, 3, 4}. In recent years, the efficient implementation of the method to problems with rotational symmetry, has attracted a lot of interest, see for example \cite{7, 10}.

In this work, our goal is to exploit these algorithms in order to solve efficiently problems which do not possess rotational symmetry. As is well-documented, one fundamental difficulty in the application of the MFS is

\footnotesize
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the choice of the pseudoboundary on which the sources are located. The theoretical predictions in the case of harmonic problems in a disk subject to Dirichlet boundary conditions [11] indicate that the error converges (exponentially) to zero as the distance of the pseudoboundary from the boundary tends to infinity. However, this theoretical prediction is of little practical value as the ill-conditioning of the MFS matrices for large distances of the pseudoboundary from the boundary leads to inaccurate results. Also, when the pseudoboundary is located too close to the boundary the MFS approximation is inaccurate as we are essentially approximating an integral with a singular integrant using simple quadrature. The optimal choice of the location of the pseudoboundary thus remains an open problem. A practical way of determining this optimal location of the pseudoboundary was proposed in [12]. In it, the problem is solved for a sequence of pseudoboundaries which are located at a distance \( \varepsilon_{\ell}, \ell \in \mathbb{N} \) from the boundary. For each \( \varepsilon_{\ell} \) the maximum error in the boundary conditions on a fixed set of points on the boundary is calculated. The optimal location of the pseudoboundary is the one for which the maximum error is minimal. The process can be rendered more efficient by using a bisection type method to locate the minimum, that is starting with a larger step and gradually reducing it. Still, in the case of rectangular domains, this algorithm requires the solution of a sequence of dense linear systems. In order to avoid this, we propose an algorithm in which most of the problems in the sequence are solved in circular domains, instead of the rectangular domain. In this way, we solve a sequence of problems in circular domains which are computationally much less expensive.

An alternative to the proposed approach is to use a conformal mapping technique which maps the domain under consideration onto the unit disk. The problem is then solved efficiently in the unit disk. This approach might, however, involve complicated evaluations of conformal mappings [6].

Although the examples presented in this work are for rectangular geometries these are easily applicable to other geometries.

An additional feature in this work is the use of the complex representation of the MFS. This has proved particularly suitable in the case of potential and elastostatics problems.
5.2 Potential Problems

Consider the solution of Laplace’s equation in the rectangle \( \Omega = (a, b) \times (c, d) \) subject to the Dirichlet boundary condition \( u = f = f_R + i f_I \). We chose to consider complex boundary conditions (and hence complex solutions) to be consistent with the corresponding elasticity problems. By doing this we are essentially solving two harmonic problems simultaneously. The solution \( u \in \mathbb{C} \) considered subsequently is approximated by the complex harmonic function

\[
u_N(\mathbf{c}, \mathbf{Q}; \mathbf{P}) = \sum_{j=1}^{N} c_j \log(z_{Q_j} - z_P)(\bar{z}_{Q_j} - \bar{z}_P), \quad \mathbf{P} \in \overline{\Omega}, \tag{5.1}\]

where \( \mathbf{c} = (c_1, c_2, \ldots, c_N)^T \in \mathbb{C}^N \) and \( \mathbf{Q} \) is a \( 2N \)-vector containing the coordinates of the singularities (sources) \( Q^\alpha_j, j = 1, \ldots, N \), which lie outside \( \overline{\Omega} \). The singularities \( Q^\alpha_j \) are fixed on the boundary \( \partial \tilde{\Omega} \) of a disk \( \tilde{\Omega} \) concentric to \( \Omega \) and defined by \( \tilde{\Omega} = \{ x \in \mathbb{R}^2 : |x| < R \} \). A set of collocation points \( \{ z_P \}_{i=1}^N \) is uniformly distributed on \( \partial \Omega \) and the positions of the singularities are given by

\[
z_{Q_j} = R e^{\frac{2(\pi - 1) + i \pi}{N}}, \quad j = 1, \ldots, N, \tag{5.2}\]

where \( 0 \leq \alpha < 1 \).

The coefficients \( c \) are determined so that the boundary condition is satisfied at the boundary points \( \{ P_i \}_{i=1}^N \):

\[
u_N(\mathbf{c}, \mathbf{Q}; P_i) = f(P_i), \quad i = 1, \ldots, N. \tag{5.3}\]

This yields a complex linear system of the form

\[
G \mathbf{c} = \mathbf{f}, \tag{5.4}\]

for the coefficients \( c \), where the elements of the matrix \( G \) are given by

\[
G_{i,j} = \log(z_{Q_j} - z_P)(\bar{z}_{Q_j} - \bar{z}_P), \quad i = 1, \ldots, N, \quad j = 1, \ldots, N. \tag{5.5}\]

The matrix \( G \) is dense with no particular structure and the system is solved by standard Gaussian elimination.

One fundamental question in the implementation of the MFS is the optimal positioning of the circle on which the singularities are to be placed.
In Dirichlet harmonic problems one may evaluate the maximum error on a fixed set of points on the boundary (not coinciding with the boundary points) for values of the radius $R$ on a certain interval. The maximum error on $\partial \Omega$ is also the maximum error in $\bar{\Omega}$ because the error in the MFS, being harmonic, satisfies the maximum principle. Therefore the value of $R$ which yields the minimum maximum error on the boundary determines the optimal circular pseudoboundary $\partial \bar{\Omega}$ [12]. This task clearly involves the solution of a sequence of MFS discrete problems for a number of different $R$ which is potentially expensive.

In order to render this task computationally efficient we propose the following algorithm:

(i) We first use the MFS for the rectangular region with an arbitrary choice of the circular pseudoboundary $\partial \bar{\Omega}$.

(ii) We evaluate the solution on a circle of radius $\rho < R$ which is concentric with the pseudoboundary $\partial \bar{\Omega}$.

(iii) We solve a sequence of harmonic problems for the disk of radius $\rho$ for pseudoboundaries with varying radius $R$. For each problem we evaluate the maximum error on a fixed set of points on the (rectangular) boundary $\partial \Omega$.

(iv) The optimal value of $R$ is the one for which the maximum error is minimized.

(v) The original problem for the rectangular domain is solved once more for the optimal location of the pseudoboundary.

This algorithm leads to substantial savings in the computational cost, as in this case the consecutive matrices $G$ (for the disk) are circulant [1, 5] and the coefficients can be obtained efficiently using Fast Fourier Transforms (FFTs). In particular, from [10], the approximate solution is evaluated on set of collocation points $\{z_{Si}\}_{i=1}^{N}$ where

$$z_{Si} = \rho e^{\frac{2(i-1)\pi}{N}}, \quad i = 1, \ldots, N.$$  \hspace{1cm} (5.6)

These now yield the boundary conditions for the problem on the disk. System (5.4) now becomes

$$Ed = g,$$  \hspace{1cm} (5.7)
where the matrix $E$ is circulant. System (5.7) is pre multiplied by the Fourier matrix $U$ ([10]) to yield

$$UEU^*d = Ug$$

(5.8)

or

$$D\hat{d} = \hat{g}$$

(5.9)

where

$$\hat{d} = Ud \quad \text{and} \quad \hat{g} = Ug.$$ \hspace{1cm} (5.10)

The solution is thus clearly,

$$\hat{d}_i = \hat{g}_i/d_i, \quad i = 1, \cdots, N,$$ \hspace{1cm} (5.11)

where $D = \text{diag}(d_i)_{i=1}^N$ is a diagonal matrix having as diagonal elements the eigenvalues of $E$. Having obtained $\hat{d}$, we can find $d$ from

$$d = U^*\hat{d}.$$ \hspace{1cm} (5.12)

Because of the circulant structure of the matrices $E$, the cost of solving each system is $O(N \log N)$ operations.

In our numerical experiments, we calculated the maximum error on a set of 100 points on $\partial\Omega$, when solving the problem using the conventional MFS and when solving the proposed algorithm.

**Example 5.2.1** In Figures 5.1-5.3, we present the maximum error obtained solving the original problem and the maximum error obtained using the proposed (modified) algorithm versus the radius of the pseudoboundary $R$ for the problem with exact solution $u = x^6$, for the case when $\Omega = (-1,1) \times (-1,1)$. In Figure 5.1, we took $\rho = 1$, in Figure 5.2, we took $\rho = \sqrt{2}$ while in Figure 5.3, we took $\rho = 2$. We observe that in all three cases the general behaviour of the maximum boundary error is reproduced accurately by the proposed algorithm.

**Example 5.2.2** In Figures 5.4-5.6, we present the corresponding results for the problem with exact solution $u = x^6$, for the case when $\Omega = (-1,1) \times (-1/2,1/2)$. In Figures 5.4, 5.5, 5.6, we took $\rho = 1$, $\rho = \sqrt{2}$, $\rho = 2$, respectively. As in the previous case, the modified approach reproduces the behaviour of the maximum boundary error accurately.
Further, in Table 5.1, we present the CPU times for various numbers of degrees of freedom $N$ when the problem is solved for one hundred steps (i.e. the system is solved one hundred times), in the case the original problem is solved and in the case the proposed algorithm is used. These times include the evaluation of the maximum error on a set of 100 points on the boundary of the domain and were recorded on an IBM PC (Processor: Intel Pentium 4, 3.4 GHz). From the table one can observe the savings in CPU time when using the proposed algorithm.

### 5.3 Elastostatics Problems

We now consider the Cauchy–Navier equations of elasticity

$$\begin{align*}
(\lambda + \mu) u_{k,ki} + \mu u_{i,kk} &= 0, \quad i = 1,2 \quad \text{in} \quad \Omega, \quad (5.13a)
\end{align*}$$

where $\Omega$ is the rectangle $(a, b) \times (c, d)$, subject to the Dirichlet (displacement) boundary conditions

$$u_i = f_i, \quad i = 1,2 \quad \text{on} \quad \partial \Omega. \quad (5.13b)$$

In equations (5.13a), the parameters $\lambda$ and $\mu$ are the Lamé elastic constants. These constants can be expressed in terms of Poisson’s ratio $\nu$ and Young’s modulus $E$ as $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$. We introduce the complex displacement $u = u_1 + bu_2$ and the complex function $f = f_1 + if_2$. Using Kelvin’s fundamental solutions [9], the displacement $u \in \mathbb{C}$ is approximated by the complex function [8]

$$u_N(c; Q; P) = \frac{1}{4\pi \mu (1+\kappa)} \left\{ \sum_{j=1}^{N} a_j \left( -\kappa \log(z_{Q_j} - z_P)(\bar{z}_{Q_j} - \bar{z}_P) + \frac{z_{Q_j} - z_P}{z_{Q_j} - \bar{z}_P} \right) \right\}$$
Figure 5.1: Maximum error versus $d$ for different values of $N$ for $\rho = 1$ in Example 2.1

$$
\nabla \sum_{j=1}^{N} b_j \left( -\kappa i \log(z_{Q_j} - z_P)(\bar{z}_{Q_j} - \bar{z}_P) - i \frac{z_{Q_j} - z_P}{\bar{z}_{Q_j} - \bar{z}_P} \right), \quad P \in \overline{\Omega}, \quad (5.14)
$$

where now $c = (c_1, c_2, \ldots, c_N)^T \in \mathbb{C}^N$, with $c_j = a_j + ib_j$, $j = 1, 2, \ldots, N$, and the singularities $Q$ are defined as before. Here $(a_j, b_j)$ is the point force loaded at the point $z_{Q_j}$ of an infinite elastic plane. In (5.14), $\kappa = 3 - 4\nu$. 

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Approximation (5.14) can also be written as

$$ u_N(c, Q; P) = \frac{1}{4\pi\mu(1+\kappa)} \left\{ \sum_{j=1}^{N} (-\kappa \log(z_Q^j - z_P^j)(\bar{z}_Q^j - \bar{z}_P^j)) c_j ight. \\
+ \left. \sum_{j=1}^{N} \left( \frac{z_Q^j - z_P^j}{\bar{z}_Q^j - \bar{z}_P^j} \right) \delta_j \right\}. $$

(5.15)

The coefficients $c$ are obtained from the system resulting from the collocation of the boundary conditions

$$ u_N(c, Q; P_\ell) = f(P_\ell), \; \ell = 1, \ldots, N. $$

(5.16)
We follow the algorithm described in the potential case. In contrast to the potential case, the resulting matrix in this case (for the circular domain) is not circulant. However, we follow the algorithm described in [7] in order to render it circulant. In particular, system (5.16) using expression (5.14) may be written as the $2N \times 2N$ real linear system $A \mathbf{c} = \mathbf{f}$ or

\[
\begin{pmatrix}
A_{11} & A_{12} & A_{13} & \cdots & A_{1N} \\
A_{21} & A_{22} & A_{23} & \cdots & A_{2N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{N1} & A_{N2} & A_{N3} & \cdots & A_{NN}
\end{pmatrix}
\begin{pmatrix}
\mathbf{c}_1 \\
\mathbf{c}_2 \\
\vdots \\
\mathbf{c}_N
\end{pmatrix}
=
\begin{pmatrix}
\mathbf{f}_1 \\
\mathbf{f}_2 \\
\vdots \\
\mathbf{f}_N
\end{pmatrix},
\]
Figure 5.4: Maximum error versus $d$ for different values of $N$ for $\rho = 1$ in Example 2.2

where

$$A_{tj} = \begin{pmatrix} G_{11}(P_t, Q_j) & G_{12}(P_t, Q_j) \\ G_{21}(P_t, Q_j) & G_{22}(P_t, Q_j) \end{pmatrix}, \quad c_t = \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \quad f_t = \begin{pmatrix} f_1(P_t) \\ f_2(P_t) \end{pmatrix},$$

and

$$G_{11}(P_t, Q_j) = \frac{1}{4\pi\mu(1+\kappa)} \Re \left\{ -\kappa \log(z_{Q_j} - z_{P_t})(\bar{z}_{Q_j} - \bar{z}_{P_t}) + \frac{z_{Q_j} - z_{P_t}}{\bar{z}_{Q_j} - \bar{z}_{P_t}} \right\},$$

$$G_{12}(P_t, Q_j) = \frac{1}{4\pi\mu(1+\kappa)} \Im \left\{ -\kappa \log(z_{Q_j} - z_{P_t})(\bar{z}_{Q_j} - \bar{z}_{P_t}) + \frac{z_{Q_j} - z_{P_t}}{\bar{z}_{Q_j} - \bar{z}_{P_t}} \right\}.$$
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\[ G_{21}(P_r, Q_j) = \frac{1}{4\pi \mu (1 + \kappa)} \Re \left\{ -\kappa i \log(z_{Q_j} - z_P)(\bar{z}_{Q_j} - \bar{z}_P) - i z_{Q_j} - z_P \right\} \]
\[ G_{21}(P_r, Q_j) = \frac{1}{4\pi \mu (1 + \kappa)} \Im \left\{ -\kappa i \log(z_{Q_j} - z_P)(\bar{z}_{Q_j} - \bar{z}_P) - i z_{Q_j} - z_P \right\} . \]

System (5.17) is the same as system (3.3) of [7], and can be rendered block circulant via an appropriate rotation. The block circulant system can then be solved efficiently using FFTs. The algorithm is described in detail in [7].
In our numerical experiments, as in the case of the Laplacian, we calculated the maximum error on a set of 100 points on $\partial \Omega$, when solving the original problem and when using the proposed (modified) algorithm.

**Example 5.3.1** In Figures 5.7-5.9, we present the maximum error obtained solving the original problem and the maximum error obtained using the modified problem versus the radius of the pseudo boundary $R_1$ for the problem with exact solution $u = e^{3z}$, for the case when $\Omega = (-1, 1) \times (-1, 1)$. In Figure 5.7, we took $\rho = 1$, in Figure 5.8, we took $\rho = \sqrt{2}$ while in Figure 5.9, we took $\rho = 2$. As in the corresponding harmonic case, the maximum boundary error is reproduced accurately by the modified approach.
Example 5.3.2 In Figures 5.10-5.12, we present the corresponding results for the problem with exact solution $u = e^{3z}$, for the case when $\Omega = (-1, 1) \times (-1/2, 1/2)$. In Figures 5.10, 5.11, 5.12, we took $\rho = 1$, $\rho = \sqrt{2}$, $\rho = 2$, respectively. As in the previous case, the modified approach reproduces the behaviour of the maximum boundary error accurately.

![Graphs showing Maximum Error versus $d$ for different values of $N$ for $\rho = 1$ in Example 3.1](image)

Figure 5.7: Maximum error versus $d$ for different values of $N$ for $\rho = 1$ in Example 3.1

5.4 Conclusions
In this work we attempted to exploit two features of the MFS. Firstly, we use complex representations of the fundamental solutions. The advantages of complex representations in both potential and elasticity theories are well
documented. In particular, the complex representations of fundamental solutions, for example Melan’s and Flamant’s solutions are more compact than their real counterparts [8]. Also, in a future extension of the approach presented herein, one may exploit the properties of analytic functions (e.g. analytic continuation, series expansions, etc). The second feature we attempted to exploit in this work is the location of the optimal boundary in conjunction with the efficient solution of circulant systems. The algorithm proposed in this work is not restricted to rectangular domains but may be used for more general geometries. The choice of the radius of the circle of testing nodes $\rho$ can influence the performance of the MFS. It was found that the closer the circle of testing nodes is to the original domain, the better
Figure 5.9: Maximum error versus $d$ for different values of $N$ for $\rho = 2$ in Example 3.1

the results. In future work we intend to use the advantages of complex representations to accurately calculate the boundary conditions on the circular domain.

Acknowledgements
The authors are grateful to the referee for his/her constructive comments.
Figure 5.10: Maximum error versus $d$ for different values of $N$ for $\rho = 1$ in Example 3.2

References


Figure 5.11: Maximum error versus $d$ for different values of $N$ for $\rho = \sqrt{2}$ in Example 3.2


Figure 5.12: Maximum error versus $d$ for different values of $N$ for $\rho = 2$ in Example 3.2


References


CHAPTER 6

The Method of Fundamental Solutions for Steady-State Nonlinear Heat Conduction

A. Karageorghis\(^1\) and D. Lesnic\(^2\)

**Abstract.** The steady-state heat conduction in heat conductors with temperature dependent thermal conductivity and mixed boundary conditions involving convection and radiation is investigated using the method of fundamental solutions. The locations of the singularities outside the solution domain are optimally determined using a nonlinear least-squares procedure. Numerical results for nonlinear materials are presented and discussed.

6.1 Introduction

In many heat transfer problems the assumption of constant thermal conductivity, i.e. that the heat conductors are homogeneous within the whole temperature variation interval, may lead to unacceptable errors in high-temperature environments or if large temperature differences are present. In the steady-state situation, the nonlinearity associated with the temperature dependence of the thermal conductivity can be removed by employing the Kirchhoff transformation, which replaces the original nonlinear partial differential equation in divergence form by the Laplace equation in the transformed space. Boundary conditions of the Dirichlet (first kind) or

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Neumann (second kind) types pose no problem for the transformation, but the Robin convective (third kind) boundary conditions become non-linear. Although this non-linearity is not strong, convergence problems may arise if radiative heat transfer (fourth kind) boundary conditions are also present, see [9]. Since all the non-linearities are transferred to the boundary conditions, the Kirchhoff transformation approach is very well-suited for applying the boundary element method (BEM), [5, 18], or the method of fundamental solutions (MFS), [19]. In the same manner these techniques can be extended to composite bodies through the subregion technique. In it, each region is dealt with separately and then the whole body is linked together by applying compatibility and equilibrium conditions along the interfaces between subregions.

Two-dimensional boundary value problems of heat conduction in non-linear single and composite materials have been the subject of several studies using the BEM, [1, 3, 4, 5, 18]. However, the implementation of the BEM becomes quite tedious, especially in three-dimensional irregular domains. Moreover, the evaluation of the gradient of the temperature solution on the boundary requires the use of finite differences or the evaluation of hypersingular integrals. In order to alleviate some of these difficulties, we propose the use of the MFS. The merits and drawbacks of the MFS over the BEM for solving elliptic boundary value problems are thoroughly discussed in [7, 10, 17, 19].

Prior to this study, the MFS was used for the solution of problems of heat conduction in linear single material with nonlinear boundary conditions, [19], and composite materials with linear boundary conditions [2]. It is the purpose of this paper to extend these analyses to nonlinear, both single and composite, materials with nonlinear boundary conditions.

### 6.2 Mathematical Formulation

Consider a bounded domain \( \Omega \subset \mathbb{R}^d, d \geq 2 \), with piecewise smooth boundary \( \partial \Omega \), formed from two (or more) subregions \( \Omega_1 \) and \( \Omega_2 \) separated by the interfacial surface \( \Gamma_{12} = \partial \Omega_1 \cap \partial \Omega_2 \). The material of subregion \( \Omega_1 \) has a temperature dependent thermal conductivity \( k_1 > 0 \) and material of subregion \( \Omega_2 \) has a different thermal conductivity \( k_2 > 0 \). The governing steady-state heat conduction equations are

\[
\nabla \cdot (k_i(T_i) \nabla T_i) = 0, \quad \text{in} \quad \Omega_i, \quad i = 1, 2,
\]
where $T_i$ is the temperature solution in domain $\Omega_i$, $i = 1, 2$, and, for the sake of simplicity, we have assumed that there is no heat generation within $\Omega$. The technique developed in this paper is valid for bodies $\Omega$ consisting of an arbitrary finite number of subregions.

Boundary conditions of the mixed type can be prescribed at the external surface $\partial\Omega$ of the composite body $\Omega = \Omega_1 \cup \Omega_2$ and they include (dropping for simplicity the region subscript $i = 1, 2$):

- **Dirichlet boundary conditions (prescribed temperature $f$)**
  \[ T = f, \quad \text{on} \quad S_1. \quad (6.2) \]

- **Neumann boundary conditions (prescribed heat flux $g$)**
  \[ -k(T) \frac{\partial T}{\partial n} = g, \quad \text{on} \quad S_2, \quad (6.3) \]
  where $\mathbf{n}$ is the outward normal to the boundary $\partial\Omega$.

- **Robin boundary conditions (prescribed heat transfer coefficient $h$)**
  \[ -k(T) \frac{\partial T}{\partial n} = h(T - T_f) - q_3, \quad \text{on} \quad S_3, \quad (6.4) \]
  where $T_f$ is the temperature of fluid exchanging heat with surface $S_3$, and $q_3$ is a given function.

- **Radiation boundary condition (prescribed fourth-order power law)**
  \[ -k(T) \frac{\partial T}{\partial n} = \sigma \varepsilon (T^4 - T_s^4) - q_4, \quad \text{on} \quad S_4, \quad (6.5) \]
  where $q_4$ is a given function, $\sigma = 5.67051 \times 10^{-8} \text{W/}(m^2\text{K}^4)$ is the Stefan-Boltzmann constant and $\varepsilon$ is the radiation interchange factor (emissivity) between the irradiated boundary $S_4$ and the environment, having a temperature $T_s$.

In (6.2)-(6.5) the boundary portions $S_j$, $j = 1, 4$, which cover the boundary $\partial\Omega$, i.e. $\partial\Omega = \bigcup_{j=1}^{4} S_j$, have no common parts, i.e. $S_i \cap S_j = \emptyset$, $i \neq j$. Also, in the above boundary conditions the nonlinearity occurs mainly due to the heat radiation (6.5), although the method of solution can also allow nonlinearities to occur from a temperature
dependent heat transfer coefficient $h(T)$, or from a temperature dependent radiation interchange factor $\varepsilon(T)$.

In addition to the above boundary conditions (6.2)-(6.5), both imperfect and ideal contact conditions can occur at the interface $\Gamma_{12}$, namely

- **Interface continuity**

  $$-k_1(T_1) \frac{\partial T_1}{\partial n^+} = k_2(T_2) \frac{\partial T_2}{\partial n^-}, \quad \text{on } \Gamma_{12},$$

  (6.6)

  where $n^+$ and $n^-$ are the outward normals to the boundaries $\partial \Omega_1 \cap \Gamma_{12}$ and $\partial \Omega_2 \cap \Gamma_{12}$, respectively, i.e. since $\Gamma_{12} = \partial \Omega_1 \cap \partial \Omega_2$ we have $n^+ = -n^-$.  

- **Ideal contact (interface temperature continuity)**

  $$T_1 = T_2, \quad \text{on } S_5,$$

  (6.7)

  and imperfect contact (interface temperature jump)

  $$T_1 = T_2 - R k_1(T_1) \frac{\partial T_1}{\partial n^+}, \quad \text{on } S_6,$$

  (6.8)

  where $R$ is the contact resistance. In (6.7) and (6.8) the interface portions $S_5$ and $S_6$ cover $\Gamma_{12}$, i.e. $S_5 \cup S_6 = \Gamma_{12}$, and have no common parts, i.e. $S_5 \cap S_6 = \emptyset$.

### 6.2.1 Kirchhoff’s transformation

The governing nonlinear partial differential equations (6.1) can be easily transformed into the Laplace equation by employing the Kirchhoff transformation defined as

$$\Psi_i = \psi_i(T_i) := \int_0^{T_i} \frac{k_i(y)}{k_0} \, dy, \quad i = 1, 2,$$

(6.9)
where \( k_i(T) = k_0_i (1 + m_i(T_i)) \), \( k_0_i \) are positive constants and \( m_i(T_i) > -1 \) are known functions.

Since \( k_i > 0 \), the inverse transformation to (6.9) exists and is given by

\[
T_i = \psi_i^{-1} (\Psi_i), \quad i = 1, 2.
\]  

(6.10)

Under (6.9), problem (6.1)-(6.8) transforms into the equivalent form

\[
\nabla^2 \Psi_i = 0, \quad \text{in } \Omega_i, \quad i = 1, 2;
\]  

(6.11)

subject to the boundary conditions (dropping for simplicity the subscript \( i = 1, 2 \))

\[
\Psi = \psi(f), \quad \text{on } S_1,
\]  

(6.12)

\[
-k_0 \frac{\partial \Psi}{\partial n} = g, \quad \text{on } S_2,
\]  

(6.13)

\[
-k_0 \frac{\partial \Psi}{\partial n} = h [\psi^{-1}(\Psi) - T_f] - q_3, \quad \text{on } S_3,
\]  

(6.14)

\[
-k_0 \frac{\partial \Psi}{\partial n} = \sigma \varepsilon [\psi^{-1}(\Psi)^4 - T_s^4] - q_4, \quad \text{on } S_4,
\]  

(6.15)

and the interface conditions

\[
-k_{0_i} \frac{\partial \Psi_1}{\partial n^+} = k_{0_j} \frac{\partial \Psi_2}{\partial n^-}, \quad \text{on } \Gamma_{12},
\]  

(6.16)

\[
\psi_1^{-1}(\Psi_1) = \psi_2^{-1}(\Psi_2), \quad \text{on } S_5,
\]  

(6.17)

\[
\psi_1^{-1}(\Psi_1) = \psi_2^{-1}(\Psi_2) - R k_{0_i} \frac{\partial \Psi_1}{\partial n^+}, \quad \text{on } S_6.
\]  

(6.18)

It can be seen that in the Kirchhoff space of the transform, the governing equation (6.11), the Dirichlet boundary condition (6.12), the Neumann boundary condition (6.13) and the flux continuity condition (6.16) are linear, whilst the convective boundary condition (6.14) and the interface temperature conditions (6.17) and (6.18) become nonlinear. The nonlinearity caused by the fourth power law radiation (6.5) is also present in the space of transform (6.15), and furthermore, the Kirchhoff transform is no longer continuous across the interface \( S_5 \), i.e. a jump in the transforms occurs there where \( \Psi_1 \neq \Psi_2 \).

The solvability of problem (6.1)-(6.8), or equivalently the transformed problem (6.11)-(6.18) depends on the form, e.g. smoothness, monotonicity, of the input data \( \Omega, \partial \Omega, \Gamma_{12}, k_i, f, g, T_f, T_s, h \) and \( \varepsilon \) and it may be
established using classical boundary integral equation methods, see e.g. [15, 23].

Once $\Psi$ has been determined, the temperature solution $T$ may be readily obtained from equation (6.10), via (6.9).

### 6.3 The method of Fundamental Solutions (MFS)

As the sources of nonlinearity are associated with the boundary conditions (6.14), (6.15), (6.17) and (6.18) only, the boundary value problem (6.11)-(6.18) for each subregion can be converted into a minimization problem, or equivalently an algebraic system of nonlinear equations, using the MFS.

From [6, 22], the MFS approximations for the solutions $\Psi_1$ and $\Psi_2$ of the Laplace equation (6.11) have the form

$$\Psi_i^N(c^i; x) = \sum_{k=1}^{N} c_i^k G_d(y_i^k, x), \quad x \in \Omega_i, \quad i = 1, 2, \quad (6.19)$$

where $N$ is the number of unknown singularities (sources) $(y^i_k)_{k=1}^N \not\in \Omega_i$, $(c^i_k)_{k=1}^N$ are unknown real coefficients and $G_d$ is a fundamental solution for the Laplace equation, given by

$$G_d(y, x) = \begin{cases} \ln |y - x|, & d = 2, \\ \frac{1}{|y - x|}, & d = 3. \end{cases} \quad (6.20)$$

The heat flux is obtained by differentiating (6.19) with respect to the outward normal $n$.

In (6.19), the coordinates of the singularities may be either preassigned, or let free and determined as part of the solution [10]. Here, we adopt the former option, where the singularities are fixed, although their location is parameterized by a single unknown parameter, as described at the end of this section. Therefore, in equation (6.19) there are $2N$ unknowns, namely, the coefficients $(c^i_k)_{k=1}^N, i = 1, 2$. These coefficients can be determined by collocating (imposing) the boundary and interface conditions (6.12)-(6.18) at $M_{\partial \Omega}$ distinct points on the boundary $\partial \Omega \setminus \Gamma_{12}$, $M_{\partial \Omega}$ distinct points on the boundary $\partial \Omega \setminus \Gamma_{12}$, and $M_{\Gamma}$ distinct points on the interface $\Gamma_{12}$. We denote the boundary points on each of the four parts $(S^i)_{i=1,4} \subset \partial \Omega$ in the following way:
• On \( S_1 \cap \partial \Omega \), we take \((x_j^i)_{j=1}^{M_6}\)

• On \( S_2 \cap \partial \Omega \), we take \((x_j^i)_{j=M_1+1, M_2+1, M_3+1, M_4+1, M_5+1}\)

• On \( S_3 \cap \partial \Omega \), we take \((x_j^i)_{j=M_1+M_2+1, M_3+M_4+1, M_5+M_6+1}\)

• On \( S_4 \cap \partial \Omega \), we take \((x_j^i)_{j=M_1+M_2+M_3+1, M_1+M_2+M_3+M_4+1, M_5+M_6+1}\)

On the interface \( \Gamma_{12} \) we take:

• On \( S_5 \) we take \((x_j^i)_{j=1, M_5+1, M_6+1}\)

and

• On \( S_6 \) we take \((x_j^i)_{j=1, M_5+M_6+1}\)

Clearly, here we have that \( M_{\partial \Omega} = M_1 + M_2 + M_3 + M_4 + M_\Gamma \).

We thus have a total number of \( 2N \) unknowns and a total of \( 2M_{\partial \Omega} + 2M_\Gamma \) conditions to be satisfied.

Let us denotes \( \Psi_{ij}^N := \Psi_{ij}^N(c^i; x^j), \Psi_{ij}^1 := \Psi_{ij}^1(c^i; x^j) \), and \( \Psi_{ij}^2 := \Psi_{ij}^2(c^i; x_j) \). Substituting (6.19) into (6.12)-(6.18), we minimize the nonlinear least-squares objective function

\[
S(c^1, c^2) := \sum_{i=1}^{2} \left\{ \sum_{j=1}^{M_6} \left[ \frac{\partial \psi_i^j}{\partial n} - f_i(x_j^i) \right]^2 + \sum_{j=M_1+1}^{M_6} \left[ -k_0 \frac{\partial \psi_i^j}{\partial n} - g_i(x_j^i) \right]^2 \right\} \\
+ \sum_{j=M_1+M_2+1}^{M_6} \left[ -k_0 \frac{\partial \psi_i^j}{\partial n} - \sigma \epsilon(x_j^i) \left( \psi_i^{1N} (\psi_i^{jN})^4 - T_i(x_j^i)^4 \right) + q_i(x_j^i) \right]^2 \\
+ \sum_{j=M_1+M_2+M_3+1}^{M_6} \left[ -k_0 \frac{\partial \psi_i^j}{\partial n} - \sigma \epsilon(x_j^i) \left( \psi_i^{1N} (\psi_i^{jN})^4 - T_i(x_j^i)^4 \right) + q_i(x_j^i) \right]^2 \\
+ \sum_{j=1}^{M_5} \left[ \psi_i^{1N} (\psi_i^{jN}) - \psi_i^{2N} (\psi_i^{jN}) \right]^2 + \sum_{j=1}^{M_5} \left[ -k_0 \frac{\partial \psi_i^{1N}}{\partial n} - k_0 \frac{\partial \psi_i^{2N}}{\partial n} \right]^2 \\
+ \sum_{j=M_5+1}^{M_6} \left[ \psi_i^{1N} (\psi_i^{jN}) - \psi_i^{2N} (\psi_i^{jN}) + R k_0 \frac{\partial \psi_i^{1N}}{\partial n} \right]^2.
\]  

(6.21)
The minimization of (6.21) is carried out using the MINPACK [14], routines lmdif or lmder which minimize the sum of the squares of nonlinear functions. In lmder the Jacobian is provided by the user, whilst in lmdif the Jacobian is calculated internally by forward finite differences. Preliminary investigations, [20, 21], found that lmder converged faster than lmdif.

Two pseudo-boundaries are taken as exterior similar deformations $\partial \Omega'_i$ of the boundaries $\partial \Omega_i$ of the original domains $\Omega_i \subset \Omega'_i$, $i = 1, 2$. A number of $N$ singularities is placed on each pseudo-boundary. An important question in the implementation of the MFS is the positioning of these pseudo-boundaries. This point is addressed by extending the approach used in [26]. In particular, the pseudo-boundaries $\partial \Omega'_i$, $i = 1, 2$ are taken at a distance $\eta > 0$ from the boundaries $\partial \Omega_i$, $i = 1, 2$, respectively. In order to determine the optimal value of $\eta$, the minimization problem is solved for various values of $\eta = \eta_0 + \ell(\delta \eta)$, $\ell = 1, L$, where $\delta \eta$ is a small positive increment. For each $\eta$, the maximum error in the boundary conditions at a selected set of uniformly spaced points on the boundary $\partial \Omega \cup \Gamma_{12}$ (different from the boundary collocation points) is calculated. The optimal value of $\eta$ is chosen the one for which the maximum error was minimized.

The MFS formulation described in this section may be viewed as a domain decomposition technique. Such approaches have been used, in conjunction with the MFS, in [2, 11].

### 6.4 Numerical Results and Discussion

In this section we present numerical results obtained from the application of the MFS described in the previous section for a single nonlinear material with radiative boundary condition. More examples including composite materials as well can be found in [20, 21].

We consider a single nonlinear heat conductor with the thermal conductivity given by

$$ k(T) = k_0(1 + aT), \quad k_0 = 1 W/mK, \quad a \in \{0, 0.1, 0.3, 0.5\} K^{-1}, \quad (6.22) $$

occupying part of an L-shaped cross-section of an industrial furnace, as depicted in Figure 6.1. We take nonlinear radiative boundary conditions
throughout $S_4 = \partial \Omega$ with $T_s = 0$ and $\varepsilon = 10^8/5.67051$, such that the boundary condition (6.5) becomes

$$k(T) \frac{\partial T}{\partial n} + |T| T^3 = q_4, \quad \text{on} \quad S_4 = \partial \Omega.$$ (6.23)

In equation (6.23), the function $q_4$ is chosen in such a way that the exact (analytical) solution of the problem

$$\nabla \cdot (k(T) \nabla T) = 0, \quad \text{in} \quad \Omega$$ (6.24)

subject to (6.23), where $k(T)$ is given by (6.22), is given by

$$T(x) = -1 + \sqrt{1 - 2a \ln|x - x'| - \frac{a}{a}}, \quad x \in \overline{\Omega},$$ (6.25)

where $x' = (0.3, -0.3) \notin \overline{\Omega}$.

For linear material, i.e. $a = 0$, this problem was investigated using a
MFS for nonlinear heat conduction

Galerkin BEM in [13]. It should be noted that the nonlinearity of type $T^4$, which occurs frequently in heat transfer problems, does not satisfy the monotonicity assumption needed for an existence theory result, [12]; in fact, $|T|^3$ was considered instead of $T^4$ in [23].

Employing the Kirchhoff transformation

$$
\Psi = \psi(T) := \int_0^T k(\xi) d\xi = T + \frac{aT^2}{2},
$$

(6.26)

problem (6.23) - (6.24) becomes

$$
\nabla^2 \Psi = 0, \quad \text{in } \Omega, (6.27)
$$

$$
\frac{\partial \Psi}{\partial n} + \left[ -1 + \sqrt{1 + 2a\Psi} \right] \left( \frac{-1 + \sqrt{1 + 2a\Psi}}{a} \right)^3 = q_4, \quad \text{on } \partial \Omega, (6.28)
$$

The analytical solution of problem (6.27) and (6.28) is given by

$$
\Psi(x) = -\ln|x-x'|, \quad x \in \Omega.
$$

(6.29)

To generate the data $q_4$ in (6.23), or (6.28), we use the derivatives

$$
\frac{\partial}{\partial x_1} (-\ln|x-x'|) = -\frac{x_1 - 0.3}{(x_1 - 0.3)^2 + (x_2 + 0.3)^2},
$$

$$
\frac{\partial}{\partial x_2} (-\ln|x-x'|) = -\frac{x_2 + 0.3}{(x_1 - 0.3)^2 + (x_2 + 0.3)^2}.
$$

(6.30)

We choose $M$ uniformly distributed points $(x_j)_{j=1}^M$ on the boundary $\partial \Omega$, $M/8$ on each $DE, EO, OA$ and $AB$, and $M/4$ on each $BC$ and $CD$, and $N$ uniformly distributed sources $(y_j)_{j=1}^N$ on the pseudo-boundary $\partial \Omega'$ which is taken at a distance $\eta > 0$ from $\partial \Omega$. Then we minimize the functional

$$
S(c) = \sum_{j=1}^M \left\{ \frac{\partial \Psi_N}{\partial n}(c;x_j) + \left[ -1 + \sqrt{1 + 2a\Psi_N(c;x_j)} \right] \left( \frac{-1 + \sqrt{1 + 2a\Psi_N(c;x_j)}}{a} \right)^3 - q_4(x_j) \right\}^2,
$$

(6.31)

where

$$
\Psi_N(c;x_j) = \sum_{k=1}^{N} c_k G_2(y_k;x_j), \quad j = 1, M.
$$
Once $\Psi_N$ has been obtained accurately, the temperature $T$ can be obtained by inverting (6.26), i.e.

$$T = \frac{-1 + \sqrt{1 + 2a\Psi_N}}{a}. \quad (6.32)$$

We first solve the problem with $a = 0$. In this case $k(T) = 1$ and there is no need to employ transformation (6.9), the problem to be solved being linear and given by

$$\nabla^2 T = 0, \quad \text{in} \quad \Omega, \quad (6.33)$$

$$\frac{\partial T}{\partial n} + |T|^3 \big|_\Omega = q_4, \quad \text{on} \quad \partial \Omega, \quad (6.34)$$

which has the analytical solution

$$T(x) = -\ln|x - x'|, \quad x \in \Omega. \quad (6.35)$$

In Figure 6.2, we present the boundary temperature along the perimeter of the L-shaped boundary, as a function of the arc-length $s \in [0, 2)$, starting from the origin and oriented counter-clockwise in the cases $a = 0, 0.1, 0.3$, and 0.5 obtained using the MFS with $M = N = 96$. From this figure very good agreement with the analytical solution (6.32) can be observed. The corresponding elongated plots are presented in Figure 6.3. In Figure 6.4, we present the maximum error calculated on a fixed set of points on the boundary (different than the boundary collocation points) versus the distance $\eta$ of the pseudoboundary from the boundary for various numbers of degrees of freedom $M = N = 32, 64$ and 96 in the case $a = 0.1$. As can be observed from this plot the accuracy improves with increasing the number of degrees of freedom. Also, the error decreases up to a certain value of $\eta$ after which it starts increasing again. This behaviour is typical of the MFS where the initial decrease of the error is consistent with the theoretical predictions [25], while the eventual increase of the error is due to the ill-conditioning of the MFS system [24]. The accuracy of the scheme was found to be independent of the values of $a$ considered.

**6.5 Conclusions**

In this paper, the application of the MFS to steady-state nonlinear heat conduction problems has been investigated. The method recasts the problem...
Figure 6.2: The analytical (−) and the MFS numerical (+) boundary temperature along the perimeter of the L-shaped domain for \( a = 0, 0.1, 0.3, 0.5 \)
as a nonlinear minimization problem. The numerical obtained results are in good agreement with the available analytical solution showing high accuracy and stable convergence, and with the BEM results of [3, 5]. However, unlike the BEM, the MFS can be easily extended to three-dimensional nonlinear steady-state heat conduction problems. Moreover, if a heat source is present in equation (6.1), then one can apply a modification of the MFS, as described in [16]. In the numerical test performed in Section 4, the relations between the dependent variables \( \Psi \) and \( T \) before and after the Kirchhoff transformation were given explicitly by equations (6.26) and (6.32). However, if the thermal conductivity would be given at some discrete nodes instead of an explicit function then one can approximate \( k(T) \) by a piecewise linear function using these nodes and then apply the Kirchhoff transformation over each linear portion, [1]. The proposed MFS technique can
be implemented in a commercial code aimed at solving general convective, radiative, steady-state, nonlinear heat transfer in composite layered (ideal or non-ideal interface contact) heat conductors. In the case of transient heat conduction problems the proposed scheme would have to be changed in order to accommodate the transient term. One way of doing this is to use the Laplace transform to remove the time dependence and then to deal with the resulting inhomogeneous Helmholtz-type problem using the method of particular solutions. A comprehensive account of the details involved in this approach may be found in [8], see also [17].
Acknowledgements
The authors would like to thank the UK Royal Society for supporting this research. D. Lesnic would like to thank the University of Cyprus for the hospitality shown to him during his outgoing short research visit. Finally, the authors wish to thank the anonymous referee for his/her constructive comments and suggestions.

References
References


CHAPTER 7

A Multi-Level Regularized Version of the Method of Fundamental Solutions

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Abstract. A regularization technique of the method of fundamental solutions is presented. Instead of the fundamental solution of the original problem, the fundamental solution of a higher order auxiliary problem is used. Thus, the computational problems caused by the singularity of the original fundamental solution is avoided. The proposed method is equivalent to a boundary interpolation technique based on the fundamental solution of the applied higher order problem. The computational cost can be significantly reduced by directly solving the interpolation problem on quadtree-based cell system using standard multi-level tools. Thus, the use of large, dense and ill-conditioned matrices can also be avoided.

7.1 Introduction

The Method of Fundamental Solutions (MFS) is a truly meshless method, which has quickly become popular because of its simplicity and accuracy. It has been applied to many types of problems e.g. elliptic partial differential equations [3, 4, 5, 10, 12]; eigenvalue problems [1, 2]; inverse problems [11] etc. The method can be considered also an approximation of the indirect Boundary Element Method. In both cases, the fundamental solutions of the original partial differential equation is needed. For instance, in the

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case of the simple 2D Laplace equation
\[ \Delta U = 0, \quad (7.1) \]
the fundamental solution has the form:
\[ \Phi(x) = \frac{1}{2\pi} \log ||x||, \quad (7.2) \]
where \( ||.|| \) denotes the Euclidean norm. In the indirect BEM, the solution is sought in the boundary integral form:
\[ U(x) = \int_{\Gamma} \sigma(y) \cdot \Phi(x - y) d\Gamma_y, \quad (7.3) \]
(with an a priori unknown function \( \sigma \)). The MFS defines the approximate solution as a finite sum:
\[ U(x) = \sum_{j=1}^{N} \alpha_j \cdot \Phi(x - \bar{x}_j), \quad (7.4) \]
where the points \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N \) (the source points) are located in the exterior of the domain of the original PDE \( \Omega \). Thus, the function \( U \) (both in (7.3) and (7.4)) automatically satisfies the PDE (7.1). The unknown function \( \sigma \) (and the coefficients \( \alpha_1, \alpha_2, \ldots, \alpha_N \), respectively) can be determined from the boundary conditions. For simplicity, assume that Dirichlet boundary condition is prescribed:
\[ U(x_k) = u_k \quad (k = 1, 2, \ldots, N), \quad (7.5) \]
where \( x_1, x_2, \ldots, x_N \) are located on the boundary \( \Gamma \) (collocation points).

After a proper boundary discretization, (7.3) leads to an algebraic system of equations similar to (7.4). (In this case, the points \( \bar{x}_j \) are located on the boundary \( \Gamma \).) Both approaches suffer from the same computational disadvantage: they lead to a linear algebraic system with large, full and often severely ill-conditioned matrix.

If the sources are located outside of the domain \( \Omega \) (like in (7.4), cf. [3, 4]), the situation is even worse as shown through the following simple example. Let \( \Omega \) be a circle with radius \( R \), centered at the origin. Then the solution of (7.1) can be expressed in the form:
\[ U(x) = \int_{\Gamma_0} \sigma(y) \cdot \Phi(x - y) d\Gamma_y, \quad (7.6) \]
where $\Gamma_0$ is a larger circle with radius $R_0 > R$. It is well known that the boundary function $u_0 := U|_{\Gamma_0}$ belongs to the Sobolev space $H^{s_0}(\Gamma_0)$ provided that $\sigma \in H^{s_0-1}(\Gamma_0)$ (for arbitrary $s_0 \in \mathbb{R}$), where $U|_{\Gamma_0}$ denotes the trace of $U$ taken on $\Gamma_0$. Expressing $u_0$ in terms of (boundary) Fourier series:

$$u_0(t) = \sum_k \hat{u}_k e^{ikt},$$

it is clear that the trace of $U$ taken at the original boundary $\Gamma$ is expressed as:

$$u(t) := U|_{\Gamma} = \sum_k \hat{u}_k \left( \frac{R}{R_0} \right)^k e^{ikt}. \quad (7.7)$$

Since $R < R_0$ and $u_0 \in H^{s_0}(\Gamma_0)$, the boundary function $u$ belongs to the Sobolev space $H^s(\Gamma_0)$ for any index $s \geq s_0$. That is, $u$ is extremely smooth. In other words, if $u$ is not extremely smooth, then $u_0$ as well as $\sigma$ become extremely irregular, which can cause serious numerical instability when implementing the solution of (7.6) (after a proper discretization e.g. applying the form (7.4)).

To illustrate the above phenomenon, consider the following two model problems. In both cases, let $\Omega$ be a circle centered at the origin, with radius $1/4$. Along the boundary $\Gamma$, Dirichlet boundary conditions are given, consistent with the exact solutions.

**Test 1:**

$$U(x, y) := -x + 2y \quad (7.8)$$

**Test 2:**

$$U(x, y) := \frac{\sqrt{4x + \sqrt{(4x)^2 + (4y+1)^2}} - \sqrt{4x + \sqrt{(4x)^2 + (4y-1)^2}}}{2} \quad (7.9)$$

Test 1 is a smooth function, while the function Test 2 has two singular points ($(0, 1/4)$ and $(0, -1/4)$), but still belongs to the Sobolev space $H^1(\Omega)$. (Note that the function is the real part of the complex-valued function $\sqrt{4z+i} - \sqrt{4z-i}$.) Table 7.1 summarizes the exactness of the approximate solutions measured in the discrete $L_2$-norm. Here $N$ denotes the number of source and the collocation points and $d$ denotes the distance of
the source points from the boundary. The asterisk indicates that the approximate solution exhibits strong irregularity outside of $\Omega$.

<table>
<thead>
<tr>
<th>$N \setminus d$</th>
<th>Test 1</th>
<th>Test 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>1.085</td>
<td>0.011</td>
</tr>
<tr>
<td>64</td>
<td>0.136</td>
<td>0.000</td>
</tr>
<tr>
<td>128</td>
<td>0.005</td>
<td>0.000</td>
</tr>
<tr>
<td>256</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Table 7.1: MFS, relative $L_2$-errors of the smooth and weakly singular test problems

As can be seen from Table 7.1, in case of non-smooth solution, the MFS produces very irregular approximate solutions; the numerical instability destroys the approximation if the sources are located too far from the boundary. This phenomenon is much more moderate when the solution is smooth; here the approximate solution is still quite exact in the interior of the domain while irregularities already appear in the vicinity of the sources. Figure 7.1 shows a typical example for this situation. Here the sources are located at a distance of 0.1 from the boundary. The number of sources was 256.

### 7.2 The Idea of the Regularized Method of Fundamental Solutions (RMFS)

As indicated in the previous section, the numerical difficulties of the MFS can be decreased by locating the source points in the vicinity of the boundary (though the accuracy also decreases in this case). However, the source and collocation points are not allowed to be identical because of the singularity of the fundamental solution $\Phi$ at the origin.

To overcome this difficulty, introduce the fundamental solution of the fourth-order Laplace-Helmholtz operator $\Delta(\Delta - c^2 I)$, where $I$ denotes the identity operator and $c$ is a positive scaling constant which will be defined later. This function has the form:

$$
\Phi(x) = -\frac{1}{2\pi c^2}(K_0(c||x||) + \log(c||x||)),
$$

(7.10)
where $K_0$ is the usual modified Bessel function of the third kind. This function is continuous at the origin, which is an immediate consequence of the well-known asymptotic expansion:

$$K_0(r) = (\log 2 - \log r - \gamma) + \frac{1}{4}(\log 2 - \log r + 1 - \gamma)r^2 +$$

$$+ \frac{1}{64}(\log 2 - \log r + \frac{3}{2} - \gamma)r^4 + ...$$

Here $\gamma$ denotes the Euler constant: $\gamma \approx 0.57721$. Moreover, since $K_0$ decreases rapidly, $\Phi$ is approximately harmonic far from the origin.

Replacing the fundamental solution of the Laplacian with the function defined in (7.10), from (7.4) we obtain the simplest form of the regularized MFS:

$$U(x) = \sum_{j=1}^{N} \alpha_j \cdot \Phi(x - \bar{x}_j), \quad (7.11)$$
where the coefficients $\alpha_j$ are determined by solving the linear system:

$$
\sum_{j=1}^{N} \alpha_j \cdot \Phi(x_k - \bar{x}_j) = u_k \quad (k = 1, 2, \ldots, N).
$$

(7.12)

Again, $x_1, \ldots, x_N$ are the collocation points on $\Gamma$ and $\bar{x}_1, \ldots, \bar{x}_N$ are the source points. Unlike the traditional MFS, the collocation and the source points are allowed to coincide. Thus, the condition number of the matrix in (7.12) can be kept at a more moderate level.

As illustrative examples, consider again the test problems presented in the previous section. We have applied the regularized method of fundamental solutions with the same scaling parameter $c := 500$. The results are summarized in Table 7.2. The numerical instability is now less than in case of MFS especially when the source and the collocation points coincide. On the other hand, the accuracy increases when the distance of the source points and the boundary grows. The goal is to achieve an acceptable compromise between the accuracy and instability.

<table>
<thead>
<tr>
<th>$N \setminus d$</th>
<th>Test 1</th>
<th>Test 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.175</td>
<td>1.074</td>
</tr>
<tr>
<td>0.01</td>
<td>0.115</td>
<td>0.002</td>
</tr>
<tr>
<td>0.05</td>
<td>0.341</td>
<td>0.002</td>
</tr>
<tr>
<td>0.10</td>
<td>1.550</td>
<td>0.000</td>
</tr>
<tr>
<td>0.00</td>
<td>0.870</td>
<td>0.131</td>
</tr>
<tr>
<td>0.01</td>
<td>1.115</td>
<td>0.000</td>
</tr>
<tr>
<td>0.05</td>
<td>0.882</td>
<td>0.146</td>
</tr>
<tr>
<td>0.10</td>
<td>2.756*</td>
<td>1.086*</td>
</tr>
<tr>
<td>0.00</td>
<td>0.870</td>
<td>0.131</td>
</tr>
<tr>
<td>0.01</td>
<td>1.115</td>
<td>0.000</td>
</tr>
<tr>
<td>0.05</td>
<td>0.882</td>
<td>0.146</td>
</tr>
<tr>
<td>0.10</td>
<td>2.756*</td>
<td>1.086*</td>
</tr>
</tbody>
</table>

Table 7.2: Regularized MFS, relative $L_2$-errors of the smooth and weakly singular test problems

### 7.3 Error Estimations

As shown in the previous section, the RMFS can be considered a method in which the solution of the original Laplace equation is approximated by a special solution of the fourth-order Laplace-Helmholtz-equation. In the following, we give some error estimations for this type of approximations.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain with a smooth boundary $\Gamma$. Let $U^*$ be the (unique) solution of the Dirichlet problem

$$
\Delta U^* = 0, \quad U^*|_{\Gamma} = u,
$$

(7.13)
where \( u \in H^{1/2}(\Gamma) \), and let \( U \) be the (unique) solution of the fourth-order Dirichlet problem

\[
\Delta(\Delta - c^2 I)U = 0, \quad U|\Gamma = u, \quad \frac{\partial U}{\partial n}|\Gamma = v, \tag{7.14}
\]

with an arbitrary function \( v \in H^{-1/2}(\Gamma) \).

**Theorem 1:** There exists a constant \( C \geq 0 \), independent of \( u \), \( v \) and \( c \) such that the following inequality holds:

\[
\|U - U^*\|_{L^2(\Omega)}^2 \leq \frac{C}{c^2} \left( \|u\|_{H^{1/2}(\Gamma)}^2 + \|v\|_{H^{-1/2}(\Gamma)}^2 \right).
\]

**Proof:** It can be assumed that \( \Omega \) is the half-stripe \((0, L) \times (0, +\infty)\) (the general case can be converted to this case by an proper coordinate transform), and the appearing boundary functions as well as the functions \( U^* \) and \( U \) are \( L \)-periodic with respect to the first variable. Expressing the function \( u \) in terms of complex Fourier series: \( u(x) = \sum \alpha_k e^{i\kappa x} \) (where \( \kappa = \frac{2\pi k}{L} \)), the solution of (7.13) has the form:

\[
U^*(x, y) = \sum \alpha_k e^{-|\kappa|y} e^{i\kappa x}.
\]

Similarly, if \( v \) is expressed in terms of Fourier series: \( v(x) = \sum \beta_k e^{i\kappa x} \), then it is easy to check that the solution of (7.14) is as follows:

\[
U(x, y) = \sum_k \left( A_k e^{-|\kappa|y} + B_k e^{-\sqrt{\kappa^2 + c^2} y} \right) e^{i\kappa x}.
\]

(Note that \( \alpha_0 = 0 \) and \( \beta_0 = 0 \), i.e. the summation is performed for the nonzero indices only.) The coefficients \( A_k, B_k \) are determined by the boundary conditions in (7.14): \( U|\Gamma(x) = U(x, 0) = \sum_k (A_k + B_k) e^{i\kappa x} = \sum_k \alpha_k e^{i\kappa x} \), whence

\[
A_k + B_k = \alpha_k.
\]

Moreover,

\[
\frac{\partial U}{\partial n}|\Gamma(x) = -\frac{\partial U}{\partial y}(x, 0) = \sum_k (|\kappa|A_k + \sqrt{\kappa^2 + c^2} B_k) e^{i\kappa x} = \sum_k \beta_k e^{i\kappa x},
\]

which implies that

\[
|\kappa|A_k + \sqrt{\kappa^2 + c^2} B_k = \beta_k.
\]
From the above two equations, \( A_k \) and \( B_k \) can be computed:

\[
A_k = \frac{\sqrt{\kappa^2 + c^2} \alpha_k - \beta_k}{\sqrt{\kappa^2 + c^2} - |\kappa|}, \quad B_k = \frac{-|\kappa| \alpha_k + \beta_k}{\sqrt{\kappa^2 + c^2} - |\kappa|}
\]

The difference of the approximate and exact solution:

\[
U(x, y) - U^*(x, y) = \sum_k \left( (A_k - \alpha_k)e^{-|\kappa|y} + B_k e^{-\sqrt{\kappa^2 + c^2}y} \right) e^{i\kappa x}
\]

Applying Parseval’s theorem and the fact that \( A_k - \alpha_k = -B_k \), we obtain:

\[
\int_0^L |U(x, y) - U^*(x, y)|^2 dx = L \cdot \sum_k |B_k|^2 \left( e^{-2|\kappa|y} - 2e^{-|\kappa|y}e^{-\sqrt{\kappa^2 + c^2}y} + e^{-2\sqrt{\kappa^2 + c^2}y} \right)
\]

Integrating from 0 to \(+\infty\) with respect to \( y \):

\[
||U - U^*||^2_{L^2(\Omega)} = L \cdot \sum_k |B_k|^2 \left( \frac{1}{2|\kappa|} - \frac{2}{|\kappa| + \sqrt{\kappa^2 + c^2}} + \frac{1}{2\sqrt{\kappa^2 + c^2}} \right)
\]

Substituting the expression of \( B_k \) into the right-hand side and applying the elementary inequality \( |z + w|^2 \leq 2|z|^2 + 2|w|^2 \), we have:

\[
||U - U^*||^2_{L^2(\Omega)} \leq 2L \cdot \sum_k \left( \frac{|\kappa|^2 |\alpha_k|^2 + |\beta_k|^2}{\sqrt{\kappa^2 + c^2} - |\kappa|} \right) \left( \frac{1}{2|\kappa|} - \frac{2}{|\kappa| + \sqrt{\kappa^2 + c^2}} + \frac{1}{2\sqrt{\kappa^2 + c^2}} \right)
\]

After some algebraic manipulations, the expression in the square brackets can be significantly simplified:

\[
||U - U^*||^2_{L^2(\Omega)} \leq \frac{L}{c^2} \sum_k \left( \frac{2\pi}{L} |\kappa| \cdot |\alpha_k|^2 + \frac{L |\beta_k|^2}{2\pi |\kappa|} \right),
\]
A multi-Level regularized version of the MFS

whence the theorem follows.

The theorem implies that if the scaling constant \( c \) is large enough, then the solution of the (second-order) problem (7.13) can be approximated by that of the fourth-order problem (7.14) with arbitrary boundary function \( v \).

In practice, the solution of the fourth-order problem (7.14) is approximated by the RMFS-form (7.11). If the source and the collocation points coincide (which is allowed, since the function \( \Phi \) is continuous everywhere), (7.11) provides a boundary type interpolation formula based on the radial basis function \( \Phi \):

\[
U(x) = \alpha_j \Phi(x - x_j), \quad \Phi(x) = -\frac{1}{2\pi c} (K_0(c|x|) + \log(c|x|)).
\]

Thus, the RMFS can be considered also a special (boundary) RBF-interpolation method.

The key issue is the proper choice of the scaling parameter \( c \) in the formula (7.18) (see [9]). If it is too low e.g. \( c = 0 \) (biharmonic interpolation), then the boundary condition is approximated fairly well (see [7], [8]), but the function defined by (7.18) can be considered harmonic only far from the boundary. If it is too high, then the function (7.18) is ‘almost’ harmonic, however, the approximation of the boundary condition is poor (numerical singularities appear at the source points). Both cases result in large errors in the approximate solution. This phenomenon is illustrates in Table 7.3. Here the smooth test function (7.8) is approximated by the RMFS-formula (7.18) using 32 boundary points.

<table>
<thead>
<tr>
<th>( c )</th>
<th>1</th>
<th>100</th>
<th>140</th>
<th>200</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative error (%)</td>
<td>7.5637</td>
<td>1.0375</td>
<td>0.4398</td>
<td>1.1869</td>
<td>3.7053</td>
<td>5.5052</td>
</tr>
</tbody>
</table>

Table 7.3: Regularized MFS, relative \( L_2 \)-errors of the smooth test problem

As a quasi-optimal compromise, the definition \( c \approx \frac{1}{h} \) (where \( h \) denotes the characteristic distance of the boundary points) makes it possible to keep both the error of the approximation of the boundary condition and the error estimated by Theorem 1 at an acceptably low level.
7.4 Improvement by Off-boundary Sources

As can be seen from the results summarized in Table 7.2, the exactness of the RMFS-approximation can be improved by locating the sources outside of the domain (off-boundary sources). If the sources are kept in the vicinity of the boundary, the condition number of the discrete problem does not grow significantly. Now we give an error estimation for such cases showing that a significant amount of improvement can be achieved.

Assume that the original domain $\Omega$ is contained in a larger domain $\Omega_\delta$, the boundary of which (denoted by $\Gamma_\delta$) lies at the distance $\delta$ from the original boundary $\Gamma$. Consider the same model problem (7.13) than in the previous section. Now denote by $U$ the solution of the following fourth-order problem:

$$
\Delta(\Delta - c^2 I)U = 0, \quad U|_{\Gamma} = u, \quad \frac{\partial U}{\partial n}|_{\Gamma_\delta} = v,
$$

(7.19)

where $U$ is defined in the larger domain $\Gamma_\delta$. Note that (7.19) is not a usual boundary value problem since the Dirichlet and Neumann data are attached to different boundaries. For simplicity, assume that both $\Omega$ and $\Omega_\delta$ are half-stripes: $\Omega = (0,L) \times (\delta, +\infty)$, $\Omega_\delta = (0, L) \times (0, +\infty)$. Assume also that the boundary data $u$ and $v$ are expressed in terms of Fourier series:

$$
u(x) = \sum_k \alpha_k e^{ikx}, \quad v(x) = \sum_k \beta_k e^{ikx},
$$

(7.20)

where $\kappa = 2\pi k/L$.

**Theorem 2:** There exists a constant $C \geq 0$, independent of $u$, $v$, $c$ and $\delta$ such that the following inequality holds:

$$
||U - U^*||^2_{L^2(\Omega)} \leq \frac{C}{c^2} \left( \sum_k |\alpha_k|^2 |k| \cdot e^{-2(\sqrt{\kappa^2 + c^2} - |\kappa|)\delta} + \frac{|\beta_k|^2}{|k|} \cdot e^{-2\sqrt{\kappa^2 + c^2}} \right).
$$

**Proof:** The technique is quite similar to the proof of Theorem 1, therefore we only outline the proof. Now the exact solution $U^*$ has the form:

$$
U^*(x,y) = \sum_k \alpha_k e^{-|\kappa|(y-\delta)} e^{ikx},
$$

while the solution of (7.19) is expressed as

$$
U(x,y) = \sum_k \left( A_k e^{-|\kappa|y} + B_k e^{-\sqrt{\kappa^2 + c^2} y} \right) e^{ikx}.$$
The coefficients $A_k, B_k$ are determined by the boundary conditions in the following way:

$$A_k = \frac{\sqrt{\kappa^2 + c^2} \alpha_k - e^{-\sqrt{\kappa^2 + c^2} \delta} \beta_k}{\sqrt{\kappa^2 + c^2} e^{-|\kappa| \delta} - |\kappa| e^{-\sqrt{\kappa^2 + c^2} \delta}},$$

$$B_k = \frac{-|\kappa| \alpha_k + e^{-|\kappa| \delta} \beta_k}{\sqrt{\kappa^2 + c^2} e^{-|\kappa| \delta} - |\kappa| e^{-\sqrt{\kappa^2 + c^2} \delta}}.$$

Computing the quantity $\| U - U^* \|_{L^2(\Omega)}^2$ via the same steps as in the proof of Theorem 1, we obtain:

$$\| U - U^* \|_{L^2(\Omega)}^2 \leq \frac{L}{c^2} \sum_k \left( \frac{\sqrt{\kappa^2 + c^2} - |\kappa| e^{-|\kappa| \delta}}{\sqrt{\kappa^2 + c^2} e^{-|\kappa| \delta} - |\kappa| e^{-\sqrt{\kappa^2 + c^2} \delta}} \right)^2 \times$$

$$\times \left[ e^{-2(\sqrt{\kappa^2 + c^2} - |\kappa| \delta)} |\kappa| \cdot |\alpha_k|^2 + e^{-2\sqrt{\kappa^2 + c^2} \delta} \frac{1}{|\kappa^2|} \cdot |\beta_k|^2 \right].$$

Now the theorem is a simple consequence of the elementary inequality:

$$\frac{\sqrt{\kappa^2 + c^2} - |\kappa| e^{-|\kappa| \delta}}{\sqrt{\kappa^2 + c^2} e^{-|\kappa| \delta} - |\kappa| e^{-\sqrt{\kappa^2 + c^2} \delta}} \leq 1.$$
where $\omega > 0$ is a (small) iteration parameter. Roughly speaking, the iteration enforces the original boundary condition $u$ along the original boundary $\Gamma$.

One can easily see that the above iteration is convergent for any sufficiently small positive parameter $\omega$. Assume for simplicity again that $\Omega = (0,L) \times (\delta, +\infty)$, $\Omega_{\delta} = (0,L) \times (0, +\infty)$. Expressing $\tilde{u}$ in terms of Fourier series:

$$\tilde{u}(x) = \sum_{k} \tilde{\alpha}_{k} e^{i\kappa x},$$

the solution of (7.21) is as follows:

$$U(x,y) = \sum_{k} (A_{k} e^{-|\kappa| y} + B_{k} e^{-\sqrt{\kappa^{2} + c^{2}} y}) e^{i\kappa x},$$

where $A_{k}, B_{k}$ are determined by (cf. the proof of Theorem 1):

$$A_{k} = \frac{\sqrt{\kappa^{2} + c^{2}} \tilde{\alpha}_{k} - \beta_{k}}{\sqrt{\kappa^{2} + c^{2}} - |\kappa|}, \quad B_{k} = \frac{-|\kappa| \tilde{\alpha}_{k} + \beta_{k}}{\sqrt{\kappa^{2} + c^{2}} - |\kappa|}.$$

Based on these expressions, the iteration (7.22) can be split into separate iterations for the Fourier coefficients $\tilde{\alpha}_{k}$:

$$\tilde{\alpha}_{k}^{\text{improved}} = \tilde{\alpha}_{k} - \omega \cdot \left( A_{k} e^{-|\kappa| \delta} + B_{k} e^{-\sqrt{\kappa^{2} + c^{2}} \delta} - \tilde{\alpha}_{k} \right)$$

$$= \left( 1 - \omega \cdot \frac{\sqrt{\kappa^{2} + c^{2}} e^{-|\kappa| \delta} - |\kappa| e^{-\sqrt{\kappa^{2} + c^{2}} \delta}}{\sqrt{\kappa^{2} + c^{2}} - |\kappa|} \right) \tilde{\alpha}_{k}$$

$$+ \omega \cdot \frac{e^{-|\kappa| \delta} - e^{-\sqrt{\kappa^{2} + c^{2}} \delta}}{\sqrt{\kappa^{2} + c^{2}} - |\kappa|} \beta_{k} + \omega \alpha_{k}$$

Since obviously

$$\sqrt{\kappa^{2} + c^{2}} e^{-|\kappa| \delta} > |\kappa| e^{-\sqrt{\kappa^{2} + c^{2}} \delta},$$

the absolute value of the factor multiplying $\tilde{u}_{k}$ is less than 1 for any sufficiently small positive parameter $\omega$. This implies that every Fourier coefficient $\tilde{\alpha}_{k}$ is convergent (the speed of convergence becomes slow for the high-frequency components). For the limit function $\tilde{u}$, it is obvious that the corresponding solution $U$ satisfies the equality $U|_{\Gamma} = u$, i.e. the iteration results in the solution of the problem (7.19).
The above iteration plays no role if the problem (7.19) is solved (approximately) by a direct RMFS-formulation. However, it can be usefully applied when a Direct Multi-Elliptic Interpolation approach is used as presented in the next section.

7.5 Combination with the Direct Multi-Elliptic Interpolation

Though the RMFS leads to a linear system which is far less ill-conditioned than the original MFS, the matrix of this system is still large and dense. Therefore the computational cost remains high. Using e.g. the traditional Gaussian elimination, the number of the arithmetic operations is proportional to the third power of $N$, which is inadmissibly high if $N$ is large. The computational cost can be significantly reduced by applying the idea of the Direct Multi-Elliptic Interpolation (see [7], [8], [9]). In this approach, instead of solving the RMFS-equations (7.12), the approximate solution is defined by directly solving the fourth-order Laplace-Helmholtz-equation:

$$\Delta(\Delta - c^2 I)U = 0 \quad \text{in} \quad \Omega_0 \setminus \{x_1, x_2, \ldots, x_N\} \quad (7.24)$$

supplied with the interpolation condition

$$U(x_k) = u_k \quad (k = 1, 2, \ldots, N). \quad (7.25)$$

Here $x_1, x_2, \ldots, x_N \in \Gamma$ are the collocation points $\Omega_0 \supset \Omega$ is a larger domain. Along the boundary $\Gamma_0 := \partial \Omega_0$, any usual boundary condition can be imposed e.g.:

$$U|_{\Gamma_0} = 0, \quad \frac{\partial U}{\partial n}|_{\Gamma_0} = 0. \quad (7.26)$$

Based on variational tools, it has been proved (see [7]) that the problem (7.24)-(7.26) has a unique solution in the Sobolev space $H^2_0(\Omega_0)$ in spite of the pointwise interpolation conditions (7.25), which destroys the well-posedness for second-order elliptic equations but not for fourth-order ones.

The connection between the Direct Multi-Elliptic Interpolation and the Method of Radial Basis Functions is given by the representation theorem of the Direct Multi-Elliptic Interpolation (see [7]), which states that the interpolation function defined by (7.24)-(7.26) is uniquely represented in the form:

$$U(x) = w(x) + \sum_{j=1}^{N} \beta_j \Phi(x - x_j),$$
where \( w \) is a function which satisfies the Laplace-Helmholtz-equation everywhere (including also the interpolation points \( x_1, \ldots, x_n \)), and \( \Phi \) is the fundamental solution of the Laplace-Helmholtz-operation defined by (7.10).

To compute the interpolation function \( U \) defined by (7.24)-(7.26), a multi-level technique can be recommended on a quadtree cell system generated by the interpolation points \( x_1, \ldots, x_N \). This cell system exhibits local refinements at the interpolation points, while the total number of cells remains moderate (\( O(N) \)). Using quite natural finite volume schemes and multigrid tools (see [6]), the overall computational cost of the solution of the interpolation (7.24)-(7.26) can be reduced to \( O(N) \). It should also be pointed out that the numerical treatment of large, dense and ill-conditioned matrices is completely avoided.

To illustrate the method, consider again the smooth and non-smooth test functions (7.8) and (7.9). The interpolation problem (7.24)-(7.26) was solved on a square \( \Omega_0 \) with sidelength 4. Starting from this square, a quadtree cellsystem was generated with 8 levels of subdivision. The subdivision process was controlled by the interpolation points, which were equi-spatially distributed along \( \Gamma \). The scaling constant in (7.24) was set to 500. Table 7.5 shows the relative \( L_2 \)-errors with different numbers of boundary points. The results are quite similar to the results summarized in Table 7.2. However, the use of dense matrices is now avoided and no numerical instability appeared. Figure 7.2 shows the approximate solutions

<table>
<thead>
<tr>
<th>( N )</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative error (%) Test 1</td>
<td>9.733</td>
<td>3.218</td>
<td>0.685</td>
<td>0.299</td>
</tr>
<tr>
<td>Relative error (%) Test 2</td>
<td>10.137</td>
<td>3.452</td>
<td>0.802</td>
<td>0.306</td>
</tr>
</tbody>
</table>

Table 7.4: Regularized MFS via Direct Multi-Elliptic Interpolation. Relative \( L_2 \)-errors of the smooth and weakly singular test problems

on the finest cell system.

Finally we show that (in the case of smooth solutions) the accuracy can be improved again by using off-boundary points similarly to the technique in Section 7.4. Let \( x_1, x_2, \ldots, x_N \in \Gamma \) be boundary interpolation points again and let \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N \) be located at a small positive distance from \( x_1, x_2, \ldots, x_N \) in the outward direction, i.e. \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N \) are located on the boundary \( \Gamma_\delta \) of a somewhat larger domain \( \Omega_\delta \supset \Omega \). The problem (7.24)-(7.25) is now
converted to a sequence of boundary interpolation problem

\[ \Delta(\Delta - c^2 I)U^{(n)} = 0 \quad \text{in} \quad \Omega_0 \setminus \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N\} \]

\[ U^{(n)}(\tilde{x}_k) = \tilde{u}_k^{(n)} \quad (k = 1, 2, \ldots, N), \]

where the interpolation conditions taken at the points \( \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N \) are defined recursively:

\[ \tilde{u}_k^{(n)} := \tilde{u}_k^{(n-1)} - \omega \cdot \left( U^{(n-1)}(\tilde{x}_k) - u_k \right) \quad (k = 1, 2, \ldots, N) \quad (7.27) \]

Here \( \omega \) is a sufficiently small positive iteration parameter and \( u_1, u_2, \ldots, u_N \) are the original interpolation conditions taken at the points \( x_1, x_2, \ldots, x_N \). Roughly speaking, the iteration (7.27) enforces the original boundary condition along the original boundary.
As an example, consider the smooth test problem (7.8). We applied 64 equally spaced boundary points. The distance between the boundary and the corresponding off-boundary points was 0.03. Table 7.5 summarizes the relative $L_2$-errors of the approximate solutions computed on a quadtree cell system with different scaling constants. The relative errors without using off-boundary points can also be seen. The maximal level of subdivision was 8. Observe that the errors become significantly smaller when off-boundary points were introduced. It can also be seen that the method is not very sensitive to the optimal choice of the scaling constant. Figure 7.3 shows the approximate solution at the value $c = 1000$. It should be pointed out that in the vicinity of the off-boundary points, the approximate solution is irregular (it exhibits numerical singularities), however, inside the original domain, it remains smooth and of good accuracy.

### 7.6 Summary and Conclusions

The main idea of the presented regularization technique of the Method of Fundamental Solutions is to replace the original fundamental solution with the fundamental solution of a higher-order partial differential operator in the MFS-formulation. In the case of the Laplace equation, this leads to the use of the fundamental solution of the fourth-order Laplace-Helmholtz operator with a carefully chosen scaling parameter. The new fundamental solution is continuous everywhere (including also the origin), which allows to locate the source points to the boundary. This reduces the computational difficulties arising necessarily in the traditional version of the MFS if the sources are far from the boundary.

The method can be considered also a boundary Laplace-Helmholtz type interpolation method. It can be implemented in an economic way by solving the Laplace-Helmholtz equation in a multi-level way using quadtrees and quadtree-based finite volume schemes. This approach makes it possi-
Figure 7.3: Regularized MFS via Direct Multi-Elliptic Interpolation, using off-boundary points. Smooth test, approximate solution

ble to completely avoid large dense and often severely ill-conditioned matrices.

If the solution of the original problem is sufficiently smooth, the exactness can be improved by using off-boundary sources which remain, however, in the vicinity of the boundary. In this case, the RMFS-procedure can be converted to a sequence of Dirichlet problems for the Laplace-Helmholtz equation. This can be easily incorporated in the multi-level solution algorithm, so that the computational cost remains relatively low.

Finally it should be pointed out that the approach can be easily generalized also to Neumann or mixed type boundary conditions. However, the local density of the boundary points should be more or less uniform on the boundary in order that a proper scaling parameter can be defined which is adequate in the whole domain.

References

[1] C. J. S. Alves and P. R. S. Antunes. The method of fundamental solutions applied to the numerical calculation of eigenfrequencies and


CHAPTER 8

The Method of Fundamental Solutions for Solving
Axi-symmetric Isothermal Gas Flow in Porous
Medium

Anita Uscilowska

Abstract. This paper presents numerical solution to a problem of the transient flow of gas within an three-dimensional porous medium. The considered problem is treated as axisymmetric one, therefore the dimension of the problem is reduced to 2D problem. A method of fundamental solution for space variables and finite difference method for time variable are employed to obtain a solution of the non-linear partial differential equation describing the flow of gas. Picard iteration is used for treating nonlinearity. The inhomogeneous term is expressed by radial basis functions and polynomials at each iteration step.

Nomenclature

ϕ - porosity
μ - viscosity
k - permeability, which is related to hydrodynamic conductivity coefficient K by: \( k = \frac{\mu}{\rho g} K \)
ρ - mass density of the fluid
g - gravity acceleration
p - pressure
q - superficial fluid velocity
T - temperature

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The MFS for axi-symmetric gas flow in porous medium

\( r, z \) - geometry variables
\( \theta \) - angle variable
\( a, b, c \) - geometry parameters
\( t \) - time
\( R_g \) - individual gas constant
\( \tau \) - dimensionless time parameter
\( R, Z \) - dimensionless geometry variables
\( D, E \) - dimensionless geometry parameters

8.1 Introduction

The method of fundamental solutions (MFS) can be understood as a method in which differential equation is satisfied exactly whereas the boundary condition are satisfy in approximate way. For case of non-linear equations the set of functions that fulfill exactly these equation is usually unknown. Then method of fundamental solution cannot be use in straight way for non-linear Boundary Value Problems (BVPs). But, there are procedures of implementing the MFS for solving the nonlinear problems. First case when it can be used is BVPs with linear equation but with non-linear boundary conditions. Examples of such applications of this method are given in papers [5, 10]. Second case known in literature is BVP with non-linear Poisson equation in the form [1, 2, 3, 8, 9]:

\[
\nabla^2 u = f(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})
\]  

(8.1)

where \( u \) is unknown function, and \( f \) is known function in which same arguments are unknown. In paper [3] the non-linear thermal explosions problem was solved by method of fundamental solutions. The radial basis functions were used for interpolation of right hand side on Picard iteration method was used to treat non-linearity. In paper [1] method called as "particular solution Trefftz method" was used. Really its is extension and improvement of ideas proposed in paper [3]. Another version of boundary element method for solution of non-linear Poisson equation was presented in paper [9]. A two-dimensional Poisson equation with spatially varying conductivity is solved by homotopy analysis method. Also steady state heat conduction problem with temperature dependent conductivity was considered in paper [8]. Combination of method fundamental solutions with Picard iteration was used for non-linear Poisson equation.
Evolutionary algorithm was applied for optimal determination of method parameters. More complicated application of the MFS was presented in paper [2] where the method of operator splitting with method of fundamental solution was used for transient non-linear Poisson problems. These problems are widely encountered in the modelling many physical phenomena and governing differential equation has a form:

$$\frac{du}{dt} = \nabla^2 u + f(u)$$  \hspace{1cm} (8.2)

where $t$ is time. The purpose of the present paper is application of the MFS to a problem of the transient flow of gas within a three-dimensional porous medium. The considered problem is axisymmetric. Unsteady gas flow through semi-infinite porous medium was considered in paper [7]. In such case problem is described by ordinary differential equation. In case of finite porous region the governing equation for pressure of gas as unknown has a form similar to (8.2). In our proposition (as in [11]) the method of fundamental solution for space variables and finite difference method for time variable are employed to obtain a solution of the non-linear partial differential equation describing the flow of gas. Picard iteration is used for treating nonlinearity. The inhomogeneous term is expressed by radial basis functions at each iteration step. The fundamental solution for axisymmetric Poisson problem is known in literature [6]. Particular solutions of axisymmetric Poisson problem for chosen radial basis functions and polynomials are proposed in paper [4].

8.2 Problem Description

Considered region of the porous medium with flowing fluid is presented on Figure 8.1. The porous medium is filled with gas under uniform pressure. The edges of considered reservoir are insulated, except one piece of edge which is opened. Pressure outside the reservoir is lower than pressure in porous medium.

The reservoir is cylindrical porous medium with radius $a$ and length $b$. The open edge is a circle of radius $c$. Due to axisymmetry of the problem the rectangular plane region is of the considered and presented in Figure 8.2.

For investigation gas flow in porous medium we introduce the following assumptions:

- The flow of gas follows Darcy’s law;
The only phase flowing is a gas of constant composition and viscosity
The gas is perfect and gas flow is isothermal
The permeability of the porous medium is constant and uniform
Gravitational forces are neglected.

8.3 Motion Equations
Darcy Law is filtration equation for fluid flow in porous media and in 3-D case has form

\[ \mathbf{q} = -\frac{k}{\mu} \text{grad} p \]  

(8.3)

which in polar coordinates is

\[ \mathbf{q} = -\frac{k}{\mu} \left( \frac{\partial p}{\partial r} e_r + \frac{\partial p}{\partial \theta} e_\theta + \frac{\partial p}{\partial z} e_z \right) \]  

(8.4)
The MFS for axi-symmetric gas flow in porous medium

and for axisymmetric case has form

$$q = -\frac{k}{\mu} \left( \frac{\partial p}{\partial r} e_r + \frac{\partial p}{\partial z} e_z \right)$$  \hspace{1cm} (8.5)

The continuity equation for porous media is

$$\text{div} (\rho q) = -\frac{\partial}{\partial t} (\varphi \rho)$$  \hspace{1cm} (8.6)

which for 3D case in cylindrical coordinates is

$$\frac{1}{r} \frac{\partial}{\partial r} (r \rho q_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho q_\theta) + \frac{\partial}{\partial z} (\rho q_z) = -\frac{\partial}{\partial t} (\varphi \rho)$$  \hspace{1cm} (8.7)
The MFS for axi-symmetric gas flow in porous medium

and for axisymmetric case

\[
\frac{1}{r} \frac{\partial}{\partial r} (r \rho q_r) + \frac{\partial}{\partial z} (\rho q_z) = -\frac{\partial}{\partial t} (\varphi \rho)
\]  
(8.8)

The gas equation for isothermal phenomena

\[
\rho = \frac{p}{R_g T}
\]  
(8.9)

and \( T \) - temperature is constant.

Applying the eq. (8.5) and eq. (8.9) to the eq. (8.8) gives

\[
\frac{\partial}{\partial x} \left( \frac{p}{RT} \frac{k}{\mu} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{p}{RT} \frac{k}{\mu} \frac{\partial p}{\partial y} \right) = \frac{\mu}{k} \frac{\partial}{\partial t} \left( \varphi \frac{p}{RT} \right)
\]  
(8.10)

Rearranging the eq. (8.10) yields to the equation

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( pr \frac{\partial p}{\partial r} \right) + \frac{\partial}{\partial z} \left( p \frac{\partial p}{\partial z} \right) = \frac{\mu}{k} \frac{\partial}{\partial t} (\varphi p)
\]  
(8.11)

The initial condition says that there is uniform pressure in porous medium:

\[
p(r,z,t) = p_0
\]  
(8.12)

for \( t = 0, \ 0 < r < a \) and \( 0 < z < b \).

The boundary condition at open edge \( \{(r,z) \mid (0 < r < c) \cap (z = b)\} \) is:

\[
p(r,z,t) = p_1 < p_0
\]  
(8.13)

for \( 0 < t < \infty \)

For the other reservoir edges

\[
\{(r,z) \mid ((0 < r < a) \cap (z = 0)) \cup ((c < r < a) \cap (z = b)) \\
\cup ((r = a) \cap (0 < z < b))\}
\]

insulated boundary condition is

\[
\frac{\partial p}{\partial n} = 0
\]  
(8.14)

and for \( \{(r,z) \mid (r = 0) \cap (0 < z < b)\} \) the symmetry condition is applied

\[
\frac{\partial p}{\partial n} = 0
\]  
(8.15)
The dimensionless variables are introduced

\[ R = \frac{r}{a}, \ Z = \frac{z}{a}, \ E = \frac{b}{a}, \ D = \frac{c}{a}, \ P = \frac{p}{p_0}, \ P_1 = \frac{p_1}{p_0}, \ \tau = \frac{kp_0}{\rho \mu a^2 t} \] (8.16)

Therefore the equation (8.11) has the dimensionless form

\[ \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial P}{\partial R} \right) + \frac{\partial}{\partial Z} \left( P \frac{\partial P}{\partial Z} \right) = \frac{\partial P}{\partial \tau} \] (8.17)

and the initial condition is

\[ P(R, Z, \tau) = 1 \] (8.18)

for \( \tau = 0, 0 < R < 1, 0 < Z < E \). The boundary conditions are the boundary condition for the open edge

\[ P(R, Z, \tau) = P_1 < 1 \] (8.19)

for \( 0 < R < D, Z = E \) and insulating and symmetry condition is

\[ \frac{\partial P}{\partial n} = 0 \] (8.20)

for the boundary

\[ \{(R, Z) \mid ((0 < R < 1) \cap (Z = 0)) \cup ((D < R < 1) \cap (Z = E)) \}

\[ \cup ((R = 0) \cap (0 < Z < E)) \cup ((R = 1) \cap (0 < Z < E)) \}\]

8.4 Algorithm for Solving Initial-boundary Problems

Assuming that time derivative term can be expanded using finite difference

\[ \frac{\partial P}{\partial \tau} = \frac{P^{(n+1)} - P^{(n)}}{\Delta \tau} \] (8.21)

for \( n = 0, 1, 2, \ldots \), eq. (8.17) can be approximated as

\[ \frac{1}{R} \frac{\partial P^{(n+1)}}{\partial R} + \frac{\partial^2 P^{(n+1)}}{\partial R^2} + \frac{\partial^2 P^{(n+1)}}{\partial Z^2} - \frac{P^{(n+1)} - P^{(n)}}{P^{(n+1)} \Delta \tau} = - \frac{1}{P^{(n+1)}} \left\{ \left( \frac{\partial P^{(n+1)}}{\partial R} \right)^2 + \left( \frac{\partial P^{(n+1)}}{\partial Z} \right)^2 \right\} \] (8.22)
The MFS for axi-symmetric gas flow in porous medium

with initial condition

\[ P^{(0)}(R, Z, \tau) = 1 \]  

(8.23)

for \( \tau = 0, 0 < R < 1, 0 < Z < E \) and boundary conditions

\[ P^{(n+1)} = P_1^{(n+1)} < 1 \]  

(8.24)

for \( 0 < R < D, Z = E \)

\[ \frac{\partial P^{(n+1)}}{\partial n} = 0 \]  

(8.25)

for the boundary

\[ \{(R, Z) \mid ((0 < R < 1) \cap (Z = 0)) \cup ((D < R < 1) \cap (Z = E)) \}

\[ \cup \{(R = 1) \cap (0 < Z < E)\} \cup \{(R = 0) \cap (0 < Z < E)\}\} \]

where \( P^{(n)} \) is dimensionless pressure at \( n \)-th time step, \( P^{(n+1)} \) is this pressure in the next time step.

The calculation of pressure in the every time steps is based also on eq. (8.22), which is transformed into Poisson equation:

\[ \frac{1}{R} \frac{\partial p^{(n+1)}}{\partial R} + \frac{\partial^2 p^{(n+1)}}{\partial R^2} + \frac{\partial^2 p^{(n+1)}}{\partial Z^2} \]

\[ = \frac{1}{\tau} \left( \frac{p^{(n)}}{p^{(n+1)}} - \frac{1}{p^{(n+1)}} \left( \left( \frac{\partial p^{(n+1)}}{\partial R} \right)^2 + \left( \frac{\partial p^{(n+1)}}{\partial Z} \right)^2 \right) \right) \]  

(8.26)

with initial and boundary conditions (8.24), (8.25), for \( n = 0, 1, 2, \ldots \). The equation is strongly non-linear with respect to \( p^{(n+1)} \), therefore, it is solved in an iterative fashion:

\[ \frac{1}{R} \frac{\partial p^{(n+1,i+1)}}{\partial R} + \frac{\partial^2 p^{(n+1,i+1)}}{\partial R^2} + \frac{\partial^2 p^{(n+1,i+1)}}{\partial Z^2} \]

\[ = \frac{1}{\tau} \left( \frac{p^{(n,i)}}{p^{(n+1,i)}} - \frac{1}{p^{(n+1,i)}} \left( \left( \frac{\partial p^{(n+1,i+1)}}{\partial R} \right)^2 + \left( \frac{\partial p^{(n+1,i+1)}}{\partial Z} \right)^2 \right) \right) \]  

(8.27)

with boundary conditions (8.24), (8.25), where \( P^{(n+1,i)} \) is the \( i \)-th iteration result at \( (n+1) \)-th time step. We introduce an initial condition for iterative
procedure e.g. trial equation in Laplace form, which is modified version of eq. (8.27):

\[
\frac{1}{R} \frac{\partial p^{n+1,1}}{\partial R} + \frac{\partial^2 p^{n+1,1}}{\partial R^2} + \frac{\partial^2 p^{n+1,1}}{\partial Z^2} = 0 \quad (8.28)
\]

with trial boundary conditions

\[
p^{(n+1,1)} = P_1 < 1 \quad (8.29)
\]

for \(0 < R < D, Z = E\) and

\[
\frac{\partial p^{(n+1,1)}}{\partial n} = 0 \quad (8.30)
\]

for the boundary

\[
\{(R, Z) \mid ((0 < R < 1) \cap (Z = 0)) \cup ((D < R < 1) \cap (Z = E))
\]

\[
\cup ((R = 0) \cap (0 < Z < E)) \cup ((R = 1) \cap (0 < Z < E))\}
\]

One extra boundary condition is added

\[
p^{(n+1,1)} = p^{(n)} \quad (8.31)
\]

for \(\{(R, Z) \mid (0 < R < 1) \cap (Z = 0)\}\) to combine the previous time step pressure distribution with the solution at the next time step.

Equation (8.28) is solved by the fundamental solution method, including the appropriate boundary conditions into calculation.

Solutions at second and next iteration steps are found by Trefftz method, based on the eq. (8.27) with its boundary conditions. Therefore, in one time step we obtain the sequence of solutions: \(P^{(n+1,1)}, P^{(n+1,2)}, \ldots\).

The iterative process is terminated when difference between solutions of two successive iteration steps is quite small, less than a chosen small parameter. We introduce \(m\), which points the iteration step number, at which solution is taken as the solution at \(n\)-th time step, noticed as \(P^{(n+1,m)} = P^{(n+1)}\).

### 8.5 The MFS for Solving Boundary Value Problems

Partial differential inhomogeneous equation

\[
Lu = f(R, Z) \quad (8.32)
\]
is considered on the region $\Omega$.

Operator $L$ is a partial differential operator, which includes Laplace operator.

The boundary condition has the general form

$$Bu = g(R, Z) \quad (8.33)$$

where $B$ is an operator imposed as boundary conditions, such Dirichlet, Neumann, and Robin.

Let us denote $\{p_{k,m} = (R_j, Z_j)\}_{j=1}^N$ to be $N$ collocations points in $\Omega \cup \partial \Omega$ of which $\{(R_j, Z_j)\}_{j=1}^{N_l}$ are interior points; $\{(R_j, Z_j)\}_{j=N_l+1}^N$ are boundary points and $N_b = N - N_l$ is a number of boundary points.

The right-hand side function $f$ is approximated as Radial Basis Functions (RBFs) as

$$f_N(R, Z) = \sum_{j=1}^N a_j \varphi_j(R, Z) + \sum_{k=0}^{l_1} \sum_{m=0}^{l_2} b_{k,m} p_{k,m}(R, Z) \quad (8.34)$$

where $\varphi_j(R, Z) = \varphi(R, Z, R_j, Z_j) : R^d \to R^+$ is a RBF; $\{p_{k,m}\}_{k=0}^{l_1} \{l_2\}_{k=0}^{l_2}$ is the complete basis for $d$-variate polynomials of degree $\leq (k-1)(m-1)$.

The coefficients $\{a_j\}$, $\{b_{k,m}\}$ can be found by solving the system

$$\sum_{j=1}^N a_j \varphi_j(R_i, Z_i) + \sum_{k=0}^{l_1} \sum_{m=0}^{l_2} b_{k,m} p_{k,m}(R_i, Z_i) = f(R_i, Z_i) \quad (8.35)$$

for $1 \leq i \leq N$,

$$\sum_{k=0}^{l_1} \sum_{m=0}^{l_2} a_j p_{k,m}(R_j, Z_j) = 0 \quad (8.36)$$

for $1 \leq j \leq N$, where $\{(R_j, Z_j)\}_{j=1}^N$ are the collocation points on $\Omega \cup \partial \Omega$.

The approximate particular solutions $u_p$ can be obtained using the coefficients $\{a_j\}$ and $\{b_{k,m}\}$ by

$$u_p(R, Z) = \sum_{j=1}^N a_j \varphi_j(R, Z) + \sum_{k=1}^{l_1} \sum_{m=1}^{l_2} b_{k,m} \psi_{k,m}(R, Z) \quad (8.37)$$

where

$$L \varphi_j = \varphi_j(R, Z) \quad (8.38)$$

$$L \psi_{k,m} = p_{k,m}(R, Z) \quad (8.39)$$
for $1 \leq j \leq N$, $0 \leq k \leq l_1$, $0 \leq m \leq l_2$. Solution of differential equation (8.32) now can be given as

$$u = u_p + v$$ \hspace{1cm} (8.40)

where $v$ is solution of boundary value problem in the form

$$Lv = 0 \quad \text{in} \quad \Omega \hspace{1cm} (8.41)$$

$$Bv = g(R,Z) - Bu_p \quad \text{on} \quad \partial \Omega \hspace{1cm} (8.42)$$

The method of fundamental solution is used to solve problem presented above, what means that

$$v(R,Z) = \sum_{j=1}^{N_s} c_j f_{S,j}(R,Z)$$ \hspace{1cm} (8.43)

where $f_S$ is the fundamental solution function and

$$f_{S,j}(R,Z) = f_S\left(\|R - R_j, Z - Z_j\|\right)$$

denotes the fundamental solution function determined for the $j$-th source point. Points $\left\{ R_j, Z_j \right\}_{j=1}^{N_s}$ are so called source points, and $N_s$ is number of source points on artificial boundary outside the considered region $\Omega$.

Putting (8.43) into boundary condition (8.42):

$$\sum_{j=1}^{N_s} c_j B f_{S,j}(R_i, Z_i) = g(R_i, Z_i) - Bu_p(R_i, Z_i) \quad \text{for} \quad 1 \leq i \leq N_b \hspace{1cm} (8.44)$$

coefficients $c_j$ are obtained. The solution of the boundary problem (8.32) and (8.33) is calculated by equation (8.40).

8.5.1 Numerical implementation

In Poisson equation differential operator is the Laplace operator:

$$L = \nabla^2$$ \hspace{1cm} (8.45)

The radial basis function for axisymmetric problem is

$$\varphi_j(R,Z) = 4E(k) \sqrt{\alpha^2 + \beta^2}$$ \hspace{1cm} (8.46)
and proper particular solution is

\[ \phi(R, Z) = \frac{1}{9} (\alpha + \beta)^{\frac{3}{2}} k^{2} (k^2 - 1) (K(k) + (4 - 2k^2) E(k)) \quad (8.47) \]

where

\[ \alpha = R^2 + R_j^2 + (Z - Z_j)^2, \quad \beta = 2RR_j, \quad k^2 = \frac{2\beta}{\alpha + \beta}, \]

and \( E(k), K(k) \) are the complete elliptic integrals of first and second kind.

For polynomials given in the form

\[ p_{k,m}(R, Z) = R^kZ^m, \quad (8.48) \]

the proper particular solutions are

\[ \psi_{k,m}(R, Z) = \sum_{j=0}^{m/2} (-1)^j m! \left( \frac{k!!}{(k + 2j + 2)!!} \right)^2 R^{k+2j+2}Z^{m-2j} \quad (8.49) \]

The fundamental solution for Poisson equation is function:

\[ f_{s,j}(R, Z) = \frac{4K(k)}{\sqrt{\alpha^2 + \beta^2}} \quad (8.50) \]

where

\[ \alpha = R^2 + R_j^2 + (Z - Z_j)^2, \quad \beta = 2RR_j, \quad k^2 = \frac{2\beta}{\alpha + \beta} \]

and \( K(k) \) is the complete elliptic integral of second kind.

### 8.6 Numerical Results

The example solved by the combination of the methods described above is presented. The considered region is a square with boundary of length equal to 1. The pressure inside the region is equal to 1 and outside pressure has value 0.5. For first time step, \( d\tau = 1.0 \) the first iteration result is presented in Figure 8.3.

It shows results obtained for auxiliary boundary value problem (8.28 - 8.30). The second iteration is calculated for Poisson equation given by (8.27) with boundary conditions (8.24), (8.25), where \( n = 0, i = 1 \). The right-hand side function is approximated by radial basis functions and polynomials. The relative error of the approximation shown in Figure 8.4 is acceptably low, to continue iterations procedure.
Finally, solution obtained for second time step is presented in Figure 8.5. Next, two time steps results at Figure 8.6 and 8.7 gas flow out from reservoir. The results are compliant with expected ones.

Values of the pressure inside the considered region are very close to value of the pressure at the open edge. The next time result shows that the pressure field is almost uniform (with tolerance $10^{-3}$) and equal to the
The MFS for axi-symmetric gas flow in porous medium

Figure 8.5: Pressure distribution at second time step.

Figure 8.6: Pressure distribution at third time step.

outside pressure.

8.7 Conclusions
In our paper the MFS is used for solving time dependent phenomena. The finite difference method is implemented to approximate differential with respect to time parameter. The technique for obtaining solution of inhomogeneous nonlinear Poisson equation is applied. The problem given in
Figure 8.7: The solution at fourth time step.

implicit form is solved using the iterative technique. Obtained result, which is gas outflow from reservoir - porous medium, agrees with expected ones.

References


References


CHAPTER 9

The Method of Fundamental Solutions for Low Reynolds Number Flows with Moving Rigid Body

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1Department of Civil Engineering, National Taiwan University, Taiwan E-mail: dlyyoun@ntu.edu.tw.
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CHAPTER 10

The Method of Fundamental Solutions for Fully Developed Laminar Flow of Power-Law non-Newtonian Fluid in Ducts of Arbitrary Cross-Section

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CHAPTER 11

Shear Deformable Plate Analysis by the Method of Fundamental Solution: Static and Dynamic

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11.1 Introduction

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CHAPTER 12

Calculation on Stress Concentration of Dull Elliptical Holes by the MFS Using Equally Dispersed Point Loads

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12.1 Introduction

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CHAPTER 13

Dipole Plate Bending Formulation for the Method of Fundamental Solutions

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Performance of the MFS in the Computation of Seismic Motion and Rotation

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A Modified Method of Fundamental Solutions for Potential Flow Problems

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MFS Analysis of Microstrip Antennas

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16.1 Introduction

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