Introduction to the Cohomology of Arithmetic Groups

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These lectures will be mainly concerned with the real or complex cohomology of an arithmetic subgroup Γ of a semisimple $\mathbb{Q}$-group G. It can be identified to the cohomology of the quotient $\Gamma \setminus X$ by Γ of the space X of maximal compact subgroups of G, which has allowed one to study the cohomology of Γ by a variety of methods: geometric constructions, de Rham and Lie algebra cohomology, harmonic and automorphic forms, harmonic analysis. I shall try to give an introduction to those and to some of the results they lead to, first when $\Gamma \setminus X$ is compact, where they appear in their simplest form, and then in the (more important) case where $\Gamma \setminus X$ is not compact, to which the main part of these lectures will be devoted. An important role is played by a compactification of $\Gamma \setminus X$ of the same homotopy type as $\Gamma \setminus X$ and by its boundary.

§1. Group, de Rham and Lie algebra cohomologies.

1.1. We recall first the definition of group (or Eilenberg-MacLane) cohomology (cf. [Br]).

Let Γ be a group, E a Γ-module. For $q \in \mathbb{N}$, let $L^q = [\Gamma^{q+1}, E]$ be the space of maps of Γ^{q+1} into E; Γ operates on it by the rule

$$\gamma \circ f(x_0, \dots, x_q) = \gamma(f(\gamma^{-1}x_0, \dots, \gamma^{-1}x_q)) \ . \tag{1}$$

Let

$$C^q = C^q(\Gamma, E) = [\Gamma^{q+1}, E]^\Gamma \tag{2}$$

i.e. the set of $f \in L^q$ which satisfy

$$f(\gamma x_0, \dots, \gamma x_q) = \gamma(f(x_0, \dots, x_q)) \ .$$

The direct sum $C^\bullet$ of the C^p is endowed with the differential d which, on C^q is the map $d_q : C^q \to C^{q+1}$ defined by

$$d_q f(x_0, \dots, x_{q+1}) = \sum (-1)^i f(x_0, \dots, \hat{x}_i, \dots, x_{q+1}) \tag{3}$$

where ˆ over a variable means that the variable should be omitted. It is readily checked that $d \cdot d = 0$, and the q-th cohomology group of Γ with coefficients in E is by definition

$$H^q(\Gamma; E) = \ker d_q / \text{im } d_{q-1} \quad (q \in \mathbb{N}) \tag{4}$$

($C^{-1} = 0$). We have

$$H^0(\Gamma; E) = \ker d_0 = E^\Gamma \ . \tag{5}$$

Remarks.

1. The differential d is also defined on the sum $L^\bullet$ of the L^q. The complex $L^\bullet$ is acyclic. More strongly, it is an injective resolution of E, so that $H^q(\Gamma; E)$ is the q-th right derived functor of the functor $E \mapsto E^\Gamma$.
2. One can define similarly the homology group $H_q(\Gamma; E)$ $(q \in \mathbb{N})$ as the q-th left derived functor of the functor $E \to E_\Gamma$, where E_Γ is the module of coinvariants, i.e. the quotient of E by the submodule generated by the differences $e - \gamma \cdot e$ $(e \in E, \gamma \in \Gamma)$, (cf. [Br]).

1.2. We recall two elementary properties of $H^\bullet(\Gamma; E)$ when E is a vector space over a field of characteristic zero.

(i) If Γ is finite, then $H^q(\Gamma; E) = 0$ for $q \neq 0$.

(ii) If Γ' is a normal subgroup of finite index of Γ, then

$$H^q(\Gamma; E) = H^q(\Gamma'; E)^{\Gamma/\Gamma'} \ .$$

(This uses the fact that the group Aut Γ of Γ operates on its cohomology, and that the inner automorphisms operate trivially.)

1.3. Eilenberg-MacLane spaces.

Let X be a space on which Γ operates freely and properly i.e. a principal Γ-bundle, (so that $\Gamma \setminus X$ is Hausdorff), and such that

$$\pi_i(X) = 0 \quad (i \neq 1) \ , \quad \pi_1(X) = \Gamma \ . \tag{1}$$

Then the quotient $\Gamma \setminus X$ is an Eilenberg-MacLane space $K(\Gamma, 1)$. Any two such spaces have the same homotopy type, hence the same cohomology. To the Γ-module E is associated on $\Gamma \setminus X$ a local system of coefficients $\tilde{E}$, which can be defined as the bundle $X \times_\Gamma E$, with discrete structure group Γ, and we have

$$H^q(\Gamma; E) = H^q(\Gamma \setminus X; \tilde{E}) \quad (q \in \mathbb{N}) \ . \tag{2}$$

In our case, X will be a manifold and the RHS may be viewed as singular cohomology. The algebraic definition 1.1 is of little use, and we shall concentrate on the RHS of (2).

Remark. By putting 1.1 ahead of 1.3, I have followed the reverse of the historical order. It was first noticed (by Hurewicz) that if X and X' are two spaces as above, then the quotients $\Gamma \setminus X$ and $\Gamma \setminus X'$ have the same homotopy type, hence their cohomology or homology depends only on Γ, so the problem arose to compute it directly from Γ. The first breakthrough is due to H. Hopf, who described the second homology group in terms of Γ. Then general formulas were soon obtained, more or less simultaneously, by Hopf, B. Eckmann, H. Freundenthal, S. Eilenberg and S. MacLane.

1.4. ¿From now on, G is a real Lie group with finitely many connected components, K a maximal compact subgroup of G (they are all conjugate), $X = G/K$, and Γ a discrete subgroup of G. The space X is always connected.

The space X is contractible and Γ operates properly on X (given $C \subset X$ compact, $\{\gamma \in \Gamma | C \cap \gamma C \neq \emptyset\}$ is finite). If Γ is torsion-free, it operates freely, hence $\Gamma \setminus X$ is a $K(\Gamma, 1)$. We shall be mostly concerned with the case where E is a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Then 1.3(2) is true even if Γ has torsion.

[To see this, consider the bundle $U \times_\Gamma X$, where U is a universal Γ-bundle (a contractible principal bundle for Γ). It has two projections

$$(1) \qquad \begin{array}{ccccc} M = & U & \times_\Gamma & X & \\ & \swarrow u & & \searrow v & \\ U/\Gamma & & & & \Gamma \setminus X \end{array}$$

v is a bundle map with fiber the contractible space X, hence (say for rational coefficients)

$$(2) \qquad H^\bullet(M) = H^\bullet(U/\Gamma) = H^\bullet(\Gamma) \; .$$

The fiber over $y \in \Gamma \setminus X$ of v is the quotient U/Γ_x, where Γ_x is the isotropy group of a point x mapping onto y. The group Γ_x is finite hence 1.2(i) shows that the fibers of v are acyclic. Therefore v^* induces an isomorphism of $H^\bullet(\Gamma \setminus X)$ onto $H^\bullet(M)$ which, combined with (2), proves our assertion, at any rate for trivial coefficients. The more general case is proved similarly.]

1.5. The groundfield L is either $\mathbb{R}$ or $\mathbb{C}$ and E is a finite dimensional vector space over L. If M is a smooth manifold, we let $A^q(M;E)$ be the space of smooth q-forms with coefficients in E, and $A^\bullet(M;E)$ the direct sum of the $A^q(M;E)$ $(q \in \mathbb{N})$. Thus

$$(1) \qquad A^\bullet(M;E) = A^\bullet(M;L) \otimes E \; .$$

As usual, d is exterior differentiation. We claim that the de Rham theorem and 1.3(2) imply

$$(2) \qquad H^\bullet(\Gamma;E) = H^\bullet(A^\bullet(X:E)^\Gamma) \; .$$

If Γ is torsion free, then $A^\bullet(X;E)^\Gamma = A^\bullet(\Gamma \setminus X;E)$ and (2) follows from the de Rham theorem on $\Gamma \setminus X$ and 1.3(2). In the case of interest to us, G is linear, Γ finitely generated, hence, by a known lemma of A. Selberg, Γ has a torsion free normal subgroup of finite index, and then (2) follows from the torsion free case and 1.2(ii). In the general case, one can view $\Gamma \setminus X$ as an orbifold, a notion introduced much earlier by I. Satake under the name of V-manifold and also studied by W. Baily. Then (2) also follows from 1.3(2) and a variant of the de Rham theorem for V-manifolds (see also VII, §2 in [BW] for another argument.). The relation (2) and its translation in relative Lie algebra cohomology, see below, will be our main starting points in the study of $H^\bullet(\Gamma;E)$.

1.6 Relative Lie algebra cohomology.

We recall it briefly, mainly to fix notation, referring to [BW] for the details.

Let L be a field of characteristic zero, $\mathfrak{g}$ a (finite dimensional) Lie algebra over L and V a vector space over L on which $\mathfrak{g}$ operates. Let

$$(1) \qquad C^\bullet(\mathfrak{g};V) = \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, V) = \wedge \mathfrak{g}^* \otimes_L V \; ,$$

endowed with the differential

$$(2) \qquad \begin{aligned} df(x_0,\dots,x_q) &= \sum_i (-1)^i x_i \cdot f(x_0,\dots,\hat{x}_i,\dots,x_q) + \\ &+ \sum_{i<j} (-1)^{i+j} f([x_i,x_j],x_0,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_q) \; . \end{aligned}$$

By definition

$$(3)\qquad H^\bullet(\mathfrak{g};V) = H^\bullet(C^\bullet(\mathfrak{g};V),d)\ .$$

Let $\mathfrak{k}$ be a reductive subalgebra of $\mathfrak{g}$ (i.e. $\mathfrak{g}$ is fully reducible under the restriction to $\mathfrak{k}$ of the adjoint representation $ad_{\mathfrak{g}}$ of $\mathfrak{g}$). The relative Lie algebra complex $C^\bullet(\mathfrak{g},\mathfrak{k};V)$ is, by definition

$$(4)\qquad C^\bullet(\mathfrak{g},\mathfrak{k};V) = \{f \in C^\bullet(\mathfrak{g};V), i_x f = \theta_x f = 0 \quad (x \in \mathfrak{k})\} = \mathrm{Hom}_{\mathfrak{k}}(\wedge^\bullet \mathfrak{g}/\mathfrak{k}, V)\ ,$$

where i_x and θ_x refer respectively to interior product and Lie derivative:

$$(5)\qquad i_x f(x_1,\dots,x_{q-1}) = f(x,x_1,\dots,x_{q-1})$$

$$(6)\qquad \theta_x f(x_1,\dots,x_q) = \sum_i f(x_1,\dots,[x,x_i],\dots,x_q) + x\cdot f(x_1,\dots,x_q)$$

($f \in C^q$). We recall the fundamental relation

$$(7)\qquad \theta_x = d\cdot i_x + i_x \cdot d \quad (x\in\mathfrak{g})$$

$C^\bullet(\mathfrak{g},\mathfrak{k};V)$ is stable under d and, by definition,

$$(8)\qquad H^\bullet(\mathfrak{g},\mathfrak{k};V) = H^\bullet(C^\bullet(\mathfrak{g},\mathfrak{k};V),d)\ .$$

1.7 We now come back to the situation of 1.5. On G fix a basis ω^i ($1 \le i \le m = \dim G$) of the space of left-invariant 1-forms (i.e. Maurer-Cartan forms). For a subset I of $I_m = \{1,2,\dots,m\}$.

$$I = \{i_1,\dots,i_q\} \quad (i_1 < \dots < i_q)$$

we let $\omega^I = \omega^{i_1}\wedge\dots\wedge\omega^{i_q}$. The ω^i define a framing of the cotangent bundle T^*G, hence also of $T^*(M)$ and the ω^I provide a framing of $\wedge^\bullet T^*(\Gamma\setminus G)$. Any smooth q-form on $\Gamma\setminus G$ can be written uniquely as

$$(1)\qquad \eta = \sum_{I\subset m} f_I\omega^I \quad f_I \in C^\infty(\Gamma\setminus G)\ .$$

Similarly $\eta \in A^q(\Gamma\setminus G;E)$ can be written as above, but with

$$(2)\qquad f_I \in C^\infty(\Gamma\setminus G;E) = C^\infty(\Gamma\setminus G)\otimes E\ .$$

This leads to an isomorphism

$$(3)\qquad A^\bullet(\Gamma\setminus G;\tilde{E}) = C^\bullet(\mathfrak{g};C^\infty(\Gamma\setminus G)\otimes E)\ .$$

It is easily seen to commute with the respective differentials, whence an isomorphism

$$(4)\qquad H^\bullet(\Gamma\setminus G;E) = H^\bullet(A^\bullet(\Gamma\setminus G;\tilde{E})) = H^\bullet(\mathfrak{g};C^\infty(\Gamma\setminus G)\otimes E)\ .$$

Let $\pi : G \to X = G/K$ be the canonical projection. We also note π the induced projection of $\Gamma \setminus G$ onto $\Gamma \setminus X$. It is readily checked that

$$\pi^* : A^\bullet(X;E)^\Gamma \to A^\bullet(\Gamma \setminus G;\tilde{E}) \ , \tag{5}$$

followed by the isomorphism (3), defines an isomorphism

$$A^\bullet(X;E)^\Gamma = C^\bullet(\mathfrak{g},\mathfrak{k};C^\infty(\Gamma \setminus G) \otimes E) \tag{6}$$

where $\mathfrak{k}$ is the Lie algebra of K, whence also, in view of 1.5(2), an isomorphism

$$H^\bullet(\Gamma;E) = H^\bullet(\mathfrak{g},\mathfrak{k};C^\infty(\Gamma \setminus G) \otimes E) \ . \tag{7}$$

Strictly speaking, this is valid if K, hence G, is connected. This would not be a serious restriction for us, but it is sometimes convenient to lift it by a minor modification of the notion of relative Lie algebra cohomology, if V is again a G-module, then K operates in a natural way on $C^\bullet(\mathfrak{g},\mathfrak{k};V)$, via the given operation on V and the adjoint representation on $\mathfrak{g}$, and we can define

$$C^\bullet(\mathfrak{g},K;V) = C(\mathfrak{g},\mathfrak{k};V)^K \tag{8}$$

$$H^\bullet(\mathfrak{g},K;V) = H^\bullet(C^\bullet(\mathfrak{g},\mathfrak{k};V)^K) = H^\bullet(\mathfrak{g},\mathfrak{k};V)^{K/K^o} \ , \tag{9}$$

where K^o is the identity component of K, so that (6) may be written

$$H^\bullet(\Gamma;E) = H^\bullet(\mathfrak{g},K;C^\infty(\Gamma \setminus G) \otimes E) \ . \tag{10}$$

Remark. The above is valid for any discrete subgroup, in particular for the trivial group. Therefore we have

$$H^i(\mathfrak{g},\mathfrak{k};C^\infty(G) \otimes E) = \begin{cases} E & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases} \tag{11}$$

In fact, $C^\infty(\Gamma \setminus G) \otimes E$ is the induced representation to G from the Γ-module E in the C^∞-set up, so that (10) can be more conceptually viewed as a Shapiro lemma in continuous or differentiable cohomology ([BW], IX), but we shall not need this interpretation.

1.8. Symmetric spaces.

(A) We now specialize to the most important case for us, and assume that G is connected, semisimple, without compact factor (of strictly positive dimension). Then X is a Riemannian symmetric space, of negative curvature, K is the fixed point set of an involution θ of $\mathfrak{g}$ (a Cartan involution) and we have the familiar decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \ , \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p} \ , \quad [\mathfrak{p},\mathfrak{p}] = \mathfrak{k} \tag{1}$$

where $\mathfrak{k}$ is the Lie algebra of K, $\mathfrak{p}$ the (-1)-eigenspace of θ. (The last relation is an equality since we assumed G to be without compact factors. Otherwise it would be an inclusion.) The restriction of the Killing form $B(\ ,\ )$ to $\mathfrak{k}$ (resp. $\mathfrak{p}$) is negative (resp. positive) definite. The last relation in (1) implies that the differential d on $C(\mathfrak{g},\mathfrak{k};V)$ simplifies to

$$df(x_0,\dots,x_q)=\sum_i(-1)^i f(x_0,\dots,\hat{x}_i,\dots,x_q)\ , \quad (x_i\in\mathfrak{p})\ . \tag{2}$$

Let (ρ,E) be a finite dimensional G-module over L and (σ,H) a unitary representation of G. We now consider the case where $V=H^\infty\otimes E$. The relative Lie algebra complex is then

$$C^\bullet=C^\bullet(\mathfrak{g},\mathfrak{k};H^\infty\otimes E)=(\wedge\mathfrak{p}^*\otimes H^\infty\otimes E)^{\mathfrak{k}}\ . \tag{3}$$

On $\wedge\mathfrak{p}^*$ we put the scalar product defined by the Killing form. We endow E with an "admissible" scalar product, i.e. one with respect to which $\mathfrak{k}$ (resp. $\mathfrak{p}$) is represented by skew-hermitian (resp. hermitian) operators, which is always possible. The tensor product of those scalar products with the given one on H^∞ defines a positive definite scalar product $(\quad)^{\mathfrak{k}}$ on $C^\bullet$. We let $\partial : C^q\to C^{q-1}$ be the adjoint of d and $\Delta=d\partial+\partial d$ the corresponding Laplacian. A form η is harmonic ($\Delta\eta=0$) if and only it is closed and coclosed ($d\eta=\partial\eta=0$).

(B) Let C be a Casimir element. Assume

$$\rho(C)=r\cdot Id\ ,\quad \sigma(C)=s\cdot Id\ , \tag{3}$$

which is in particular the case when ρ and σ are irreducible. By a formula of Kuga

$$\Delta\eta_I=(r-s)\eta_I\quad (\eta=\sum\eta_I\omega^I) \tag{4}$$

([BW],II,2.5). This has two immediate consequences ([BW],II,3.1,3.2):

(i) if $r\neq s$, then $H^\bullet(\mathfrak{g},\mathfrak{k};H^\infty\otimes E)=0$.

Indeed, if η is a cocycle, then $\Delta\eta=d\partial\eta$ and we see that

$$\eta=(r-s)^{-1}\Delta\eta=(r-s)^{-1}d\partial\eta \tag{5}$$

is a coboundary.

(ii) If $r=s$, then every cochain is harmonic, hence closed and coclosed, and we have

$$H^\bullet(\mathfrak{g},\mathfrak{k};H^\infty\otimes E)=C^\bullet(\mathfrak{g},\mathfrak{k};H^\infty\otimes E)=\mathrm{Hom}_{\mathfrak{k}}(\wedge\mathfrak{p},H^\infty\otimes E)\ . \tag{6}$$

We apply this to the case where ρ is the trivial representation, hence $\rho(C)=0$ and E is irreducible. Then, since $r=0$ if and only if E is trivial, we get

$$H^\bullet(\mathfrak{g},\mathfrak{k};E)=\begin{cases}0 & \text{if } E \text{ is non trivial}\\ (\wedge\mathfrak{p}^*)^K & \text{if } E \text{ is trivial}\ .\end{cases} \tag{7}$$

(C) For simplicity of notation, we let $I_G^\bullet=A^\bullet(X;L)^G$ be the algebra of G-invariant forms on X. Each such form is determined by its value on $T(X)_1$, which is invariant under K, and conversely

$$I_G^\bullet=(\wedge^\bullet\mathfrak{p})^K=H^\bullet(\mathfrak{g},\mathfrak{k};L) \tag{8}$$

consists of harmonic forms.

(D) Let G_u be a maximal compact subgroup of the complexification G_C of G which contains K. The quotient $X_u = G_u/K$ is a compact symmetric space, the compact dual of X. We have

$$\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p} \ , \tag{9}$$

hence the isotropy representations of K in $T(X_u)_1$ and $T(X)_1$ are the same, whence

$$A^\bullet(X_u)^G \cong A^\bullet(X)^G \cong I_G^\bullet \ . \tag{10}$$

On the other hand, by averaging, it is seen that

$$H^\bullet(X_u; L) = H^\bullet(A^\bullet(X_u)^G) \cong I_G^\bullet \ . \tag{11}$$

In particular, $A^\bullet(X_u)^G$ consists of harmonic forms. The equality (11) is due to E. Cartan, and was the starting point of E. Cartan's introduction of exterior differential forms into algebraic topology.

References

[BW] A. Borel and N. Wallach, Continuous cohomology, discrete subgroups and representations of reductive groups, 2nd edition, Mathematical Surveys and monographs **67**, AMS 1999.

[Br] K. Brown, Cohomology of groups, GTM **87**, Springer 1982.

§2. The cocompact case.

In this section, G is assumed to be semisimple, linear, connected and Γ to be a cocompact discrete subgroup of G. As before $L = \mathbb{R}, \mathbb{C}$, E is a finite dimensional vector space over L and a G-module.

We shall review here various ways to get information on $H^\bullet(\Gamma; E)$ using : a) differential geometry and curvature assumptions, b) representation theory, c) a combination of both, and finally d) some geometric constructions.

A) The use of curvature · Matsushima's theorem.

2.1. The main object here is a theorem of Matsushima (2.3). We use §1 freely. We consider the case where $E = L$, and omit the mention of the coefficients. Since $I_G^\bullet$ consists of closed (even harmonic) forms, the inclusion $I_G^\bullet \subset A^\bullet(X)$ induces a homomorphism

(1) $$j_\Gamma^\bullet : I_G^\bullet \to H^\bullet(\Gamma \setminus X) = H^\bullet(\Gamma) \ .$$

In view of 1.8(11), it can also be viewed as a homomorphism

(2) $$j_\Gamma^\bullet : H^\bullet(X_u) \to H^\bullet(\Gamma) \ .$$

It is of course defined for any Γ, cocompact or not. However, if it is (as we assume here), there $j_\Gamma^\bullet$ always injective, because in a compact manifold, a non-zero harmonic form is not a coboundary. [If η is harmonic and $\eta = d\sigma$, then

(3) $$(\eta, \eta) = (\eta, d\sigma) = (\partial\eta, \sigma) = 0 \ .]$$

The theorem of Matsushima to be stated here gives a range in which $j_\Gamma^\bullet$ is surjective, hence bijective.

2.2. Recall that the curvature tensor $R(x, y)$ at $\mathfrak{p} = T(X)_1$ is given by

(1) $$R(x, y) = -ad[x, y]|_{\mathfrak{p}} \quad (x, y \in \mathfrak{p}) \ .$$

Fix an orthonormal basis x_i $(1 \leq i \leq n = \dim X)$ of $\mathfrak{p}$. Then the components of the curvature tensor on $T(X)_1$ are

(2) $$R_{ijkl} = B([x_i, x_j], [x_k, x_l]) \ .$$

On $\mathfrak{p} \otimes \mathfrak{p}$, consider for $q \in \mathbb{N}$, the symmetric bilinear form

(3) $$F_{\mathfrak{g}}^q(\xi, \eta) = (A/q) \sum \xi_{ij}\eta_{ij} + \sum R_{ijkl}\xi_{il}\eta_{jk} \ .$$

Here A is a constant contained in (0, 1], defined in term of the restriction of the Killing form to $\mathfrak{k}$ operating on $\mathfrak{p}$. Let

(4) $$m(\mathfrak{g}) = \min\{0, q \text{ such that } F_{\mathfrak{g}}^q \text{ is positive definite}\}$$

to be called the Matsushima constant.

2.3 Theorem (Matsushima)**.** *Let η be a Γ-invariant harmonic q-form on X. If $q \leq m(\mathfrak{g})$, then η is G-invariant. The homomorphisme j^g_Γ is bijective for $q \leq m(\mathfrak{g})$.*

Since every cohomology class of a compact manifold is represented by a harmonic form by Hodge theory, the second assertion follows from the first one and the last remark in 2.1. For details on the proof of the first one, see [M1] or [BW], II.2.3. We just indicate the starting point. Given a Γ-invariant q form $\eta = \sum_I \eta_I \omega^I$, one has to prove that $y \cdot \eta = 0$ for all $y \in \mathfrak{g}$. In view of the relation $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, it suffices to show this for $y \in \mathfrak{p}$. To this effect, Matsushima starts from the expression

(1) $$\Phi(\eta) = \sum_{i,j,I} \int_{\Gamma \backslash X} ([x_i x_j]\eta_I, [x_i, x_j]\eta_I)_x dv_x$$

$(1 \leq i, j \leq n, I \subset I_q)$, where dv_x is the Riemannian volume element at x and $(\ ,\ )_x$ the usual scalar product defined by the invariant metric. By a series of ingenious manipulations, he arrives at the inequality

(2) $$F^q_{\mathfrak{g}}(x_i \cdot \eta_I) \leq 0 \quad (1 \leq i \leq n, I \subset I_q)\ .$$

This is for any q, but, if $q \leq m(\mathfrak{g})$, it implies $x_i \cdot \eta_I = 0$ and the theorem.

Remark. For this theorem to be of any use, one has to have information on $m(\mathfrak{g})$. Since A is rather explicit and easily estimated, the main point is to compute the eigenvalues of the second quadratic form in the right hand side of 2.2(3). This was done in various cases in [B], [M1], [M2] and [KN]. In view of a theorem to be discussed in 2.5, it is particularly interesting to see when $m(\mathfrak{g}) \geq rk_{\mathbb{R}}\mathfrak{g}$. This occurs in finitely many cases, listed in II,8.7 of [BW].

2.4. Some history. Around 1960, E. Calabi considered compact quotients of n-dimensional hyperbolic space, and proved that the hyperbolic structure is locally rigid for $n \geq 3$, using some analogue of Φ in 2.3(1). He did not publish it. A bit later, A. Weil was trying to establish a conjecture of Selberg, according to which a cocompact discrete subgroup of G was locally rigid, provided G was simple, not of dimension three. He saw Calabi's argument in some notes by Kodaira of a seminar lecture by Calabi, and this suggested quickly to him to use an analogue of Φ [W1]. In [W2], he showed that his local rigidity was equivalent to:

(1) $$H^1(\Gamma; \mathfrak{g}) = 0\ ,$$

where Γ acts on $\mathfrak{g}$ by the adjoint representation. Then, following Weil, Matsushima studied similarly $H^1(\Gamma; \mathbb{C})$ in [M1]. It was expected that it would be zero in most cases where $rk_{\mathbb{R}}\mathfrak{g} \geq 2$. Now $I^1_G = 0$ for any semisimple group, so that such a vanishing could be viewed as the surjectivity of j^1_Γ. This led Matsushima to the more general question discussed above.

The vanishing or non-vanishing of the first Betti number of Γ has since been investigated in many ways. Some references will be given in 2.9.

B) Use of representation theory.

2.5. Since $Y = \Gamma \setminus X$ is compact, so is $\Gamma \setminus G$ and $C^\infty(\Gamma \setminus G) \subset L^2(\Gamma \setminus G)$. More precisely

(1) $$C^\infty(\Gamma \setminus G) = L^2(\Gamma \setminus G)^\infty \ ,$$

where, as usual, $L^2(\Gamma \setminus G)^\infty$ denotes the space of smooth vectors in $L^2(\Gamma \setminus G)$, viewed as a unitary G-module. By compactness, it is well known that $L^2(\Gamma \setminus G)$ is a Hilbert direct sum of irreducible unitary G-modules, each type occurring with finite multiplicity $m(\pi, \Gamma)$:

(2) $$L^2(\Gamma \setminus G) = \tilde{\bigoplus_{\pi \in \hat{G}}} m(\pi, \Gamma)\pi \ ,$$

where $\hat{G}$ is the unitary dual of G. Let again (ρ, E) be a complex finite dimensional G-module. We have, as seen before

(3) $$H^\bullet(\Gamma; E) = H^q(\mathfrak{g}, \mathfrak{k}; C^\infty(\Gamma \setminus G) \otimes E) = H^\bullet(\mathfrak{g}, \mathfrak{k}; L^2(\Gamma \setminus G)^\infty \otimes E) \ .$$

Using (2), we can replace the last argument by the space of smooth vectors in an infinite Hilbert direct sum. However the left hand side is finite dimensional and one deduces that

(4) $$H^\bullet(\Gamma; E) = \oplus\ m(\pi, \Gamma, E) \cdot H^\bullet(\mathfrak{g}, \mathfrak{k}; H_\pi^\infty \oplus E) \ ,$$

where the right hand side is a (finite) algebraic direct sum (cf. [BW],VIII,3.2) and, in view of 1.8B), the sum is restricted to the $\pi \in \hat{G}$ such that

(5) $$\pi(C) = \rho(C) \ .$$

In particular

(6) $$H^\bullet(\Gamma; \mathbb{C}) = \bigoplus_{\pi \in \hat{G}, \pi(C)=0} m(\pi, \Gamma) \cdot H^\bullet(\mathfrak{g}, \mathfrak{k}; H_\pi^\infty)$$

a formula due to Y. Matsushima [M3].

Remark. For the reader who wishes to compare (4),(6) and 1.8B) with the corresponding statements in [BW],II, where they are phrased in terms of infinitesimal characters. Recall that the center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$ acts by scalars on an irreducible representation, say (σ, V) and this defines a homomorphism of $\mathcal{Z}(\mathfrak{g})$ into $\mathbb{C}$ called the infinitesimal character of σ, to be denoted χ_σ. Let now (π, H_π^∞) and (ρ, E) be as before. By a lemma of Wigner (cf. [BW]), if $H^\bullet(\mathfrak{g}, \mathfrak{k}; H_\pi^\infty \otimes E) \neq 0$, then $\chi_\pi = \chi_{\rho^*}$, where ρ^* is the contragredient representation to ρ. This of course implies $\pi(C) = \rho(C)$. However, by Wigner's lemma and 1.8B) we have

(∗) Assume that $\mathrm{Hom}_k(\wedge^\bullet \mathfrak{p}, H_\pi^\infty \otimes E) \neq 0$. Then

$$\pi(C) = \rho(C) \Leftrightarrow \chi_\pi = \chi_{\rho^*} \ .$$

2.6. A first problem is then to study $H^\bullet(\mathfrak{g}, \mathfrak{k}; H_\pi^\infty \otimes E)$ for G simple, E irreducible finite dimensional and π unitary irreducible. Using Langlands' classification of irreducible admissible representations, it was shown in [BW], V,3.3 that H^q vanishes if $0 < q < rk_\mathbb{R}\mathfrak{g}$ and π is not trivial, a result also proved independently by G. Zuckerman [Z].

Let us denote by $c(\mathfrak{g})$ the maximum of the q's for which $j^q_\Gamma : I^q_G \to H^q(\Gamma;\mathbb{C})$ is an isomorphism. Then $c(\mathfrak{g}) \geq m(\mathfrak{g})-1$ by definition. We see now that $c(\mathfrak{g}) \geq rk_{\mathbb{R}}\mathfrak{g}-1$. A complete description of the π for which this cohomology is not zero and of the cohomology groups is given for $\mathbf{SO}(n,1)$ and $\mathbf{SU}(n,1)$ in [BW], V and for any simple group in [VZ] (cf. [BW],VI,5.3). This then leads to better estimates for the constant $c(\mathfrak{g})$ just introduced. See [BW],II,10.3.

The second question raised by (5) is to see when $m(\pi,\Gamma) \neq 0$. Usually, given Γ and π, one is interested in whether Γ has a subgroup Γ' of finite index such that $m(\pi,\Gamma') \neq 0$. This is a problem which is of interest to analyze $L^2(\Gamma \setminus G)$, regardless of whether π contributes to cohomology or not. The best studied case is when π belongs to the discrete series (assuming G has one). If π is integrable, the construction of Poincaré series shows that $m(\pi,\Gamma') \neq 0$ for suitable Γ'. If π is not, it can be proved that the limit of $m(\pi,\Gamma')/[\Gamma \cdot \Gamma']$ tends to the formal degree of π, as Γ' runs through a "tower", i.e. a decreasing family of subgroups Γ' whose intersection reduces to the identity, so that, eventually, $m(\pi_1,\Gamma') \neq 0$. If $\pi(C) = 0$, then π contributes to the cohomology in the middle dimension ([BW],II,5.3,5.4).

Not much is known otherwise. See below and [dG-W] for some results.

C) Combination of A) and B).

2.7. As was pointed out earlier, $m(\mathfrak{g})$ is bigger in some cases than the constant $c(\mathfrak{g})$ obtained by representation theoretic methods. The question of combining the two approaches arose. It was noticed about 25 years ago that it was possible to adapt the Matsushima argument to relative Lie algebra cohomology, and this was in fact the starting point of the seminar which led to [BW]. Matsushima's argument may indeed be transcribed to prove that if H is an irreducible non trivial unitary representation of G, then

(1) $$H^q(\mathfrak{g},\mathfrak{k};H^\infty) = 0 \quad \text{for } q \leq m(\mathfrak{g})$$

(cf. [BW],II, §8).

D) Geometric constructions.

2.8. We consider real cohomology. Here again, one is interested in showing the existence of Γ' with non-zero cohomology, in the form of the fundamental class of some subvariety. The general idea is the following. One looks for a semisimple (or reductive) subgroup H of G such that $\Gamma_H := \Gamma \cap H$ is cocompact in H. Since the symmetric space X_H of maximal compact subgroups of H is embedded in X, one is led to a natural map of $\Gamma_H \setminus X_H$ into $\Gamma \setminus X$. One wants then to find Γ' so that a) this map is an embedding onto a closed orientable submanifold and b) this manifold is not homologous to zero in $\Gamma \setminus X$.

This method was initiated by J. Millson, in order to prove that certain compact hyperbolic n-manifolds have a finite covering with non-zero first Betti number [Mi1].

Let $G = \mathbf{SO}(n,1)$ $(n \geq 2)$. Then $K = \mathbf{O}(n)$ and $X = H^n$ is the hyperbolic n-space. We take for Γ the group of units of an indefinite quadratic form F of index $(n,1)$ defined over a real quadratic field L and anisotropic over L. To have a concrete example (but [Mi1] is of course more general), let $L = \mathbb{Q}(\sqrt{2})$

(1) $$F = x_1^2 + \cdots x_n^2 - \sqrt{2}x_{n+1}^2 \ ,$$

and Γ be the group of elements in $\mathbf{SO}(F)$ with coefficients in the ring $\mathfrak{o}_L$ of algebraic integers in L. Let H be the subgroup of G leaving stable the hyperplane Z given by $x_1 = 0$. Then $Z = X_H$ where H is the orthogonal group of the restriction F' of F to Z and Γ_H is the group of units of F' and is cocompact. Let $Y_H = \Gamma_H \setminus X_H$ and $Y = \Gamma \setminus X$. We have a natural map $\tau : Y_H \to Y$. The image is closed, but τ need not be injective because, a priori, the intersection with X_H of an orbit of Γ may consist of several orbits of Γ_H. However, it is shown to be injective here, after some minor adjustments. First, by passing to a subgroup of finite index, we may assume Γ to be torsion-free. Moreover, H is the fixed point set of an involution σ of G, and, again passing to a subgroup of finite index, we may assume Γ invariant under σ. Then a remark of Jaffee shows that τ is injective: Assume that $\gamma \cdot x = y$, where $\gamma \in \Gamma$ and $x, y \in Z$. We want to show that $\gamma \in H$, i.e. ${}^{\sigma}\gamma = \gamma$. Apply σ to the equality $\gamma x = x$. We get

$$\gamma^{\sigma} \cdot x = y = \gamma \cdot x$$

hence $\gamma^{-1} \cdot \gamma^{\sigma} = 1$ since it fixes x and since Γ is torsion free. Going over to a subgroup Γ', on may also arrange that Y_H is orientable. The hard part of the proof is then to show that, again after passage to a suitable subgroup of finite index, Y_H does not separate Y, i.e. $Y - Y_H$ is connected. I shall not try to summarize the rather delicate arithmetic considerations. Then it is clear that one can find a loop C whose intersection with Y_H is transversal and consists of one point. Therefore $[Y_H] \cdot [C] = 1$ and $H_1(X_H) \neq 0$.

2.9. Millson also proved the existence of Γ' with arbitrary large Betti numbers b_q, $1 \leq q < n$. Now for each $q \leq n/2$, there is one representation J_q with non-zero cohomology exactly in dimensions q and $n - q$, (two if $q = n/2$), ([BW],VI,§4). In the setup of B), this shows that every representation J_q occurs in $L^2(\Gamma \setminus G)$ for some Γ.

Millson's theorem confirms in this case a conjecture of W. Thurston, to the effect that any compact hyperbolic n-manifold has a finite covering with non-zero first Betti number.

In the previous argument, an essential role is played by a totally geodesic hyperbolic submanifold of codimension one of $\Gamma \setminus X$. This allows one handle the arithmetically defined subgroup of $\mathbf{SO}(n,1)$ associated to quadratic forms. The most general theorem is due to A. Lubotzky [L] which deduces from the existence of such a submanifold that Γ has a subgroup of finite index mapping onto a free group of prescribed rank.

But, for n odd, there is a quite different family of arithmetic subgroups, attached to skew-hermitian forms over quaternionic spaces, for which such hyperplanes do not exist. Still, a similar theorem could be proved by J.S. Li, J.S. Li and Millson [LM] and M.S. Raghunathan-T.N. Venkataramana [RV], by completely different methods. This takes care of all arithmetic subgroups except for certain arithmetic groups arising from triality forms in the case $n = 7$.

It is relatively easy to deduce a similar theorem for the non-arithmetic cocompact subgroups defined by M. Gromov and I. Piatetski-Shapiro [MP]. However, the general case is still open, as far as I know.

For non-compact hyperbolic manifolds of finite invariant volume this existence theorem is much easier to prove, by making use of the structure of the cusps.

2.10. The results in B) or C) show that $b_1(\Gamma) = 0$ at least for G simple of real rank ≥ 3. But there is a stronger theorem due to D. Kazhdan: if G is simple, non compact, Γ a discrete subgroup of finite covolume, and the trivial representation of G is isolated in $\hat{G}$, (endowed with the so-called Fell topology), then Γ is finitely generated and the commutator subgroup of Γ is of finite index in Γ, hence in particular $b_1(\Gamma) = 0$. It was also shown that Kazhdan's condition is fulfilled if $rk_{\mathbb{R}} G \geq 2$ and even in two cases of real rank 1, when $G = \mathbf{SP}(n,1)$, i.e. G is the unitary group of a hermitian form on quaternionic $(n+1)$-space of index $(n,1)$, or the real rank one form of $\mathbf{F}_4$. There remain then as candidates for $b_1(\Gamma) \neq 0$ cocompact subgroups of $\mathbf{SO}(n,1)$ and $\mathbf{SU}(n,1)$. The case of $\mathbf{SO}(n,1)$ was discussed above. The first examples of groups $\Gamma \subset \mathbf{SU}(n,1)$ with non-zero first Betti number were given by D. Kazdhan, cf. [BW],VIII,§5 for references, and a more general theorem.

The existence of the Kähler form implies that $b_{2i}(\Gamma) \geq 1$ for all $i \leq n$. I do not know whether more information about the Betti numbers of finite coverings of $\Gamma \setminus X$ has been obtained for those Γ's.

2.11. The construction in 2.9 suggests by itself a procedure in more general cases: find two reductive subgroups H, M such that X_H and X_M are transversal and span X, $\Gamma_H \setminus X_H$ and $\Gamma_M \setminus X_M$ embed as closed orientable submanifolds of complementary dimension and try to prove the intersection of their fundamental classes is not zero, all that of course by passing to subgroups of finite index of Γ. For a number of examples, see [MR].

For the first step in 2.9 we used a remark of Jaffee. The same argument works if H is the fixed point set of a finite group of automorphism. More generally, Raghunathan has shown for Γ arithmetic that one can always achieve it by going over to a sufficiently small subgroup of finite index (unpublished).

One can also try to proceed similarly when $\Gamma \setminus X$ is not compact, and Γ is arithmetic, by finding a pair H, M such that $\Gamma_H \setminus X_H$ and $\Gamma_M \setminus X_M$ are also arithmetic quotients, one compact and the other one non-compact. This allows one in particular to show that "generalized modular symbols" define non-zero cohomology classes in many cases.

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§3. A compactification of $\Gamma \setminus X$. First applications.

3.1. We recall first the usual compactification of $\Gamma \setminus X$ when X is the upper-half plane, $\Gamma \subset G = \mathbf{SL}_2(\mathbb{R})$ is not compact, but of finite covolume and (for simplicity) torsion-free (see e.g. [B1],§3). The action of G on X extends continuously to the boundary $Z = \mathbb{R} \cup \{\infty\}$, and the isotropy group of $z \in Z$ is a parabolic subgroup, conjugate to the group of upper triangular matrices P_o of determinant one. A point $z \in Z$ is Γ-cuspidal if its isotropy group Γ_Z is infinite or, equivalently if $N_P \cap \Gamma$ is infinite, where P is the isotropy group of z in G (in which case P is called Γ-cuspidal) and N_P is the unipotent radical of P. The set of cuspidal points is the union of finitely many orbits of Γ. One considers the union X^* of X and the set of cuspidal points, endowed with a suitable topology. The quotient $\Gamma \setminus X^* = S$ is a compact Riemann surface, thus $\Gamma \setminus X$ appears as the complement of a finite set of points in S. This compactification is useful in many ways. However, its fundamental group is different from Γ (it is trivial if S has genus zero, e.g.), so not of much use to study Γ. We want a compactification $\Gamma \setminus \overline{X}$ which has the same homotopy type as $\Gamma \setminus X$ and will serve again as a compact $K(\Gamma, 1)$. This can be achieved here by deleting from S small open discs, one for each cuspidal point. The complement is then a compactification of $\Gamma \setminus X$ which is a compact manifold with a boundary consisting of finitely many circles, one for each Γ-equivalence class of cuspidal points, of which $\Gamma \setminus X$ is a deformation retract. The compactification to be introduced here, called the Borel-Serre compactification, is a generalization for arithmetic groups, which specializes to the one just described when $G = \mathbf{SL}_2(\mathbb{R})$ and Γ is of finite index in $\mathbf{SL}_2(\mathbb{Z})$.

3.2. Before introducing it, I rephrase the previous construction.

Assume first that ∞ is a cuspidal point. Let x, y be the usual coordinates on X. In the construction of X^*, one *common* cuspidal point is added to the half lines $x = c$, $y > t$. Now, let us add one point to *each* such half line. The isotropy group of ∞ is P_o and its unipotent radical N is the group of translations in the x direction. Therefore we are adding at ∞ a copy of an orbit of N, or of N itself.

More invariantly, fix z on the boundary and let P be its isotropy group. Recall the Iwasawa decomposition $G = N_P \cdot A \cdot \mathbf{SO}_2$, where A is the identity component in a split Cartan subgroup, i.e. conjugate to the group of diagonal matrices $\mathrm{diag}(t, t^{-1})$ with $t > 0$. To fix it completely, we assume its Lie algebra to be orthogonal to that of $\mathbf{SO}_2$. This gives a canonical isomorphism $X = N_P \times A$. Write $e(P)$ for $X/A = N_P$. Define $X(P) = X \cup e(P)$ with the obvious topology: $(n_j, a_j) \to n$ if $n_j \to n$ and a_j tends to infinity. Given Γ as before, let $\mathcal{P}_\Gamma$ be the set of Γ-cuspidal subgroups. Then, by definition, the completion $\overline{X}$ of X is the union of X and the $e(P)$'s, or equivalently, of the $X(P)$ glued on X. The group Γ acts freely and properly on it and $\Gamma \setminus \overline{X}$ is the sought for compactification.

3.3. ¿From now on, we assume familiarity with the structure theory of linear algebraic groups over a field (mostly, $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$) and review some facts, mainly to fix notation. For simplicity of notation, unless otherwise stated, an algebraic group defined over a subfield of $\mathbb{R}$ will be denoted by its group of real points. For later use, we intercalate here a decomposition of a connected $\mathbb{Q}$-group. Let H be one, N its unipotent radical and L a Levi $\mathbb{Q}$-subgroup. Then $H = L \cdot N$ (semi-direct). As usual, $X(H)_\mathbb{Q}$ denotes the group of rational characters (morphisms into $\mathbf{GL}_1$) which are defined over $\mathbb{Q}$. The restriction to L defines an isomorphism of $X(H)$ onto $X(L)$. Let S be a maximal $\mathbb{Q}$-split torus of the center $\mathcal{Z}L$ of L and

$A = S^o$ be its identity component (in ordinary topology). The restriction map $X(L)_{\mathbb{Q}} \to X(S)_{\mathbb{Q}} = X(S)$ is injective and identifies $X(L)_{\mathbb{Q}}$ to a subgroup of finite index of $X(S)_{\mathbb{Q}}$. Let

$$M = \bigcap_{\chi \in X(L)_{\mathbb{Q}}} \ker \chi^2 \ . \tag{1}$$

Then

$$L = A \times M \ . \tag{2}$$

[Note that S is the product of A by a product of $\mathbb{Z} \setminus 2\mathbb{Z}$, and, by putting χ^2 in (1) rather than χ we absorb the torsion of S in M. It follows that any compact subgroup of L is contained in M.] Since $H = L \cdot N$, we have

$$H = N \cdot A \cdot M \ , \tag{3}$$

whence also an isomorphism of manifolds $H \cong N \times A \times M$.

3.4. Let G be semisimple $\mathbb{Q}$-group, which, for simplicity, is assumed to be simple over $\mathbb{Q}$. Let P be a parabolic $\mathbb{Q}$-subgroup of G, N_P its unipotent radical. The Levi $\mathbb{Q}$-subgroups of P are the centralizers of the maximal $\mathbb{Q}$-split tori in the radical of P. If $L = \mathcal{Z}S$ is one, we have, by 3.3

$$P = N_P \cdot (A \times M) \ . \tag{1}$$

So far, L, A, M are defined up to conjugacy under $N_P(\mathbb{Q})$. But we need to fix them, at the cost of giving up rationality over $\mathbb{Q}$.

Choose a maximal compact subgroup K of G and let θ be the associated Cartan involution. There is a unique conjugate of L which is stable under θ. From now on use reserve the notation A_P and M_P for the unique conjugates of the groups denoted A and M in (2) which are stable under θ. Then

$$G = N_P \cdot A_P \cdot M_P \cdot K \tag{3}$$

is the Langlands decomposition of G with respect to P (and K). The intersection $P \cap K$ is equal to $M_P \cap K$ and is a maximal compact subgroup of M_P or P. The quotient

$$X_{M_P} = X_P = M_P/(K \cap M_P) \tag{4}$$

is the symmetric space of non-compact type of M_P. The relation $G = P \cdot K$ and (2) show the existence of an isomorphism of manifolds

$$\sigma_P : N_P \times X_P \times A_P \xrightarrow{\sim} X \ . \tag{5}$$

This is the *horospherical decomposition* of X (with respect to K and P). If $x \in X$, then the components of $\sigma_P^{-1}(x)$ with respect to this decomposition are called the horospherical coordinates of x.

Let $\Phi(A_P, P)$ be the set of roots of P with respect to A_P, i.e. the set of characters of A_P in $\mathfrak{n}_P$. It consists of positive integral linear combinations of $\dim A_P$ of them, the set $\Delta(A_P, P)$ of simple roots of P with respect to A_P. The value of $\alpha \in \Phi(A_P, P)$ on $a \in A$ is denoted a^α.

[I am using here a notation and terminology more familiar for real parabolic subgroups in representation theory than in the theory of linear algebraic groups. To relate the two, fix a minimal parabolic $\mathbb{Q}$-subgroup P_o contained in P and a maximal $\mathbb{Q}$-split torus S_o in P_o containing A_P. Let $\Phi(S, G)$ be the system of roots of G with respect to S, Φ^+ the set of positive roots for the ordering defined by P_o and $\Delta(S, G)$ the set of simple roots. Then $\Phi(A_P, P)$ (resp. $\Delta(A_P, P)$) is the set of non-zero restrictions to A_P of elements in Φ^+ (resp. $\Delta(S, G)$.]

3.5. The completion $\overline{X}$ of X.

We now generalize the construction sketched in 3.2. Given P as before, we let $e(P) = N_P \times X_P$. Let $\mathcal{P}_G$ be the set of parabolic $\mathbb{Q}$-subgroups of G. We include G in it, hence $e(G) = X$. By definition, $\overline{X}$ is, set theoretically

(1) $$\overline{X} = \coprod_{P \in \mathcal{P}_G} e(P) \ .$$

We have to define a topology, which we do by means of convergent sequences (see remarks below). Each $e(P)$ carries its natural topology and X is, by definition open in $\overline{X}$. Let $(n, z) \in e(P)$ $(n \in N_P, z \in X_P)$. A sequence $\{x_j\} \in X$ converges to (n, z) if, the horospherical coordinates (n_j, z_j, a_j) of the x_j's (cf. 3.3(4)) satisfy:

(2) $$n_j \to n \ , \ z_j \to z \ , \ a_j^\alpha \to \infty \quad (\alpha \in \Delta(A_P, P)) \ .$$

If $rk_{\mathbb{Q}} G = 1$, this makes $\overline{X}$ a manifold with boundary consisting of the $e(P)$ $(P \neq G)$. In particular the $e(P)$, $(P \neq G)$, are closed. Assume now G to be of higher $\mathbb{Q}$-rank. For $Q \in \mathcal{P}_G$, we have by definition, in analogy with (1)

(3) $$\overline{e(Q)} = \coprod_{P \subset Q} e(P) \ ,$$

and the convergence of a sequence in $e(Q)$ to a point of $e(P)$ is defined similarly. In fact, I could say is the same if I had started from any group G such that $G(\mathbb{C})$ is connected rather than from a $\mathbb{Q}$-simple one. Since I did not, let me sketch it. Let $P \in \mathcal{P}_Q$. Then $P' = P \cap M_Q$ is a parabolic subgroup of M_Q and $P = P' \cdot N_Q$, (semi-direct). In particular, P and P' have the same symmetric space: $X_P = X_{P'}$. We can write

(4) $$e(Q) = N_Q \times X_Q \quad e(P) = N_Q \times N_{P'} \times X_{P'} = N_Q \times e(P') \ .$$

The definition of the convergence of a sequence in X_Q to a point of $e(P')$ is the same as before and, by (4), gives the convergence of a sequence in $e(Q)$ to a point of $e(P)$. This completes the definition of a topology in which $e(P) \subset \overline{e(Q)}$ if and only if $P \subset Q$. In particular

$$\overline{e(P)} \cap \overline{e(Q)} = \overline{e(R)} \text{ if } R = P \cap Q \text{ is parabolic, and is empty otherwise .}$$

Let

$$X(P) = \coprod_{Q \supset P} e(Q) \ . \tag{5}$$

It is an open neighborhood of $e(P)$ [open, since its complement is clearly closed by the definition of the topology.] We want to give at least some heuristic argument to indicate that it has a natural structure of manifold with corners. (See [BS,Appendix] for more details about that notion.) A model of a corner is the positive quadrant C^m in $\mathbb{R}^m$ of elements on which the coordinates are ≥ 0. A manifold with corners has a chart of open subsets, each of which is a product $\mathbb{R}^{n-m} \times C^m$. Topologically, it is a manifold with boundary, but the point here is that $\overline{X}$ is in a natural way a differential (even real analytic) manifold with corners, where the maximum of m is the $\mathbb{Q}$-rank of G. Fix P and let $m = \dim A_P$. In order to visualize the corner structure, identify A to the interior of C^m by mapping a to the point with coordinates $a^{-\alpha}$ $(\alpha \in \Delta(A_P, P))$ and write $\overline{A_P}$ for its closure C^m. Then $X(P)$ may be identified to $e(P) \times \overline{A_P}$. To see this, note first that if $Q \supset P$, then $A_Q \subset A_P$. Let $A_{P,Q}$ be the subgroup of A_P with Lie algebra orthogonal to that of A_Q. Using 3.3(5) we see that we can write

$$e(Q) = e(P) \times A_{P,Q} \tag{6}$$

(here non-canonically, but this can be made more intrinsic). Then one can see that $\overline{A_P}$ is the union of A_P and of the $A_{P,Q}$, $(Q \supset P)$ which then yields (5). Conjugation by $g \in G(Q)$ leaves $\mathcal{P}_G$ stable, and one sees easily that the action of $G(\mathbb{Q})$ on X extends continuously to $\overline{X}$.

Remarks. I have of course slurred over many technical points. For a complete treatment, see [BS]. Here, I have also used the presentation of [BJ].

The topology has been defined just by specifying convergent sequences. In order to be sure this yields a topological space, certain axioms have to be verified, but this offers no difficulty. More serious *a priori* is the fact that the choice of K underlies the whole construction. However it can be phrased differently, so that $\overline{X}$ may be shown to be independent of that choice, as is done in [BS].

3.6. The boundary $\partial\overline{X}$.

We have $e(P) \subset \overline{e(Q)}$ if and only if $P \subset Q$. This reminds one of the Tits building $\mathcal{T}_G$ of proper parabolic $\mathbb{Q}$-subgroups of G. In fact, $\partial\overline{X}$ has the same homotopy type as $\mathcal{T}_G$ ([BS],8.4.2). Recall that $\mathcal{T}_G$ is a simplicial complex, the apartments of which are $(l-1)$-spheres where $l = rk_{\mathbb{Q}}G$. As a consequence, $\partial\overline{X}$ has the homotopy type of a "bouquet" of $(l-1)$-spheres, i.e. a collection of $(l-1)$-spheres with one common point.

For a connected space Y and a ring of coefficients L, Let $\tilde{H}_i(Y;L)$ be the reduced i-th singular homology group, i.e. the kernel of the homomorphism $H_i(Y;L) \to H_i(pt;L)$, hence

$$\tilde{H}_0(Y;L) = 0 \ , \ \tilde{H}_i(X;L) = H_i(X;L) \quad (i \neq 0) \ . \tag{1}$$

Then we have

$$H_i(\partial\overline{X};\mathbb{Z}) = 0 \ (i \neq l-1) \ , \ H_{l-1}(\partial\overline{X};\mathbb{Z}) = I \tag{2}$$

where I is a free module, which is obviously a $G(\mathbb{Q})$-module.

3.7. Let H be a $\mathbb{Q}$-group. A subgroup Γ of $H(\mathbb{Q})$ is arithmetic if, given an embedding $\sigma : H \subset \mathbf{GL}_N$ defined over $\mathbb{Q}$, the group $\sigma(\Gamma)$ is commensurable with $\sigma(H) \cap \mathbf{GL}_N(\mathbb{Z})$, a condition which is independent of the $\mathbb{Q}$-embedding. The intersection of Γ with any $\mathbb{Q}$-subgroup of H is arithmetic. Assume $H(\mathbb{C})$ to be connected. Then Γ has finite covolume if and only if $X(H)_{\mathbb{Q}} = \{1\}$. If H is reductive semisimple, then Γ is cocompact if and only if $rk_{\mathbb{Q}} H = 0$, or also if and only if Γ consists of semisimple elements (cf. [B] for proofs and references).

Let now G be as before and Γ an arithmetic subgroup. If $P \in \mathcal{P}_G$ we let $\Gamma_P = \Gamma \cap P$, $\Gamma_{N_P} = \Gamma \cap N_P$. Let $\pi_P : P \to L = P/N_P$ be the canonical projection. It maps as any Levi subgroup of P isomorphically onto L. Then we define

$$\Gamma_L = \pi_P(\Gamma_P) \cong \Gamma_P/\Gamma_{N_P} \ . \tag{1}$$

[Note that π_P identifies M_P to the $\mathbb{Q}$-subgroup defined by 3.3(1), but even if M_P is defined over $\mathbb{Q}$, π_P maps $\Gamma \cap M_P$ onto a subgroup of finite index of Γ_L, which may be proper. Also, the intersection of Γ with a Levi $\mathbb{Q}$-subgroup depends on the choice of the Levi subgroup.]

It follows from reduction theory that Γ operates properly on $\overline{X}$ and that $\Gamma \backslash \overline{X}$ is compact [BS]. The stability group of $e(P)$ is Γ_P. Let $\pi : \overline{X} \to \Gamma \backslash \overline{X}$ be the natural projection. We let $e'(P) = \pi(e(P))$. Then $e'(P) = \Gamma_P \backslash e(P)$. The quotient $\mathcal{P}_G/\Gamma$ is finite and

$$\Gamma \backslash \overline{X} = \bigcup_{P \in \mathcal{P}_G/\Gamma} e'(P) \ . \tag{2}$$

If Γ is torsion-free, it acts freely on $\overline{X}$, hence $\overline{Y} = \Gamma \backslash \overline{X}$ is a compact $K(\Gamma, 1)$. The projection π is a local diffeomorphism, therefore Y inherits the structure of manifold with corners of $\overline{X}$.

3.8. Relations with reduction theory.

We go back to the horospherical decomposition 3.4(5), which in the notation of 3.5, can be written as $e(P) \times A_P$. For $t > 0$, let

$$A_{P,t} = \{a \in A | a^\alpha \geq t, (\alpha \in \Delta(A_P, P)) \ .$$

A *Siegel set* $\mathfrak{S}_{\omega,t}$ (with respect to P) in X is the image under σ_P of $\omega \times A_{P,t}$, where ω is relatively compact in $e(P)$. Its inverse image in G is a Siegel set in G. Let $z \in e(P)$. Then the Siegel sets $\mathfrak{S}_{\omega,t}$, where ω runs through a fundamental set of neighborhoods of z in $e(P)$ and $t \to \infty$ are the intersections of X with a fundamental system of neighborhoods of z in $\overline{X}$. Their main properties are: a) they have finite invariant volume, b) $\Gamma \backslash X$ is covered by the projections of finitely many Siegel sets, which may be assumed to be associated to minimal parabolic $\mathbb{Q}$-subgroups, c) given $\mathfrak{S}_{\omega,t}$,

$$\{\gamma \in \Gamma | \gamma(\mathfrak{S}_{\omega,t}) \cap \mathfrak{S}_{\omega,t} \neq \emptyset\}$$

is finite (cf. [B]). These properties, or in fact slightly stronger ones, are at the basis of the proofs of the properties of $\overline{X}$ and $\Gamma \backslash \overline{X}$.

3.9. Some properties of Γ.

We give here some consequences of the existence of $\Gamma \setminus \overline{X}$.

(i) Γ is finitely presented. Its fine subgroups form finitely many conjugacy classes.

Since $\Gamma \setminus \overline{X}$ is compact there exists a compact subset $C \subset \overline{X}$ mapping onto $\Gamma \setminus \overline{X}$ under π. The group Γ operates properly, therefore

$$\Gamma_C = \{\gamma \in \Gamma | \gamma(C) \cap C \neq \phi\}$$

is finite. Let now F be a finite subgroup of Γ. It has a fixed point on X, hence is conjugate to a subgroup F' leaving a point of C fixed. Therefore F' belongs to Γ_C. The second assertion follows. The first one is a familiar consequence of the fact that $\Gamma \setminus \overline{X}$ is compact, connected and locally simply connected.

(ii) Let Γ be torsion-free. Then its cohomological dimension $cd(\Gamma)$ is finite, equal to $n - l$ ($n = \dim X$, $\rho = rk_{\mathbb{Q}} G$).

[The cohomological dimension of a discrete group Γ is the maximum of the i's for which there exists a Γ-module M such that $H^i(\Gamma; M) \neq 0$, it may be infinite.] Since $\Gamma \setminus \overline{X}$ is a compact $K(\Gamma, 1)$, and a manifold with boundary, we have at first $cd(\Gamma) \leq n = \dim X$. To bring it down to $n - l$ requires 3.6(2), some more homological algebra and sheaf theory, (cf. [BS],§11).

(iii) The group Γ, being linear, finitely generated, always has torsion-free subgroups of finite index. It can be shown that their cohomological dimensions are the same. By definition, the common value of those is the virtual cohomological dimension $vcd(\Gamma)$ of Γ. Thus $vcd(\Gamma) = n - l$ for any arithmetic subgroup of G.

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§4. Generalization of a theorem of Matsushima. Application.

If Γ is cocompact, 2.3 asserts the existence of a computable constant $m(\mathfrak{g})$ such that the natural homomorphism

$$j^\bullet_\Gamma : I^\bullet_G \to H^\bullet(\Gamma;\mathbb{C})$$

is an isomorphism in degrees $\leq m(\mathfrak{g})$. We want to establish a similar theorem (with possibly another constant) when Γ is arithmetic, not cocompact. First we review some facts about L^2-forms.

4.1. L^2-forms.

Let (M,g) be a connected oriented smooth Riemannian m-manifold. We let $A^q(M)$ be the space of complex valued q-forms on M. The metric defines a scalar product on the exterior powers of the tangent or cotangent spaces at every point and we let

$$(1) \qquad (\eta,\sigma) = \int_M (\eta_x,\sigma_x)dv_x \ .$$

It may be infinite. Let

$$(2) \qquad A^q_2 = \{\eta \in A^q(M), (\eta,\eta) < \infty\}$$

be the space of square integrable q-forms and $A^\bullet_2(M)$ their direct sum. Let $* : A^q(M) \xrightarrow{\sim} A^{m-q}(M)$ be the usual star operator. Set

$$(3) \qquad \partial = (-1)^{m(q+1)+1} * \circ d \circ * \ , \quad \Delta = \partial d + d\partial \ .$$

We can also write the right hand side of (1) as $\int_M \eta \wedge (*\sigma)$.

A form $\eta \in A^q_2(M)$ is harmonic if $d\eta = \partial\eta = 0$, which implies $\Delta\eta = 0$. The space of harmonic square integrable q-forms is denoted $\mathcal{H}^q_2(M)$. By a theorem of Kodaira (cf. [B], §2 for references), if $\eta \in A^q_2$ and $d\eta = 0$, there exists $H\eta \in \mathcal{H}^q_2(M)$ and $\sigma \in A^{q-1}(M)$ such that $\eta = H\eta + d\sigma$, (but σ is not necessarily L^2).

Assume now that (M,g) is complete (our main case of interest). Then the Stokes formula

$$(4) \qquad (\sigma, d\eta) = (\partial\sigma,\eta) \quad (\sigma \in A^q_2, \eta \in A^{q-1}_2) \ .$$

is valid (at least) if σ, $\partial\sigma$, η, $d\eta$ are all in L^2. Moreover, η is harmonic if and only if $\Delta\eta = 0$.

It may well be that a L^2-harmonic q-form η is cohomologous to zero. However, and this will be decisive for us, it cannot be the coboundary of a L^2-form unless it is zero. In view of (4), the proof is the same as in the compact case (2.1).

References for all these statements are given in [B], §2.

4.2. We now go back to our situation: G is a $\mathbb{Q}$-simple $\mathbb{Q}$-group of strictly positive $\mathbb{Q}$-rank l, and Γ is an arithmetic subgroup. Let $\sigma,\eta \in A^q(\Gamma\setminus X)$, identified to $C^q(\mathfrak{g},\mathfrak{k};C^\infty(\Gamma\setminus G))$ as before. Write them

$$\sigma = \sum \sigma_I \cdot \omega^I \quad \eta = \sum \eta_I \cdot \omega^I \ .$$

Then

$$(\sigma, \eta) = \sum_I (\sigma_I, \eta_I) = \sum_I \int_{\Gamma\backslash G} \sigma_{I,x} \cdot \overline{\eta}_{I,x} dx \ .$$

In particular, if $\eta \in I_G^q$, then the coefficients η_I are constant, hence I_G^q is square integrable, (and harmonic): $I_G^q \subset \mathcal{H}_2^q(\Gamma \setminus X)$, $(q \in \mathbb{N})$. The generalization of 2.3 to the present case is divided into two parts:

a) the inclusion $I_G^q \subset \mathcal{H}_2^q$ is an isomorphism for $q \leq m(\mathfrak{g})$;
b) there exists a computable constant $c(\mathfrak{g})$ such that

j_Γ^q is injective and the natural map $\mathcal{H}_2^q(\Gamma \backslash X) \to H^q(\Gamma \backslash X)$ is surjective for $q \leq c(\mathfrak{g})$.

It then follows that j_Γ^q is an isomorphism for $q \leq \min(m(\mathfrak{g}), c(\mathfrak{g}))$. Part a) is obtained by a relatively simple adaptation of Matsushima's argument (see 4.3). In the cocompact case, that was sufficient since part b) was a given. Here it is not so anymore, b) requires some new arguments and the use of $\Gamma \setminus \overline{X}$. This is described in 4.4, 4.5.

4.3. Proposition. Let $\eta \in \mathcal{H}_2^q(\Gamma \setminus X)$. If $p \leq m(\mathfrak{g})$, then $\eta \in I_G^q$.

Matsushima's argument starts from

(1) $$\Phi(\eta) = \sum_{i,j,I} ([x_i, x_j]\eta_I, [x_i, x_j]\eta_I)$$

and, after a certain number of transformations, arrives at

(2) $$F_{\mathfrak{g}}^p(x_i \cdot \eta_I) \leq 0 \ .$$

A first, this cannot be carried out here, since these manipulations use Stokes formula, and feature the Lie derivatives $x_i \eta_I$, which, a priori, are not L^2.

To obviate to that, a trick is used which should be familiar from the courses on automorphic forms or representation theory: convolution with a K-invariant element of $C_c^\infty(G)$.

Recall that if $f \in L^2(\Gamma \setminus G)$ and $\alpha \in C_c^\infty(G)$, then the convolution $f * \alpha$, defined by

(3) $$(f * \alpha)(x) = \int_G f(x \cdot y^{-1}) \cdot \alpha(y) \ dy$$

is smooth, L^2 as well as all its derivatives Df $(D \in U(\mathfrak{g}))$. Assume α to be K-invariant and let

(4) $$\eta_\alpha = \eta * \alpha = \sum (\eta_I * \alpha)\omega^I \ .$$

Then all the operations carried out in Matsushima's proof remain valid and show that $\eta_\alpha \in I_G^q$ if $q \leq m(\mathfrak{g})$. Take now a Dirac sequence $\{\alpha_j\}$ and let $\eta_j = \eta * \alpha_j$. Then $\eta_j \to \eta$ in $L^2(\Gamma \setminus G)$. Since I_G^q is finite dimensional, it is closed in $L^2(\Gamma \setminus G)$, and we get $\eta \in I_G^q$.

4.4. The invariant metric on a Siegel set.

Let ds^2_X be the invariant metric on X, normalized by requiring it to be equal to the Killing form on $\mathfrak{p}$, to fix the ideas. I want to describe it in horospherical coordinates, i.e. to give $\sigma_P^*(ds^2_X)$. This is not strictly needed here, since detailed proofs are not given, but it is worth considering, since it underlies a number of proofs of results to be stated later.

We have the decomposition

$$(1) \qquad \mathfrak{n}_P = \bigoplus_{\beta \in \Phi(A_P,P)} \mathfrak{n}_\beta \quad (\mathfrak{n}_\beta = \{x \in \mathfrak{n}_P \ , \ \mathrm{Ad}\ a(x) = a^\beta \cdot x \ (a \in A_P)\} \ .$$

Let h_β be the restriction of $-B(x, \theta y)$ to $\mathfrak{n}_\beta$. It is a positive definite scalar product. Note that M_P acts on $\mathfrak{n}_\beta$ be the adjoint representation and $K \cap M_P$ acts isometrically, so that, up to isometry, the action by $m \in M_P$ depends only on its image z in X_P under the natural projection. Then, for $y = (n, z, a)$, we have

$$(2) \qquad (\sigma_P^* ds^2_X) T(Y)_y = \bigoplus_\beta 2^{-1} \cdot a^{-2\beta} \cdot \mathrm{Ad}\ m^{-1}(h_\beta)_n \oplus (ds^2_{X_P})_z \oplus (ds^2_{A_P})_a \ .$$

where $m \in M_P$ is in the inverse image of z. Here $ds^2_{X_P}$ is the invariant metric on X_P, suitably normalized, and $ds^2_{A_P}$ is the A-invariant metric on A_P. If we use the a^α $(\alpha \in \Delta(A_P, P))$ as coordinates, it can be written

$$(3) \qquad ds^2_{A_P} = \bigoplus_{\alpha,\beta \in \Delta(A_P,P)} c_{\alpha\beta} \frac{d\alpha}{\alpha} \cdot \frac{d\beta}{\beta} \ ,$$

where the $c_{\alpha\beta}$ are constant and define a positive definite quadratic form.

The invariant volume element dv_y is, up to some constant

$$(4) \qquad dv_y = a^{-2\rho} \cdot dv_N \wedge dv_X \wedge dv_A \ ,$$

where

$$(5) \qquad 2\rho = \sum_\beta \beta \cdot \dim \mathfrak{n}_\beta \ .$$

The relation (2) is an immediate consequence of the following formula, which describes the action of P on X in horospherical coordinates, i.e. if l_p denotes left translation by p on X, the composition $\sigma_P^{-1} \circ l_p \circ \sigma_P$, namely:

Let y be as before and $p = (n', m', a')$, where $n' \in N_P$, $m' \in M_P$, $a' \in A_P$. Then

$$(6) \qquad p \cdot y = (n' \cdot^{m' \cdot a'} n, m' \cdot z, a' \cdot a) \ .$$

If we consider the restriction of ds^2_X to a Siegel set $\mathfrak{S}_{\omega,t}$ (see 3.7) then z varies in a bounded set, so that, up to quasi isometry, Ad z may be neglected in (2). What is important here is that the metric on X_P goes without significant change to X_P, viewed as a subspace of $e(P) \subset \partial\overline{X}$, whereas, because of the factors $a^{-2\beta}$. The metric on N_P tends to zero.

These computations are all carried out in [B] §4. The formulas differ from those above by the order of the factors and some signs, due to the fact that, in [B], Γ operates on the right and K on the left.

4.5. There are so far two ways to handle b) in 4.2, one in [B], outlined below and one due to S. Zucker, to be discussed in §5.

We are interested in putting some conditions on the growth of the norm of differential forms at x when approaches $\partial\overline{X}$. The form $\eta = \sum \eta_I \omega^I$ is said to be of logarithmic growth if for every parabolic $\mathbb{Q}$-subgroup we can find Siegel sets $\mathfrak{S}_{\omega,t}$ such that

$$|\eta_I(z,a)| \le P(\log a^{\alpha_1}, \dots, \log a^{\alpha_s})$$

where $z \in \omega$, $\Delta(A_P, P) = \{\alpha_1, \dots, \alpha_s\}$, and P is a polynomial in $s = \dim A_P$ variables with real coefficients.

Since an element in I_G^q has constant coefficients, it is clearly of logarithmic growth for any q.

If the metric is described in some local coordinates by the matrix (g_{ij}), the scalar product on the cotangent bundle is given by the inverse matrix (g^{ij}) and the norm of differential forms involves products of g^{ij}. From (2), we see that for some coordinates this will bring in coefficients $a^{2\beta}$, which tend to infinity as one approaches the boundary in a Siegel set. On the other hand, the volume element has the factor $a^{-2\rho}$ which tends to zero. Altogether, if d_o is the dominant root, it is then not difficult to prove that $\eta \in C^q$ is square integrable if

$$\rho - q \cdot d_o > 0 \ ,$$

so we can define $c(\mathfrak{g})$ as the maximum of the q's for which $\rho > q \cdot d_o$. It can be estimated ([B],9.1). In particular, if $G = \mathbf{SL}_{n+1}$, then $c(\mathfrak{g})$ is equal to the greatest integer which is strictly smaller that $[n/2]$. This and a remark above take care of (i) and (ii) in the following:

Theorem. *Let $C^\bullet$ be the subcomplex of $A^\bullet(\Gamma \setminus X)$ consisting of forms which, together with their exterior derivatives, have logarithmitic growth near the boundary of $\overline{X}$. Then*

(i) *For $q \le c(\mathfrak{g})$, C^q consists of L^2-forms.*
(ii) *$I_G^\bullet \subset C^\bullet$.*
(iii) *The inclusion $C^\bullet \to A^\bullet(\Gamma \setminus X)$ induces an isomorphism in cohomology.*

Assuming it, we first prove (b) in 4.1.

Let $\eta \in I_G^q$. If it is cohomologous to zero, then it is already so in $C^\bullet$ by (iii) hence is zero by the last remark in 4.1. By (iii) an element of $H^q(\Gamma \setminus X)$ has a representative cocycle in C^q, hence in L^2. By Kodaira's theorem (see 4.1), it has also a representative in $\mathcal{H}_2^q(\Gamma \setminus X)$, whence the second assertion of b).

Assume now $q \le \min(c(\mathfrak{g}), m(\mathfrak{g}))$. Then $I_G^q = \mathcal{H}_2^q(\Gamma \setminus X)$ by 4.2, and it follows that j_Γ^q is an isomorphism for $q \le \min(m(\mathfrak{g}), c(\mathfrak{g}))$.

We still have to explain how (iii) is established. To this effect, sheaf theory is used on $\Gamma \setminus \overline{X}$. (See e.g. [Br] or [G] for the facts used here).

Consider on $\Gamma \setminus \overline{X}$ the presheaf $\mathcal{C}^\bullet$ which associates to the open subset U the forms of $C^\bullet$ defined on $U \cap (\Gamma \setminus X)$. It is clearly a sheaf and, by construction

$$\mathcal{C}^\bullet(\Gamma \setminus \overline{X}) = C^\bullet \ .$$

The main point is then to show that $\mathcal{C}^\bullet$ is a fine resolution of $\mathbb{R}$ on $\Gamma \setminus \overline{X}$. If $\eta \in \mathcal{C}^q(U)$ and $\alpha \in C_c^\infty(U)$, then α and $d\alpha$ have bounded coefficients, and it is readily seen that $\alpha\eta \in \mathcal{C}^q(U)$. It follows that $\mathcal{C}^\bullet$ is fine. Let $\mathcal{H}^\bullet\mathcal{C}^\bullet$ be its derived sheaf. Since $\mathcal{C}^\bullet(U)$ contains the constants, it is clear that $\mathcal{H}^0\mathcal{C}^\bullet = \mathbb{R} \times (\Gamma \setminus \overline{X})$. There remains to show that $\mathcal{H}^q\mathcal{C}^\bullet = 0$ for $q > 0$. If U is relatively compact and a ball in $\Gamma \setminus X$, then $\mathcal{C}^\bullet(U) = A^\bullet(U)$ and this follows from the Poincaré lemma. If now U is a neighborhood of a point on $\partial\overline{X}$, we may assume that $U \cap \Gamma \setminus X$ is a Siegel set $\mathfrak{S}_{\omega,t}$, where ω is a ball, so that $\mathfrak{S}_{\omega,t}$ is diffeomorphic to euclidean space. One has then to make sure that the proof of the Poincaré lemma can be arranged so as to be valid for elements of $\mathcal{C}^q(U)$, (cf. [B],§7).

The same argument, only simpler, shows that the direct image $i_*\mathcal{A}^\bullet$ of the sheaf $\mathcal{A}^\bullet$ of germs of differential forms on $\Gamma \setminus X$, with respect to the inclusion $i : \Gamma \setminus X \to \Gamma \setminus \overline{X}$, is a fine resolution of $\mathbb{R}$ on $\Gamma \setminus \overline{X}$. Then (iii) follows from the uniqueness theorem of sheaf theory.

4.6. Stable cohomology.

For a given group, the range of degrees for which the theorem holds is small. However, it grows with the $\mathbb{Q}$-rank, and this leads to "stable cohomology". I'll describe the simplest example, the groups $\mathbf{SL}_n(\mathbb{Z})$ as $n \to \infty$. Let $G_n = \mathbf{SL}_n(\mathbb{R})$ and $I_n^\bullet = I_G^\bullet$ for $G = G_n$. For $m > n$, the consider the obvious embedding $G_n \to G_m$. The restriction gives a map $I_m^\bullet \to I_n^\bullet$. These spaces and maps are known and one sees that

$$I_\infty^\bullet = \varprojlim I_n^\bullet = \wedge\{x_5, x_9, \dots\}$$

is a exterior algebra over generators x_j of degrees $j = 4i + 1$, $(i = 1, \dots)$. Let $\mathbf{SL}\ \mathbb{Z}$ be the inductive limit of the $\mathbf{SL}_n\mathbb{Z}$. Then $I_\infty^\bullet$ is the real cohomology of $\mathbf{SL}\ \mathbb{Z}$. Given q, $H^q(\mathbf{SL}_n\ \mathbb{Z};\mathbb{R})$ is isomorphic to $I_\infty^\bullet$ up to q, for all n sufficiently big (e.g. $n > 4q$). In that sense, the real cohomology of $\mathbf{SL}_n\ \mathbb{Z}$ is "stable".

There are similar results for $\mathbf{SL}_n\mathfrak{o}_k$, where $\mathfrak{o}_k$ is the ring of integers of a number field k, which are of interest in algebraic K-theory, and for other series of classical arithmetic groups [B;§§11,12].

4.7. A theorem on rational cohomology.

The compact dual $X_{n,u}$ of $X_n = \mathbf{SL}_n(\mathbb{R})/\mathbf{SO}_n$ is the space $\mathbf{SU}_n/\mathbf{SO}_n$ and we have seen in 1.8 that its real cohomology may also be identified with $I_n^\bullet$. Fix q and let n be big enough. Then this isomorphism and the theorem provide an isomorphism

$$\mu_q : H^q(X_{n,u};\mathbb{R}) \xrightarrow{\sim} H^q(\mathbf{SL}_n\mathbb{Z};\mathbb{R})\ .$$

The two sides are cohomology of spaces and have therefore rational structures, and one may ask what is the behavior of μ_q with respect to those. To do this, one restricts oneself to spaces of "primitive elements". If X is a space, L a field, then the space $P^q(X;L)$ of primitive elements in $H^q(X;L)$ is the quotient of $H^q(X;L)$ by the subspace spanned by cup products of elements of strictly smaller dimensions. Of course, $P^m(X;\mathbb{R}) = P^m(X;\mathbb{Q}) \otimes \mathbb{R}$. In our case, P^q is one-dimensional for $q = 4m + 1$, and zero otherwise. Via the isomorphisms with I_n^q we have therefore a canonical isomorphism

$$\nu_m : P^{4m+1}(X_{n,u};\mathbb{Q}) \otimes \mathbb{R} \xrightarrow{\sim} P^{4m+1}(\mathbf{SL}_n\mathbb{Z};\mathbb{Q}) \otimes \mathbb{R}\ ,\ (n \gg 0)\ .$$

Fix non-zero elements u_m and v_m in $P^{4m+1}(X_{n,u};\mathbb{Q})$ and $P^{4m+1}(\mathbf{SL}_n\mathbb{Z};\mathbb{Q})$. There exists then $R_m \in \mathbb{R}^*$ such that

$$\nu_m(u_m) = R_m \cdot v_m \ . \tag{1}$$

R_m depends on the choice of u_m and v_m up to multiplication by a non-zero rational number, therefore its class $[R_m]$ in $\mathbb{R}^*/\mathbb{Q}^*$ is well-defined. It can be proved that

$$[R_m] = [\zeta(2m+1) \cdot \pi^{-(2m+1)}] \ , \tag{2}$$

where ζ denotes Riemann's zeta-function. $[R_m]$ is an example of a higher regulator in algebraic K-theory. Here, too, there are generalizations to algebraic integers of number fields (cf. [B1],[B2]).

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§5. L^2-cohomology of arithmetic groups.

In this section, we first introduce L^2-cohomology of a Riemannian manifold, and then give some indications on a theorem of S. Zucker [Z] on the L^2-cohomology of $\Gamma \setminus X$. It gives another, more direct, approach to the part b) of the proof of the theorem in 4.2. In 5.3 we relate the image of L^2-cohomology into $H^\bullet(\Gamma; E)$ to the relative Lie algebra cohomology with respect to $L^2(\Gamma \setminus G)^\infty \otimes E$.

5.1. Let (M, g) be a Riemannian manifold. Let

$$A^q_{(2)}(M) = \{\eta \in A^q(M)\ ,\ \eta\ ,\ d\eta \in A^\bullet_2(M)\}\ .$$

and $A^\bullet_{(2)}(M)$ be the direct sum of the $A^q_{(2)}(M)$. It is a subcomplex and, by definition, the L^2-cohomology of M is

$$H^\bullet_2(M) = H^\bullet(A^\bullet_{(2)}(M), d)\ . \tag{1}$$

Here the forms are either real or complex. In the latter case, it is convenient to extend the definition to forms with values in a flat hermitian vector bundle E over M. The scalar product defined on $\wedge^\bullet T^*(M)$ by g and the scalar product on E define at each point a scalar product on $A^\bullet(M; E) = A^\bullet(M) \otimes E$, whence also an L^2-norm. Then $H^\bullet_2(M; E)$ is defined similarly.

The concept of L^2-cohomology became explicit in the late seventies, though it has its origin in earlier work of K. Kodaira and M. Gaffney. It became important notably because of the, at first conjectural, connections with intersection cohomology.

There are also Hilbert space definitions of that notion. For instance, let $L^q_{(2)}(M; E)$ be the space of q-forms with values in E and measurable coefficients which are L^2, together with their exterior differentials, in the sense of distribution. Then d extends to an operator $\overline{d}$ on $L^\bullet_{(2)}(M; E)$ and the cohomology of this complex is also a candidate for L^2-cohomology. Fortunately, the natural map of $A^\bullet_{(2)}$ into $L^\bullet_{(2)}$ is an isomorphism in cohomology (cf. [C],§8).

This cohomology can be infinite dimensional. If $H^q_2(M; E)$ is finite dimensional, then it is equal to the space $\mathcal{H}^q_2(M; E)$ of L^2-harmonic forms.

5.2. We now come back to the case where $M = \Gamma \setminus X$.

Zucker's theorem states the existence of a constant $z(\mathfrak{g})$ such that the natural map $H^q_2(\Gamma \setminus X) \to H^q(\Gamma \setminus X)$ is an isomorphism for $q \leq z(\mathfrak{g})$.

Here again, the constant $z(\mathfrak{g})$ can be estimated in terms of roots and weights. There is a variant of the theorem for E-valued forms, but I shall limit myself here to complex valued forms. Assume this, Then, since $H^q(\Gamma)$ is finite dimensional, a previous remark implies that $\mathcal{H}^q_2(\Gamma \setminus X) \to H^q(\Gamma \setminus X)$ is an isomorphism for $q \leq z(\mathfrak{g})$. On the other hand, 4.3 implies that $I^q_G \to \mathcal{H}^q_2(\Gamma \setminus X)$ is an isomorphism for $q \leq m(\mathfrak{g})$, hence γ^q_Γ is an isomorphism for $q \leq \min(m(\mathfrak{g}), z(\mathfrak{g}))$.

To prove Zucker's theorem on might first try an approach similar to that of 4.5. One defines a sheaf $\mathcal{A}^\bullet_{(2)}$ on $\Gamma \setminus \overline{X}$ such that $\mathcal{A}^\bullet_{(2)}(U)$ is $A^\bullet_{(2)}(U \cap (\Gamma \setminus X))$. The theorem would be established if it could be shown that $A^\bullet_{(2)}$ is fine and is a resolution of $\mathbb{C}$ at least up to some degree. However, $\mathcal{A}^\bullet_{(2)}$ is *not* fine.

To see this, consider a neighborhood U of a point on $\partial\overline{X}$, such that $U \cap (\Gamma \backslash X)$ is a Siegel set. Let $\eta \in A^q_{(2)}(U)$ and $\alpha \in C^\infty_c(U)$. We have to see whether $\alpha\eta \in A^q_{(2)}(U)$. Clearly, $\alpha\eta$ and $\alpha \cdot d\eta$ are L^2. However, the L^2-norm of $d\alpha$ may include factors $a^{2\beta}$, hence may tend to infinity as a approaches the boundary, so that $d\alpha \cdot \eta$ and therefore $d(\alpha\eta)$, may not be L^2. However, the factors $a^{2\beta}$ occur only with coordinates along the fibers $\Gamma_{N_P} \setminus N_P$. Zucker's idea is then to collapse them to points and define another compactification of $\Gamma \setminus X$. I shall denote $\Gamma \setminus \hat{X}$. We have the fibrations

$$N_P \to e(P) \to X_P \ .$$

Dividing out by $\Gamma_P = \Gamma \cap P$, we get a fibration

$$\Gamma_{N_P} \setminus N_P \to e'(P) \to \Gamma_{M_P} \setminus X_P \ ,$$

where $\Gamma_{M_P} = \Gamma_P / \Gamma_{N_P}$.

Write now $\hat{e}(P)$ for $X(P)$, and $\hat{e}'(P)$ for $\Gamma_{M_P} \setminus X_P$. The new compactification of $\Gamma \setminus X$ is the union of the $\hat{e}'(P)$, with the understanding that $\hat{e}(G) = X$, and $\hat{e}'(G) = \Gamma \setminus X$. As in the previous case, we can also define $\hat{X} = \amalg\, \hat{e}(P)$, the convergence of $x_j = (n_j, z_j, z_j)$ to $z \in \hat{e}(P)$ being defined by $z_j \to z$, $a_j^\alpha \to \infty$ for $\alpha \in \Delta(A_P, P)$, and no condition on the n_j. Then, we can view this compactification X as the quotient of $\hat{X}$ by Γ, but the action of Γ is not proper anymore. Then

$$\Gamma \setminus \hat{X} = \amalg_{P \in \mathcal{P}_G/\Gamma}\ \hat{e}'(P) \ . \tag{1}$$

This is usually called the reductive Borel-Serre compactification. The projections $e(P) \to \hat{e}(P)$ provide natural projections $\overline{X} \to \hat{X}$ and $\Gamma \setminus \overline{X} \to \Gamma \setminus \hat{X}$. This compactification is called "reductive" because only locally symmetric spaces are added at ∞.

[For the reader familiar with Satake compactifications, $\Gamma \setminus \hat{X}$ is closely related to the maximal Satake compactification. In the latter case, one adds at infinity, the instead of $\hat{e}'(P)$, a quotient of the symmetric space of the *semisimple part* of M_P. The two compactifications are the same if $rk_{\mathbb{Q}}G = rk_{\mathbb{R}}G$. Otherwise, there is a canonical projection of $\Gamma \setminus \overline{X}$ onto the maximal Satake compactification, some fibers of which are (topological) tori.]

Define $\mathcal{A}^\bullet_{(2)}$ as before, but on $\Gamma \setminus \hat{X}$. Then, the previous difficulty related to the growth of $d\alpha$ disappears and this sheaf is indeed fine. Let $\mathcal{A}^\bullet$ be the sheaf of germs of differential forms and $i_*\mathcal{A}^\bullet$ its direct image on $\Gamma \setminus \hat{X}$. It is also fine. We have again

$$\mathcal{A}^\bullet_{(2)}(\Gamma \setminus \hat{X}) = A^\bullet_{(2)} \ , \quad i_*\mathcal{A}^\bullet(\Gamma \setminus \hat{X}) = A^\bullet(\Gamma \setminus X) \ . \tag{2}$$

We have to show that the inclusion $A^\bullet_{(2)} \to A^\bullet$ induces an isomorphism up to some degree $z(\mathfrak{g})$. Since these are sections of fine sheaves, we can again use sheaf theory. However, $\mathcal{A}^\bullet_{(2)}$ and $i_*\mathcal{A}^\bullet$ are not resolutions anymore and one has to use a more sophisticated version of the fundamental theorem of sheaf theory which implies: if the homomorphism of the derived sheaves $\mathcal{H}^\bullet\mathcal{A}^\bullet_{(2)} \to \mathcal{H}^\bullet i_*\mathcal{A}^\bullet$ is an isomorphism for degrees $q \leq c$, where c is some constant, then it induces an isomorphism of $H^q(A^\bullet_{(2)})$ onto $H^q(A^\bullet)$ for $q < c$.

We are then reduced to studying the stalks of the derived sheaves at points $x \in \Gamma \setminus \hat{X}$. If $x \in \Gamma \setminus X$, there is no problem. In fact both derived sheaves are acyclic. However, if $x \in \Gamma \setminus \partial\hat{X}$, the stalks are not at all acyclic, and the computations on the L^2-side quite involved.

Assume $x \in \hat{e}'(P)$. Then $(\mathcal{H}^\bullet i_*\mathcal{A}^\bullet)_x$ is the cohomology of $\Gamma_{N_P} \setminus N_P$. By a theorem of van Est, it is equal to the Lie algebra cohomology of $\mathfrak{n}_P$. The latter has been described as a module over a Levi subgroup of P by Kostant, and this gives a very explicit description of that cohomology. The stalk of $\mathcal{H}^\bullet\mathcal{A}^\bullet_{(2)}$ at x is basically the L^2-cohomology of a Siegel set $\mathfrak{S}_{\omega,t}$, where one may assume ω to be a small ball, and the metric is given as in 4.4. This is a long and involved computation I could not try to describe without entering into many technicalities and I shall not try, referring to [Z].

5.3. L^2-cohomology from the point of view of harmonic analysis.

Recall (cf. §1, to which we refer for the notation) that

$$H^\bullet(\Gamma; E) = H^\bullet(A^\bullet(\Gamma \setminus X; E)) = H^\bullet(\mathfrak{g}, \mathfrak{k}; C^\infty(\Gamma \setminus G) \otimes E)\ . \tag{1}$$

$L^2(\Gamma\setminus G)^\infty$ is a G-submodule and also a $(\mathfrak{g}, \mathfrak{k})$-submodule of $C^\infty(\Gamma\setminus G)$. Its inclusion in $C^\infty(\Gamma \setminus G)$ yields a map

$$H_2^\bullet(\Gamma; E) := H^\bullet(\mathfrak{g}, \mathfrak{k}; L^2(\Gamma \setminus G)^\infty \otimes E) \to H^\bullet(\Gamma; E)\ . \tag{2}$$

Identify $C^\bullet(\mathfrak{g}, \mathfrak{k}; L^2(\Gamma \setminus G)^\infty \otimes E$ with a subcomplex of $A^\bullet(X; E)^\Gamma$ by the inverse map of 1.7(6). Then it consists of differential forms, the coefficients of which are L^2 as well as all derivatives under $U(\mathfrak{g})$. In particular it is contained in $A^\bullet_{(2)}(X; E)$. We have the natural homomorphisms

$$H_2^\bullet(\Gamma; E) \xrightarrow{\mu} H_2^\bullet(\Gamma \setminus X; E) \xrightarrow{\nu} H^\bullet(\Gamma; E)\ . \tag{3}$$

We shall see here that ν and $\nu \circ \mu$ have the same image.

I recall some known properties of $L^2(\Gamma \setminus G)$. It is a direct sum

$$L^2(\Gamma \setminus G) = L^2(\Gamma \setminus G)_d + L^2(\Gamma \setminus G)_{ct}\ . \tag{4}$$

The first summand on the right is the span of the closed irreducible G-submodules. It is a direct sum of unitary irreducible G-modules with finite multiplicities. The second summand is the orthogonal complement of the first and is a finite sum of direct integrals of representations.

Let H be an irreducible unitary representation of G. The complex

$$C^\bullet(\mathfrak{g}, \mathfrak{k}; H^\infty \otimes E) = \mathrm{Hom}_{\mathfrak{k}}(\wedge^\bullet\mathfrak{p}, H^\infty \otimes E) \tag{5}$$

may also be written

$$C^\bullet(\mathfrak{g}, \mathfrak{k}; H^\infty \otimes E) = \mathrm{Hom}_{\mathfrak{k}}(\wedge^\bullet p \otimes E^*, H^\infty)\ . \tag{6}$$

Let J be the set of $\mathfrak{k}$-types occurring in $\wedge\mathfrak{p} \otimes E^*$. It is finite. The complex (6) is identically zero if H_π does not contain a $\mathfrak{k}$-type in J. Let us denote by $\hat{G}_J$ the set of equivalence classes of irreducible unitary representations of G which have at least one $\mathfrak{k}$-type in J and let $L^2(\Gamma \setminus G)_{d,J}$ be the span of the irreducible representations belonging to $\hat{G}_J$. Then

$$H^\bullet(\mathfrak{g}, \mathfrak{k}; L^2(\Gamma \setminus G)^\infty_d \otimes E) = H^\bullet(\mathfrak{g}, \mathfrak{k}; L^2(\Gamma \setminus G)^\infty_{d,J} \otimes E) \; . \tag{7}$$

Recall from 1.8 that if (π, H) is irreducible, unitary, then $H^\bullet(\mathfrak{g}, \mathfrak{k}; H^\infty \otimes E) = 0$ if $\pi(C) \neq \rho(C)$ and that

$$H^\bullet(\mathfrak{g}, \mathfrak{k}; H^\infty \otimes E) = C^\bullet(\mathfrak{g}, \mathfrak{k}; H^\infty \otimes E)$$

consists of harmonic forms if $\pi(C) = \rho(C)$. It follows from [BG], 4.3 that the eigenvalues in $L^2(\Gamma \setminus G)_d$ of C (or rather of a self-adjoint extension) are *discrete with finite multiplicities and tend to* $-\infty$. In particular, the set of $\pi \in \hat{G}_J$ occurring in $L^2(\Gamma \setminus G)_d$ for which $\pi(C) = \rho(C)$ is finite. Call it $F(J, E)$. Each element $\pi \in F(J, E)$ occurs in $L^2(\Gamma \setminus G)_d$ with a finite multiplicity. Let V_π be the corresponding isotypic subspace. We claim that

$$H^\bullet(\mathfrak{g}, \mathfrak{k}; L^2(\Gamma \setminus G)^\infty_d \otimes E) = \bigoplus_{\pi \in F(J,E)} H^\bullet(\mathfrak{g}, \mathfrak{k}; V^\infty_\pi \otimes E) \; . \tag{8}$$

The left-hand side will also be denoted $H^\bullet_{dis}(\Gamma; E)$. Let M be the direct sum of the V_π
($\pi \in F(J, E)$) and $M^\perp$ its orthogonal complement in $L^2(\Gamma \setminus G)_{d,J}$. In view of (7), it suffices to prove that $H^\bullet(\mathfrak{g}, \mathfrak{k}; M^{\perp,\infty} \otimes E) = 0$.

It follows from Kuga's formula and our assumption on M that Δ^{-1} extends to a bounded operator on $M^\perp \otimes E$. Therefore if $\eta \in C^q(\mathfrak{g}, \mathfrak{k} : M^\perp \otimes E)$ is closed then

$$\eta = \Delta \cdot \Delta^{-1} \cdot \eta = d\partial\Delta^{-1}\eta$$

is a coboundary (cf. [BG],§§4,5).

Let now $\xi \in H^q(\Gamma; E)$ be in the image of V. It is represented by a closed L^2-form, hence also, in view of Kodaira's theorem, by a harmonic one, say η. It follows from 5.5 in [BG] that the space of square integrable harmonic q-forms is finite dimensional. Their coefficients form a finite dimensional space invariant under K and the center $\mathcal{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$. By a theorem of Harish-Chandra, they belong to $L^2(\Gamma \setminus G)^\infty_d$, hence ζ is also in the image of $\nu \circ \mu$. Consequently, the space $\nu(H^\bullet_2(\Gamma \setminus X; E))$ is the image of $H^\bullet_{dis}(\Gamma; E)$ as claimed. Moreover

$$H^\bullet_{dis}(\Gamma; E) = \bigoplus_{\pi \in F(J;E)} \mathrm{Hom}_\mathfrak{k}(\wedge^\bullet_\mathfrak{p} \otimes E^*, V^\infty_\pi) = \mathcal{H}^\bullet_2(\Gamma \setminus X; E) \; . \tag{9}$$

The cohomology with respect to the continuous spectrum plays therefore no role on the study of the cohomology of Γ. It is zero in many cases, in particular if G and K have the same real rank. If it is not, then it is infinite dimensional [BC].

Remark. The previous result is all that one needs to discuss the relations between $H^\bullet(\Gamma)$ and $H^\bullet_2(\Gamma \setminus X)$. The equality Im $\nu \circ \mu =$ Im ν also follows more directly from the fact that μ is an isomorphism, which is proved in [B].

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§6. $H^\bullet(\Gamma; E)$ **and automorphic forms.**

6.1. Γ and E are as before. Moreover, we let

$$(1) \qquad Y = \Gamma \setminus G \quad \overline{Y} = \Gamma \setminus \overline{X} \quad Z = \Gamma \setminus \partial\overline{X} \ .$$

We have the long exact sequence in cohomology

$$(2) \qquad \cdots \to H^q_c(Y;E) \xrightarrow{s^\bullet} H^q(Y;E) = H^q(\overline{Y};E) \xrightarrow{r^\bullet} H^q(Z;E) \to H^{q+1}_c(Y) \to \cdots$$

The image of $s^\bullet$ is called the interior cohomology, and is denoted $H^\bullet_!(Y;E)$. We want to relate these groups to automorphic forms. In particular, we shall see that $s^\bullet$ is connected with cusp forms and $r^\bullet$ with Eisenstein series.

6.2. Recall the definition of the constant term along the parabolic $\mathbb{Q}$-subgroup P of a function on f on $\Gamma \setminus G$:

$$(1) \qquad f_P(x) = \int_{\Gamma_{N_P} \setminus N_P} f(n \cdot x) \cdot dn \ ,$$

the Haar measure dn being normalized by giving $\Gamma_{N_P} \setminus N_P$ the volume 1. The function f is cuspidal if $f_P = 0$ for all $P \in \mathcal{P}$, $P \neq G$. Let, as usual

$$(2) \qquad {}^oL^2(\Gamma \setminus G) = \{f \in L^2(\Gamma \setminus G) \ , \ f \text{ cuspidal}\} \ .$$

The space ${}^oL^2(\Gamma \setminus G)$ is contained in $L^2(\Gamma \setminus G)_d$. If $\alpha \in C^\infty_c(G)$, then $*\alpha$ is of trace class on ${}^oL^2(\Gamma \setminus G)$ by a theorem of Gelfand and Piatetski-Shapiro, and on $L^2(\Gamma \setminus G)_d$ by a more recent theorem proved first by W. Mueller if α is K-finite and then, independently by W. Mueller and Lizhen Ji for any $\alpha \in C^\infty_c(G)$.

Let us denote by $H^\bullet_{\text{cusp}}(\Gamma;E)$ (resp. $H^\bullet_{\text{dis}}(\Gamma;E)$, $H^\bullet_2(\Gamma;E)$) the $(\mathfrak{g},\mathfrak{k})$-cohomology with respect to ${}^oL^2(\Gamma \setminus G)^\infty \otimes E$ (resp. $L^2(\Gamma \setminus G)^\infty_d \otimes E$, resp. $L^2(\Gamma \setminus G)^\infty \otimes E$). We have the natural homomorphisms

$$(3) \qquad H^\bullet_{\text{cusp}}(\Gamma;E) \xrightarrow{\alpha} H^\bullet_{\text{dis}}(\Gamma;E) \xrightarrow{\beta} H^\bullet_2(\Gamma;E) \xrightarrow{\gamma} H^\bullet_2(Y;E) \xrightarrow{\delta} H^\bullet(\Gamma;E) \ .$$

We have seen in §5:

$$(4) \qquad \text{Im } \delta = \text{Im}(\delta \circ \gamma \circ \beta)$$

$$(5) \qquad H^\bullet_{\text{dis}}(\Gamma;E) = \mathcal{H}^\bullet_2(Y) \ .$$

We want to prove

$$(6) \qquad \delta \circ \gamma \circ \beta \circ \alpha \quad \text{is injective} \ .$$

and the existence of a natural map

$$(7) \qquad H^\bullet_{\text{cusp}}(\Gamma;E) \to H^\bullet_c(Y;E) \ .$$

Then (6) and (7) will show that $H^\bullet_{\text{cusp}}(\Gamma;E)$ may be identified to a subspace of $H^\bullet_!(Y;E)$.

6.3. Let $\|g\|$ be the Hilbert-Schmidt norm of $g \in G$ for a fixed $\mathbb{Q}$-embedding of G. A continuous function f on $\Gamma \setminus G$ is or moderate growth if there exists $m \in \mathbb{N}$ such that $|f(x)| < \|x\|^m$. It is known that this is equivalent to requiring that f be of moderate growth on any Siegel set, i.e. given $\mathfrak{S}_{\omega,t}$, we have $|f(z,a)| \prec a^\lambda$ for $a \in A_t$ and some $\lambda \in \mathfrak{a}^*$.

The function $f \in C^\infty(\Gamma \setminus G)$ is said to be *fast decreasing* if

$$f(x) < \|x\|^m \text{ for any } m \in \mathbb{Z}\ . \tag{1}$$

We let $C^\infty_{fd}(\Gamma \setminus G)$ be the space of functions which are fast decreasing together with all their derivatives Df $(D \subset U(\mathfrak{g}))$. It is know that

$$^oL^2(\Gamma \setminus G)^\infty \subset C^\infty_{fd}(\Gamma \setminus G)\ . \tag{2}$$

This follows from two facts: by the Dixmier-Malliavin theorem, any element of the left-hand side is a finite sum of convolutions $f * \alpha$ $(f \in {}^oL^2(\Gamma \setminus G),\ \alpha \in C^\infty_c(G))$ and if $f \in L^2(\Gamma \setminus G)$, then $f * \alpha$ has moderate growth. See [HC] or [BJ] for more details. $C^\infty_{fd}(\Gamma \setminus G)$ is a smooth G-module. We claim that

$$H^\bullet_c(Y;E) = H^\bullet(\mathfrak{g}, \mathfrak{k}; C^\infty_{fd}(\Gamma \setminus G) \otimes E)\ . \tag{3}$$

The complex $C^\bullet(\mathfrak{g}, \mathfrak{k}; C^\infty_{fd}(\Gamma \setminus G) \otimes E)$ (resp. $C^\bullet(\mathfrak{g}, \mathfrak{k}; C^\infty_c(\Gamma \setminus G) \otimes E)$) may be identified with the complex of differential forms on Y with fast decreasing (resp. compactly supported) coefficients. Then (3) follows from the more precise:

Proposition. *The inclusion $A^\bullet_c(Y;E) \to A^\bullet_{fd}(Y;E)$ induces an isomorphism in cohomology.*

This is again proved by sheaf theory on $\overline{Y}$. Let $\mathcal{A}^\bullet_c$ (resp. $\mathcal{A}^\bullet_{fd}$) be the differential graded presheaf on Y which assigns to an open subset U the complex $A^\bullet_c(U \cap Y;E)$ (resp. $A_{fd}(U \cap Y;E)$. It is a sheaf and we have

$$\mathcal{A}^\bullet_c(\overline{Y}) = A^\bullet_c(Y;E)\ , \quad \mathcal{A}^\bullet_{fd}(\overline{Y}) = A^\bullet_{fd}(Y;E)\ . \tag{3}$$

Let $j : A^\bullet_c \to A^\bullet_{fd}$ be the inclusion. We want to prove that it induces an isomorphism of $H^\bullet(A^\bullet_c(Y))$ onto $H^\bullet(A_{fd}(Y))$. For this, it suffices to show that $\mathcal{A}^\bullet_c$ and $\mathcal{A}^\bullet_{fd}$ are fine and that j induces an isomorphism of the derived sheaves. It is clear that if η is an element of $\mathcal{A}^\bullet_c(U)$ (resp. $\mathcal{A}^\bullet_{fd}(U)$, then so is $\alpha\eta$ for any $\alpha \in C^\infty_c(U)$, whence the fineness. Let $y \in Y$. Then the stalks of $\mathcal{A}^\bullet_c$ and $\mathcal{A}^\bullet_{fd}$ at j are the same, (and equal to the complex of germs of differential forms at y), hence the derived sheaves are equal at y. Let now $y \in e'(P)$, for some proper parabolic $\mathbb{Q}$-subgroup. Then y is not in the support of any element of $A^\bullet_c(U)$, hence $(\mathcal{H}^\bullet\mathcal{A}^\bullet_c)_y = 0$, so we have to see that $(\mathcal{H}^\bullet\mathcal{A}^\bullet_{fd})_y = 0$, too. In degree 0, this comes from the fact that $\mathcal{A}^0_{fd}(U)$ does not contain any non-zero constant function around y. To prove that $(\mathcal{H}^q\mathcal{A}^\bullet_{fd})_y = 0$ for $q > 0$, one checks that, in a suitable Siegel set, the homotopy operator in the proof of the Poincaré lemma transforms fast decreasing forms into fast decreasing ones, which is not difficult ([B]).

Then 6.2(7) is obtained by composition of the homomorphism $H^\bullet_{\text{cusp}}(\Gamma;E) \to H^\bullet_{fd}(\Gamma;E)$ with the inverse of the isomorphism provided by the previous proposition.

6.4. We let $C^\infty_{mg}(\Gamma\backslash G)$ be the space of $f \in C^\infty(\Gamma\backslash G)$ such that Df is of moderate growth for every $D \in U(\mathfrak{g})$.

Let $\mathcal{A}^\bullet_{mg}$ be the differential graded presheaf on $\overline{Y}$ which assigns to the open set U the complex $A^\bullet(U \cap Y; E)$ of E-valued differential forms with coefficients of moderate growth on $U \cap Y$. This is a sheaf and one sees again that it is a fine resolution of the direct image of the locally constant sheaf defined by E. Consequently the inclusion $C^\infty_{mg}(\Gamma \setminus G) \to C^\infty(\Gamma \setminus G)$ yields an isomorphism

$$(1) \qquad H^\bullet(\mathfrak{g}, \mathfrak{k}; C^\infty_{mg}(\Gamma \setminus G) \otimes E) \xrightarrow{\sim} H^\bullet(\Gamma; E) \ .$$

6.5 Corollary. *The homomorphism $\delta \circ \gamma \circ \beta \circ \alpha : H^\bullet_{\text{cusp}}(\Gamma; E) \to H^\bullet(\Gamma; E)$ is injective.*

An element of the left hand side is represented by a harmonic form η with fast decreasing coefficients. A fast decreasing function can always be integrated against a function of moderate growth. In fact, if σ and $d\sigma$ are fast decreasing and $\tau, \partial\tau$ are of moderate growth, then we have again a Stokes formula.

$$(1) \qquad (\sigma, \partial\tau) = (d\sigma, \tau) \ .$$

Now if η maps to zero, then it is already the coboundary of a form of moderate growth and we get $\eta = 0$ by the usual argument (see 4.1 for its first occurrence).

In analogy with the role of cusp forms among automorphic forms, the cuspidal cohomology appears to be a basic constituent of the cohomology, but, as is the case for cusp forms, one which is difficult to investigate. See [BLS] for some results and references to others.

6.6. A smooth function f on $\Gamma \setminus G$ is said to be of *uniform moderate growth* if there exists $m \in \mathbb{Z}$ such that

$$|Df(x)| \prec \|x\|^m \quad (x \in \Gamma \setminus G)$$

for all $D \in U(\mathfrak{g})$.

[The difference with C^∞_{mg} is that for $f \in C^\infty_{mg}(\Gamma \setminus G)$ the exponent m may vary with D.]

Let $C^\infty_{umg}(\Gamma \setminus G)$ be the space of functions of uniform moderate growth. It is a $(\mathfrak{g}, \mathfrak{k})$-module, contained in $C^\infty_{mg}(\Gamma \setminus G)$.

Theorem. *The inclusion $C^\infty_{umg}(\Gamma \setminus G) \to C^\infty_{mg}(\Gamma \setminus G)$ induces an isomorphism*

$$(1) \qquad H^\bullet(\mathfrak{g}, \mathfrak{k}; C^\infty_{umg}(\Gamma \setminus G) \otimes E) \xrightarrow{\sim} H^\bullet(\mathfrak{g}, \mathfrak{k}; C^\infty_{mg}(\Gamma \setminus G) \otimes E) \ .$$

Combined with 6.5(1) this shows that $H^\bullet(\Gamma; E)$ can be computed by means of forms with coefficients of uniform moderate growth.

The proof of this theorem is completely different from the previous one and does not use sheaf theory. Some indications will be given later. We pass now to a decomposition of $C^\infty_{umg}(\Gamma \setminus G)$ due to R. Langlands and to some consequences for $H^\bullet(\Gamma; E)$.

6.7. The constant term f_P of a function of uniform moderate growth will be viewed as a function on $\Gamma_{M_P} \setminus M_P$. It is also of uniform moderate growth. The function f will be said to be *negligible along* P if f_P is orthogonal to the cuspidal functions on $\Gamma_{M_P} \setminus M_P$. (Note that since a cuspidal function can be integrated against a function of moderate growth, this requirement makes sense.)

Recall that two parabolic $\mathbb{Q}$-subgroups P, Q are *associate* if P and a conjugate of Q have a common Levi subgroup. Then they have the same dimension and A_P and A_Q are conjugate. The dimension of A_P is usually called the parabolic rank of P, denoted $prk(P)$. Let Ass or Ass_G be the set of classes of associate parabolic $\mathbb{Q}$-subgroups. It is a partition of $\mathcal{P}_G$. The common rank of the elements in $\mathcal{P}$ is denoted $prk(\mathcal{P})$. For $\mathcal{P} \in Ass$, let

$$C^\infty_{umg}(\Gamma \setminus G)_{\mathcal{P}} = \{f \in C^\infty_{umg}(\Gamma \setminus G)\ ,\ f \text{ is negligible along any } Q \in \mathcal{P}_G,\, Q \notin \mathcal{P}\}\ .$$

If P is minimal and f is negligible along P, then $f_P = 0$, since in that case $\Gamma_{M_P} \setminus M_P$ is compact and all smooth functions on $\Gamma_{M_P} \setminus M_P$ are cuspidal. Let $f \in C^\infty_{umg}(\Gamma \setminus G)_{\mathcal{P}}$. Using descending induction on prk, it is easily checked that if $\mathcal{Q} \in Ass$, and no element of $\mathcal{Q}$ contains one of $\mathcal{P}$, then $f_Q = 0$ for all $Q \in \mathcal{Q}$. This holds in particular if $\mathcal{Q} \neq \mathcal{P}$ and $prk(\mathcal{Q}) \geq prk(\mathcal{P})$. It also implies that f_P is cuspidal for all $P \in \mathcal{P}$ and these constant terms determine all other constant terms of f.

We also see that if $\mathcal{P} = \{G\}$ consists of G itself, and $f \in C^\infty_{umg}(\Gamma \setminus G)_{\mathcal{P}}$, then $f_P = 0$ for all $P \in \mathcal{P}_G$, $P \neq G$. As a consequence, *if f is negligible along all $P \in \mathcal{P}_G$, then $f = 0$.*

6.8 Theorem (R. Langlands)**.** *The space $C^\infty_{umg}(\Gamma \setminus G)$ is an orthogonal direct sum*

(1) $$C^\infty_{umg}(\Gamma \setminus G) = \bigoplus_{\mathcal{P} \in Ass} C^\infty_{umg}(\Gamma \setminus G)_{\mathcal{P}}\ .$$

(See [BLS] for a proof and earlier references.)

The spaces $C^\infty_{umg}(\Gamma \setminus G)_{\mathcal{P}}$ are $(\mathfrak{g}, \mathfrak{k})$-modules. Let

(2) $$H^\bullet_{\mathcal{P}}(\Gamma; E) = H^\bullet(\mathfrak{g}, \mathfrak{k}; C^\infty_{umg}(\Gamma \setminus G)_{\mathcal{P}} \otimes E)\ .$$

Then 6.4(1), 6.6(1) and 6.8 imply

(3) $$H^\bullet(\Gamma; E) = \bigoplus_{\mathcal{P} \in Ass} H^\bullet_{\mathcal{P}}(\Gamma; E)\ .$$

We already noticed that $C^\infty_{umg}(\Gamma \setminus G)_{\{G\}}$ consists of cuspidal functions. Its elements are fast decreasing, hence are square integrable on $\Gamma \setminus G$. Therefore, in view of 6.3.2

(4) $${}^oL^2(\Gamma \setminus G)^\infty = C^\infty_{umg}(\Gamma \setminus G)_{\{G\}})$$

hence

(5) $$H^\bullet_{\text{cusp}}(\Gamma; E) = H^\bullet_{\{G\}}(\Gamma; E)\ ,$$

so that (3) provides another proof of 6.2(6), and gives moreover a direct complement for $H^\bullet_{\text{cusp}}(\Gamma; E)$ in $H^\bullet(\Gamma; E)$.

6.9. As we have seen, $H^\bullet(\Gamma; E)$ can be computed using functions in $C^\infty_{umg}(\Gamma \setminus G)$. In fact, we might as well restrict to the subspace $C^\infty_{umg}(\Gamma \setminus G)_{(K)}$ of K-finite functions, since the complex defining $(\mathfrak{g}, \mathfrak{k})$-cohomology consists of K-finite functions, by definition. Therefore, we are using K-finite functions of uniform moderate growth. If such a function is moreover $\mathcal{Z}$-finite then it is an automorphic form. This led me to ask whether one could replace $C^\infty_{umg}(\Gamma \setminus G)$ by the still smaller space $\mathcal{A}(\Gamma \setminus G)$ of automorphic forms on $\Gamma \backslash G$, in other words, does the inclusion $\mathcal{A}(\Gamma\backslash G) \subset C^\infty(\Gamma\backslash G)$ yield an isomorphism

$$H^\bullet(\mathfrak{g}, \mathfrak{k} : \mathcal{A}(\Gamma \setminus G) \otimes E) \xrightarrow{\sim} H^\bullet(\Gamma; E) \ ?$$

A positive answer was provided by J. Franke [F]. The proof is much more difficult than those of the results previously discussed here.

In a similar vein, G. Harder has conjectured, in analogy with Hodge theory, that any cohomology class has a harmonic representative. I do not know the present status of this conjecture.

6.10. We now give some indications on the proof of the theorem in 6.6. For simplicity of writing, we assume $E = \mathbb{C}$. The sheaf theoretic approach used repeatedly earlier fails completely, already because if U is open in $\overline{Y}$, $f \in C^\infty_{umg}(U \cap Y)$ and $\alpha \in C^\infty_c(U)$, then αf need not be of uniform moderate growth on $U \cap Y$. The proof has to proceed along different lines.

First a general remark. Assume $C^\bullet$ is a differential graded complex and $D^\bullet$ a subcomplex. To show that $D^\bullet \to C^\bullet$ induces an isomorphism in cohomology, one procedure is to construct a morphism $R : C^\bullet \to D^\bullet$ and a linear map $A : C^\bullet \to C^\bullet$ decreasing the degree by one, such that

$$R\eta - \eta = dA\eta + A \cdot d\eta \quad (\eta \in C^q) \ .$$

It is an elementary exercise to see that this implies the desired conclusion.

We want to apply this principle on $\Gamma \setminus G$. Denote by $A^\bullet_{mg}(\Gamma \setminus G)$ (resp. $A^\bullet_{umg}(\Gamma \setminus G)$) the space of differential forms on $\Gamma \setminus G$ with coefficients in $C^\infty_{mg}(\Gamma \setminus G)$ (resp. $C^\infty_{umg}(\Gamma \setminus G)$). Thus

$$A^\bullet_{mg}(\Gamma \setminus G) = C^\bullet(\mathfrak{g}; C^\infty_{mg}(\Gamma \setminus G))$$
$$A^\bullet_{umg}(\Gamma \setminus G) = C^\bullet(\mathfrak{g}; C^\infty_{umg}(\Gamma \setminus G))$$

hence

$$H^\bullet(A^\bullet_{mg}) = H^\bullet(\mathfrak{g}; C^\infty_{mg}(\Gamma \setminus G)) \ , \ H^\bullet(A^\bullet_{umg}) = H^\bullet(\mathfrak{g}; C^\infty_{umg}(\Gamma \setminus G)) \ .$$

Fix $\alpha \in C^\infty_c(G)$ with total integral one and with support contained in an exponential neighborhood U of the identity, i.e. such that exp is an isomorphism onto U of some neighborhood V of 0 in $\mathfrak{g}$. If $f \in C^\infty_{mg}(\Gamma \setminus G)$, then $f * \alpha \in C^\infty_{umg}(\Gamma \setminus G)$. Therefore, if $\eta \in A^q_{mg}(\Gamma \setminus G)$, then

$$R_\alpha \eta = \eta * \alpha \ , \quad \text{i.e.}$$
$$R_\alpha \eta(y) = \int_G \alpha(x^{-1}) \cdot r_x(\eta)(y) dx \quad (x \in G, y \in \Gamma \setminus G)$$

belongs to $A^q_{umg}(\Gamma\backslash G)$ and R_α defines a morphism of $A^\bullet_{mg}(\Gamma\backslash G)$ into $A^\bullet_{umg}(\Gamma\backslash G)$. Since the integral of $\alpha(x)$ or $\alpha(x^{-1})$ is one, we have

$$R_\alpha\eta - \eta = \int_G \alpha(x^{-1})(r_x\eta - \eta)dx \ .$$

If $x \in U$, putting $x = e^X(X \in \mathfrak{g})$, we can write

$$R_\alpha\eta - \eta = \int_G \alpha(x^{-1})dx \int_0^1 dt\ r(e^{tX}) \cdot \theta_X\eta \ .$$

But $\theta_X = d\cdot i_X + i_X \cdot d$, so that we get

$$R_\alpha\eta - \eta = d\cdot E_\alpha \cdot \eta + E_\alpha \cdot d\eta \ ,$$

where

$$E_\alpha = \int_G \alpha(x^{-1})dx \int_0^1 dt\ r(e^{tX}) \cdot i_X \cdot \eta$$

decreases the degrees by one. It follows that $A^\bullet_{umg}(\Gamma\setminus G) \to A^\bullet_{mg}(\Gamma\setminus G)$ induces an isomorphism

$$H^\bullet(\mathfrak{g}; C^\infty_{umg}(\Gamma\setminus G)) \to H^\bullet(\mathfrak{g}; C^\bullet_{mg}(\Gamma\setminus G)) \ .$$

However, we are really interested in Y, i.e. in the subcomplexes $C^\bullet(\mathfrak{g},\mathfrak{k} : C^\infty_{mg}(\Gamma\backslash G)$ and $C^\bullet(\mathfrak{g},\mathfrak{k} : C^\infty_{umg}(\Gamma\setminus G))$. We would be through if R_α would leave them stable. They are defined by the two conditions $\theta_X\eta = 0 = i_X$ for $X \in \mathfrak{k}$. The morphism R_α respects the first one if α is K-invariant, but it does not seem possible to define R_α compatible with the second one.

To conclude we then use a comparison theorem for spectral sequences. If Γ is torsion-free, which we assume, $\Gamma\backslash G$ is a principal K-bundle over $Y = \Gamma\backslash X$. There are algebraic analogues in our setting of the spectral sequence of a fiber bundle. There is a filtration of $C^\bullet(\mathfrak{g}; C^\infty_{umg}(\Gamma\setminus G))$ (resp. $C^\bullet(\mathfrak{g}; C^\infty_{mg}(\Gamma\setminus G))$ leading to a spectral sequence (E_r) (resp. (E'_r)) in which

$$E_2^{p,q} = H^p(\mathfrak{g},\mathfrak{k}; C^\infty_{umg}(\Gamma\setminus G)) \otimes H^q(\mathfrak{k}) \ ,$$
$$E_2'^{p,q} = H^p(\mathfrak{g},\mathfrak{k}; C^\infty_{mg}(\Gamma\setminus G)) \otimes H^q(\mathfrak{k}) \ ,$$

The inclusion $C^\infty_{umg} \to C^\infty_{mg}$ induces a morphism of spectral sequences. At the E_2-level, it is the tensor product of the identity on $H^\bullet(\mathfrak{k})$ by the natural morphism

$$\gamma^\bullet : H^\bullet(\mathfrak{g},\mathfrak{k}; C^\infty_{umg}(\Gamma\setminus G)) \to H^\bullet(\mathfrak{g},\mathfrak{k}; C^\infty_{mg}(\Gamma\setminus G)) \ .$$

We already know that the map in cohomology of the total complexes is an isomorphism. It is also one on the fiber terms $H^\bullet(\mathfrak{k})$. The comparison theorem for spectral sequences implies then that γ^* is an isomorphism, too, which proves the theorem. Note that this is obvious if K is totally non-homologous to zero in $\Gamma\setminus G$, i.e. if $E_2 = E_\infty$ and $E'_2 = E'_\infty$.

6.11. The restriction map. $r^\bullet : H^\bullet(Y) \to H^\bullet(Z)$ (cf. 6.1. for the notation).

The boundary is the disjoint union of the faces

$$e'(P) = \Gamma_P \setminus (N_P \times X_P) \ .$$

Let $r_P^\bullet : H^\bullet(Y) \to H^\bullet(e'(P))$ be the composition of $r^\bullet$ and of the restriction to $e'(P)$. Group theoretically, it is the restriction map $H^\bullet(\Gamma) \to H^\bullet(\Gamma_P)$.

We have $X = e(P) \times A_P$ and $\Gamma_P \setminus X = e'(P) \times A_P$. Let η be a closed q-form on Y, and $\overline{\eta}$ its cohomology class. The constant term η_P may be viewed as a N_P-invariant form on $\Gamma_P \setminus X$. The submanifolds $e'(P)_a = e'(P) \times a$ of $\Gamma_P \setminus X$ are all cohomologous to one another, hence the cohomology class $\overline{\eta}_P$ of the restriction of η_P to $e'(P)_a$ is independent of a and yields an element of $H^\bullet(e'(P))$. We claim that

$$r_P^\bullet(\overline{\eta}) = \overline{\eta}_P \ . \tag{1}$$

Let $\sigma : \Gamma_P \setminus X \to Y$ be the canonical projection and $r_a^\bullet$ be the restriction to $e'(P)_a$. We have to show that

$$r_a^\bullet \cdot \sigma^*(\overline{\eta}) = \overline{r_a^\bullet(\eta_P)}$$

or that $\sigma^*(\eta)$ and η_P take the same value on any compact cycle C in $e'(P)_a$. The differential form $\sigma^*\eta - \eta_P$ decreases fast as $a^\alpha \to \infty$ for $\alpha \in \Delta(A_P, P)$, hence the value of $\sigma^*\eta - \eta_P$ on $C_a = C \times a$ tends to zero. On the other hand, it is constant since the cycles C_a are cohomologous to one another for all $a \in A_P$. Therefore this value is equal to zero.

6.12. Thus the consideration of the classes $\overline{\eta}_P$ bears some analogy with the formation of the constant terms of an automorphic form. Now, if f is orthogonal to cusp forms, it is completely determined by its constant terms. The homomorphism $r^\bullet$ maps $H^\bullet(\Gamma)/H_!^\bullet(Y)$ injectively into $H^\bullet(Z)$. One may ask whether an element of that quotient is completely determined by the classes of its constant terms. In other words, if $\overline{\eta}_P = 0$ for all proper $P \in \mathcal{P}_G$, is $r^\bullet\overline{\eta} = 0$. I have proposed to call *ghost class* a class η such that $r^\bullet\eta \neq 0$ but $r_P^\bullet\eta = 0$ for all proper P.

The problem of their existence does not arise in $\mathbb{Q}$-rank 1, because Z is then a disjoint union of the $e'(P)$. The first example in higher rank has been given by G. Harder (unpublished, as far as I know). J. Schwermer has proved there are none if $G = \mathbf{Sp}_4$ and Γ is a congruence subgroup of the Siegel modular group. On the other hand, two students of J. Franke claim to have found many examples, including one which would contradict Schwermer's result, so the situation is somewhat murky.

6.13. Let $\mathcal{P}$ be a proper associate class and let η represent an element of $H^q(\Gamma)_{\mathcal{P}}$. Then, as we saw, the η_P ($P \in \mathcal{P}$) are cusp forms and all constant terms are determined by those. Thus, the problem of finding cuspidal classes in the image of $r_P^\bullet$ is basic. In order to get information on Im $r_P^\bullet$, the idea of Harder was to transpose the formalism of Eisenstein series: starting from a class in $H^\bullet_{\text{cusp}}(\Gamma_P)$, try to construct by means of Eisenstein series a class in $H^\bullet(\Gamma)$ and investigate its constant term. This meets with considerable technical difficulties and has been carried out only in a few cases. I shall just outline the strategy and mention some cases where it has led to precise results. Harder's approach was at first differential geometric, but it was translated in representation theoretic terms and I shall use that framework.

6.14. We need first some information on $H^\bullet(e'(P))$. Since this does not complicate matters, I shall again consider cohomology with respect to a G-module. More precisely, we let E_λ be an irreducible G-module with highest weight λ, and want to describe $H^\bullet(e'(P); E_\lambda)$. We fix P and suppress the indexes P. We have the fibration

$$\Gamma_N \setminus N \to e'(P) \to Y_M = \Gamma_M \setminus X_M \ , \tag{1}$$

where we write X_M for our previous X_P. By a theorem of van Est,

$$H^\bullet(\Gamma_N \setminus N; E_\lambda) = H^\bullet(\mathfrak{n}; E_\lambda) \ . \tag{2}$$

The RHS has been studied by B. Kostant. Let W be the Weyl group of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{b}_c = \mathfrak{b}_c \oplus \mathfrak{a}_c$, where $\mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{m}$. Let W_P be its intersection with P. It is naturally identified with $W(\mathfrak{b}_c, \mathfrak{m}_c)$. The quotient W/W_P has a canonical set of representatives W^P (once a suitable ordering of the roots has been introduced), consisting of elements of smallest length in their cosets modulo W_P.

Given $s \in W$ and $\mu \in \mathfrak{h}_c^*$ we let $s \circ \mu = s(\mu + \rho) - \rho$, where ρ is, as usual, half the sum of the positive roots. Then Kostant has proved

$$H^q(\mathfrak{n}; E_\lambda) = \bigoplus_{w \in W^P, (w) = q} F_{s \circ \lambda} \tag{3}$$

where F_μ is an irreducible representation of M with extremal weight μ, it being understood that, by abuse of notation, the index $s \circ \lambda$ stands for its restriction to $\mathfrak{b}_c$. Moreover, Kostant has given a set of harmonic representatives for the cohomology classes. Using those, one sees easily that the spectral sequence of the fibration $e'(P) \to Y_M$ degenerates at E_2, so that

$$H^\bullet(e'(P); E_\lambda) = H^\bullet(Y_M; H^\bullet(\mathfrak{n}; E_\lambda)) \ . \tag{4}$$

The RHS is the cohomology of Γ_M with respect to $H^\bullet(\mathfrak{n}; E_\lambda)$. For $\pi \in \hat{M}$, let V_π be the *isotypic* subspace of ${}^oL^2(\Gamma \setminus M)$ corresponding to π. Then, by (3), (4) and §5, the cuspidal part of $H^\bullet(\Gamma_M; H^\bullet(\mathfrak{n}, E_\lambda))$ is a direct summand of $H^\bullet(e'(P); E_\lambda)$ which can be written

$$H^\bullet_{\text{cusp}}(\Gamma_M; H^\bullet(\mathfrak{n}, E_\lambda)) = \bigoplus_{\pi \in \hat{M}, s \in W^P} H^\bullet(\mathfrak{m}, \mathfrak{k}_M; V_\pi^\infty \otimes F_{s\circ\lambda}[l(s)]) \tag{5}$$

where $[l(s)]$ indicates that $F_{s\circ\lambda}$ has to be given the degree $l(s)$. The summand of the RHS assigned to π and s will be denoted $F^\bullet_{\pi,s}$. For it to be $\neq 0$, V_π^∞ must have a $\mathfrak{k}_M$-type in common with $\wedge(\mathfrak{p}_M) \otimes F^*_{s\circ\lambda}$ and χ_π should be equal to the infinitesimal character of $F^*_{s\circ\lambda}$. (Here $\mathfrak{k}_M$ and $\mathfrak{p}_M$ refer to the Cartan decomposition of M with respect to $K_M = K \cap M$).

6.15. For $\mu \in \mathfrak{a}_c^*$, we let $\mathbb{C}_\mu$ denote $\mathbb{C}$ on which A acts via the character defined by μ. View then $V_\pi^\infty \otimes \mathbb{C}_{\wedge+\rho}$ ($\wedge \in \mathfrak{a}_c^*$) as a representation of MA, and then as a representation of P trivial on N. We let

$$I_{\pi,\wedge} = I_P^G(V_\pi^\infty \otimes \mathbb{C}_{\wedge+\rho}) \ , \tag{1}$$

where I_P^G stands for induced representation in the C^∞ framework (cf. [BW], III). (Note that one should write ρ_P instead of ρ, but ρ and ρ_P are *equal on* $\mathfrak{a}_c^*$).

The space $H^\bullet(\mathfrak{g}, \mathfrak{k}; I_{\pi,\wedge} \otimes E_\lambda)$ is determined in [BW], III, 3.3. Let $\mu_\pi \in \mathfrak{a}_c^*$ be such that $\chi_\pi = \chi_{\mu_\pi}$. Then this space is non zero only if there exists $s \in W^P$, such that

$$s \circ \lambda + \wedge + \mu_\pi = 0 \ . \tag{2}$$

Such an s is necessarily unique, hence $\wedge$ is also determined by π and λ, if we require, as we do, that space to be $\neq 0$. If (2) is fulfilled, then

$$H^\bullet(\mathfrak{g}, \mathfrak{k}; I_{\pi,\wedge} \otimes E_\lambda) = F^\bullet_{\pi,s} \otimes \wedge^\bullet \mathfrak{a}_c \ . \tag{3}$$

The construction and analytic continuation of Eisenstein series yield an intertwining operator

$$E(\wedge) : I_{\pi,\wedge} \to \mathcal{A}(\Gamma \setminus G) \tag{4}$$

which is meromorphic in $\wedge$, whence also a homomorphism

$$H^\bullet(E(\wedge)) : H^\bullet(\mathfrak{g}, \mathfrak{k}; I_{\pi,\wedge} \otimes E_\lambda) \to H^\bullet(\Gamma; E_\lambda) \ . \tag{5}$$

For it to be interesting, the LHS should be $\neq 0$. By (2), this is possible for a unique value of $\wedge$. Assume that $E(\wedge)$ is holomorphic at that point. Then by (3), we get a homomorphism of $F^\bullet_{\pi,s} \otimes \wedge^\bullet \mathfrak{a}_c$ into $H^\bullet(\Gamma; E_\lambda)$. W. Casselman and B. Speh have proved, independently, that this map is zero on $F^\bullet_{\pi,s} \otimes \wedge^i \mathfrak{a}_c$ for $i \geq 1$. We obtain then a map of a direct summand $F^\bullet_{\pi,s}$ of $H^\bullet_{\text{cusp}}(\Gamma_M; E_\lambda)$ into $H^\bullet(\Gamma; E_\lambda)$, call $\tilde{F}^\bullet_{\pi,s}$ its image. A first hope would be that $F^\bullet_{\pi,s}$ be the constant term at P of $\tilde{F}^\bullet_{\pi,s}$. But the theory of the constant term indicates that other terms should be involved.

Let (P', A') be associate to (P, A). Let $W(A, A')$ be the set of isomorphisms of A onto A' induced by inner automorphisms of G. To each $w \in W(A, A')$ is associated an intertwining operator

$$C(w, \wedge) : {}^o\mathcal{A}(\Gamma_M \setminus M) \to {}^o\mathcal{A}(\Gamma_{M'} \setminus M') \tag{6}$$

meromorphic in $\wedge$. Assume they are holomorphic at $\wedge$. Then if f is an automorphic form on $\Gamma \setminus G$, $f_{P'}$ is a sum of terms $C(w, \wedge) \cdot f_P$. Assume now the $C(\omega, \wedge)$ are holomorphic at the value of $\wedge$ under consideration. Then $(\tilde{F}^\bullet_{\pi,s})_{P'}$ will be a sum of terms $C(w, \wedge) \cdot F^\bullet_{\pi,s}$. Consider the case where $P' = P$ and assume $W(A, A)$ has at least one element besides the identity. Then the sum of the $C(w, \wedge) \cdot F^\bullet_{\pi,s}$ contains $F^\bullet_{\pi,s}$, but may still be zero, since the summands may not be linearly independent. This difficulty already occurs if the $C(w, \wedge)$ are holomorphic at $\wedge$. If they have poles, then residues must be taken.

The residue of $E(\wedge)$ may define a homomorphism of a quotient of $I_{\pi,\wedge}$ into $\mathcal{A}(\Gamma \setminus G)$, whence also a map in cohomology, which may lead to a class with constant term related to the starting point, but this is even more delicate than in the holomorphic case. Altogether, this analysis is quite difficult, all the more since little information is available on the poles of the $C(w, \wedge)$ or of $E(\wedge)$. When it works, it leads to substantial information on Im $r^\bullet_P$. Some of the main cases in which this has been carried out are:

(a) $G = \mathbf{SL}_2$ viewed as defined over a number field k, and $\Gamma = \mathbf{SL}_2(\mathfrak{o}_k)$, where $\mathfrak{o}_k$ is the ring of algebraic integers in k, [H1].
(b) G has $\mathbb{Q}$-rank 1. Then G. Harder constructs a supplement $H^\bullet_{\text{Eis}}$ to $H^\bullet_!$ which maps isomorphically onto Im $r^\bullet$, [H2].
(c) $G = \mathbf{SL}_n$, Γ is a congruence subgroup of $\mathbf{SL}_n(\mathbb{Z})$ and P is proper maximal (Schwermer [S2]).

We refer to [S1] for an exposition of this construction and to [S3] for a survey of results.

6.16. In this introduction, I have mostly concentrated on a number of general theorems and left out many more special and interesting questions. Also, I have emphasized the analytic point of view, but much of the theory of automorphic forms is developed in view of applications to number theory, in particular in the Langlands program. Adelic constructions have also to be applied to cohomology.

To restore some perspective, I have added, under "Further references" a list of papers compiled in 1997 by J. Schwermer. (Note by the editors: For an comprehensive and updated list of references, see the paper by J.S.Li and J.Schwermer *Automorphic representations and cohomology of arithmetic groups* in **Challenges for the 21st century (Singapore, 2000)**, 102–137, World Sci. Publishing, River Edge, NJ, 2001.)

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