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Local conserved charges in principal chiral models

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ABSTRACT

We investigate local conserved charges in principal chiral models in 1+1 dimensions. There is a classically conserved local charge for each invariant tensor of the underlying group. We prove that these are always in involution with the non-local Yangian charges and we study their classical Poisson bracket algebra. For the groups $U(N)$, $O(N)$ and $Sp(N)$ we show that there are infinitely many commuting local charges, and we further identify finite sets of mutually commuting charges with spins equal to the Lie algebra exponents. We elaborate on arguments of Goldschmidt and Witten for conservation of some local charges at the quantum level, and we briefly discuss the implications for the multiplet structures. We comment on the possible existence of a version of Dorey's rule for the quantum principal chiral models.

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1 Introduction

Integrable lagrangian field theories in 1+1 dimensions exhibit various kinds of higher-spin conserved quantities which are crucial in constraining their quantum behaviour and which ultimately determine their exact S-matrices. These exotic symmetries can usually be traced to underlying mathematical structures which incorporate Lie algebras in some way. Beyond these broad similarities, however, one encounters many examples with profound physical and mathematical differences.

In some theories the coupling constant is small at low energies, so that perturbation theory and semi-classical techniques can be used to find the particle spectrum. Well-known examples are the sine-Gordon theory and its affine Toda generalizations. The defining Lie algebra data for an affine Toda field theory (ATFT) are just the simple roots plus the lowest root. For an ATFT with a real coupling constant, the complete tree-level mass spectrum can be derived directly from the classical lagrangian, whilst in the more complicated case when the coupling is imaginary, there are also classical soliton solutions. In the quantum theory, mass ratios of solitons (semiclassically) and particles (to one-loop order) are unchanged for simply-laced algebras, while they vary with the coupling constant for the nonsimply-laced cases. Exact S-matrices can then be found which incorporate all these results and place them in a richer context.

In complete contrast, there are other models which are strongly-coupled in the infra-red, so that information about the quantum theory is much harder to extract from the lagrangian. Amongst these are non-linear sigma-models with compact target manifolds, and in particular the principal chiral models (PCMs), which are the subject of this paper, whose target manifolds are Lie groups. The classical lagrangians for these theories are scale-invariant, and their masses therefore arise purely quantum-mechanically through a complicated strong coupling effect. The pattern of masses which emerges for PCMs (from exact S-matrices) is nevertheless identical to the set of masses of the ATFTs in the cases of simply-laced algebras. For the nonsimply-laced algebras there is a more subtle connection, in which the PCM mass ratios coincide with those of tree-level ATFTs based on twisted affine algebras.

It is surprising, given the rather different physical attributes of these two classes of theories, that they can still turn out to have much in common at the quantum level—particularly the patterns of masses and couplings. The ubiquitous presence of Lie algebras seems at first to offer hope for understanding the common attributes of these otherwise disparate models, but this is confounded by the fact that the Lie algebras make their appearance in completely different guises. Moreover, the conserved charges which emerge can also differ greatly in character, depending on whether they are *local*, or *non-local*.

Each ATFT possesses a set of conserved charges with spins equal to the exponents of the underlying Lie algebra modulo the Coxeter number h [3]:

$$
\begin{array}{rcc}
a_n = su(n{+}1) & 1,2,3,...,n & h = n+1 \\
b_n = so(2n{+}1) & 1,3,5,..,2n{-}1 & h = 2n \\
c_n = sp(2n) & 1,3,5,...,2n{-}1 & h = 2n \\
d_n = so(2n) & 1,3,5,...,2n{-}3\,,\, n{-}1 & h = 2n{-}2
\end{array}
$$

These charges are *local*, so that they are additive on asymptotic, multi-particle states, and single particle states may be chosen to be simultaneous eigenstates of them. Their existence places strong constraints on the possible three-point couplings, but, as was made clear in the elegant construction of Dorey [4], a solution can always be found. The conservation of each charge can be described by a triangle (the dual Feynman diagram) with sides of lengths equal to the charges. In Dorey's construction, each three-point coupling corresponds to a triangle of roots, and the projection of this onto various planes through root space gives the triangles for the charges.

The particle states which appear in the PCMs, as well as those corresponding to ATFT solitons, occur in multiplets which are naturally acted on by a set of *non-local* charges. In marked distinction to the local charges, these non-local charges are not additive on multiparticle states and they have either indefinite or non-integral spin. An apparently unrelated set of three-point couplings then arises from fusions of these particle multiplets. Mathematically, there is an underlying quantum group symmetry generated by the non-local charges, which is associated with non-trivial solutions of the Yang-Baxter equation. The quantum group structure which emerges is a Yangian [7, 8] in the case of the PCM, whereas it is a quantized affine algebra for the ATFT [5]. In either case, the S-matrix fusions are to this algebra what the Clebsch-Gordan rule is to Lie algebras: they are its rule for tensor products of representations.

The non-local charges and their role in determining the PCM S-matrices have been extensively studied. It has actually been known for some time that the PCMs contain local conserved charges too, but these have received comparatively little attention. Our aim in this paper is to rectify this omission by investigating the local charges in some detail, including their relationship with the non-local charges. This is clearly an interesting question in itself, but there are also more general motives for such an analysis.

It is natural to wonder whether there is some more all-embracing point-of-view regarding the local and non-local charges, and what, if any, is the relationship between them. One striking connection which has long been conjectured and recently been proved [6] is that the quantum group fusion rule relevant to the non-local charges and Dorey's rule obtained through local charges are the same. The proof which has been given is exhaustive rather than enlightening, however, and from the quantum group point of view it remains something of a mystery why it is so. A broader aim of our work is simply to find out more about models, like the PCMs, in which the two types of charges co-exist. These results may then help to confirm or refute any conjectured relationships between them.

The majority of the paper—sections 2, 3 and 4—elucidates the classical properties of the local charges appearing in PCMs. We shall see that there exists a conserved local charge for each invariant tensor or Casimir of the underlying Lie algebra, and we shall prove that they always commute with the non-local Yangian charges. We then investigate in detail the Poisson bracket algebra of the local charges amongst themselves, which turns out to be surprisingly subtle in some cases. For the algebras $u(N)$, $so(N)$ and $sp(N)$, we are able to find infinitely many 'simple' commuting local charges (whereas the Poisson brackets close in a more complicated way for $su(N)$). It is also natural to consider certain finite sets of 'primitive' currents, from which all others arise as polynomials. The spins of the primitive charges are exactly the exponents of the algebra, suggesting that some analogue of Dorey's construction may be relevant. We give general formulas for combinations of simple currents which we conjecture (based on substantial calculations) lead to a full set of mutually commuting charges with spins equal to the exponents.

In section 5 we review, with some small improvements and additions, what is known about the quantum behaviour of these charges. We also examine briefly the implications for the multiplet structure in the quantum theory and show how these are consistent with the exact S-matrices [1]. We conclude with some comments on how Dorey's rule might apply to the quantum PCMs, based on the local charges we have studied.

2 The classical principal chiral model

2.1 The lagrangian and its symmetries

The principal chiral model is defined by the lagrangian

$$\mathcal{L} = \mathrm{Tr}\left(\partial_\mu g^{-1} \partial^\mu g\right) , \tag{2.1}$$

where the field $g(x^\mu)$ takes values in some compact Lie group $\mathcal{G}$. It has a global continuous symmetry

$$\mathcal{G}_L \times \mathcal{G}_R \quad : \qquad g \mapsto U_{\mathrm{L}}\, g\, U_{\mathrm{R}}^{-1} \tag{2.2}$$

with conserved currents which can be defined by

$$j^L_\mu = \partial_\mu g\, g^{-1}, \qquad j^R_\mu = -g^{-1}\partial_\mu g \tag{2.3}$$

and which take values in the Lie algebra $\mathcal{A}$ of $\mathcal{G}$. The equations of motion following from (2.1) correspond to the conservation of these currents:

$$\partial^\mu j_\mu(x,t) = 0\,. \tag{2.4}$$

They also obey

$$\partial_\mu j_\nu - \partial_\nu j_\mu - [j_\mu, j_\nu] = 0 \tag{2.5}$$

identically, as a consequence of their definitions in terms of the field g. Here and elsewhere we will adopt the convention that any equation written for a current j_μ without a label holds true for both L and R currents. It is significant that the last condition above can be interpreted as a zero-curvature condition for a connection with covariant derivative

$$\nabla_\mu X = \partial_\mu X - [j_\mu, X] \tag{2.6}$$

acting on any X in $\mathcal{A}$. Here we view $-j_\mu$ as a two-dimensional gauge field, its definition in terms of g implying that it is pure-gauge. The two conditions (2.4, 2.5) capture the entire algebraic structure of the PCM.

We shall always take the Lie algebra $\mathcal{A}$ to be in its *defining* representation, so that the elements are $N \times N$ real or complex matrices X which obey:

$$u(N) \quad X^\dagger = -X; \qquad su(N) \quad X^\dagger = -X, \quad \mathrm{Tr}(X) = 0;$$

$$so(N) \quad X^T = -X; \qquad sp(N) \quad X^\dagger = -X, \quad X^T = -JXJ^{-1} \tag{2.7}$$

where in the last case N is even and J is some chosen symplectic form. In each case we introduce a basis of anti-hermitian generators t^a for $\mathcal{A}$ with real structure constants f^{abc} and normalizations given by

$$[t^a, t^b] = f^{abc} t^c \,, \qquad \mathrm{Tr}(t^a t^b) = -\delta^{ab} \,. \tag{2.8}$$

(Lie algebra indices will always be taken from the beginning of the alphabet.) For any $X \in \mathcal{A}$ we write

$$X = t^a X^a, \qquad X^a = -\mathrm{Tr}(t^a X) \tag{2.9}$$

In addition to the continuous symmetries comprising $\mathcal{G}_L$ and $\mathcal{G}_R$, there are also important discrete symmetries of the principal chiral model. For any PCM there is a symmetry

$$\pi \,:\, g \mapsto g^{-1} \qquad \Rightarrow \qquad j^L \leftrightarrow j^R \,, \tag{2.10}$$

which exchanges $\mathcal{G}_L$ and $\mathcal{G}_R$ and which we shall therefore refer to as $\mathcal{G}$-parity. (In the PCM effective field theory description of strong interactions in four dimensions, the physical parity operator is our $\mathcal{G}$-parity together with spatial reflection.) Additional discrete symmetries arise as outer automorphisms of $\mathcal{G}$ acting on the the field g. Thus we have

$$\sigma \,:\, g \mapsto g^* \qquad \Rightarrow \qquad j^L \leftrightarrow -(j^L)^T \,, \qquad j^R \leftrightarrow -(j^R)^T \,, \tag{2.11}$$

which exchanges complex-conjugate representations. This map is trivial (up to conjugation) if the group has only real (or pseudo-real) representations. For $\mathcal{G} = O(N)$ we also have

$$\tau \,:\, g \mapsto M g M^{-1} \qquad \Rightarrow \qquad j^L \mapsto M j^L M^{-1} \,, \qquad j^R \mapsto M j^R M^{-1} \,,$$

where M is an $O(N)$ matrix with determinant -1. When N is even this exchanges the inequivalent spinor representations. The maps σ and τ coincide (up to conjugation) for the family $O(4m+2)$.

Space-time symmetries will also play an important role in what follows. The classical lagrangian is conformally-invariant, and as a result the energy momentum tensor

$$T_{\mu\nu} = \mathrm{Tr}(j_\mu j_\nu) - \eta_{\mu\nu} \mathrm{Tr}(j_\rho j^\rho) \tag{2.12}$$

is not only conserved and symmetric but also traceless. The Poincaré subgroup of the conformal group consists of the momenta

$$P_\mu = \int_{-\infty}^{\infty} T_{0\mu}(x,t)\, dx$$

and the Lorentz boost generator

$$M \equiv M_{01} = \int_{-\infty}^{\infty} (\, x\, T_{00}(x,t) - t\, T_{01}(x,t)\,)\, dx \quad .$$

In terms of standard Minkowski coordinates $x^0 = t$ and $x^1 = x$ in two dimensions, our spacetime conventions are $\eta_{00} = -\eta_{11} = 1$ and $\epsilon_{01} = 1$. We shall also make extensive use of light-cone coordinates and their derivatives defined by

$$x^{\pm} = \frac{1}{2}(t \pm x)\,, \qquad \partial_{\pm} = \partial_t \pm \partial_x\,.$$

The equations (2.4, 2.5) can then be written

$$\partial_- j_+ = -\partial_+ j_- = -\frac{1}{2}[j_+, j_-]\,, \tag{2.13}$$

whilst the energy-momentum tensor takes the familiar form

$$T_{\pm\pm} = \mathrm{Tr}(j_{\pm} j_{\pm}), \qquad T_{+-} = T_{-+} = 0\,, \tag{2.14}$$

with

$$\partial_- T_{++} = \partial_+ T_{--} = 0\,. \tag{2.15}$$

2.2 Canonical Formalism

The canonical Poisson brackets for the theory are

$$\begin{aligned}
\left\{ j_0^a(x,t), j_0^b(y,t) \right\} &= f^{abc} j_0^c(x,t)\delta(x-y) \\
\left\{ j_0^a(x,t), j_1^b(y,t) \right\} &= f^{abc} j_1^c(x,t)\delta(x-y) + \delta^{ab}\frac{\partial}{\partial x}\delta(x-y) \\
\left\{ j_1^a(x,t), j_1^b(y,t) \right\} &= 0\,.
\end{aligned} \tag{2.16}$$

(where $j = j^a t^a$, as in (2.9)). Note the derivative of the delta function in the second bracket; this classical analogue of a Schwinger term is known as 'non-ultralocal'. These expressions hold for *either* of the currents j^L or j^R separately, while the algebra of j^L with j^R involves only non-ultralocal terms in the brackets of space- with time-components ([9]; we shall not need them and so do not give them here), in keeping with the direct product structure of $\mathcal{G}_L \times \mathcal{G}_R$.

The standard Poisson bracket relations for the Poincaré generators, namely

$$\begin{aligned} \{P_\mu, P_\nu\} &= 0\,, \\ \{M, P_\mu\} &= \epsilon_{\mu\nu} P^\nu\,, \\ \{P_\mu, j_\nu\} &= \partial_\mu j_\nu\,, \\ \text{and} \qquad \{M, j_\mu\} &= \frac{1}{2}\epsilon_{\rho\sigma}\left(x^\rho \partial^\sigma - x^\sigma \partial^\rho\right) j_\mu + \epsilon_\mu{}^\sigma j_\sigma\,, \end{aligned}$$

can now readily be confirmed using the previous definitions of these generators in terms of the underlying currents.

There are various ways to derive the expressions (2.16) for the Poisson brackets written above. One economical approach [9] deals directly with the currents rather than with the underlying field g. One selects j_1 as the only independent dynamical variable, and then one can regard j_0 as a function of it which is determined through the relation (2.5). The price to be paid is the introduction of an operator ∇_1^{-1} which is non-local in space, being the inverse of the operator ∇_1 defined on any lie-algebra-valued quantity X by $\nabla_1 X = \partial_1 X - [j_1, X]$. The definition of this inverse operator assumes suitable boundary conditions for the fields at spatial infinity. If we accept this then we can clearly write

$$j_0 = \nabla_1^{-1} \frac{\partial j_1}{\partial t}$$

We now substitute this expression in the lagrangian and find

$$\mathcal{L} = \mathrm{Tr}(j_0^2 - j_1^2) = \mathrm{Tr}\left[(\nabla_1^{-1} \frac{\partial j_1}{\partial t})^2 - j_1^2 \right]$$

Since j_1 is an unconstrained dynamical coordinate, we may define its conjugate momentum in the usual way: $\pi_1 = \partial\mathcal{L}/\partial \dot{j}_1 = -\nabla_1^{-2} \dot{j}_1$. and we deduce that

$$j_0 = -\nabla_1 \pi_1$$

This uses the property $\nabla_1^{-1}(A)B = -A\nabla_1^{-1}(B)$ up to boundary terms which vanish on integrating over space; the adoption of suitable boundary conditions on the fields is crucial in this respect. By using the standard equal-time Poisson brackets for j_1 and π_1, the expressions (2.16) can now be recovered after a short calculation. A lengthier but more routine derivation which avoids the use of the operator ∇_1^{-1} can be found in the appendix.

3 Classical conserved charges

3.1 Non-local charges

There exist infinitely many conserved non-local charges in the PCM, which are generated by the obvious local charge

$$Q^{(0)a} = \int_{-\infty}^{\infty} j_0^a dx \tag{3.1}$$

and the first non-local charge

$$Q^{(1)a} = \int_{-\infty}^{\infty} j_1^a dx - \frac{1}{2} f^{abc} \int_{-\infty}^{\infty} j_0^b(x) \int_{-\infty}^{x} j_0^c(y)\, dy\, dx\,, \tag{3.2}$$

and whose Poisson brackets form a Yangian $Y(\mathcal{A})$ [10]. In fact there are two infinite sequences of such charges constructed from both j_μ^L and j_μ^R in (2.3), and so the model has a charge algebra $Y_L(\mathcal{A}) \times Y_R(\mathcal{A})$. (It can be checked that Y_L and Y_R commute.) These charges can be extracted from the monodromy matrix by a power series expansion in the spectral parameter, or equivalently can be constructed by an iterative procedure [11].

Classically, these non-local charges are Lorentz scalars: applying the boost operator M we obtain

$$\left\{M, Q^{(0)a}\right\} = \left\{M, Q^{(1)a}\right\} = 0 \quad . \tag{3.3}$$

Because the charges are non-local they will not be additive on products of states. Their action is given by the coproduct

$$\begin{aligned} \Delta\left(Q^{(0)a}\right) &= Q^{(0)a} \otimes 1 + 1 \otimes Q^{(0)a} \\ \text{and} \quad \Delta\left(Q^{(1)a}\right) &= Q^{(1)a} \otimes 1 + 1 \otimes Q^{(1)a} + \frac{1}{2} f^{abc} Q^{(0)b} \otimes Q^{(0)c} \quad , \end{aligned} \tag{3.4}$$

which has the usual interpretation in the quantum theory, and may also be interpreted classically as giving the values of the charges on widely-separated, localized configurations [12].

3.2 Local charges

In any conformally-invariant theory, the conservation of the energy-momentum tensor (2.15) immediately implies a series of higher-spin conservation laws, simply by taking powers:

$$\partial_-(T_{++}^n) = \partial_+(T_{--}^n) = 0 \tag{3.5}$$

(these correspond to the higher-grade generators of the classical Virasoro algebra). But the PCM has more basic conservation laws which depend on the precise form of the equations of motion of the currents rather than on conformal invariance alone. The simplest examples are

$$\partial_- \mathrm{Tr}(j_+^m) = \partial_+ \mathrm{Tr}(j_-^m) = 0 \tag{3.6}$$

which follow easily from (2.13) ($m = 2$ gives conservation of the energy-momentum tensor).

More generally, we may consider any rank-m, symmetric, invariant tensor $d_{a_1 a_2 \dots a_m}$ associated with a Casimir operator

$$\mathcal{C}_m = d_{a_1 a_2 \dots a_m} t^{a_1} t^{a_2} \dots t^{a_m} \tag{3.7}$$

where

$$[\mathcal{C}_m, t_b] = 0 \qquad \Leftrightarrow \qquad d_{c(a_1 a_2 \dots a_{m-1}} f_{a_m)bc} = 0 \,. \tag{3.8}$$

(and as usual $(\dots)$ means 'symmetrize on the enclosed indices'). It is then easy to check that invariance of d implies

$$\partial_\mp (\, d_{a_1 a_2 \dots a_m} j_\pm^{a_1} j_\pm^{a_2} \dots j_\pm^{a_m} \,) = 0 \ . \tag{3.9}$$

The corresponding conserved charges will be denoted

$$\begin{aligned} q_s \; &= \; d_{a_1 a_2 \dots a_m} \int_{-\infty}^{\infty} j_+^{a_1}(x) \dots j_+^{a_m}(x) \, dx \\ \text{and} \quad \bar{q}_s \; &= \; d_{a_1 a_2 \dots a_m} \int_{-\infty}^{\infty} j_-^{a_1}(x) \dots j_-^{a_m}(x) \, dx \end{aligned} \tag{3.10}$$

with spins $s = m - 1$. This behaviour under Lorentz transformations can be inferred from the number of $\pm$ signs labelling the currents, or, more formally, we can calculate the action of the boost generator and find

$$\{M, q_s\} = s q_s \, , \qquad \{M, \bar{q}_s\} = -s \bar{q}_s \, . \tag{3.11}$$

Invariance of the d-tensor also implies that the same local conservation laws are obtained using either of the currents j^L or j^R, so there is just a single copy of these local charges, unlike the two-fold L and R copies of the non-local charges. Also in contrast to the non-local charges, we note that any local charge must be additive on multi-particle states, which we can also express by saying that such a charge has a trivial co-product:

$$\Delta(q_s) = q_s \otimes 1 + 1 \otimes q_s \, , \qquad \Delta(\bar{q}_s) = \bar{q}_s \otimes 1 + 1 \otimes \bar{q}_s \, .$$

The currents in (3.6) correspond to the particular choice

$$d_{a_1 a_2 \dots a_m} = \mathrm{STr}(t^{a_1} t^{a_2} \dots t^{a_m}) \tag{3.12}$$

with 'STr' denoting the trace of a completely symmetrized product of matrices. We shall refer to these invariant tensors and their associated currents and charges as being of *simple* type, and we introduce a convenient notation for the currents

$$\mathcal{J}_m = \mathrm{Tr}(j_+^m)\,. \tag{3.13}$$

There are infinitely many simple tensors, but only finitely many of them are independent. Moreover, it is well-known that for each algebra $\mathcal{A}$ there are exactly $rank(\mathcal{A})$ independent or *primitive* d-tensors and Casimirs (see, *e.g.* [13]) and that all others can be expressed as polynomials in these and the structure constants f_{abc}. For the classical algebras these primitive d-tensors may be chosen to be simple in our terminology (*i.e.* symmetrized traces in the defining representation) with the one exception of the Pfaffian invariant for the algebra $so(2n)$, which can be written

$$d_{a_1\ldots a_n} = \epsilon_{i_1 j_1 \ldots i_n j_n} t^{a_1}_{i_1 j_1} \ldots t^{a_n}_{i_n j_n} \tag{3.14}$$

(though this can be related to a trace in the spinor representation). We shall denote the corresponding current by

$$\mathcal{P} = \epsilon_{i_1 j_1 \ldots i_n j_n} (j_+)_{i_1 j_1} \ldots (j_+)_{i_n j_n} \tag{3.15}$$

We shall also refer to the currents and charges associated to the primitive d-tensors as being of *primitive* type. The spins of the primitive charges $s = m - 1$ are precisely the exponents of the algebra $\mathcal{A}$ which we have already listed in the introduction.

We should emphasize that our definitions of simple and primitive tensors depend on our choice of the defining representation of the algebra. Taking the generators in different representations would amount to a re-definition of each simple d-tensor by various polynomials in d-tensors of lower rank. The special importance of the defining representation for our purposes will become clear shortly.

Last of all we should point out that there are infinitely many more conserved currents which arise as arbitrary *differential* polynomials in those already discussed. For example, $\partial_-\Big(\mathrm{Tr}(j_+^p)\partial_+^r \mathrm{Tr}(j_+^q)\Big) = 0$ follows immediately from (3.6). We shall not be directly concerned with the properties of these more general currents in this paper.

3.3 Commutation of local with non-local charges

We now show that all local charges of the type (3.10) are in involution with the non-local charges generated by $Q^{(0)a}$ and $Q^{(1)a}$ — that is, they commute with one another:

$$\{q_s, Q^{(0)b}\} = \{\bar{q}_s, Q^{(0)b}\} = 0 \tag{3.16}$$

$$\{q_s, Q^{(1)b}\} = \{\bar{q}_s, Q^{(1)b}\} = 0\,. \tag{3.17}$$

The conditions (3.16) follow immediately from invariance of the d-tensors; they say simply that the charges q_s and $\bar{q}_s$ are singlet with respect to the Lie algebra. The calculations of the brackets in (3.17) are much more delicate, involve a cancellation between terms originating from ultralocal and non-ultralocal expressions.

Let us consider for definiteness $\{q_s, Q^{(1)b}\}$. Taking the first term in the expression (3.2) for $Q^{(1)b}$ we find

$$\begin{aligned}
\{q_s, \int dy\, j_1^b(y)\} &= d_{a_1 a_2 \ldots a_m} \int\!\!\int dx\, dy\, \{j_+^{a_1}(x) \ldots j_+^{a_m}(x), j_1^b(y)\} \\
&= d_{a_1 a_2 \ldots a_m} \int dx\, \Big(f^{a_m b d} j_+^{a_1}(x) \ldots j_+^{a_{m-1}}(x) j_1^d(x) \\
&\quad + f^{a_{m-1} b d} j_+^{a_1}(x) \ldots j_+^{a_{m-2}}(x) j_1^d(x) j_+^{a_m}(x) \;+\; \ldots.. \Big) \\
&= d_{a_1 a_2 \ldots a_{m-1} c} f^{a_m c b} \int j_+^{a_1}(x) \ldots j_+^{a_{m-1}}(x) j_1^{a_m}(x) \;\;+\;\; (m{-}1) \text{ others}\;;
\end{aligned}$$

the ('non-ultralocal') terms arising from integration of the derivative of the δ-function vanish, again because $j \to 0$ as $x \to \pm\infty$.

Next we consider the second, non-local term in (3.2). Here we must be explicit about the limits of the spatial integration, which we shall take to be $\pm L$, letting L become large. Letting $\theta(x)$ be the step function,

$$\begin{aligned}
&-\frac{1}{2}\{q_s, \int_{-L}^{L}\int_{-L}^{y} dz\, dy\, f^{bcd} j_0^c(y) j_0^d(z)\} \\
&\quad = -\frac{1}{2} d_{a_1 a_2 \ldots a_m} f^{bcd} \int_{-L}^{L}\int_{-L}^{L}\int_{-L}^{y} dz\, dy\, dx\, \{j_+^{a_1}(x) \ldots j_+^{a_m}(x)\,,\; j_0^c(y) j_0^d(z)\} \\
&\quad = -\frac{1}{2} d_{a_1 a_2 \ldots a_m} f^{bcd} \int_{-L}^{L}\int_{-L}^{L}\int_{-L}^{L} dz\, dy\, dx \\
&\qquad\quad .\; \theta(y-z) j_+^{a_1}(x) \ldots j_+^{a_{m-1}}(x) \left(j_0^d(z) \delta^{a_m c} \frac{\partial}{\partial x}\delta(x-y) + j_0^c(y) \delta^{a_m d} \frac{\partial}{\partial x}\delta(x-z) \right) \\
&\qquad\quad + (m{-}1) \;\text{ others } \;+\; \text{ultralocal terms, which vanish}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2} d_{a_1 a_2 \ldots a_{m-1} c} f^{bcd} \int_{-L}^{L} \int_{-L}^{L} \int_{-L}^{L} dz\, dy\, dx \\
&\qquad . \, j_+^{a_1}(x) \ldots j_+^{a_{m-1}}(x) j_0^d(z) \left(\theta(y-z) - \theta(z-y)\right) \frac{\partial}{\partial x} \delta(x-y) \\
&\qquad + (m{-}1) \text{ others} \\
&= -\frac{1}{2} d_{a_1 a_2 \ldots a_{m-1} c} f^{bcd} \int_{-L}^{L} \int_{-L}^{L} dz\, dx \\
&\qquad . \, j_+^{a_1}(x) \ldots j_+^{a_{m-1}}(x) j_0^d(z) \left(\delta(x-z) - \delta(x+L) - \delta(x-L) + \delta(x-z)\right) \\
&\qquad + (m{-}1) \text{ others} \\
&= d_{a_1 a_2 \ldots a_{m-1} c} f^{a_m cb} \int j_+^{a_1}(x) \ldots j_+^{a_{m-1}}(x) j_0^{a_m}(x) \quad + (m{-}1) \text{ others} ,
\end{aligned}
$$

where in the penultimate step the middle two δs do not contribute, again because $j \to 0$ as $x \to \pm\infty$.

The sum of the two expressions calculated above is now zero due to the identity (3.8) and the symmetry of the integrand. We have thus established (3.16) and (3.17) and we emphasize that our calculation did not assume that the local charge was either of simple or primitive type.

4 Classical algebra of local charges

We now wish to investigate the classical Poisson bracket algebra of the local charges (3.10). We first show that

$$
\{q_s, \bar{q}'_r\} = 0 \, . \tag{4.1}
$$

This can be calculated directly from the Poisson brackets (2.16). The non-ultralocal terms (*i.e.* those involving derivatives of δ-functions) always cancel. The local terms, on the other hand, give rise to integrals of expressions of the form

$$
d_{a_1 \ldots a_s a} \, d'_{b_1 \ldots b_r b} \, j_+^{a_1} \ldots j_+^{a_s} j_-^{b_1} \ldots j_-^{b_r} \, f^{abc} j_\pm^c
$$

where all fields are taken at the same argument. But such terms vanish by invariance of either d or d'.

Next we consider brackets of the form $\{q_s, q'_r\}$ (those of type $\{\bar{q}_s, \bar{q}'_r\}$ are similar), which turn out to be rather more subtle. The ultra-local terms still give vanishing contributions. This is because they produce integrals of the expressions

$$
d_{a_1 \ldots a_s a} \, d'_{b_1 \ldots b_r b} \, j_+^{a_1} \ldots j_+^{a_s} j_+^{b_1} \ldots j_+^{b_r} \, f^{abc} j_\pm^c \; ,
$$

where again all fields are taken at the same argument. A term involving j_+^c vanishes directly by invariance of d or d'. A term involving j_-^c can be rearranged, using invariance of d' to exchange the Lie algebra indices on the last two currents, to yield a result proportional to

$$d_{a_1\ldots a_s a}\, d'_{b_1\ldots b_r b}\, j_+^{a_1}\ldots j_+^{a_s} j_+^{b_1}\ldots j_-^{b_r}\, f^{abc} j_+^c\,.$$

This new expression contains $d_{a_1\ldots a_s a} f^{abc}$ totally symmetrized on $(a_1, a_2, \ldots, a_s, c)$ and it therefore vanishes by invariance of d. Unlike the previous bracket (4.1), however, there are still contributions from the non-ultralocal terms in the current Poisson brackets, giving a result

$$\{q_s, q'_r\} = (const)\int_{-\infty}^{\infty} dx\, d_{ca_1\ldots a_s} j_+^{a_1}\ldots j_+^{a_s} d'_{cb_1\ldots b_r}\partial_1(j_+^{b_1}\ldots j_+^{b_r})\,. \tag{4.2}$$

Note that the result is always anti-symmetric in s and r by integration by parts.

If $r = 1$ and $d'_{bc} = \delta_{bc}$ then the integrand in (4.2) becomes a total derivative and hence the Poisson bracket vanishes. This simply says that all the local charges (3.10) commute with energy and momentum, so that they are invariant under translations in space and time. For $s, r > 1$ however, the situation is more complicated because the integrand above does not automatically vanish, nor can it be re-expressed as a total derivative by using just the equations of motion and invariance of the d-tensors alone.

4.1 Algebra of simple and primitive charges

To make further progress we will first consider the cases in which the currents are simple, of the form (3.13), so that (4.2) may be written as

$$\{q_s, q'_r\} = (const)\int_{-\infty}^{\infty} dx\, \mathrm{Tr}(t^c j_+^s)\,\partial_1 \mathrm{Tr}(t^c j_+^r)\,. \tag{4.3}$$

The strategy is to simplify this by using the completeness condition

$$Y = -t^a\,\mathrm{Tr}(t^a Y)$$

for any Y in $\mathcal{A}$. If X is some given matrix in $\mathcal{A}$ then we can apply the completeness condition to $Y = X^n$ provided this also lies in $\mathcal{A}$, which will hold for certain powers n by virtue of specific properties of the defining representation of the algebra $\mathcal{A}$, as we discuss below. The crucial point is that if j_+^s lies in $\mathcal{A}$, we can use the completeness condition with $Y = j_+^s$, and it is then easy to see that the integrand in (4.3) can be re-written as a total derivative, so that $\{q_s, q'_r\} = 0$.

Consider first the case $\mathcal{A} = u(N)$. If X is in $u(N)$ then so is X^n for n odd, or iX^n for n even. Either way the argument above can be applied, and the expression in (4.3) vanishes on carrying out the integral. A similar argument applies to each of the algebras $b_n = so(2n+1)$, $c_n = sp(2n)$ and $d_n = so(2n)$. In all these cases the exponents s and r appearing in (4.2,4.3) are odd numbers. But if X satisfies the defining conditions for any of these algebras, then so does X^n when n is odd. So once again we can use the completeness condition to re-write the integrand as a total derivative and we deduce that the Poisson bracket vanishes.

We conclude that all the simple charges have vanishing Poisson brackets with one another for each of the algebras $u(N)$, $so(N)$ and $sp(N)$. If we now ask about the algebra of the *primitive* charges, there is just the case of the Pfaffian charge in $so(2n)$ still to be considered. The Poisson bracket of charges q_s and q_{n-1} defined from the currents $\mathcal{J}_{s+1}$ and $\mathcal{P}$ respectively is given by the general formula (4.2). This cannot be simplified using our previous techniques, but it can be calculated by other means, as sketched in one of the appendices. The result is

$$\{q_s, q_{n-1}\} = (const) \int_{-\infty}^{\infty} dx\, \mathcal{P} \partial_1 \mathcal{J}_{s-1} \tag{4.4}$$

which certainly does not vanish.

It may seem strange that we have chosen to deal with $u(N)$ above, rather than with $su(N)$. The brackets of simple charges in $su(N)$ turn out to be rather more complicated, and they are non-zero in general. The reason for this different behaviour is that powers of matrices in the defining representation of $su(N)$ will in general violate the trace-free condition, and so will lie outside the algebra. Because of this, our previous arguments do not enable us to re-write the integrand as a total derivative. They can, however, be used to simplify the answers that we get. If we consider extending a basis of generators t^a for $su(N)$ to a basis of generators for $u(N)$ by appending the identity matrix, then the completeness condition in $u(N)$ allows us to re-write the expression (4.2) for $su(N)$ as

$$\{q_s, q_r\} = (const) \int_{-\infty}^{\infty} dx\, \mathrm{Tr}(j_+^s)\, \partial_1 \mathrm{Tr}(j_+^r) \tag{4.5}$$

We have thus discovered non-trivial Poisson brackets for all the simple charges in $su(N)$ and for the primitive Pfaffian charge in $so(2n)$, whereas brackets amongst all other simple (or primitive) charges vanish. If conserved charges fail to commute, they must yield other conserved charges. So it is reassuring that we have succeeded in re-expressing the non-vanishing brackets as (4.4) and (4.5) above, which clearly involve differential polynomials in the simple or primitive currents and which are hence conserved. In this respect the results accord with our general expectations.

4.2 Modifying the primitive charges

Given the non-vanishing Poisson brackets for the $su(N)$ charges and the Pfaffian $so(2n)$ charge, it is natural to ask whether there are homogeneous (non-differential) polynomials in the primitive currents which would obey a simpler algebra, and perhaps even commute. The first non-trivial cases to consider arise in $su(N)$ and involve currents of spin 3 and spin 4 (since we need two distinct charges beyond energy and momentum). A short calculation reveals that the charges constructed from $\mathrm{Tr}(j_+^3)$ and $\mathrm{Tr}(j_+^4) - \frac{3}{2N}(\mathrm{Tr}(j_+^2))^2$ do in fact commute. Since these are the only charges for $su(4)$, they constitute a complete set of commuting primitive charges in this theory. But attempts to carry through a similar construction for higher-spin charges in $su(N)$ fail, and we believe that no such modification of the primitive charges is possible for $N > 4$.

Undaunted, we can try the same procedure for $so(2n)$. The simplest case $so(6)$ must work, since this is isomorphic to $su(4)$, with $\mathcal{P}$ in the former description corresponding to the current $\mathcal{J}_3$ in the latter. For the general case $so(2n)$, we consider modifying each current $\mathcal{J}_{2m}$ by any terms which are homogeneous of the same degree in currents, and we calculate the conditions necessary for commutation with the Pfaffian charge. This fixes the relative coefficients uniquely, with results for the first four cases:

$$\begin{aligned}
q_3 &= \int \left(\mathcal{J}_4 - \frac{1}{2}\left(\frac{6}{h}\right)(\mathcal{J}_2)^2 \right) dx \\
q_5 &= \int \left(\mathcal{J}_6 - \frac{3}{4}\left(\frac{10}{h}\right)\mathcal{J}_4\mathcal{J}_2 + \frac{1}{8}\left(\frac{10}{h}\right)^2 (\mathcal{J}_2)^3 \right) dx \\
q_7 &= \int \left(\mathcal{J}_8 - \frac{2}{3}\left(\frac{14}{h}\right)\mathcal{J}_6\mathcal{J}_2 - \frac{1}{4}\left(\frac{14}{h}\right)(\mathcal{J}_4)^2 + \frac{1}{4}\left(\frac{14}{h}\right)^2 \mathcal{J}_4(\mathcal{J}_2)^2 - \frac{1}{48}\left(\frac{14}{h}\right)^3 (\mathcal{J}_2)^4 \right) dx \\
q_9 &= \int \left(\mathcal{J}_{10} - \frac{5}{8}\left(\frac{18}{h}\right)\mathcal{J}_8\mathcal{J}_2 - \frac{5}{12}\left(\frac{18}{h}\right)\mathcal{J}_6\mathcal{J}_4 + \frac{5}{24}\left(\frac{18}{h}\right)^2 \mathcal{J}_6(\mathcal{J}_2)^2 \right. \\
&\qquad \left. + \frac{5}{32}\left(\frac{18}{h}\right)^2 (\mathcal{J}_4)^2\mathcal{J}_2 - \frac{5}{96}\left(\frac{18}{h}\right)^3 \mathcal{J}_4(\mathcal{J}_2)^3 + \frac{1}{384}\left(\frac{18}{h}\right)^4 (\mathcal{J}_2)^5 \right) dx \,. \qquad (4.6)
\end{aligned}$$

where $h = 2n-2$, the Coxeter number of $so(2n)$. These commute with the Pfaffian charge by construction, but what are their brackets amongst themseves? We should like to demand that these brackets also vanish, but this is of course a highly over-determined set of conditions on the coefficients (in fact the condition for commutation with the Pfaffian already over-determines the coefficients for q_9). Remarkably, we find after lengthy calculation that the brackets of these quantities do vanish!

The expressions written above actually admit a natural (though not obvious) generalization. We introduce a family of polynomials $P_m(t_1, \ldots, t_m)$ defined by the following property. Let X be any anti-symmetric matrix with skew-eigenvalues x_a; then

$$P_m(\mathrm{Tr}(X^2), \mathrm{Tr}(X^4), \ldots, \mathrm{Tr}(X^{2m})) = \sum_{a_1 < a_2 < \ldots < a_m} x_{a_1}^2 x_{a_2}^2 \ldots x_{a_m}^2 \, .$$

(These polynomials may be familiar from studies of anomalies or index theorems on curved manifolds [25]; if X is interpreted as a matrix of curvature 2-forms, then P_m are the Pontryagin classes.) We now define

$$\mathcal{K}_{2m} = P_m(\, (2m{-}1)\alpha\mathcal{J}_2\, ,\, (2m{-}1)\alpha\mathcal{J}_4\, , \ldots ,\, (2m{-}1)\alpha\mathcal{J}_{2m}\,)$$

which yield precisely the correct coefficients in (4.6) for $m \leq 5$.[4] Note that while these expressions are homogeneous in the eigenvalues x_a, the polynomials $P_m(t_i)$ are *not* homogeneous in the arguments t_i, so that the relative factors in each argument for different values of m are important. The constant α is arbitrary at the moment. We now conjecture that for any of the algebras $so(N)$ or $sp(N)$ the charges

$$q_{2m-1} = \int \mathcal{K}_{2m} dx$$

all have mutually vanishing Poisson brackets *for any value of the constant* α. This claim is made on the basis of explicit calculations for the first four examples, which already provide a very stringent test. Thus we have a one-parameter family of charges in involution. Notice that if we take the parameter $\alpha \to 0$ we recover our earlier primitive charges, since then $\mathcal{K}_{2m} = (const)\mathcal{J}_{2m}$. In the special case of $so(2n)$ the value of α is fixed by requiring that these charges also commute with the Pfaffian, which gives $\alpha = 2/h$.

There are a number of well-known ways to think of the P_m introduced above, the underlying mathematics being the elementary theory of symmetric polynomials [26]. However, none of these points of view seems to offer an obvious way of proving our conjecture. In its support, we would like to emphasize once again the very delicate nature of the cancellations required to verify its truth in even the simplest cases. We hope to return to this issue in future work.

[4] The relative coefficients should also be compared with those in (A.7) of [25]

4.3 Summary and comments

We have shown that the classical charges based on the simple currents $\mathcal{J}_m$ have vanishing Poisson brackets for the algebras $u(N)$, $so(N)$ and $sp(N)$. These include all the primitive charges, with the exception of the Pfaffian in $so(2n)$. We have conjectured general formulas for one-parameter families of commuting charges for the cases $so(N)$ and $sp(N)$, and we have further conjectured that these will commute with the Pfaffian charge in $so(2n)$ on making a specific choice for the parameter. The conjecture therefore implies the existence of a set of commuting charges with spins equal to the exponents for each of these algebras.

The situation for the algebra $su(N)$ is more complicated, and we have shown that the simple charges do not commute in general. (Of course it is still possible for the particles to be simultaneous eigenstates of the charges, but this would have to be a consequence of the detailed manner in which the currents are represented on them.) It should be emphasized, however, that much information can be deduced indirectly for $su(N)$ from the simpler behaviour of the model based on $u(N)$. This is because the $u(N)$ PCM is precisely the $su(N)$ PCM accompanied by a non-interacting, massless scalar field (both classically and quantum-mechanically). Thus, for example, the infinite number of commuting classical charges in the $u(N)$ theory might be viewed as responsible for its classical integrability, which would imply in turn integrability of the classical $su(N)$ sub-theory.

5 Quantum conserved charges

The central questions which we would like to answer are whether the conserved quantities we have been discussing so far in the classical PCM survive in the quantum theory, and if so whether their properties are modified in any significant ways. Unfortunately the subtle relationship between the classical PCM lagrangian and the quantum theory it defines makes many of these questions rather difficult to address directly.

The non-local charges are known to survive quantization. Their quantum behaviour has been studied [7, 8] using a point-splitting regularization of $Q^{(1)a}$, and they were found to obey a quantum version of the classical Yangian symmetry $Y_L(\mathcal{A}) \times Y_R(\mathcal{A})$. One important novel feature, however, is that the Poisson bracket $\{M, Q^{(1)}\}$, which vanishes classically, develops a term at order $\hbar^2$. This is essential to the construction of the quantum S-matrices, which would otherwise be trivial.

By contrast, very little is known about the quantum behaviour of the local charges. Point-splitting quantization would be a very much more complicated undertaking for these, since they involve products of many currents taken at a single point. A second approach, again likely to be rather complex, would be to regularize the model on a lattice, and we shall have a little more to say about this later. In the absence of any tractable explicit quantization procedure, we shall consider here an indirect approach which can yield sufficient (but not necessary) conditions for the quantum conservation of some local charges [14]. Our aim is to review this evidence in the light of the work we have carried out in earlier sections.

The commutators of charges in the quantum theory will be given to lowest order by the classical Poisson brackets we have already calculated. Whether or not these receive quantum corrections at higher orders is of course an open question. Nevertheless we should say that, even though no calculations have ever been carried out, it would be rather surprising if the vanishing commutators between the non-local and local charges received quantum modifications: it is difficult to see, for example, how the known S-matrices could be consistent if the local charges did not commute with the Yangian.

5.1 Goldschmidt-Witten anomaly counting

In the method of Goldschmidt and Witten [14] one simply counts the possible anomaly terms which might appear as modifications to a given classical conservation equation, and then counts the number of derivative terms with the same symmetry properties. (This is reminiscent of Zamolodchikov's approach [15] to perturbed conformal field theories.) We illustrate this by considering a general simple charge in a PCM.

The classical conservation equation $\partial_- \mathrm{Tr}(j_+^m) = 0$ cannot be expected to survive unscathed in the quantum theory because conformal invariance is broken. Nevertheless, the only quantum modifications which can appear are operators constructed from the currents and their derivatives with the same mass dimension and the same behaviour under all continuous and discrete symmetries. We can enumerate a linearly-independent set of such operators A_i with $i = 1, \ldots, p$, say; then the most general possible quantum modification is

$$\partial_- \mathrm{Tr}(j_+^m) = \sum_{i=1}^{p} \alpha_i A_i \,, \tag{5.1}$$

where the α_i are unknown constants. In deciding which As are independent, we are free to use the classical equations of motion, because any quantum modifications appearing in the Heisenberg equations will again correspond to operators with the correct dimensions and invariance properties, so that they will already occur in our list.

If the combination of As which appears in (5.1) can be written as a total space-time derivative, then we still have a conservation law, albeit of a modified form. In the absence of any explicit calculation, Goldschmidt and Witten observed (see also [17]) that it is sometimes sufficient to simply count the number of linearly-independent total-derivative terms B_j, with $j = 1, \ldots, q$, which again have the same quantum numbers as $\partial_- \mathrm{Tr}(j_+^m)$. Since each B can be written as a combination of As, $q \leq p$. But if $q = p$ then, conversely, every anomaly can be written as a total divergence, and we can deduce that the conservation law survives. In counting the Bs, we are also free to make use of the classical equations of motion, as explained in [16].

Moreover, we can consider not only the conservation of the simple current $\mathrm{Tr}(j_+^m)$ but also any others of the same spin like $\mathrm{Tr}(j_+^r)\mathrm{Tr}(j_+^s)$ with $r + s = m$ and products of traces corresponding to higher partitions of m. In fact the most general possibility is a differential polynomial in such quantities, with m being the total number of j_+s and ∂_+s. We should ensure, however, that we do not count *total* ∂_+ derivatives, since these are trivially conserved. If the total number of such classically conserved currents is n, and if $p < q+n$, then there must be at least one linear combination of currents which is still conserved quantum mechanically, corresponding to the fact that there is at least one linear combination of anomaly terms which can be written as a space-time divergence.

When Goldschmidt and Witten considered PCMs they wrote down As and Bs as functions of the field g. In confirming and generalizing their analysis, however, we have found it essential to use the currents j_μ so as to clarify the all-important question of which As and Bs are really independent. We also found it useful to set up some notation to enable us to keep track of these currents' behaviour under the discrete symmetries. First recall (2.10), and thus that

$$\pi \ : \qquad j_+^L \mapsto -g^{-1} j_+^L \, g \, . \tag{5.2}$$

We shall henceforth consider only j^L, and drop the label L. One finds also that $j_{++} \equiv \partial_+ j_+ \mapsto -g^{-1} \, j_{++} \, g$, but unfortunately the situation is more complicated for higher derivatives. To introduce combinations of currents which have simple behaviour under this discrete symmetry, we define $j_r = j_{++\ldots+}$ (r + indices) recursively by

$$j_{r+1} \equiv \partial_+ j_r \, + \, \frac{1}{2}[j_+, j_r] \, , \tag{5.3}$$

It is then easy to show that

$$\pi \ : \qquad j_n \mapsto -g^{-1}\, j_n\, g$$

$$\sigma \ : \qquad j_n \mapsto j_n^* = -j_n^T \tag{5.4}$$

$$\tau \ : \qquad j_n \mapsto M j_n M^{-1} \qquad \mathcal{A} = so(N)\,. \tag{5.5}$$

so that we have succeeded in identifying currents with simple behaviour under *all* discrete symmetries. Note that, for $O(N)$, all the charges are even under τ except (when $N = 2n$) for the spin-$(n-1)$ Pfaffian charge, which is odd.

The first example of a conservation law is to take $m = 2$ in (5.1). The classical current $\mathrm{Tr}(j_+j_+)$ is nothing but the energy-momentum tensor, and we certainly expect this to survive quantization. Indeed, there is only one possible anomaly $A_1 = \mathrm{Tr}(j_-j_{++})$, only one derivative $B_1 = \partial_+\mathrm{Tr}(j_-j_+)$, and in fact $A_1 = B_1$. This modification reflects the non-vanishing of the trace of the energy-momentum tensor quantum mechanically, corresponding to the breaking of conformal symmetry.

The next case $m = 3$ (or $s = 2$) is relevant only to $SU(N)$. The current $\mathrm{Tr}(j_+^3)$ is odd under both π and σ. Here again there is one A and one B,

$$A_1 = \mathrm{Tr}(j_2\{j_-, j_+\})\,, \qquad\qquad B_1 = \partial_+\mathrm{Tr}(j_-j_+^2)\,.$$

So conservation again survives quantization, and $A_1 = B_1$.

The next, less trivial case is $m = 4$, corresponding to $s = 3$, which is an exponent for all the classical algebras. In addition to $\mathrm{Tr}(j_+^4)$ there is a second current $(\mathrm{Tr}(j_+^2))^2$, and these are both even under each of the discrete symmetries. In this case we find

$$\begin{aligned}
A_1 &= \mathrm{Tr}(j_-j_4) & B_1 &= \partial_+\mathrm{Tr}(j_-j_3)\\
A_2 &= \mathrm{Tr}(j_-j_+)\mathrm{Tr}(j_+j_2) & B_2 &= \partial_+\left(\mathrm{Tr}(j_-j_+)\mathrm{Tr}(j_+^2)\right)\\
A_3 &= \mathrm{Tr}(j_-j_2)\mathrm{Tr}(j_+^2) & B_3 &= \partial_+\mathrm{Tr}(j_-j_+^3)\\
A_4 &= \mathrm{Tr}(j_+^2\{j_-, j_2\}) & B_4 &= \partial_-\mathrm{Tr}(j_2^2)\\
A_5 &= \mathrm{Tr}(j_-j_+j_2j_+) & &
\end{aligned}$$

(Note that another apparent possibility among the Bs, $\partial_-\mathrm{Tr}(j_+j_3)$, is not allowed; it is not independent of B_3.) All other potential terms are disallowed, either (for $SU(N)$) because they are odd under σ, or (for other groups) because they involve traces of elements of $\mathcal{A}$.

Thus $p = 5$, $q = 4$ but the number of classical currents is $n = 2$. We conclude that there is a linear combination of $\mathrm{Tr}(j_+^4)$ and $(\mathrm{Tr}(j_+^2))^2$ which will survive quantization.

We have reached the same conclusions as [16] (though they did not consider the case of $Sp(N)$) by showing the existence of spin-3 (where appropriate) and spin-4 currents in each of the PCMs based on a classical algebra. In comparing our lists of anomalies and derivatives with theirs, however, we should point out that there are some discrepancies. There seem to be some misprints and/or errors in eqns. (19) and (23) of [16]: the terms A_4 in (19) and A_2 in (23) do not have the correct behaviour under discrete symmetries. Furthermore, the terms B_1 and B_2 in eqn. (24) of [16] are not independent, since they can be related using the equations of motion. Any obvious modification of the term A_2 in eqn. (23) to give it the correct symmetry can similarly be related to A_1, confirming our counting above of one anomaly and one derivative for the spin-3 current, rather than two of each, as claimed in [16]. This underscores our opinion that it is clearest to work with quantities valued in the Lie algebra – *i.e.* currents j_μ rather than the field g. We mention these details simply because the Goldschmidt-Witten counting arguments provide the only means we know for proving the existence of higher-spin local charges in the quantum theory, and so it is obviously important to ensure that the counting is absolutely correct.

It is natural to wonder whether the same method can be used to establish the existence of more combinations of local charges in the quantum theory. We have investigated the situation for currents of spin 5 and 6, but the counting in these cases is inconclusive. For $m = 5$ ($SU(N)$ only) the classical currents are $\mathrm{Tr}(j_+^5)$ and $\mathrm{Tr}(j_+^2)\mathrm{Tr}(j_+^3)$ which are both odd under π, σ. The numbers of anomalies and derivatives are $p = 8$, $q = 6$, and so with $n = 2$ quantum conservation is not guaranteed. For $SU(N)$ and $m = 6$, there are $n = 5$ classical currents which are even under π, σ. The numbers of anomalies and derivatives are $p = 25$ and $q = 18$, so the argument again fails. For $Sp(N)$ and $O(N)$, the numbers are $n = 4$, $p = 23$, $q = 17$. There is also a single classical spin-6 current which is odd, but (here for $SU(N)$) $p = 21$ and $q = 10$, so again conservation is not guaranteed. Some details of these cases are given in an appendix.

The only other instance in which the counting arguments guarantee a quantum conservation law is the case of the Pfaffian charge in $SO(2n)$. It was shown in [16] that this always survives in the quantum theory; we do not reproduce the details here.

We should re-iterate at this point that the Goldschmidt-Witten counting arguments are sufficient conditions for quantum conservation, but they are by no means necessary. The fact that they fail in most cases should certainly not be interpreted as meaning that there are no more quantum conservation laws. Indeed, it is believed that one additional conserved charge of higher-spin is all that is required to guarantee integrability and factorization of the S-matrix [27]. For this reason alone, it would actually be a great surprise if, one charge being conserved, the others were not.

The counting arguments above guarantee the existence of some quantum charges, but unfortunately the method can give us no information about *which* combinations of simple charges might survive. For $O(2n)$ we would expect them to be those of (4.6), and we might well expect similar combinations for the B and C series. However, since the primitive charges already have the necessary classical commutation properties in these cases, we have no means of discriminating among the possibilities.

Finally, let us point out an important consequence of the transition from classical holomorphic-type conservation laws (3.6) to quantum non-holomorphic conservation laws of type (2.4). In the case of holomorphic laws, we can multiply the densities of two conserved charges to give a new density which, integrated, yields a new conserved local charge with a spin which may lie outside the set of exponents and which would therefore vitiate Dorey's rule. However, in the case of non-holomorphic laws this is not possible: the time-derivative of the product of two such densities cannot now be written as an overall space-derivative. Thus, if a full set of charges with spins equal to the exponents is conserved in the quantum theory, any local charge potentially obtained by taking a polynomial in their densities certainly is not.

5.2 Particle multiplets and $\mathcal{G}$-parity

The fundamental representations V of $Y(\mathcal{A})$ are generally reducible representations of $\mathcal{A}$, with one component being a fundamental representation V_i of $\mathcal{A}$ [1, 21, 22]. For the a_n and c_n series, $V = V_i$. The particle multiplets form representations of $Y_L \times Y_R$, which in [1] were taken to be $V \otimes \bar{V}$. Although it is possible to advance various arguments for this multiplet structure, none of them seem entirely conclusive, although the assumption certainly leads to consistent S-matrices.

Of course, the particles also represent the local charges, of which – since the local charges commute with the Yangian charges – these multiplets will be singlets. Now let us consider $\mathcal{G}$-parity (2.10). Since $\mathcal{G}$-parity is a symmetry of the lagrangian, the particles should form multiplets under it. The symmetry $g \mapsto g^{-1}$ exchanges $\mathcal{G}_L$ and $\mathcal{G}_R$, and so its action on $Y_L \times Y_R$ multiplets must be of the form $V \otimes W \mapsto W \otimes V$. In terms of the local charges, note that the charges in (3.10) are even/odd under $\mathcal{G}$-parity precisely as m is even/odd. But we have already pointed out that the integers m, for a given algebra $\mathcal{A}$, are its exponents plus one. For algebras whose representations are all self-conjugate, the exponents are all odd, and the charges are then all even. In the absence of odd-parity local charges, we expect only the $\mathcal{G}$-parity singlets $V \otimes V$, as proposed in [1].

However, if an odd-parity local charge q^- exists (and is non-zero on a physical state), it must yield a different physical state with opposite $\mathcal{G}$-parity, and the Yangian multiplets will appear in $\mathcal{G}$-parity doublets. Even exponents (and thus odd $\mathcal{G}$-parity local charges) exist precisely when there are complex representations of $\mathcal{A}$ (and thus of $Y(\mathcal{A})$), and the $\mathcal{G}$-parity doublets will lie in $V \otimes \bar{V} \oplus \bar{V} \otimes V$, again as in [1]. Note that the eigenstates of q^- are $V \otimes \bar{V}$ and $\bar{V} \otimes V$ (in distinction to the $\mathcal{G}$-parity eigenstates), with eigenvalues $\pm q^-_V$, whilst $q^- = 0$ on states with $V = \bar{V}$. In contrast, even-parity charges take the same values on $V \otimes \bar{V}$ and $\bar{V} \otimes V$. This is precisely the form of the values taken by the charges in ATFTs, where it is the even-spin charges which enable states to be distinguished from their (mass-degenerate) conjugates [28].

Finally let us note for $O(N)$ how π and τ fit together. For n even, the Pfaffian charge q_{n-1} is π-even and τ-odd. The particle multiplets are $S^+ \otimes S^+$ and $S^- \otimes S^-$, and the τ-doublets, exchanged by q_{n-1}, lie in $S^+ \otimes S^+ \oplus S^- \otimes S^-$. For n odd, q_{n-1} is odd under both parities, the particle multiplets are $S^+ \otimes S^-$ and $S^- \otimes S^+$, and the parity doublets lie in $S^+ \otimes S^- \oplus S^- \otimes S^+$.

6 Discussion

In this paper we have considered local charges in classical principal chiral models which are associated with invariant tensors or Casimirs of the underlying Lie algebra. We have shown that such local charges always commute with the Yangian non-local charges. Further, for the models based on the algebras $u(N)$, $so(N)$ and $sp(N)$ we have demonstrated that there exist infinitely many 'simple' local charges which mutually commute. The corresponding charges for the model based on $su(N)$ obey a more complicated algebra, however.

We have also studied in detail a finite set of invariant tensors and charges which we have called 'primitive'. These naturally arise as a generating set for all invariant tensors, and also, because they lead to charges with spins equal to the exponents of the algebra, in the context of Dorey's rule. For $u(N)$, $so(2n+1)$ and $sp(N)$ they are found among the simple charges, but the case of $so(2n)$ is particularly subtle, since there exists a primitive tensor which is not simple. Nevertheless we have been able to conjecture formulas for a full commuting set of charges in this theory too (based on calculations for low spin).

6.1 Alternative approaches to classical integrability

The existence of infinitely many conserved charges in involution is clearly a pre-requisite for the classical integrability of any field theory. In this connection, it is interesting to note that a much more radical approach to the integrability of classical PCMs has been advocated by Faddeev and Reshetikhin [18]. They were concerned that (2.16) do not allow the r-matrix formalism of classical inverse scattering, and they proceeded on the expectation [8] that the quantum current algebra should be covariant, in contrast to (2.16). How, then, do (2.16) arise in the classical limit? Their discovery was that either (2.16) or covariant Poisson brackets without non-ultralocal terms (which would lead to charges in involution via an r-matrix) could be obtained, depending on how the classical limit is taken. The point is that in any quantized form of the theory there are two limits to be taken: $\hbar \to 0$, and another, in their case $S \to \infty$ where S is a spin scale in the XXX model. For quantization on a lattice [19], the second limit would be $\Delta \to 0$ where Δ is the lattice step length. The different orders in which the two limits can be taken then (with a suitable rescaling of the space component of the current) give the two types of Poisson bracket: taking $\Delta \to 0$ first leads to covariant, ultralocal brackets, whereas taking $\hbar \to 0$ first leads to (2.16).

It seems to us that, following our results, the classical motivation for this approach is less clear. The canonical Poisson brackets (2.16) do not allow for a straightforward r-matrix approach to classical integrability, yet in a more subtle way they seem to provide for classically commuting charges.

6.2 Dorey's Rule

We have already mentioned that in affine Toda theories (ATFTs) it is particularly significant that one encounters conserved charges with spins equal to the exponents of the underlying Lie algebra, since these lead to the elegant rule for particle fusings discovered by Dorey [4]. That the same patterns of exponents appear as the spins of classical conserved charges in the PCMs is therefore highly suggestive that Dorey's rule should apply here too.

In ATFTs, particle fusings appear in tree-level three-point couplings as well as in exact S-matrices. For PCMs, in contrast, our knowledge of them comes only from the Yangian-invariant S-matrices. For simply-laced algebras Dorey's rule is relatively straightforward: the ATFT fusings are the same at tree-level as in the exact S-matrices, and agree with the particle fusings deduced from the Yangian structure [6]—that is, from the PCM exact S-matrices. It is therefore quite natural to expect that they should emerge from the presence of local charges in the PCM too.

For nonsimply-laced algebras, however, Dorey's rule is rather subtle. The ATFTs appear in dual pairs based on an untwisted and a twisted algebra. Let us consider the example of the pair of algebras $c_n^{(1)}$ and $d_{n+1}^{(2)}$. The charges have spins equal to the exponents of c_n, but their values depend on the coupling constant: in the weak limit these values are associated with c_n, while in the strong limit they are associated with $d_{n+1}^{(2)}$. Dorey's construction does not allow for such a coupling constant dependence, and gives the tree-level couplings either for c_n, the set of which we shall call $D(c_n)$, or (when suitably generalized [23]) for $d_{n+1}^{(2)}$, which we shall call $D(d_{n+1}^{(2)})$. It is then the intersection of the two sets, $D'(c_n) \equiv D(c_n) \cap D(d_{n+1}^{(2)})$, which gives the correct fusings for the bootstrap principle applied to the quantum S-matrices.

The c_n PCM S-matrices [24] also have $D'(c_n)$ fusings, rather than either of the two ATFT tree-level constituents mentioned above. But in the PCM S-matrices there can be no coupling constant dependence: the mass ratios are rigid and are in fact those of the tree-level $d_{n+1}^{(2)}$ ATFT, suggesting that the other charges should have similarly fixed ratios. Thus the local charges' surviving quantization would appear to allow all the $D(d_{n+1}^{(2)})$ fusings. We have no indication of what further restricts the fusings to $D'(c_n)$.

Let us return to the broader question of the apparent ubiquity of Dorey's rule in integrable field theories and the co-existence of local with non-local charges. It is worth remarking that it is precisely the properties of the local charges which have been used to deduce properties of the S-matrix in two [27] and four [29] dimensions. The non-local charges escape such theorems, but nevertheless lead to factorized S-matrices through their quasitriangular Hopf algebra ('quantum group') structure: and if they co-exist with local charges, perhaps their unusual properties are a false trail after all. In other words, Dorey's construction may entirely transcend the quantum group approach.

How might the two sets of charges be related? The Yangian $Y(\mathcal{A})$ has a trivial centre [30], so it seems that we cannot hope to construct local charges which commute as 'Casimirs' in $Y(\mathcal{A})$ by taking polynomials or series in the Yangian charges. Another avenue would be to consider the transfer matrix which generates the non-local charges, but it is far from clear to us how the local charges might emerge from this. ¿From yet another point of view, there are hints of a connection of the type we seek in recent work of Frenkel and Reshetikhin on deformed W-algebras ([31] and references therein).

Finally we note that many of the features we have identified in this paper also appear in the supersymmetric principal chiral model, with which we shall deal in a forthcoming paper.

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7 Appendix: An alternative derivation of Poisson brackets

The PCM can be regarded as a special case of a general σ-model with lagrangian

$$\mathcal{L} = \frac{1}{2}\, g_{ij}(\phi)\, \partial_\mu \phi^i\, \partial^\mu \phi^j$$

where ϕ^i are coordinates on some target manifold with metric $g_{ij}(\phi)$. The momenta conjugate to the fields ϕ^i are

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^i} = g_{ij}(\phi)\, \dot{\phi}^j$$

with the standard non-vanishing equal-time PBs

$$\{\, \phi^i(x),\, \pi_j(y)\,\} = \delta^i{}_j\, \delta(x-y)\,.$$

We may now consider a current of the form

$$J^a_\mu = E^a_i\, \partial_\mu \phi^i$$

where $E_i^a(\phi)$ are vielbeins on the target manifold satisfying

$$E_i^a E_j^a = g_{ij}$$

(Whether these currents are actually conserved or not is irrelevant for the arguments here.) In terms of the canonical coordinates ϕ^i and π_i we have

$$J_0^a = E_i^a g^{ij} \pi_j \qquad J_1^a = E_i^a \phi'^{\,i} \, .$$

The PB algebra of these currents can now be calculated routinely, although the general result requires a little effort and is not particularly illuminating.

Dramatic simplification occurs for the special case of a group manifold, with currents defined by the vielbeins

$$E_i^{R\,a} = \mathrm{Tr}(t^a g^{-1} \partial_i g) \, , \qquad E_i^{L\,a} = -\mathrm{Tr}(t^a \partial_i g g^{-1})$$

where

$$E_i^R = -g^{-1} \partial_i g \, , \qquad E_i^L = \partial_i g g^{-1}$$

are the left-invariant and right-invariant forms on the group respectively (so our labels L and R signify the symmetries under which the vielbeins *transform*). To simplify the current algebra calculations it is necessary to use only the properties

$$\partial_{[i} E_{j]} = E_{[i} E_{j]}$$

(the Maurer-Cartan relations) which are easily verified from the definitions above. The Poisson brackets (2.16) then follow.

8 Appendix: Computing brackets of the Pfaffian charge

Here we derive (4.4) from (4.2). At each point in space the matrix j_+ is antisymmetric; let its skew eigenvalues be λ_i, so that there exist an orthogonal U and block diagonal D with

$$U j_+ U^{-1} = D \qquad\qquad D = \mathrm{diag}\left(\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \ldots \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} \right) .$$

Clearly

$$\mathrm{Tr}(j_+^s) = 2 \sum_i \lambda_i^s \quad (s \text{ even}), \qquad \mathcal{P} = 2^n \lambda_1 \ldots \lambda_n$$

From (4.2) we find an integrand proportional to

$$\epsilon_{i_1 j_1 i_2 j_2 \ldots i_n j_n}(j_+)_{i_2 j_2} \ldots (j_+)_{i_n j_n} t^c_{i_1 j_1} \partial_x \mathrm{Tr}(t^c j_+^s) = -\epsilon_{i_1 j_1 i_2 j_2 \ldots i_n j_n}(j_+)_{i_2 j_2} \ldots (j_+)_{i_n j_n} \partial_x (j_+^s)_{i_1 j_1}$$

(by completeness). Now observe that

$$\partial_x (j_+^s) = U \partial_x (D^s) U^{-1} + U[U^{-1}\partial_x U, D^s]U^{-1} .$$

On substituting this into the previous expression, the term involving the commutator vanishes by antisymmetry of the matrix $U^{-1}\partial_x U$ in conjunction with the block-diagonal structure of D. We therefore obtain

$$\epsilon_{i_1 j_1 i_2 j_2 \ldots i_n j_n} D_{i_2 j_2} \ldots D_{i_n j_n} \partial_x (D^s)_{i_1 j_1} .$$

Since $\partial_x(\lambda^s) = (const)\, \lambda\, \partial_x(\lambda^{s-1})$ we find, up to a constant,

$$\lambda_1 \ldots \lambda_n \partial_x \sum \lambda_i^{s-1} ,$$

which gives the desired result.

9 Appendix: Further details of anomalies

We list below the As and Bs for the highest-spin cases.

Spin-5 ($m = 5$) currents for $SU(N)$: $\mathrm{Tr}(j_+^5)$ and $\mathrm{Tr}(j_+^2)\mathrm{Tr}(j_+^3)$, both π- and σ-odd

$$\begin{aligned}
A_1 &= \mathrm{Tr}(j_-\{j_+, j_4\}) \\
A_2 &= \mathrm{Tr}(j_-\{j_2, j_3\}) \\
A_3 &= \mathrm{Tr}(j_+^3\{j_-, j_2\}) \\
A_4 &= \mathrm{Tr}(j_- j_+^2 j_2 j_+ + j_+ j_2 j_+^2 j_-) \\
A_5 &= \mathrm{Tr}(j_- j_+)\mathrm{Tr}(j_+^2 j_2) \\
A_6 &= \mathrm{Tr}(j_- j_2)\mathrm{Tr}(j_+^3) \\
A_7 &= \mathrm{Tr}(j_- j_+^2)\mathrm{Tr}(j_+ j_2) \\
A_8 &= \mathrm{Tr}(j_-\{j_+, j_2\})\mathrm{Tr}(j_+^2)
\end{aligned}$$

$$\begin{aligned}
B_1 &= \partial_+ \mathrm{Tr}(j_-\{j_+, j_3\}) \\
B_2 &= \partial_+ \mathrm{Tr}(j_- j_2^2) \\
B_3 &= \partial_- \mathrm{Tr}(j_+ j_2^2) \\
B_4 &= \partial_+ \mathrm{Tr}(j_- j_+^4) \\
B_5 &= \partial_+(\, \mathrm{Tr}(j_- j_+)\mathrm{Tr}(j_+^3)\,) \\
B_6 &= \partial_+(\, \mathrm{Tr}(j_- j_+^2)\mathrm{Tr}(j_+^2)\,)
\end{aligned}$$

Note that the additional term $\partial_- \mathrm{Tr}(j_+^2 j_3)$ is proportional to B_3.

Spin-6 ($m = 6$) classical currents: $\mathrm{Tr}(j_+^6)$, $\mathrm{Tr}(j_+^4)\mathrm{Tr}(j_+^2)$, $(\mathrm{Tr}(j_+^2))^3$, $(\partial_+ \mathrm{Tr}(j_+^2))^2$ and, for

$SU(N)$, $(\mathrm{Tr}(j_+^3))^2$, are all π-, σ-even. With these symmetries we find

$$
\begin{aligned}
A_1 &= \mathrm{Tr}(j_+^4\{j_2, j_-\}) \\
A_2 &= \mathrm{Tr}(j_+^3(j_2 j_+ j_- + j_- j_+ j_2)) \\
A_3 &= \mathrm{Tr}(j_+^2 j_2 j_+^2 j_-) \\
A_4' &= \mathrm{Tr}(j_+^3)\mathrm{Tr}(j_+\{j_2, j_-\}) \\
A_5' &= \mathrm{Tr}(j_+^2 j_2)\mathrm{Tr}(j_+^2 j_-) \\
A_6 &= \mathrm{Tr}(j_+^2)\mathrm{Tr}(j_+ j_2 j_+ j_-) \\
A_7 &= \mathrm{Tr}(j_+^2)\mathrm{Tr}(j_+^2\{j_2, j_-\}) \\
A_8 &= \mathrm{Tr}(j_+ j_2)\mathrm{Tr}(j_+^3 j_-) \\
A_9 &= \mathrm{Tr}(j_+ j_-)\mathrm{Tr}(j_+^3 j_+) \\
A_{10} &= \mathrm{Tr}(j_2 j_-)\mathrm{Tr}(j_+^4) \\
A_{11} &= \mathrm{Tr}(j_+ j_2)\mathrm{Tr}(j_+ j_-)\mathrm{Tr}(j_+^2) \\
A_{12} &= (\mathrm{Tr}(j_+^2))^2\mathrm{Tr}(j_2 j_-) \\
A_{13} &= \mathrm{Tr}(j_+^2\{j_-, j_4\}) \\
A_{14} &= \mathrm{Tr}(j_+ j_- j_+ j_4) \\
A_{15} &= \mathrm{Tr}(j_2 j_3 j_+ j_- + j_- j_+ j_3 j_2) \\
A_{16} &= \mathrm{Tr}(j_2 j_+ j_3 j_- + j_- j_3 j_+ j_2) \\
A_{17} &= \mathrm{Tr}(j_+ j_2 j_3 j_- + j_- j_3 j_2 j_+) \\
A_{18} &= \mathrm{Tr}(j_2^3 j_+) \\
A_{19} &= \mathrm{Tr}(j_+ j_-)\mathrm{Tr}(j_+ j_4) \\
A_{20} &= \mathrm{Tr}(j_+ j_-)\mathrm{Tr}(j_2 j_3) \\
A_{21} &= \mathrm{Tr}(j_2 j_-)\mathrm{Tr}(j_2^2) \\
A_{22} &= \mathrm{Tr}(j_2 j_-)\mathrm{Tr}(j_+ j_3) \\
A_{23} &= \mathrm{Tr}(j_3 j_-)\mathrm{Tr}(j_+ j_2) \\
A_{24} &= \mathrm{Tr}(j_4 j_-)\mathrm{Tr}(j_+^2) \\
A_{25} &= \mathrm{Tr}(j_6 j_-)
\end{aligned}
\qquad
\begin{aligned}
B_1 &= \partial_+ \mathrm{Tr}(j_+^5 j_-) \\
B_2' &= \partial_+ \left(\mathrm{Tr}(j_+^3)\mathrm{Tr}(j_+^2 j_-)\right) \\
B_3 &= \partial_+ \left(\mathrm{Tr}(j_+ j_-)\mathrm{Tr}(j_+^4)\right) \\
B_4 &= \partial_+ \left(\mathrm{Tr}(j_+^2)\mathrm{Tr}(j_+^3 j_-)\right) \\
B_5 &= \partial_+ \left(\mathrm{Tr}(j_+ j_-)(\mathrm{Tr}(j_+^2)^2\right) \\
B_6 &= \partial_+ \mathrm{Tr}(j_+^2\{j_3, j_-\}) \\
B_7 &= \partial_+ \mathrm{Tr}(j_+ j_3 j_+ j_-) \\
B_8 &= \partial_+ \mathrm{Tr}(j_2^2\{j_+, j_-\}) \\
B_9 &= \partial_+ \mathrm{Tr}(j_2 j_+ j_2 j_-) \\
B_{10} &= \partial_- \mathrm{Tr}(j_+^2 j_2^2) \\
B_{11} &= \partial_- \mathrm{Tr}(j_+^3 j_3) \\
B_{12} &= \partial_+ \left(\mathrm{Tr}(j_+ j_-)\mathrm{Tr}(j_2^2)\right) \\
B_{13} &= \partial_+ \left(\mathrm{Tr}(j_+ j_-)\mathrm{Tr}(j_+ j_3)\right) \\
B_{14} &= \partial_+ \left(\mathrm{Tr}(j_2 j_-)\mathrm{Tr}(j_+ j_2)\right) \\
B_{15} &= \partial_+ \left(\mathrm{Tr}(j_3 j_-)\mathrm{Tr}(j_+^2)\right) \\
B_{16} &= \partial_- \left(\mathrm{Tr}(j_+^2)\mathrm{Tr}(j_2^2)\right) \\
B_{17} &= \partial_+ \mathrm{Tr}(j_5 j_-) \\
B_{18} &= \partial_- \mathrm{Tr}(j_+ j_5)
\end{aligned}
$$

(The terms marked A' and B' vanish for all cases but $SU(N)$.) The further derivative terms $\partial_- \mathrm{Tr}(j_+ j_2 j_+ j_2)$, $\partial_- \left(\mathrm{Tr}(j_+^2)\mathrm{Tr}(j_+ j_3)\right)$, $\partial_- \mathrm{Tr}(j_2 j_4)$ and $\partial_- \mathrm{Tr}(j_3^2)$ are not independent of those listed.

For $SU(N)$ here is also a single independent classical spin-6 current $\mathrm{Tr}(j_+^3)\mathrm{Tr}(j_+ j_2)$ which is π-, σ-odd. In this case, $p = 21$, $q = 10$, but we omit the details.

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