Termination of rewriting strategies: a generic approach

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ABSTRACT: We propose a generic termination proof method for rewriting under strategies, based on an explicit induction on the termination property. Rewriting trees on ground terms are modeled by proof trees, generated by alternatively applying narrowing and abstracting steps. The induction principle is applied through the abstraction mechanism, where terms are replaced by variables representing any of their normal forms. The induction ordering is not given a priori, but defined with ordering constraints, incrementally set during the proof. Abstraction constraint can be used to contain the narrowing mechanism, well known to easily diverge. The generic method is then instantiated for the innermost, outermost and local strategies.

1. INTRODUCING THE PROBLEM

Rewriting techniques are now widely used in automated deduction, especially to handle equality, as well as in programming, in functional, logical or rule-based languages. Termination of rewriting is a crucial property, important in itself to guarantee a result in a finite number of steps, but it is also required to decide properties like confluence and sufficient completeness, or to allow proofs by consistency. Existing methods for proving termination of term rewriting systems (TRS in short) essentially tackle the universal termination problem: they work on free term algebras and prove termination for any term of these algebras. Most are based on syntactic or semantic noetherian orderings containing the rewriting relation induced by the TRS [Kamin and Lévy 1980; Dershowitz 1982; Lankford 1979; Ben Cherifa and Lescanne 1987; Dershowitz and Hoot 1995]. Other methods consist in transforming the termination problem of a TRS R into the termination problem of another TRS R', provable with techniques of the previous category. Examples are semantic labelling [Zantema 1995], and the dependency pair method [Arts and Giesl 2000; Giesl et al. 2003]. For most approaches, finding an appropriate ordering is the key problem, that often comes down to solving a set of ordering constraints.

In the context of proof environments for rule-based programming languages, such as ASF+SDF [Klint 1993], Maude [Clavel et al. 1996], CafeOBJ [Futatsugi and Nakagawa 1997], ELAN [Borovanský et al. 1998], or TOM [Moreau et al. 2003], where

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a program is a term rewriting system and the evaluation of a query consists in rewriting a ground expression, more specific termination proof tools are required, to allow termination proofs under specific reduction strategies. There are still few results in this domain. To our knowledge, methods have only been given for the innermost strategy [Arts and Giesl 1996; Giesl and Middeldorp 2003] and for the context-sensitive rewriting which involves particular kinds of local strategies [Lucas 2001]. In previous works, we have also obtained termination results for the innermost strategy [Gnaedig et al. 2001; Fissore et al. 2002a], for general local strategies on the operators [Fissore et al. 2001], and for the outermost strategy [Fissore et al. 2002b].

In this paper, we propose a generic proof principle, based on an explicit induction mechanism on the termination property, which is a generalization of our three previous results. We then show how it can be instantiated to give an effective termination proof algorithm for the innermost strategy, the outermost strategy, and local strategies on operators. This generalizing work allowed not only to propose a generic version of our proof method, but also to considerably simplify the technical features of the algorithms initially designed for the different strategies.

The three above strategies have been chosen for their relevance to programming languages. The most widely used innermost strategy consists in rewriting always at the lowest possible positions. It is often used as a built-in mechanism in evaluation of rule-based or functional languages. In addition, for non-overlapping or locally confluent overlay systems [Gramlich 1995], or systems satisfying critical peak conditions [Gramlich 1996], innermost termination is equivalent to standard termination (i.e. termination for standard rewriting, which consists in rewriting without any strategy). As proved in [Krishna Rao 2000], termination of rewriting is equivalent for the leftmost innermost and the innermost strategies.

The outermost strategy for evaluating expressions in the context of programming is essentially used when one knows that computations can be non-terminating. The intuition suggests that rewriting a term at the highest possible position gives more chance than with another strategy to lead to an irreducible form. Indeed, outermost rewriting may succeed when innermost fails, as illustrated by the expression second(dec(1), 0), with the rewrite rules second(x, y) → y and dec(x) → dec(x - 1) on integers. Innermost rewriting fails to terminate, because it first evaluates dec(1) into dec(0), dec(-1), and so on. Outermost rewriting, however, gives 0 in one rewriting step. Moreover, outermost derivations are often shorter: in our example, to reduce second(u, v), one does not need to reduce u, which can lead to infinite computations or, at least, to a useless evaluation. This advantage makes the outermost strategy an interesting strategy for rule-based languages, by allowing the interpreters to be more efficient, as well as for theorem proving, by allowing the rewriting-based proofs to be shorter.

Outermost computations are of interest in particular for functional languages, where interpreters or compilers generally involve a strategy for call by name. Often, lazy evaluation is used instead: operators are labelled in terms as lazy or eager, and the strategy consists in reducing the eager subterms only when their reduction allows a reduction step higher in the term [Nguyen 2001]. However, lazy evaluation may diverge while the outermost computation terminates, which gives an additional motivation for studying outermost termination. For instance, let
us consider the evaluation of the expression \( inf(0) \) with the following two rules:

\[
\text{cons}(x, \text{cons}(y, z)) \rightarrow \text{big}, \quad \text{inf}(x) \rightarrow \text{cons}(x, \text{inf}(s(x))).
\]

If \( inf \) is labelled as eager, \( inf(0) \) is reduced to \( \text{cons}(0, \text{inf}(s(0))) \), and then, since application of the first rule fails, the sub-expression \( \text{inf}(s(0)) \) has to be evaluated before considering the whole expression, which leads to an infinite evaluation. Evaluated in an outermost manner, \( inf(0) \) is also reduced to \( \text{cons}(0, \text{inf}(s(0))) \), but then \( \text{inf}(s(0)) \) is reduced to \( \text{cons}(s(0), \text{inf}(s(s(0)))) \), and the whole expression is reduced to \( \text{big} \). Lazy termination of functional languages has already been studied (see for example [Panitz and Schmidt-Schauss 1997]), but to our knowledge, no termination proof tool exists for specifically proving outermost termination of rewriting.

Local strategies on operators are used in particular to force the evaluation of expressions to terminate. A famous example is the evaluation of a recursive function defined with an \texttt{if\_then\_else} expression, which can diverge if the first argument is not evaluated first.

This kind of strategy is allowed by languages such as OBJ3, CafeOBJ or Maude, and studied in [Eker 1998] and [Nakamura and Ogata 2000]. It is defined in the following way: to any operator \( f \) is attached an ordered list of integers, giving the positions of the subterms to be evaluated in a given term, whose top operator is \( f \). For example, the TRS

\[
\begin{align*}
    f(i(x)) & \rightarrow \text{if\_then\_else}(\text{zero}(x), g(x), f(h(x))) \\
    \text{zero}(0) & \rightarrow \text{true} \\
    \text{zero}(s(x)) & \rightarrow \text{false} \\
    \text{if\_then\_else}(\text{true}, x, y) & \rightarrow x \\
    \text{if\_then\_else}(\text{false}, x, y) & \rightarrow y \\
    h(0) & \rightarrow i(0) \\
    h(x) & \rightarrow s(i(x))
\end{align*}
\]

using the conditional expression, does not terminate for the standard rewriting relation, but does with the following strategy: \( LS(\text{ite}) = [1; 0] \), \( LS(f) = LS(\text{zero}) = LS(h) = [1; 0] \) and \( LS(g) = LS(i) = [1] \), where \( \text{ite} \) denotes \texttt{if\_then\_else} for short.

Local strategies are to be compared with context-sensitive rewriting. In both cases, rewriting is allowed only at some specified positions in the terms, but local strategies specify in addition an ordering on these rewriting positions.

The termination problem for these various strategies is always different: in [Fiscare et al. 2002c], the interested reader can find examples showing that termination for one of these strategies does not imply termination for any other of them. A better knowledge of these differences would be interesting, and could help to choose the good one when programming in these languages.

Despite of these distinct behaviours, the termination proofs we propose rely on a generic principle and a few common concepts, that are emphasized in this paper. Our termination proof method for rewriting on ground terms is based on an explicit induction mechanism on the termination property. The main idea is to proceed by induction on the ground term algebra with a noetherian ordering \( \succ \), assuming that for any \( t' \) such that \( t \succ t' \), \( t' \) terminates, i.e. there is no infinite derivation chain starting from \( t' \). The general proof principle relies on the simple idea that for establishing termination of a ground term \( t \), it is enough to suppose that subterms of \( t \) are smaller than \( t \) for this ordering, and that rewriting the context only leads to terminating chains. Iterating this process until a non-reducible context is obtained
establishes termination of \( t \).

Unlike classical induction proofs, where the ordering is given, we do not need to define it \textit{a priori}. We only have to check its existence by ensuring satisfiability of ordering constraints incrementally set along the termination proof. Thanks to the power of induction, the generated constraints are often simpler to solve than for other approaches, and even, in many cases, do not need any constraint solving algorithm.

Directly using the termination notion on terms has also been proposed in [Goubault-Larrecq 2001], for inductively proving well-foundedness of binary relations, among which path orderings. The approach differs from ours in that it works on general relations, that can then be used on TRSs, whereas we directly handle the termination proof of a given TRS.

In order to explain the basic idea of this work, let us consider the classical example, due to Toyama, of a TRS that does not terminate, but terminates with the innermost strategy:

\[
\begin{align*}
    f(0, 1, x) & \rightarrow f(x, x, x) \\
    g(x, y) & \rightarrow x \\
    g(x, y) & \rightarrow y.
\end{align*}
\]

Let us prove by induction on the set \( T(\mathcal{F}) \) of ground terms built on \( \mathcal{F} = \{0, 1, f, g\} \) with a noetherian ordering \( \succ \), that any term \( t \) innermost terminates (i.e. there is no infinite innermost rewriting chain starting from \( t \)). The terms of \( T(\mathcal{F}) \) are 0, 1, or terms of the form \( f(t_1, t_2, t_3) \), or \( g(t_1, t_2) \), with \( t_1, t_2, t_3 \in T(\mathcal{F}) \).

The terms 0 and 1 are obviously terminating.

Let us now prove that \( f(t_1, t_2, t_3) \) is innermost terminating. First, \( f(t_1, t_2, t_3) \succ t_1, t_2, t_3 \) for any term ordering with the subterm property (i.e. any term is greater than any of its subterms). Then, by induction hypothesis, assume that \( t_1, t_2 \) and \( t_3 \) innermost terminate. Let \( t_1 \downarrow, t_2 \downarrow, t_3 \downarrow \) be respectively any of their normal forms. The problem is then reduced to innermost termination of all \( f(t_1 \downarrow, t_2 \downarrow, t_3 \downarrow) \). If \( t_1 \downarrow = 0 \), \( t_2 \downarrow = 1 \), then \( f(0, 1, t_3 \downarrow) \) only rewrites at the top position into \( f(t_3 \downarrow, t_3 \downarrow, t_3 \downarrow) \), which is in normal form. Else \( f(t_1 \downarrow, t_2 \downarrow, t_3 \downarrow) \) is already in normal form.

Let us finally prove that \( g(t_1, t_2) \) is innermost terminating. First, \( g(t_1, t_2) \succ t_1, t_2 \). Then, by induction hypothesis, assume that \( t_1 \) and \( t_2 \) innermost terminate. Let \( t_1 \downarrow, t_2 \downarrow \) be respectively any of their normal forms. It is then sufficient to prove that \( g(t_1 \downarrow, t_2 \downarrow) \) is innermost terminating. The term \( g(t_1 \downarrow, t_2 \downarrow) \) rewrites either into \( t_1 \downarrow \) or \( t_2 \downarrow \), at the top position, with both \( t_1 \downarrow \) and \( t_2 \downarrow \) in normal form. Remark that for \( \succ \) in this proof, any ordering having the subterm property is convenient. Our goal is to provide a procedure implementing such a reasoning.

The paper is organized as follows: in Section 2, the background is presented. Section 3 introduces the inductive proof principle of our approach. Section 4 gives the basic concepts of our inductive proof mechanism based on abstraction and narrowing, and the involved constraints. Section 5 presents the generic termination proof procedure that is further applied to different rewriting strategies. In Section 6, the mechanism is instantiated for the case of innermost termination. In Section 7, the procedure is applied to outermost termination. Finally, in section 8, the same method is adapted to the case of local strategies.
2. THE BACKGROUND

We assume that the reader is familiar with the basic definitions and notations of term rewriting given for instance in [Dershowitz and Jouannaud 1990]. $T(F, X)$ is the set of terms built from a given finite set $F$ of function symbols $f$ having arity $n \in \mathbb{N}$ (denoted $f : n$), and a set $X$ of variables denoted $x, y, \ldots$. $T(F)$ is the set of ground terms (without variables). The terms reduced to a symbol of arity 0 are called constants. Positions in a term are represented as sequences of integers. The empty sequence $\epsilon$ denotes the top position. The symbol at the top position of a term $t$ is written $\text{top}(t)$. Let $p$ and $p'$ be two positions. The position $p$ is said to be (a strict) prefix of $p'$ (and $p'$ suffix of $p$) if $p' = p \lambda$, where $\lambda$ is a non-empty sequence of integers. Given a term $t$, $\text{Var}(t)$ is the set of variables of $t$. The set of terms built from a given finite set $T$ is denoted $\text{tr}$. The set of terms built from a given finite set $F$ is denoted $T(F)$.

A substitution is an assignment from $X$ to $T(F, X)$, written $\sigma = (x \mapsto t) \ldots (y \mapsto u)$. It uniquely extends to an endomorphism of $T(F, X)$. The result of applying $\sigma$ to a term $t \in T(F, X)$ is written $\sigma(t)$ or $\sigma t$. The domain of $\sigma$, denoted $\text{Dom}(\sigma)$ is the finite subset of $X$ such that $\sigma x \neq x$. The range of $\sigma$, denoted $\text{Ran}(\sigma)$, is defined by $\text{Ran}(\sigma) = \bigcup_{x \in \text{Dom}(\sigma)} \text{Var}(\sigma x)$. We have in addition $\text{Dom}(\sigma) \cap \text{Ran}(\sigma) = \emptyset$. A ground substitution or instantiation is an assignment from $X$ to $T(F)$. $I d$ denotes the identity substitution. The composition of substitutions $\sigma_2 \circ \sigma_1$. Given a subset $X_1$ of $X$, we write $\sigma_{X_1}$ for the restriction of $\sigma$ to the variables of $X_1$, i.e. the substitution such that $\text{Dom}(\sigma_{X_1}) \subseteq X_1$ and $\forall x \in \text{Dom}(\sigma_{X_1}) : \sigma_{X_1} x = \sigma x$.

Given a set $R$ of rewrite rules (a set of pairs of terms of $T(F, X)$, denoted $l \rightarrow r$, such that $\text{Var}(r) \subseteq \text{Var}(l)$) or term rewriting system on $T(F, X)$, a function symbol in $F$ is called a constructor if it does not occur in $R$ at the top position of a left-hand side of rule, and is called a defined function symbol otherwise. The set of defined function symbols of $F$ for $R$ is denoted by $\text{Def}_R$ ($R$ is omitted when there is no ambiguity).

The rewriting relation induced by $R$ is denoted by $\Rightarrow_R$ (→ if there is no ambiguity on $R$), and defined by $s \rightarrow t$ if there exists a substitution $\sigma$ and a position $p$ in $s$ such that $s[p] = \sigma t$ for some rule $l \rightarrow r$ of $R$, and $t = s[\sigma r]_p$. This is written $s \rightarrow_R^{p \leftarrow r \sigma} t$ where either $p$ or $l \rightarrow r$ or $\sigma$ or $R$ may be omitted; $s[p]$ is called a redex. The reflexive transitive closure of the rewriting relation induced by $R$ is denoted by $\Rightarrow_R$. If $t \not\Rightarrow t'$ and $t'$ cannot be rewritten anymore, then $t'$ is called a normal form of $t$ and denoted by $\rightarrow_N t$. Remark that given $t$, $\rightarrow_N$ may not be unique.

Let $R$ be a TRS on $T(F, X)$. A term $t$ is narrowed into $t'$, at the non-variable position $p$, using the rewrite rule $l \rightarrow r$ of $R$ and the substitution $\sigma$, when $\sigma$ is a most general unifier of $t[p]$ and $l$, and $t' = \sigma(t[p])$. This is denoted $t \rightarrow_R^{p \leftarrow r \sigma} t'$ where either $p$, or $l \rightarrow r$ or $\sigma$ may be omitted. It is always assumed that there is no variable in common between the rule and the term, i.e. that $\text{Var}(l) \cap \text{Var}(t) = \emptyset$.

An ordering $\succ$ on $T(F, X)$ is said to be noetherian iff there is no infinite decreasing chain for this ordering. It is $F$-stable iff for any pair of terms $t, t'$ of $T(F, X)$, for
any context \( f(\ldots, t, \ldots) \), \( t \succ t' \) implies \( f(\ldots, t, \ldots) \succ f(\ldots, t', \ldots) \). It has the subterm property iff for any \( t \in T(\mathcal{F}, \mathcal{X}) \), \( f(\ldots, t, \ldots) \succ t \). Observe that, for \( \mathcal{F} \) and \( \mathcal{X} \) finite, if \( \succ \) is \( \mathcal{F} \)-stable and has the subterm property, then it is noetherian [Kruskal 1960]. If, in addition, \( \succ \) is stable under substitution (for any substitution \( \sigma \), any pair of terms \( t, t' \in T(\mathcal{F}, \mathcal{X}), t \succ t' \) implies \( \sigma t \succ \sigma t' \)), then it is called a simplification ordering. Let \( t \) be a term of \( T(\mathcal{F}) \); let us recall that \( t \) terminates if and only if any rewriting derivation (or derivation chain) starting from \( t \) is finite.

Rewriting strategies are in general aimed at reducing the derivation tree (for standard rewriting) of terms. The following definition expresses that rewriting a term with a strategy \( S \) can only give a term that would be obtained with the standard rewriting relation.

**Definition 2.1 (rewriting strategy).** Let \( \mathcal{R} \) a TRS on \( T(\mathcal{F}, \mathcal{X}) \). A rewriting strategy \( S \) for \( \mathcal{R} \) is a mapping \( S : T(\mathcal{F}, \mathcal{X}) \rightarrow T(\mathcal{F}, \mathcal{X}) \) such that for every \( t \in T(\mathcal{F}, \mathcal{X}) \), \( S(t) = t' \) (we write \( t \rightarrow^S t' \)) where \( t' \) is such that \( t \rightarrow_{\mathcal{R}} t' \).

**Definition 2.2 (innermost/outermost strategy).** Let \( \mathcal{R} \) a TRS on \( T(\mathcal{F}, \mathcal{X}) \). The innermost (resp. outermost) strategy is a rewriting strategy written \( S = Innermost \) (resp. \( S = Outermost \)) such that for any term \( t \in T(\mathcal{F}, \mathcal{X}) \), if \( t \rightarrow^S t' \), the rewriting position \( p \) in \( t \) is such that there is no suffix (resp. prefix) position \( p' \) of \( p \) such that \( t \) rewrites at position \( p' \).

Rewriting strategies may be more complex to define. This is the case for local strategies on operators, used in the OBJ-like languages. We use here the notion of local strategy as expressed in [Goguen et al. 1992] and studied in [Eker 1998].

**Definition 2.3 (LS-strategy).** An LS-strategy is given by a function \( LS \) from \( \mathcal{F} \) to the set of lists of integers \( \mathcal{L}(\mathbb{N}) \), that induces a rewriting strategy as follows.

Given a LS-strategy such that \( LS(f) = \{p_1, \ldots, p_k\}, p_i \in [0..\text{arity}(f)] \) for all \( i \in [1..k] \), for some symbol \( f \in \mathcal{F} \), normalizing a term \( t = f(t_1, \ldots, t_m) \in T(\mathcal{F}, \mathcal{X}) \) with respect to \( LS(f) = \{p_1, \ldots, p_k\} \), consists in normalizing all subterms of \( t \) at positions \( p_1, \ldots, p_k \) successively, according to the strategy. If there exists \( i \in [1..k] \) such that \( p_1, \ldots, p_{i-1} \neq 0 \) and \( p_i = 0 \) (0 is the top position), then

— if the current term \( t' \) obtained after normalizing \( t|_{p_1}, \ldots, t|_{p_{i-1}} \) is reducible at the top position into a term \( g(u_1, \ldots, u_n) \), then \( g(u_1, \ldots, u_n) \) is normalized with respect to \( LS(g) \) and the rest of the strategy \( \{p_{i+1}, \ldots, p_k\} \) is ignored,

— if \( t' \) is not reducible at the top position, then \( t' \) is normalized with respect to \( p_{i+1}, \ldots, p_k \).

Let \( t \) be a term of \( T(\mathcal{F}) \); we say that \( t \) terminates (w.r.t. to the strategy \( S \)) if and only if every rewriting derivation (or derivation chain) (w.r.t. to the strategy \( S \)) starting from \( t \) is finite. Given a term \( t \), we call normal form (w.r.t. to the strategy \( S \)) or S-normal form of \( t \), and we note it \( t^S \), any irreducible term, if it exists, such that \( t \rightarrow^S t^S \).

### 3. THE INDUCTIVE PROOF PROCESS

#### 3.1 Lifting rewriting trees into proof trees

For proving that a term \( t \) of \( T(\mathcal{F}) \) terminates (for the considered strategy), we proceed by induction on \( T(\mathcal{F}) \) with a noetherian ordering \( \succ \), assuming that for any
such that \( t \succ t' \), \( t' \) terminates. To warrant non emptiness of \( \mathcal{T}(\mathcal{F}) \), we assume that \( \mathcal{F} \) contains at least a constructor constant.

The main intuition is to observe the rewriting derivation tree (for the considered strategy) starting from a ground term \( t \in \mathcal{T}(\mathcal{F}) \) which is any instance of a term \( g(x_1, \ldots, x_m) \), for some defined function symbol \( g \in \text{Def} \), and variables \( x_1, \ldots, x_m \). Proving termination on ground terms amounts proving that all rewriting derivation trees have only finite branches, using the same induction ordering \( \succ \) for all trees.

Each rewriting derivation tree is simulated, using a lifting mechanism, by a proof tree, developed from \( g(x_1, \ldots, x_m) \) on \( \mathcal{T}(\mathcal{F}, \mathcal{X}) \), for every \( g \in \text{Def} \), by alternatively using two main operations, namely narrowing and abstraction, adapted to the considered rewriting strategy. More precisely, narrowing schematizes all rewriting possibilities of terms. The abstraction process simulates the normalization of subterms in the derivations, according to the strategy. It consists in replacing these subterms by special variables, denoting one of their normal forms, without computing them. This abstraction step is performed on subterms that can be assumed terminating by induction hypothesis.

The schematization of ground rewriting derivation trees is achieved through constraints. The nodes of the developed proof trees are composed of a current term of \( \mathcal{T}(\mathcal{F}, \mathcal{X}) \), and a set of ground substitutions represented by a constraint progressively built along the successive abstraction and narrowing steps. Each node in a proof tree schematizes a set of ground terms: the ground instances of the current term, that are solutions of the constraint.

The constraint is in fact composed of two kinds of formulas: ordering constraints, set to warrant the validity of the inductive steps, and abstraction constraints combined to narrowing substitutions, which effectively define the relevant sets of ground terms. The latter are actually useful for controlling the narrowing process, well known to easily diverge.

The termination proof procedures given in this paper are described by deduction rules applied with a special control \( \text{Strat-Rules}(S) \), depending on the studied rewriting strategy \( S \). To prove termination of \( \mathcal{R} \) on any term \( t \in \mathcal{T}(\mathcal{F}) \) w.r.t. the strategy \( S \), we consider a so-called reference term \( t_{\text{ref}} = g(x_1, \ldots, x_m) \) for each defined symbol \( g \in \text{Def} \), and empty sets \( \top \) of constraints. Applying the deduction rules according to the strategy \( \text{Strat-Rules}(S) \) to the initial state \( (\{g(x_1, \ldots, x_m)\}, \top, \top) \) builds a proof tree, whose nodes are the states produced by the inference rules. Branching is produced by the different possible narrowing steps.

Termination is established when the procedure terminates because the deduction rules do not apply anymore and all terminal states of all proof trees have an empty set of terms.

### 3.2 A generic mechanism for strategies

As said previously, we consider any term of \( \mathcal{T}(\mathcal{F}) \) as a ground instance of a term \( t \) of \( \mathcal{T}(\mathcal{F}, \mathcal{X}) \) occurring in a proof tree issued from a reference term \( t_{\text{ref}} \). Using the termination induction hypothesis on \( \mathcal{T}(\mathcal{F}) \) naturally leads us to simulate the rewriting relation by two mechanisms:

—first, some subterms \( t_j \) of the current term \( t \) of the proof tree are supposed to
have only terminating ground instances, by induction hypothesis, if $\theta_{t_{\text{ref}}} \succ \theta_{t_j}$ for the induction ordering $\succ$ and for every $\theta$ solution of the constraint associated to $t$. They are replaced in $t$ by abstraction variables $X_j$ representing respectively one of their normal forms $t_j \downarrow$. Reasoning by induction allows us to only suppose the existence of the $t_j \downarrow$ without explicitly computing them.

—second, narrowing (w.r.t. to the strategy $S$) the resulting term $u = t[X]_{j \in \{i_1, \ldots, i_p\}}$ (where $i_1, \ldots, i_p$ are the positions of the abstracted subterm $t_j$ in $t$) into terms $v$, according to the possible instances of the $X_j$. This corresponds to rewriting (w.r.t. to the strategy $S$) the possible ground instances of $u$ (characterized by the constraint associated to $u$) in all possible ways.

In general, the narrowing step of $u$ is not unique. We obviously have to consider all terms $v$ such that $\theta u$ rewrites into $\theta v$, which corresponds to considering all narrowing steps from $u$.

Then the termination problem of the ground instances of $t$ is reduced to the termination problem of the ground instances of $v$. If $\theta_{t_{\text{ref}}} \succ \theta v$ for every ground substitution $\theta$ solution of the constraint associated to $v$, by induction hypothesis, $\theta v$ is supposed to be terminating. Else, the process is iterated on $v$, until getting a term $v'$ such that either $\theta_{t_{\text{ref}}} \succ \theta v'$, or $\theta v'$ is irreducible.

We introduce in the next section the necessary concepts to formalize and automate this technique.

4. ABSTRACTION, NARROWING, AND THE INVOLVED CONSTRAINTS

4.1 Ordering constraints

The induction ordering is constrained along the proof by imposing constraints between terms that must be comparable, each time the induction hypothesis is used in the abstraction mechanism. As we are working with a lifting mechanism on the proof trees with terms of $T(F, X)$, we directly work with an ordering $\succ_P$ on $T(F, X)$ such that $t \succ_P u$ implies $\theta t \succ \theta u$, for every $\theta$ solution of the constraint associated to $u$.

So inequalities of the form $t > u_1, \ldots, u_m$ are accumulated, which are called ordering constraints. Any ordering $\succ_P$ on $T(F, X)$ satisfying them and which is stable under substitution fulfills the previous requirements on ground terms. The ordering $\succ_P$, defined on $T(F, X)$, can then be seen as an extension of the induction ordering $\succ$, defined on $T(F)$. For convenience, the ordering $\succ_P$ will also be written $\succ$.

It is important to remark that, for establishing the inductive termination proof, it is sufficient to decide whether there exists such an ordering.

**Definition 4.1.1 (Ordering Constraint).** An ordering constraint is a pair of terms of $T(F, X)$ noted $(t > t')$. It is said to be satisfiable if there exists an ordering $\succ$, such that for every instantiation $\theta$ whose domain contains $\text{Var}(t) \cup \text{Var}(t')$, we have $\theta t \succ \theta t'$. We say that $\succ$ satisfies $(t > t')$.

A conjunction $C$ of ordering constraints is satisfiable if there exists an ordering satisfying all conjuncts. The empty conjunction, always satisfied, is denoted by $\top$.
Satisfiability of a constraint conjunction $C$ of this form is undecidable. But a sufficient condition for an ordering $\succ$ to satisfy $C$ is that $\succ$ is stable under substitution and $t \succ t'$ for any constraint $t \succ t'$ of $C$.

4.2 Abstraction

To abstract a term $t$ at positions $i_1, \ldots, i_p$, where the $t|_j$ are supposed to have a normal form $t|_j\sigma$, we replace the $t|_j$ by abstraction variables $X_j$ representing respectively one of their possible normal forms. Let us define these special variables more formally.

**Definition 4.2.1.** Let $X_A$ be a set of variables disjoint from $X$. Symbols of $X_A$ are called abstraction variables. Substitutions and instantiations are extended to $T(F, X \cup X_A)$ in the following way: let $X \in X_A$; for any substitution $\sigma$ (resp. instantiation $\theta$) such that $X \in \text{Dom}(\sigma)$, $\sigma X$ (resp. $\theta X$) is in $S$-normal form.

**Definition 4.2.2 (term abstraction).** The term $t[t|_j]\subset {i_1, \ldots, i_p}$ is said to be abstracted into the term $u$ (called abstraction of $t$) at positions $\{i_1, \ldots, i_p\}$ iff $u = t[X_j]|_{j \in \{i_1, \ldots, i_p\}}$, where the $X_j, j \in \{i_1, \ldots, i_p\}$ are fresh distinct abstraction variables.

Termination on $T(F)$ is proved by reasoning on terms with abstraction variables, i.e. on terms of $T(F, X \cup X_A)$. Ordering constraints are extended to pairs of terms of $T(F, X \cup X_A)$. When subterms $t|_j$ are abstracted by $X_j$, we state constraints on abstraction variables, called abstraction constraints to express that their instances can only be normal forms of the corresponding instances of $t|_j$. Initially, they are of the form $t| = X$ where $t \in T(F, X \cup X_A)$, and $X \in X_A$, but we will see later how they are combined with the substitutions used for the narrowing process.

4.3 Narrowing

After abstraction of the current term $t$ into $t[X_j]|_{j \in \{i_1, \ldots, i_p\}}$, we check whether the possible ground instances of $t[X_j]|_{j \in \{i_1, \ldots, i_p\}}$ are reducible, according to the possible values of the instances of the $X_j$. This is achieved by narrowing $t[X_j]|_{j \in \{i_1, \ldots, i_p\}}$.

The narrowing relation depends on the considered strategy $S$ and needs to be refined. The first idea is to use innermost (resp. outermost) narrowing. Then, if a position $p$ in a term $t$ is a narrowing position, a suffix (resp. prefix) position of $p$ cannot be a narrowing position too. However, if we consider ground instances of $t$, we can have rewriting positions $p$ for some instances, and $p'$ for some other instances, such that $p'$ is a suffix (resp. a prefix) of $p$. So, when narrowing at some position $p$, the set of relevant ground instances of $t$ is defined by excluding the ground instances that would be narrowable at some suffix (resp. prefix) position of $p$, that we call $S$-better position: a position $S$-better than a position $p$ in $t$ is a suffix position of $p$ if $S$ is the innermost strategy, a prefix position of $p$ if $S$ is the outermost strategy. Note that local strategies are not of the same nature, and there is no $S$-better position in this case.

Moreover, to preserve the fact that a narrowing step of $t$ schematizes a rewriting step of possible ground instances of $t$, we have to be sure that an innermost (resp. outermost) narrowing redex in $t$ corresponds to the same rewriting redex in a ground instance of $t$. This is the case only if, in the rewriting chain of the ground instance of $t$, there is no rewriting redex at a suffix position of variable of $t$ anymore.
So before each narrowing step, we schematize the longest rewriting chain of any ground instance of \( t \), whose redexes occur in the variable part of the instantiation, by a linear variable renaming. Linearity is crucial to express that, in the previous rewriting chain, ground instances of the same variables can be reduced in different ways. For the innermost strategy, abstraction of variables performs this schematization. For the outermost strategy, a reduction renaming will be introduced. For local strategies however, this variable renaming is not relevant.

The \( S \)-narrowing steps applying to a given term \( t \) are computed in the following way. After applying the variable renaming to \( t \), we look at every position \( p \) of \( t \) such that \( t|_p \) unifies with the left-hand side of a rule using a substitution \( \sigma \). The position \( p \) is a \( S \)-narrowing position of \( t \), iff there is no \( S \)-better position \( p' \) of \( t \) such that \( \sigma|_{p'} \) unifies with a left-hand side of rule. Then we look for every \( S \)-better position \( p' \) than \( p \) in \( t \) such that \( \sigma|_{p'} \) narrows with some substitution \( \sigma' \) and some rule \( l' \rightarrow r' \), and we set a constraint to exclude these substitutions. So the substitutions used to narrow a term have in general to satisfy a set of disequalities coming from the negation of previous substitutions. To formalize this point, we need the following notations and definitions.

In the following, we identify a substitution \( \sigma = (x_1 \mapsto t_1) \ldots (x_n \mapsto t_n) \) on \( \mathcal{T}(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A) \) with the finite set of solved equations \( (x_1 = t_1) \land \ldots \land (x_n = t_n) \), also denoted by the equality formula \( \bigwedge_i (x_i = t_i) \), with \( x_i \in \mathcal{X} \cup \mathcal{X}_A, t_i \in \mathcal{T}(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A) \), where \( = \) is the syntactic equality. Similarly, we call negation \( \overline{\sigma} \) of the substitution \( \sigma \) the formula \( \bigvee_i (x_i \neq t_i) \).

**Definition 4.3.1 (Constrained Substitution).** A constrained substitution \( \sigma \) is a formula \( \sigma_0 \land \bigwedge_j \bigvee_{i_j} (x_{i_j} \neq t_{i_j}) \), where \( \sigma_0 \) is a substitution.

**Definition 4.3.2 (\( S \)-Narrowing).** A term \( t \in \mathcal{T}(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A) \) \( S \)-narrow into a term \( t' \in \mathcal{T}(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A) \) at the non-variable position \( p \) of \( t \), using the rule \( l \rightarrow r \in \mathcal{R} \) with the constrained substitution \( \sigma = \sigma_0 \land \bigwedge_{j \in [1..k]} \overline{\sigma_j} \), which is written \( t \sim_{p, l \rightarrow r, \sigma} t' \) iff

\[
\sigma_0(l) = \sigma_0(t|_p) \text{ and } t' = \sigma_0(t[\overline{r}]_{p})
\]

where \( \sigma_0 \) is the most general unifier of \( t|_p \) and \( l \) and \( \sigma_j, j \in [1..k] \) are all most general unifiers of \( \sigma_0|_{l'} \) and a left-hand side \( l' \) of a rule of \( \mathcal{R} \), for all position \( p' \) which are \( S \)-better positions than \( p \) in \( t \).

It is always assumed that there is no variable in common between the rule and the term, i.e. that \( \text{Var}(l) \cap \text{Var}(t) = \emptyset \). This requirement of disjoint variables is easily fulfilled by an appropriate renaming of variables in the rules when narrowing is performed. The most general unifier \( \sigma_0 \) used in the above definition can be taken such that its range only contains fresh variables. Since we are interested in the narrowing substitution applied to the current term \( t \), but not in its definition on the variables of the left-hand side of the rule, the narrowing substitutions can be restricted to the variables of the narrowed term \( t \).

The following lifting lemma, generalized from [Middeldorp and Hamoen 1994], ensures the correspondence between the narrowing relation, used during the proof, and the rewriting relation.
Lemma 4.3.2 (S-lifting Lemma). Let $\mathcal{R}$ be a TRS. Let $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $\alpha$ a ground substitution such that $\alpha s$ is $S$-reducible at a non variable position $p$ of $s$, and $\mathcal{Y} \subseteq \mathcal{X}$ a set of variables such that $\text{Var}(s) \cup \text{Dom}(\alpha) \subseteq \mathcal{Y}$. If $\alpha s \overset{S}{\rightarrow}_{p,t} t'$, then there exist a term $s' \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and substitutions $\beta, \sigma = \sigma_0 \land \bigwedge_{j \in [1..k]} \overline{\sigma_j}$ such that:

1. $s \overset{S}{\rightarrow}_{p,t,r,\sigma} s'$,
2. $\beta s' = t'$,
3. $\beta \sigma_0 = \alpha[\mathcal{Y}]$
4. $\beta$ satisfies $\bigwedge_{j \in [1..k]} \overline{\sigma_j}$

where $\sigma_0$ is the most general unifier of $s|_p$ and $t$ and $\sigma_j, j \in [1..k]$ are all most general unifiers of $\sigma_0|_{p'}$ and a left-hand side $t'$ of a rule of $\mathcal{R}$, for all position $p'$ which are $S$-better positions than $p$ in $s$.

4.4 Cumulating constraints

Abstraction constraints have to be combined with the narrowing constrained substitutions to characterize the ground terms schematized by the proof trees. A narrowing step is applied to a current term $t$ if the narrowing constrained substitution $\sigma$ effectively corresponds to a rewriting step of ground instances of $u$, i.e. if $\sigma$ is compatible with the abstraction constraint formula $A$ associated to $u$ (i.e. $A \land \sigma$ is satisfiable). So the narrowing constraint attached to the narrowing step is added to $A$. This motivates the introduction of abstraction constraint formulas.

**Definition 4.4.1 (Abstraction Constraint Formula).** An abstraction constraint formula (ACF in short) is a formula $\bigwedge_i (t_i \downarrow = t'_i) \land \bigwedge_j (x_j = t_j) \land \bigwedge_k \mathcal{V}_{t_k}(u_k \neq v_{l_k})$, where $t_i, t'_i, j, u_k, v_{l_k} \in \mathcal{T}(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A)$, $x_j \in \mathcal{X} \cup \mathcal{X}_A$.

**Definition 4.4.2 (Satisfiability of an ACF).** An abstraction constraint formula $\bigwedge_i (t_i \downarrow = t'_i) \land \bigwedge_j (x_j = t_j) \land \bigwedge_k \mathcal{V}_{t_k}(u_k \neq v_{l_k})$, is satisfiable iff there exists at least one instantiation $\theta$ such that $\bigwedge_i (\theta t_i \downarrow = \theta t'_i) \land \bigwedge_j (\theta x_j = \theta t_j) \land \bigwedge_k \mathcal{V}_{t_k}(\theta u_k \neq \theta v_{l_k})$. The instantiation $\theta$ is then said to satisfy the ACF $A$ and is called solution of $A$.

Applying a constrained substitution $\sigma = \sigma_0 \land \bigwedge_j \mathcal{V}_{t_j}(x_j \neq t_j)$ to an ACF $A$ is done by adding the formula defining $\sigma$ to $A$, thus giving the formula $A \land \sigma$.

While new abstraction constraints are initially of the form $t_i \downarrow = X_i$, we can propagate $\sigma$ into $A$ (by applying $\sigma_0$ to $A$), thus getting instantiated abstraction constraints of the form $t_i \downarrow = t'_i$.

An ACF $A$ is attached to each term $u$ in the proof trees; its solutions characterize the interesting ground instances of this term, i.e. the $\theta u$ such that $\theta$ is a solution of $A$. When $A$ has no solution, the current node of the proof tree represents no ground term. Such nodes are then irrelevant for the termination proof. So we have the choice between generating only the relevant nodes of the proof tree, by testing satisfiability of $A$ at each step, or stopping the proof on a branch on an irrelevant node, by testing unsatisfiability of $A$. These are both facets of the same question, but in practice, they are handled in different ways.

Checking satisfiability of $A$ is in general undecidable. The disequality part of an ACF is a particular instance of a disunification problem (a quantifier free equational formula), whose satisfiability has been addressed in [Conon 1991], that provides
rules to transform any disunification problem into a solved form. Testing satisfiability of the equational part of an ACF is undecidable in general, but sufficient conditions can be given, relying on a characterization of normal forms.

Unsatisfiability of $A$ is also undecidable in general, but simple sufficient conditions can be used, very often applicable in practice. They rely on reducibility, unifiability, narrowing and constructor tests.

According to Definition 4.4.2, an ACF $\bigwedge_i (t_i \downarrow = t_i') \land \bigwedge_j (x_j = t_j) \land \bigwedge_k \bigvee_k (u_k \neq v_k)$ is unsatisfiable if for instance, one of its conjunct $t_i \downarrow = t_i'$ is unsatisfiable, i.e. is such that $\theta t_i'$ is not a normal form of $\theta t_i$ for any ground substitution $\theta$. Hence, we get four automatable conditions for unsatisfiability of an abstraction constraint $t \downarrow = t'$:

**Case 1.** $t \downarrow = t'$, with $t'$ reducible. Indeed, in this case, any ground instance of $t'$ is reducible, and hence cannot be a normal form.

**Case 2.** $t \downarrow = t' \land \ldots \land t' \downarrow = t''$, with $t'$ and $t''$ not unifiable. Indeed, any ground substitution $\theta$ satisfying the above conjunction is such that (1) $\theta t \downarrow = \theta t'$ and (2) $\theta t' \downarrow = \theta t''$. In particular, (1) implies that $\theta t'$ is in normal form and hence (2) imposes $\theta t' = \theta t''$, which is impossible if $t'$ and $t''$ are not unifiable.

**Case 3.** $t \downarrow = t'$ where $\text{top}(t)$ is a constructor, and $\text{top}(t) \neq \text{top}(t')$. Indeed, if the top symbol of $t$ is a constructor $s$, then any normal form of any ground instance of $t$ is of the form $s(u)$, where $u$ is a ground term in normal form. The above constraint is therefore unsatisfiable if the top symbol of $t'$ is $g$, for some $g \neq s$.

**Case 4.** $t \downarrow = t'$ with $t, t' \in T(F, X_A)$ not unifiable and $\bigwedge \forall v \in v \downarrow = t'$ unsatisfiable. This criterion is of interest if unsatisfiability of each conjunct $v \downarrow = t'$ can be shown with one of the four criteria we present here.

So both satisfiability and unsatisfiability checks need to use sufficient conditions. But in the first case, the proof process stops with failure as soon as satisfiability of $A$ cannot be proved. In the second one, it can go on, until $A$ is proved to be unsatisfiable, or until other stopping conditions are fulfilled.

Let us now come back to ordering constraints. If we check satisfiability of $A$ at each step, we only generate states in the proof trees, that represent non empty sets of ground terms. So in fact, the ordering constraints of $C$ have not to be satisfied for every ground instance, but only for those instances that are solution of $A$, hence the following definition, that can be used instead of Definition 4.1.1, when constraints of this definition cannot be proved satisfiable, and solutions of $A$ can easily be characterized.

**Definition 4.4.3 (Constraint Problem).** Let $A$ be an abstraction constraint formula and $C$ a conjunction of ordering constraints. The constraint problem $C/A$ is satisfied by an ordering $\succ$ iff for every instantiation $\theta$ satisfying $A$, then $\theta t \succ \theta t'$ for every conjunct $t > t'$ of $C$. $C/A$ is satisfiable iff there exists an ordering $\succ$ as above.

Note that $C/A$ may be satisfiable even if $A$ is not.

### 4.5 Relaxing the induction hypothesis

It is important to point out the flexibility of the proof method that allows the combination with auxiliary termination proofs using different techniques: when the
induction hypothesis cannot be applied on a term $u$, i.e. when it is not possible to decide whether the ordering constraints are satisfiable, it is often possible to prove termination (for the considered strategy) of any ground instance of $u$ by another way. In the following we use a predicate $\text{TERM}(S, u)$ that is true iff every ground instance of $u$ terminates for the considered strategy $S$.

In particular, $\text{TERM}(S, u)$ is true when every instance of $u$ is in normal form. This is the case when $u$ is not narrowable, and all variables of $u$ are in $X_A$. Indeed, by Lemma 4.3.1 and Definition 4.2.1, every instance of $u$ is in normal form. This includes the cases where $u$ itself is an abstraction variable, and where $u$ is a non-narrowable ground term.

Every instance of a narrowable $u$ whose variables are all in $X_A$, and whose narrowing substitutions are not compatible with $A$, is also in normal form. As said in Section 4.4, these narrowing possibilities do not represent any reduction step for the ground instances of $u$, which are then in normal form.

Otherwise, in many cases, for proving that $\text{TERM}(S, u)$ is true, the notion of usable rules [Arts and Giesl 1996] is relevant. Given a TRS $R$ on $\mathcal{T}(F, X)$ and a term $t \in \mathcal{T}(F, X \cup X_A)$, the usable rules of $t$ are a subset of $R$, which is a computable superset of the rewrite rules that are likely to be used in any rewriting chain (for the standard strategy) starting from any ground instance of $t$, until its ground normal forms are reached, if they exist.

Proving termination of any ground instance of $u$ then comes down to proving termination of its usable rules, which is in general much easier than proving termination of the whole TRS $R$. In general, we try to find a simplification ordering $\succ_N$ that orients these rules. Thus any ground instance $\alpha t$ is bound to terminate for the standard rewriting relation, and then for the rewriting strategy $S$. Indeed, if $\alpha t \rightarrow t_1 \rightarrow t_2 \rightarrow \ldots$, then, thanks to the previous hypotheses, $\alpha t \succ_N t_1 \succ_N t_2 \succ_N \ldots$ and, since the ordering $\succ_N$ is noetherian, the rewriting chain cannot be infinite. As a particular case, when a simplification ordering can be found to orient the whole TRS, it also orients the usable rules of any term, and our inductive approach can also conclude to termination. If an appropriate simplification ordering cannot be found, termination of the usable rules may also be proved with our inductive process itself. The fact that the induction ordering used for usable rules is independent of the main induction ordering, makes the proof very flexible. Complete results on usable rules for the innermost strategy are given in Section 6.2. For the outermost and local strategies, this is developed in [Fissore et al. 2002b] and [Fissore et al. 2001].

5. THE TERMINATION PROOF PROCEDURE

5.1 Strategy-independent proof steps

We are now ready to describe the different steps of the proof mechanism presented in Section 3.

The proof steps transform 3-tuples $(T, A, C)$ where

$- T$ is a set of terms of $\mathcal{T}(F, X \cup X_A)$, containing the current term $u$ whose termination has to be proved. $T$ is either a singleton or the empty set. For local strategies, the term is enriched by the list of positions where $u$ has to be evaluated, $LS(top(u))$. This is denoted by $u^{LS(top(u))}$. 


—A is a conjunction of abstraction constraints. At each abstraction step, constraints of the form $u \downarrow = X, u \in T(F, X \cup X_A), X \in X_A$ are stated for each subterm term $t$ abstracted into a new abstraction variable $X$. At each narrowing step with narrowing substitution $\sigma$, $A$ is replaced by $A \land \sigma$.

—$C$ is a conjunction of ordering constraints stated by the abstraction steps.

Starting from initial states ($T = \{t_{ref} = g(x_1, \ldots, x_m)\}, A = \top, C = \top$), where $g \in Def$, the proof process consists in iterating the following generic steps:

—The first step abstracts the current term $t$ at given positions $i_1, \ldots, i_p$. If the conjunction of ordering constraints $\bigwedge_{j \in \{i_1, \ldots, i_p\}} t_{ref} > t_{ij}$ is satisfiable for some $j \in \{i_1, \ldots, i_p\}$, we suppose, by induction, the existence of irreducible forms for the $t_{ij}$. We must have $\text{TERMIN}(S, t_{ij})$ for the other $t_{ij}$. Then, $t_{i_1}, \ldots, t_{i_p}$ are abstracted into abstraction variables $X_{i_1}, \ldots, X_{i_p}$. The abstraction constraints $t_{i_1} \downarrow = X_{i_1}, \ldots, t_{i_p} \downarrow = X_{i_p}$ are added to the ACF $A$. We call that step the abstract step.

—The second step narrows the resulting term $u$ in one step with all possible rewrite rules of the rewrite system $R$, and all possible substitutions $\sigma$, into terms $v$, according to Definition 4.3.2. This step is a branching step, creating as many states as narrowing possibilities. The substitution $\sigma$ is added to $A$. This is the narrow step.

—We then have a stop step halting the proof process on the current branch of the proof tree, when $A$ is detected to be unsatisfiable, or when the ground instances of the current term can be stated terminating for the considered strategy. This happens when the whole current term $u$ can be abstracted, i.e. when the induction hypothesis applies on it, or when we have $\text{TERMIN}(S, u)$.

The satisfiability and unsatisfiability tests of $A$ are integrated in the previously presented steps. If testing unsatisfiability of $A$ is chosen, the unsatisfiability test is integrated in the stop step. If testing the satisfiability of $A$ is chosen, the test is made at each attempt of an abstraction or a narrowing step, which are then effectively performed only if $A$ can be proved satisfiable. Otherwise, the proof cannot go on anymore and stops with failure.

As we will see later, for a given rewriting strategy $S$, these generic proof steps are instantiated by more precise mechanisms, depending on the strategy $S$, and taking advantage of its specificity. We will define these specific instances by inference rules.

5.2 Discussion on abstraction and narrowing positions

There are different ways to simulate the rewriting relation on ground terms, using abstraction and narrowing.

For example, the abstraction positions can be chosen so that the abstraction mechanism captures the greatest possible number of rewriting steps. For that, we abstract the greatest subterms in the term, that are the immediate subterms of the term. Then, if a narrowing step follows, the abstracted term has to be narrowed in all possible ways at the top position only. This strategy may yield a deadlock if some of the direct subterms cannot be abstracted. We can instead abstract all greatest possible subterms of $t = f(t_1, \ldots, t_n)$. More concretely, we try to abstract $t_1, \ldots, t_n$ and, for each $t_i = g(t'_1, \ldots, t'_{p_i})$ that cannot be abstracted, we try to
abstract \(t'_1, \ldots, t'_p\), and so on. In the worst case, we are driven to abstract leaves of the term, which are either variables, that do not need to be abstracted if they are abstraction variables, or constants.

On the contrary, we can choose in priority the smallest possible subterms \(u_i\), that are constants or variables. The ordering constraints \(t > u_i\) needed to apply the induction hypothesis, and then to abstract the term, are easier to satisfy than in the previous case since the \(u_i\) are smaller.

Between these two cases, there are a finite but possibly big number of ways to choose the positions where terms are abstracted. Anyway it is not useful to abstract the subterms, whose ground instances are in normal form. Identifying these subterms is made in the same way that for the study of \(\text{TERMIN}(S, u)\) (see Section 4.5).

From the point of view of the narrowing step following the abstraction, there is no general optimal abstracting strategy either: the greater the term to be narrowed, the greater is the possible number of narrowing positions. On another side, more general the term to be narrowed, greater is the possible number of narrowing substitutions for a given redex.

5.3 How to combine the proof steps

The previous proof steps, applied to every reference term \(t_{\text{ref}} = g(x_1, \ldots, x_m)\), where \(x_1, \ldots, x_m \in \mathcal{X}\) and \(g \in \text{Def}\), can be combined in the same way whatever \(S \in \{\text{Innermost}, \text{Outermost}, \text{Local}−\text{Strat}\}\):

\[
\text{Strat−Rules}(S) = \text{repeat}^* (\text{try(abstract)}, \text{try(narrow)}, \text{try(stop)}).
\]

"\(\text{repeat}^*(T_1, \ldots, T_n)\)" repeats the strategies of the set \(\{T_1, \ldots, T_n\}\) until it is not possible anymore. The operator "\(\text{try}\)" is a generic operator that can be instantiated, following \(S\), by \(\text{try}−\text{skip}(T)\), expressing that the strategy or rule \(T\) is tried, and skipped when it cannot be applied, or by "\(\text{try}−\text{stop}(T)\)\", stopping the strategy if \(T\) cannot be applied.

5.4 The termination theorem

For each strategy \(S \in \{\text{Innermost}, \text{Outermost}, \text{Local}−\text{Strat}\}\), we write \(\text{SUCCESS}(g, \succ)\) if the application of \(\text{Strat−Rules}(S)\) on \(((g(x_1, \ldots, x_m)), \top, \top)\) gives a finite proof tree, whose sets \(C\) of ordering constraints are satisfied by a same ordering \(\succ\), and whose leaves are either states of the form \((\emptyset, A, C)\) or states whose set of constraints \(A\) is unsatisfiable.

**Theorem 5.4.1.** Let \(R\) be a TRS on a set \(\mathcal{F}\) of symbols containing at least a constructor constant. If there exists an \(\mathcal{F}\)-stable ordering \(\succ\) having the subterm property, such that for each symbol \(g \in \text{Def}\), we have \(\text{SUCCESS}(g, \succ)\), then every term of \(T(\mathcal{F})\) terminates with respect to the strategy \(S\).

We are now ready to instantiate this generic proof process, according to the different rewriting strategies.
6. THE INNERMOST CASE

6.1 Abstraction and narrowing

When rewriting a ground instance of the current term according to the innermost principle, the ground instances of variables in the current term have to be normalized before a redex appears higher in the term. So the variable renaming performed before narrowing corresponds here to abstracting variables in the current term. Then, here, narrowing is only performed on terms of $T(\mathcal{F}, \mathcal{X}_A)$.

Moreover for the most general unifiers $\sigma$ produced during the proof process, all variables of $\text{Ran}(\sigma)$ are abstraction variables. Indeed, by Definition 4.2.1, if $X \in \text{Dom}(\sigma)$, $\sigma X$ is in normal form, as well as $\theta X$ for any instantiation $\theta$. By definition of the innermost strategy, this requires that variables of $\sigma X$ can only be instantiated by terms in normal form, i.e. variables of $\sigma X$ are abstraction variables.

Then, since before the first narrowing step, all variables are renamed into variables of $X A$, and the narrowing steps only introduce variables of $X A$, it is superfluous to rename the variables of the current term after the first narrowing step.

6.2 Relaxing the induction hypothesis

To establish $\text{TERMIN}($Innermost, $u))$, a simple narrowing test of $u$ can first be tried. Except for the initial state, the variables of $u$ are in $X A$. So if $u$ is not narrowable, or if $u$ is narrowable with a substitution $\sigma$ that is not compatible with $A$, then every ground instance of $u$ is in innermost normal form. Else, we compute the usable rules.

When $t$ is a variable of $\mathcal{X}$, the usable rules of $t$ are $R$ itself. When $t \in X A$, the set of usable rules of $t$ is empty, since the only possible instances of such a variable are ground terms in normal form.

Definition 6.2.1 Usable rules. Let $R$ be a TRS on a set $\mathcal{F}$ of symbols. Let $\text{Rls}(f) = \{ l \rightarrow r \in R \mid \text{root}(l) = f \}$. For any $t \in T(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A)$, the set of usable rules of $t$, denoted $\mathcal{U}(t)$, is defined by:

- $\mathcal{U}(t) = R$ if $t \in \mathcal{X}$,
- $\mathcal{U}(t) = \emptyset$ if $t \in \mathcal{X}_A$,
- $\mathcal{U}(f(u_1, \ldots, u_n)) = \text{Rls}(f) \cup \bigcup_{i=1}^{n} \mathcal{U}(u_i) \cup \bigcup_{l \rightarrow r \in \text{Rls}(f)} \mathcal{U}(r)$.

Lemma 6.2.1. Let $R$ be a TRS on a set $\mathcal{F}$ of symbols and $t \in T(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A)$. Whatever at ground instance of $t$ and at $\ldots \rightarrow p_{n-1} l_n \rightarrow r_n \ldots l_2 \rightarrow \ldots \rightarrow p_1 l_1 \rightarrow r_1 \in \mathcal{U}(t)$, $\forall i \in [1..n]$.

A sufficient criterion for ensuring standard termination (and then innermost termination) of any ground instance of a term $t$ can be given.

Proposition 6.2.1. Let $R$ be a TRS on a set $\mathcal{F}$ of symbols, and $t$ a term of $T(\mathcal{F}, \mathcal{X} \cup \mathcal{N})$. If there exists a simplification ordering $\succ$ such that $\forall l \rightarrow r \in \mathcal{U}(t) : l \succ r$, then any ground instance of $t$ is terminating.

6.3 The innermost termination proof procedure

The inference rules Abstract, Narrow and Stop instantiate respectively the proof steps abstract, narrow, and stop defined in Section 5.1. They are given in Table I.
Table I. Inference rules for the innermost strategy

<table>
<thead>
<tr>
<th>Rule</th>
<th>Precondition</th>
<th>Postcondition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>{t}, A, C \cap t[i_1] = X_{i_1} \ldots \cap t[i_p] = X_{i_p}, C \land H_C(t[i_1]) \ldots \land H_C(t[i_p])</td>
<td>where t is abstracted into u at positions i_1, \ldots, i_p \neq \epsilon if \text{COND-ABSTRACT}</td>
</tr>
<tr>
<td>Narrow</td>
<td>{t}, A, C \cap t \sim^{\text{inn}, \sigma}_R u and \text{COND-NARROW}</td>
<td>if \text{COND-NARROW}</td>
</tr>
<tr>
<td>Stop</td>
<td>\emptyset, A \land H_A(t), C \land H_C(t)</td>
<td>if \text{COND-STOP}</td>
</tr>
</tbody>
</table>

H_A(t) = \begin{cases} \top & \text{if any ground instance of } t \\ t[i] = X & \text{is in normal form} \end{cases}

H_C(t) = \begin{cases} \top & \text{if } \text{TERMIN}(\text{Innermost}, t) \\ t_{\text{ref}} > t & \text{otherwise} \end{cases}

Table II. Conditions for inference rules dealing with satisfiability of A

<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>COND-ABSTRACT :</td>
<td>( (A \land t[i_1] = X_{i_1} \ldots \land t[i_p] = X_{i_p}) \land (C \land H_C(t[i_1]) \ldots \land H_C(t[i_p])) ) are satisfiable</td>
</tr>
<tr>
<td>COND-NARROW :</td>
<td>( A \land \sigma ) is satisfiable</td>
</tr>
<tr>
<td>COND-STOP :</td>
<td>( (A \land H_A(t)) \land (C \land H_C(t)) ) are satisfiable</td>
</tr>
</tbody>
</table>

Their application conditions depend on whether satisfiability of A or unsatisfiability of A is checked. These conditions are specified in Tables II and III respectively.

As said above, the ground terms whose termination is studied are defined by the solutions of A. When satisfiability of A is checked at each inference step, the nodes of the proof tree exactly model the ground terms generated during the rewriting derivations. Satisfiability of A, although undecidable in general, can be proved by exhibiting a ground substitution satisfying the constraints of A.

When satisfiability of A is not checked, nodes are generated in the proof tree, that can represent empty sets of ground terms, so the generated proof trees can have branches that do not represent any derivation on the ground terms. The unsatisfiability test of A is only used to stop the development of meaningless branches as soon as possible, with the sufficient conditions presented in Section 4.4.

Once instantiated, the generic strategy \text{Strat-Rules}(S) simply becomes:

```
repeat*(try-skip(\text{Abstract}); try-stop(\text{Narrow}); try-skip(\text{Stop}))
```

with conditions of Table II, and
Table III. Conditions for inference rules dealing with unsatisfiability of $A$

| Cond-Abstract | $C \land H_C(t_{i_1}) \ldots \land H_C(t_{i_p})$ is satisfiable |
| Cond-Narrow | true |
| Cond-Stop | $(C \land H_C(t))$ is satisfiable or $A$ is unsatisfiable |

Table IV. Conditions for inference rules dealing with satisfiability of $A$

| Cond-Abstract | $(C \land H_C(t_{i_1}) \ldots \land H_C(t_{i_p}))$ is satisfiable |
| Cond-Narrow | $A \land \sigma$ is satisfiable |
| Cond-Stop | $(C \land H_C(t))$ is satisfiable |

repeat* (try-skip(Abstract); try-skip(Narrow); try-skip(Stop))

with conditions of Table III. Note that Narrow with conditions of Table II is the only rule stopping the proof procedure when it cannot be applied: in this case, when $A \land \sigma$ is satisfiable, the narrowing step can be applied, while, if satisfiability of $A \land \sigma$ cannot be proved, the procedure must stop.

The procedure can diverge, with infinite alternate applications of Abstract and Narrow. With conditions of Table II, it can stop on Narrow with at least in a branch of the proof tree, a state of the form $(\{t\} \neq \emptyset, A, C)$. In both cases, nothing can be said on termination. Termination is proved when, for all proof trees, the procedure stops with an application of Stop on each branch, generating only final states of the form $(\emptyset, A, C)$.

According to the strategy $\text{Strat-Rules(Innermost)}$, testing satisfiability of $A$ in conditions of Table II can be optimized on the basis of the following remarks. In the first application of Abstract for each initial state, $(A \land t_{i_1} \downarrow = X_{i_1} \ldots \land t_{i_p} \downarrow = X_{i_p}) = (\top \land x_1 \downarrow = X_1 \ldots \land x_m \downarrow = X_m)$, which is always satisfiable, since the signature admits at least a constructor constant. Moreover, the following possible current application of Abstract comes after an application of Narrow, for which it has been checked that $A \land \sigma$ is satisfiable. So $(A \land \sigma \land t_{i_1} \downarrow = X_{i_1} \ldots \land t_{i_p} \downarrow = X_{i_p})$ is also satisfiable since $X_{i_1}, \ldots, X_{i_p}$ are fresh variables, not used in $A \land \sigma$. So it is useless to verify satisfiability of $(A \land t_{i_1} \downarrow = X_{i_1} \ldots \land t_{i_p} \downarrow = X_{i_p})$ in Cond-Abstract.

In a similar way, as Stop is applied with a current abstraction constraint formula $A$, which is satisfiable, $A \land t \downarrow = X$ is also satisfiable since $X$ is a fresh variable, not used in $A$. So it is also useless to verify that $A \land t \downarrow = X$ is satisfiable in Cond-Stop.

This leads to the conditions expressed in Table IV, simplifying those of Table II.

6.4 Examples

For a better readability, when a constrained substitution $\sigma$ is added to the ACF $A$, we propagate the new constraint $\sigma$ into $A$ in applying the substitution part $\sigma_0$ of
σ to A.

Example 6.4.1. Let $R$ be the previous example of Toyama. We prove that $R$ is innermost terminating on $T(F)$, where $F = \{f:3, g:2, 0:0, 1:0\}$.

\[
f(0,1,x) \rightarrow f(x,x,x)
g(x,y) \rightarrow x
g(x,y) \rightarrow y
\]

The defined symbols of $F$ are here $f$ and $g$. Applying the rules on $f(x_1, x_2, x_3)$, we get:

\[
\begin{align*}
t_{ref} &= f(x_1, x_2, x_3) \\
A &= T \\
C &= T
\end{align*}
\]

Abstract

\[
\begin{align*}
f(x_1, x_2, x_3) \\
A &= (x_1 = X_1 \land x_2 = X_2 \land x_3 = X_3) \\
C &= (f(x_1, x_2, x_3) > x_1, x_2, x_3)
\end{align*}
\]

Narrow

\[
\begin{align*}
f(x_1, x_2, x_3) \\
A &= (x_1 = X_1 \land x_2 = X_2 \land x_3 = X_3) \\
C &= (f(x_1, x_2, x_3) > x_1, x_2, x_3)
\end{align*}
\]

Stop

\[
\begin{align*}
\emptyset \\
A &= (x_1 = X_1 \land x_2 = X_2 \land x_3 = X_3) \\
C &= (f(x_1, x_2, x_3) > x_1, x_2, x_3)
\end{align*}
\]

Abstract applies since $f(x_1, x_2, x_3) > x_1, x_2, x_3$ is satisfiable by any simplification ordering.

If we are using the conditions for inference rules dealing with satisfiability of $A$ given in Table IV, we have to justify the Narrow application. Here, Narrow applies because $A \land \sigma = (x_1 = 0 \land x_2 = 1 \land x_3 = X_3)$, where $\sigma = (X_1 = 0 \land X_2 = 1)$, is satisfiable by any ground instantiation $\theta$ such that $\theta x_1 = 0$, $\theta x_2 = 1$ and $\theta x_3 = 0$.

Then Stop applies because $f(x_3, X_3, X_3)$ is a non narrowable term whose all variables are abstraction variables, and hence we have $\text{TERMIN}(\text{Innermost}, f(X_3, X_3, X_3))$.

Considering now $g(x_1, x_2)$, we get:
\[ t_{\text{ref}} = g(x_1, x_2) \]
\[ A = T \]
\[ C = T \]

**Abstract**

\[ g(x_1, x_2) \]
\[ A = (x_1 \downarrow = x_1 \land x_2 \downarrow = x_2) \]
\[ C = (g(x_1, x_2) > x_1, x_2) \]

**Narrow**

\[ X_1 \]
\[ A = (x_1 \downarrow = x_1 \land x_2 \downarrow = x_2) \]
\[ C = (g(x_1, x_2) > x_1, x_2) \]

\[ X_2 \]
\[ A = (x_1 \downarrow = x_1 \land x_2 \downarrow = x_2) \]
\[ C = (g(x_1, x_2) > x_1, x_2) \]

**Stop**

\[ \emptyset \]

\[ A = (x_1 \downarrow = x_1 \land x_2 \downarrow = x_2) \]
\[ C = (g(x_1, x_2) > x_1, x_2) \]

**Abstract** applies since \( g(x_1, x_2) > x_1, x_2 \) is satisfiable by any simplification ordering.

Again, we have to justify the **Narrow** application. Here, **Narrow** applies because \( A \land \sigma = (x_1 \downarrow = x_1 \land x_2 \downarrow = x_2) \), where \( \sigma = \text{Id} \), is satisfiable by any ground instantiation \( \theta \) such that \( \theta x_1 = \theta X_1 = 0 \) and \( \theta x_2 = \theta X_2 = 0 \).

Then **Stop** applies on both branches because \( X_1 \) and \( X_2 \) are abstraction variables, hence we trivially have \( \text{TERMIN(Innermost, } X_1) \) and \( \text{TERMIN(Innermost, } X_2) \).

**Example 6.4.2.** Let us now give an example illustrating how the usable rules can be helpful and why detecting unsatisfiability of \( A \) can be important. Let us consider the following system \( \mathcal{R} \):

\[
\begin{align*}
\text{plus}(x, 0) & \rightarrow x & (1) \\
\text{plus}(x, s(y)) & \rightarrow s(\text{plus}(x, y)) & (2) \\
f(0, s(0), x) & \rightarrow f(x, \text{plus}(x, x), x) & (3) \\
g(x, y) & \rightarrow x & (4) \\
g(x, y) & \rightarrow y & (5)
\end{align*}
\]

Let us first remark that \( \mathcal{R} \) is not terminating, as illustrated by the following cycle, where successive redexes are underlined:

\[
\begin{align*}
f(0, s(0), g(0, s(0))) & \rightarrow (3) f(g(0, s(0)), \text{plus}(g(0, s(0)), g(0, s(0))), g(0, s(0))) \\
& \rightarrow (4) f(0, \text{plus}(g(0, s(0)), g(0, s(0))), g(0, s(0))) \\
& \rightarrow (5) f(0, \text{plus}(g(0, s(0)), g(0, s(0))), g(0, s(0))) \\
& \rightarrow (4) f(0, \text{plus}(g(0, s(0)), g(0, s(0))) \\
& \rightarrow (1) f(0, s(0), g(0, s(0))) \\
& \rightarrow (3) \ldots
\end{align*}
\]

Let us prove the innermost termination of \( \mathcal{R} \) on \( T(\mathcal{F}) \), where \( \mathcal{F} = \{ 0 : 0, s : 1, \text{plus} : 2, g : 2, f : 3 \} \). The defined symbols of \( \mathcal{F} \) are \( f, \text{plus} \) and \( g \).
Let us apply the inference rules checking unsatisfiability of $A$, whose conditions are given in Table III. Applying the rules on $f(x_1, x_2, x_3)$, we get:

\[
\begin{align*}
A &= \top \\
C &= \top \\
\text{Abstract} \\
A &= (x_1 \downarrow = X_1 \wedge x_2 \downarrow = X_2 \wedge x_3 \downarrow = X_3) \\
&= f(x_1, x_2, x_3) > x_1, x_2, x_3 \\
\sigma &= (X_1 = 0 \wedge X_2 = s(0)) \\
\text{Narrow} \\
A &= (x_1 \downarrow = 0 \wedge x_2 \downarrow = s(0) \wedge x_3 \downarrow = X_3) \\
C &= f(x_1, x_2, x_3) > x_1, x_2, x_3 \\
\sigma &= (X_3 = 0 \wedge X_4 = s(0)) \\
\text{Narrow} \\
\sigma &= (0, \text{plus}(0, 0), 0) \\
A &= (x_1 \downarrow = 0 \wedge x_2 \downarrow = s(0) \wedge x_3 \downarrow = 0 \wedge \text{plus}(0, 0) \downarrow = s(0)) \\
C &= f(x_1, x_2, x_3) > x_1, x_2, x_3
\end{align*}
\]

The first Abstract applies since $f(x_1, x_2, x_3) > x_1, x_2, x_3$ is satisfiable by any simplification ordering.

Since we are using the inference rules checking unsatisfiability of $A$ given in Table III, we do not have to justify the Narrow applications.

The second Abstract by using the TERMIN predicate. Indeed, the usable rules of plus($X_3, X_3$) consist of the system \{plus($x, 0$) $\rightarrow$ $x$, plus($x, s(y)$) $\rightarrow$ s(plus($x, y$))\}, that can be proved terminating with any precedence based ordering with the precedence $\text{plus} \succ s$, which ensures the property TERMIN(Innermost, plus($X_3, X_3$)). Without abstraction here, the process would have generated a branch containing an infinite number of Narrow applications.

Finally, Stop applies because the constraint $A$ becomes unsatisfiable. Indeed, it contains the abstraction constraint $\text{plus}(0, 0) \downarrow = s(0)$, which is not satisfiable since the unique normal form of plus($0, 0$) is 0. Note that if we would have chosen to apply the inference rules checking satisfiability of $A$, whose conditions are given in Table IV, then the last narrowing step would not have applied, and would have been replaced by a Stop application.

Considering now $g(x_1, x_2)$, we get:
\[ t_{ref} = g(x_1, x_2) \]
\[ A = T \]
\[ C = T \]

**Abstract**

\[ g(X_1, X_2) \]
\[ A = (x_1 \models X_1 \land x_2 \models X_2) \]
\[ C = (g(x_1, x_2) > x_1, x_2) \]

**Narrow**

\[ A = (x_1 \models X_1 \land x_2 \models X_2) \]
\[ C = (g(x_1, x_2) > x_1, x_2) \]

\[ X_1 \]
\[ A = (x_1 \models X_1 \land x_2 \models X_2) \]
\[ C = (g(x_1, x_2) > x_1, x_2) \]

\[ X_2 \]
\[ A = (x_1 \models X_1 \land x_2 \models X_2) \]
\[ C = (g(x_1, x_2) > x_1, x_2) \]

**Stop**

\[ A = (x_1 \models X_1 \land x_2 \models X_2) \]
\[ C = (g(x_1, x_2) > x_1, x_2) \]

**Abstract** applies since \( g(x_1, x_2) > x_1, x_2 \) is satisfiable the previous precedence based ordering with \( \text{plus} \succ \neq \cdot \). **Stop** applies on both branches because \( X_1 \) and \( X_2 \) are abstraction variables, hence we trivially have \( \text{TERMIN}(\text{Innermost}, X_1) \) and \( \text{TERMIN}(\text{Innermost}, X_2) \).

Let us finally apply the inference rules of Table III on \( \text{plus}(x_1, x_2) \):

\[ \text{plus}(x_1, x_2) \]
\[ A = T \]
\[ C = T \]

**Abstract**

\[ \text{plus}(X_1, X_2) \]
\[ A = (x_1 \models X_1 \land x_2 \models X_2) \]
\[ C = \text{plus}(x_1, x_2) > x_1, x_2 \]

**Narrow**

\[ \text{plus}(X_1, X_3) \]
\[ A = (x_1 \models X_1 \land x_2 \models X_2) \]
\[ C = \text{plus}(x_1, x_2) > x_1, x_2 \]

\[ \sigma = (x_2 = 0) \]
\[ \sigma = (x_2 = s(X_3)) \]

**Stop**

\[ A = (x_1 \models X_1 \land x_2 \models X_2) \]
\[ C = \text{plus}(x_1, x_2) > x_1, x_2 \]

\[ A = (x_1 \models X_1 \land x_2 \models X_2) \]
\[ C = \text{plus}(x_1, x_2) > x_1, x_2 \]

**Abstract** applies since \( g(x_1, x_2) > x_1, x_2 \) is satisfiable by the previous precedence based ordering. **Stop** applies on the left branch because \( X_1 \) is an abstraction variable, hence we trivially have \( \text{TERMIN}(\text{Innermost}, X_1) \). **Stop** applies on the right branch by using the \( \text{TERMIN} \) predicate. Indeed, the usable rules of \( s(\text{plus}(X_1, X_3)) \) consist of the previous terminating system \( \{ \text{plus} \ldots \} \).
\[ (x, 0) \rightarrow x, \text{plus}(x, s(y)) \rightarrow s(\text{plus}(x, y)) \].

7. THE OUTERMOST CASE

7.1 Abstraction

According to the outermost strategy, abstraction can be performed on subterms \( t_i \) only if during their normalization, the \( t_i \) do not introduce outermost redexes higher in the term \( t \). More formally, the induction hypothesis is applied to the subterms \( t[p_1, \ldots, p_n] \) of the current term \( t \), provided \( \alpha t |_{p_1, \ldots, p_n} \succ \alpha t[p_1, \ldots, p_n] \) for every ground substitution \( \alpha \), for the induction ordering \( \succ \) and provided \( u = t[y_1|p_1, \ldots, y_n|p_n] \) is not narroable at prefix positions of \( p_1, \ldots, p_n \), for the outermost narrowing relation defined below.

As already mentioned in Section 4.5, if in addition, the variables of \( u \) are all in \( \mathcal{X}_A \), and \( u \) is not narroable, then every ground instance of the term \( u \) outermost terminates.

7.2 The narrowing mechanism

Outermost narrowing is defined by Definition 4.3.2, where a \( S \)-better position is a prefix position. In order to support intuition, let us consider for instance the system \( \{ f(g(a)) \rightarrow a, f(f(x)) \rightarrow b, g(x) \rightarrow f(g(x)) \} \). With the standard narrowing relation used at the outermost position, \( f(g(x_1)) \) only narrows into \( a \) with the first rule and the substitution \( \sigma = (x_1 = a) \). With the outermost narrowing relation, \( f(g(x_1)) \) narrows into \( a \) with the first rule and \( \sigma = (x_1 = a) \), and into \( f(f(g(x_2))) \) with the third rule and the constrained substitution \( \sigma = (x_1 = x_2 \land x_2 \neq a) \).

The variables of the narrowed terms are in \( \mathcal{X} \cup \mathcal{X}_A \): as we will see, renaming variables of \( \mathcal{X} \) still gives variables of \( \mathcal{X} \), and abstraction, replacing subterms by variables of \( \mathcal{X}_A \), may not cover all variables of \( \mathcal{X} \) in the abstracted term.

In the outermost termination proof, the variable renaming performed before the narrowing step has a crucial meaning for the schematization of outermost derivations. This renaming, applied on the current term \( g(x_1, \ldots, x_m) \), replaces the variable occurrences \( x_1, \ldots, x_m \) by new and all different variables \( x'_1, \ldots, x'_m \), defined as follows. Given any ground instance \( \alpha g(x_1, \ldots, x_m) \) of \( g(x_1, \ldots, x_m) \), the \( x'_1, \ldots, x'_m \) represent the first reduced form of \( \alpha x_1, \ldots, \alpha x_m \) generating an outermost reduction higher in the term (here, at the top), in any outermost rewriting chain starting from \( \alpha g(x_1, \ldots, x_m) \). This replacement is memorized in a reduction formula before applying a step of outermost narrowing to \( g(x'_1, \ldots, x'_m) \). The abstraction variables are not renamed: since their ground instances are in normal form, they are not concerned by the rewriting chain schematized by the variable renaming.

Formally, the definition of the variable replacement performed before a narrowing step is the following.

**Definition 7.2.1.** Let \( t \in \mathcal{T}(\mathcal{F}, \mathcal{X}) \) be a term whose variable occurrences from left to right in \( t \) are \( x_1, \ldots, x_m \). The reduction renaming of \( t \), noted \( \rho = (x_1 \rightarrow^* x'_1) \ldots (x_m \rightarrow^* x'_m) \), consists in replacing the \( x_i \) by new and all different variables \( x'_i \) in \( t \), giving a term \( t^\rho \). This is denoted by the so-called reduction formula

\[ R(t) = t \rightarrow^* t^\rho. \]
Notice that the reduction renaming linearizes the term. For instance, the two occurrences of $x$ in $g(x, x)$ are respectively renamed into $x_1'$ and $x_2'$, and $g(x, x) \rightarrow^* g(x_1', x_2')$.

**Definition 7.2.2.** Let $t \in T(\mathcal{F}, \mathcal{X})$ be a term whose variable occurrences from left to right are $x_1, \ldots, x_m$, at positions $p_1, \ldots, p_m$ respectively. A ground substitution $\theta$ satisfies the reduction formula $R(t) = t \rightarrow^* t^\theta$, where $\rho = (x_1 \rightarrow^* x_1') \ldots (x_m \rightarrow^* x_m')$, iff there exists an outermost rewriting chain $\theta t \rightarrow^*_{p \in \mathcal{C}(t)} \theta t^\rho \rightarrow^*_{p \in \mathcal{C}(t)} u$, i.e. such that:

— either $t[\theta x_1|p_1| \ldots [\theta x_m]|p_m]$ is the first reduced form of $\theta t = t[\theta x_1|p_1| \ldots [\theta x_m]|p_m]

— or $\theta x_1 = (\theta x_1|)| \ldots , \theta x_m = (\theta x_m|)$ if there is no such position.

Before going on, a few remarks on this definition can be made. In the second case of satisfiability, $t[\theta x_1]| \ldots [\theta x_m]|_m$ is in normal form. In any case, $R(t)$ is always satisfiable: it is sufficient to take a ground substitution $\theta$ such that $t[\theta x_1|p_1| \ldots [\theta x_m]|p_m]$ has an outermost rewriting position at a non variable position of $t$, and then to extend its domain $\{x_1, \ldots, x_m\}$ to $\{x_1, \ldots, x_m, x_1', \ldots, x_m'\}$ by choosing for each $i \in \{1, \ldots, m\}$, $\theta x_i' = \theta x_i$. If such a substitution does not exist, then every ground instance of $t$ has no outermost rewriting position at a non variable position of $t$, and it is sufficient to take a ground substitution $\theta$ such that $\theta x_1 = \ldots = \theta x_m = \theta x_1' = \ldots = \theta x_m' = u$, with $u$ any ground term in normal form.

However, there may exist several instantiations solution of such constraints. Let us consider for instance the rewrite system $R = \{ f(a) \rightarrow f(c), b \rightarrow a \}$ and the reduction formula $R(f(x)) = f(x) \rightarrow^* f(x')$. The substitution $\theta_1(x) = \theta_1(x') = a$ and $\theta_2(x) = b, \theta_2(x') = a$ are two distinct solutions. With the substitution $\theta_2$, $f(a)$ is the first reduced form of $f(b)$ having an outermost rewriting position at a non variable position of $f(x)$ (here at top).

Notice also that if $t$ is outermost reducible at position $p$, variables of $t$ whose position is a suffix of $p$ are not affected by the reduction renaming.

Indeed, if $t$ is reducible at position $p$, a ground instance $\alpha x$ of $t$ cannot be outermost reduced in the instance of $x$, whose positions are suffix of $p$. So $x'$, representing the first reduced form of $\alpha x$ in any outermost rewriting chain starting from $\alpha t$, such that the reduction is performed higher in the current term, is equal to $x$.

To illustrate this, let us consider the system $(g(x) \rightarrow x, f(x, x) \rightarrow x)$ (the right-hand sides of the rules are not important here). Then, since $f(x, g(y))$ outermost rewrites at the position of $g$, the variable $y$ does not need to be renamed. So $R(f(x, g(y))) = (f(x, g(y)) \rightarrow^* f(x', g(y)))$.

Because of the previously defined renaming process, the formula $A$ for cumulating constraints has to be completed in the following way.

**Definition 7.2.3.** A renaming-abstraction constraint formula (RACF for short) is a formula

$\bigwedge_m u \rightarrow^* u^\rho \bigwedge_j (t_j = t_j') \bigwedge_{i \in k} \bigvee_{i \in k} (u_i \neq v_i)$, where $u, u', t, t', t_j, u_1, v_1 \in T(\mathcal{F, X} \cup \mathcal{A}), x_j, x_j' \in \mathcal{X} \cup \mathcal{A}$. The empty formula is denoted $\top$.

**Definition 7.2.4.** A renaming-abstraction constraint formula $\bigwedge_m u \rightarrow^* u^\rho \bigwedge_j (t_j = t_j') \bigwedge_{i \in k} \bigvee_{i \in k} (u_i \neq v_i)$ is said to be satisfiable
iff there exists at least one instantiation $\theta$ such that $\bigwedge_i (\theta t_i \downarrow = \theta t'_i) \land \bigwedge_j (\theta x_j = \theta t_j) \land \bigvee_k (\theta u_k \neq \theta v_k)$ and $\theta$ satisfies $\bigwedge_m u_m \rightarrow^* u'_m$.

In practice, one can solve the equality and disequality part of the constraint and then check whether the solution $\theta$ satisfies the reduction formulas. This is trivial when $\theta$ only instantiates the $x'_i$, since it can be extended by setting $\theta(x_i) = \theta(x'_i)$.

Unfortunately, when $\theta$ also instantiates the $x_i$, we get the undecidable problem of reachability: given two ground terms $t$ and $t'$, can $t$ be transformed into $t'$ by repeated application of a given set of rewriting rules?

So here again, we can either test satisfiability of the formula of cumulated constraints, or unsatisfiability. As satisfiability is in general more difficult to show than in the innermost case, we only present here inference rules checking unsatisfiability.

7.3 Inference rules for the outermost case

The inference rules $\text{Abstract}$, $\text{Narrow}$ and $\text{Stop}$ instanciate respectively the proof steps $\text{abstract}$, $\text{narrow}$, and $\text{stop}$.

They work as follows:

—The narrowing step is expressed by a rule $\text{Narrow}$ applying on $\{(t), A, C\}$: the variables of $t$ are renamed as specified in Definition 7.2.1. Then $t^\rho$ is outermost narrowed in all possible ways in one step, with all possible rewrite rules of the rewrite system $R$, into terms $u$. For any possible $u$, we generate the state $\{(u), R(t) \land A \land \sigma, C\}$ where $\sigma$ is the constrained substitution allowing outermost narrowing of $t^\rho$ into $u$.

—The rule $\text{Abstract}$ works as in the innermost case, except that the abstraction positions are such that the abstracted term is not narroable at prefix positions of the abstraction positions.

—The rule $\text{Stop}$ also works as in the innermost case.

To prove outermost termination of $R$ on every term $t \in T(F)$, for each defined symbol $g \in \text{Def}$, we apply the rules on the initial state $\{(t_{\text{ref}} = g(x_1, \ldots, x_m), \top, \top)\}$, with the strategy:

$$\text{Strat-Rules(Outermost)} = \text{repeat}* (\text{try-skip}(\text{Abstract}); \text{try-skip}(\text{Narrow}); \text{try-skip}(\text{Stop})).$$

There are three cases for the behavior of the strategy: either there is a branch in the proof tree with infinite applications of $\text{Abstract}$ and $\text{Narrow}$, in which case we cannot say anything about termination, or the procedure stops on each branch with the rule $\text{Stop}$. Then, outermost termination is established, if all proof trees are finite.

According to the remark following Definition 7.2.2, the reduction formulas in $A$ may often be reduced to simple variable renamings. In this case, $A$ only contains variable renamings and constrained substitutions, that can be used to show that the ordering constraint needed to apply $\text{Abstract}$ or $\text{Stop}$ is satisfiable (see Examples B.1 and B.4 in [Fissore et al. 2002c]). The following lemma can also be used, if satisfiability of $C$ is considered with Definition 4.4.3 (see Examples B.2, B.3 and B.4 in [Fissore et al. 2002c]).
Table V. Inference rules for the outermost strategy

<table>
<thead>
<tr>
<th>Rule</th>
<th>Abstract</th>
<th>Narrow</th>
<th>Stop</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>${t}, A, C$</td>
<td>${u}, A \land t</td>
<td>\iota_{1} \cdots \land t</td>
</tr>
</tbody>
</table>
|               | where $t$ is abstracted into $u$ at positions $i_1, \ldots, i_p \neq \epsilon$ | if $C \land H_{C}(t|\iota_{1}) \cdots \land H_{C}(t|\iota_{p})$ is satisfiable and $u$ is not narrowable at prefix positions of $i_1, \ldots, i_p$ | if $C \land H_{C}(t)$ is satisfiable or $A$ is unsatisfiable and $H_{A}(t) = \begin{cases} 
\top & \text{if any ground instance of } t \\
 t|\iota = X & \text{otherwise.} 
\end{cases}$, $H_{C}(t) = \begin{cases} 
\top & \text{if } \text{TERTMIN}(\text{Outermost}, t) \\
t_{\text{ref}} > t & \text{otherwise.} 
\end{cases}$ |

Lemma 7.3.1. Let $(\{t_i\}, A_i, C_i)$ be the $i^{th}$ state of any branch of the derivation tree obtained by applying the strategy $S$ on $(\{t_{\text{ref}}\}, \top, \top)$, and $\succ$ an $F$-stable ordering having the subterm property. If every reduction formula in $A_i$ can be reduced to a formula $\bigwedge_j x_j = x'_j$, then we have:

for all variable $x$ of $t_i$ in $X$: $(t_{\text{ref}} > x)/A_i$ is satisfiable by $\succ$.

7.4 Examples

Example 7.4.1. Consider the previous example $R = \{f(g(a)) \rightarrow a, f(f(x)) \rightarrow b, g(x) \rightarrow f(g(x))\}$, that is outermost terminating, but not terminating for the standard rewriting relation. We prove that $R$ is outermost terminating on $T(F)$ where $F = \{f : 1, g : 1, a : 0, b : 0\}$.

The defined symbols of $F$ for $R$ are $f$ and $g$. Applying the rules on $f(x_1)$, we get:
The first Stop is applied because \( a \) is in normal form, the second Stop because \( b \) is in normal form. Applying the rules on \( g(x_1) \), we get:

\[
\begin{align*}
g(x_1) & \quad A = T, \quad C = T \\
& \quad \text{Narrow} \\
& \quad A = (f(g(x_1)) \\
& \quad \sigma = \text{Id} \\
& \quad A = (f(x_1) \rightarrow f(x_1')) \\
& \quad A = (x_1 = g(a)) \\
& \quad C = T \\
& \quad \text{Stop} \\
& \quad \emptyset \\
& \quad A = (x_1 = a) \\
& \quad C = T
\end{align*}
\]

There is no reduction renaming before the Narrow steps, since \( g(x_1) \), \( f(g(x_1)) \) and \( f(f(g(x_1))) \) are reducible at prefix positions of the position of \( x_1 \).

When narrowing \( f(g(x_1)) \), we first try the top position, and find a possible unification with the first rule (the left branch). One also must consider the third rule if \( x_1 \) is such that \( x_1 \neq a \) (second branch). Stop is applied on \( a \) and \( b \) as previously.

Example 7.4.2. Let \( R \) be the TRS cited in the introduction, built on \( F = \{ \text{cons} : 2, \text{inf} : 1, \text{big} : 0 \} : \)

\[
\begin{align*}
\text{cons}(x, \text{cons}(y, z)) & \rightarrow \text{big} \\
\text{inf}(x) & \rightarrow \text{cons}(x, \text{inf}(s(x)))
\end{align*}
\]
Applying the inference rules on $\text{inf}(x_1)$, we get:

\[
\text{inf}(x_1) \\
A = \top, C = \top
\]

\[\sigma = \text{Id}
\]

Narrow

\[
\text{cons}(x_1, \text{inf}(s(x_1))) \\
A = \top, C = \top
\]

\[\sigma = \text{Id}
\]

Narrow

\[
\text{cons}(x_1', \text{inf}(s(x_1))) \\
A = (\text{cons}(x_1, \text{inf}(s(x_1))) \rightarrow^* \text{cons}(x_1', \text{inf}(s(x_1)))) \\
C = \top
\]

\[\sigma = \text{Id}
\]

Narrow

\[\text{big}
\]

\[
A = (\text{cons}(x_1, \text{inf}(s(x_1))) \rightarrow^* \text{cons}(x_1', \text{inf}(s(x_1)))) \\
C = \top
\]

Stop

\[\emptyset
\]

\[
A = (\text{cons}(x_1, \text{inf}(s(x_1))) \rightarrow^* \text{cons}(x_1', \text{inf}(s(x_1)))) \\
C = \top
\]

Applying the inference rules on $\text{cons}(x_1, x_2)$, we get:

\[
\text{cons}(x_1, x_2) \\
A = \top, C = \top
\]

\[\sigma = (x_2 = \text{cons}(x_3, x_4))
\]

Narrow

\[\text{big}
\]

\[
A = (\text{cons}(x_1, x_2) \rightarrow^* \text{cons}(x_1', x_2')) \\
\wedge x_2' = \text{cons}(x_3, x_4) \\
C = \top
\]

Stop

\[\emptyset
\]

\[
A = (\text{cons}(x_1, x_2) \rightarrow^* \text{cons}(x_1', x_2')) \\
\wedge x_2' = \text{cons}(x_3, x_4) \\
C = \top
\]

Other examples can be found in [Fissore et al. 2002c].

8. LOCAL STRATEGIES ON OPERATORS

We now address the termination problem for rewriting with local strategies on operators.
8.1 Abstraction and narrowing

The information that variables are abstraction variables can be very important to conclude the proofs here: if the current term is an abstraction variable, its strategy is set to $\emptyset$ in the Narrow step, and then the Stop step applies. This information can be easily deduced when new variables are introduced: the abstracting process directly introduces abstraction variables, by definition. But the resulting term may still have variables of $X$ since the abstracted subterms of a term may not cover all variables of the term.

Moreover, narrowing is performed on terms of $T(F, X \cup X_A)$. Indeed, there is no variable renaming before the narrowing steps, that could transform all variables into abstraction variables. In addition, even if the variables of a narrowed term are all in $X_A$, the range of the narrowing substitution can introduce variables of $X$, according to the LS-strategies, if these variables do not appear at LS-positions.

However some variable occurrences can be particularized into variables of $X_A$ in the narrowing process: the narrowing substitution $\sigma$, whose range only contains new variables of $X$, can be transformed into a new substitution $\sigma_A$ by replacing some of these variables by abstraction variables. Let us consider an equality of the form $X = u$, introduced by the narrowing substitution $\sigma$, where $X \in X_A$, and $u \in T(F, X')$. As $X$ is an abstraction variable, every ground instance of $u$ must be in normal form. So the variables in $u$ that occur at an LS-position can be re-placed by abstraction variables. Let now $\mu$ be the substitution $(x_i = X_i)$, for all $x_i \in \text{Var}(u)$ such that $X = u$ is an equality of $\sigma$ with $X \in X_A$, $u \in T(F, X' \cup X_A)$, and $x_i$ occurs at an LS-position in $u$. Then $\sigma_A = \mu \sigma$.

Combining abstraction and narrowing is achieved here in the following way. The abstraction positions are chosen so that the abstraction mechanism captures the greatest possible number of rewriting steps: we try to abstract the immediate subterms of the current term. If the abstraction is possible, then a narrowing step is applied, only at the top position, which limits the number of narrowing steps, more complicated here than for the other strategies, since, as we will see later, they involve complementary branches.

If Abstract cannot be applied at all LS-positions of the term, the process is stopped, and nothing can be concluded about termination.

8.2 The termination proof procedure for local strategies

The inference rules Abstract, Narrow and Stop instanciate respectively the proof steps abstract, narrow, and stop. They work in the following way on a state $\{(t[p_1,\ldots,p_n]), A, C\}$, where $\text{top}(t) = f$ and $\text{LS}(f) = [p_1,\ldots,p_n]$.

—The rule Abstract processes the abstracting step. It can apply:

—when there exists $k \in [2..n]$, $p_j \neq 0$ for $1 \leq j \leq k - 1$ and $p_k = 0$. The term $t$ is abstracted at positions $p_j \neq 0$ for $1 \leq j < k$ if there exists an $F$-stable ordering having the subterm property and such that $C \land (t_{\text{ref}} > t|_{p_j}, 1 \leq j < k)$ is satisfiable. Indeed, by induction hypothesis, all ground instances of $t|_{p_j}, 1 \leq j < k$ LS-terminate. We can instead have $\text{TERMIN}(\text{Local–Strat}, t|_{p_j})$ for some of the previous $t|_{p_j}$. The list of positions then becomes $[0, p_{k+1},\ldots,p_n]$. 


—when there is no position 0 in the strategy of the current term. Any ground instance of the term obtained after abstraction is irreducible, by definition of the LS-strategy, which ends the proof on the current derivation chain. The set containing the current term is then replaced by the empty set.

—when \( p_1 = 0 \). The rule applies but does not change the state on which the narrow step can be applied.

—The rule \textbf{Narrow} works as follows:

—if the current term \( t \) is narrowable at position 0, \( t \) is narrowed in all possible ways in one step, with all possible rewrite rules of the rewrite system \( R \), and all possible substitutions \( \sigma_i \), into \( u_i \), \( i \in [1..l] \). Then from the state \( \langle \{ t[0\ldots p_n] \}, A, C \rangle \) we generate the states \( \langle \{ u_i^{LS(top(u_i))} \}, A \land \sigma_i, C \rangle, i \in [1..l] \), where the \( \sigma_i \) are all most general unifiers allowing narrowing of \( t \) into terms \( u_i \), such that \( A \land \sigma_i \) is satisfiable. This narrowing step means that \( \sigma_1 t, \ldots, \sigma_l t \) are all most general instances of \( t \) that are reducible at the top position. As a consequence, if \( \Phi = \sigma_1 \land \ldots \land \sigma_l \) is satisfiable, for each instantiation \( \mu \) satisfying \( \Phi \), \( \mu t \) is not reducible at the top position. Then, as these \( \mu t \) have to be reduced at positions \( p_1, \ldots, p_n \), we also generate the complementary state \( \langle \{ t[p_1\ldots p_n] \}, A \land \bigwedge_{i=1}^l \sigma_i, C \rangle \).

Let us also notice that if \( u_i \) is a variable \( x \in X \), we cannot conclude anything about termination of ground instances of \( x \). Setting \( LS(x) \) to \( [0] \) or \( [] \) would wrongly lead to conclude, with the rule \textbf{Narrow}, that ground instances of \( x \) are terminating. So we force the proof process to stop in setting \( LS(x) \) to a particular symbol \( \sharp \). However, if \( u_i = X \in X_A, LS(X) \) is set to \( [] \), which is coherent with the fact that any ground instance of \( X \) is in LS-normal form.

—if \( t \) is not narrowable at position 0 or is narrowable with a substitution that is not compatible with the current constraint formula \( A \), then no narrowing is applied and the current term is evaluated at positions following the top position in the strategy. The list of positions then becomes \( [p_1, \ldots, p_n] \).

—We also can check for the current term whether there exists an ordering having the subterm property such that \( C \land t_{ref} > t \) is satisfiable. Then, by induction hypothesis, any ground instance of \( t \) terminates for the LS-strategy, which ends the proof on the current derivation chain. The \textbf{Stop} rule then replaces the set containing the current term by the empty set.

The rule \textbf{Stop} also allows to stop the inference process when the list of positions is empty.

The set of inference rules is given in Table VI. In the conditions of these rules, satisfiability of \( A \) is checked. Working with unsatisfiability of \( A \) would be more technical to handle here than in the innermost case, because of the complementary branches generated by the \textbf{Narrow} rule.

The strategy for applying these rules is:

\begin{verbatim}
repeat*
  (try-stop(\textbf{Abstract}); try-stop(\textbf{Narrow}); try-skip(\textbf{Stop}))
\end{verbatim}

There are here also three cases for the behavior of the proof process. It can diverge as previously, or stop and the states in the leaves have then to be considered. The good case is when the process stops and all final states of all proof trees are of
Table VI. Inference rules for \( t_{\text{ref}} \) LS-termination

Abstract:

\[
\begin{align*}
\left\{ t[p_1, \ldots, p_n] \right\}, &\quad A, \quad C \\
\{ u^S \}, &\quad A \land \bigwedge_{j \in \{i_1, \ldots, i_p\}} (t|_j \downarrow = X_j), \quad C \land \bigwedge_{j \in \{i_1, \ldots, i_p\}} H_C(t|_j)
\end{align*}
\]

where \( t \) is abstracted into \( u \) at the positions \( i_1, \ldots, i_p \in \text{POS} \)

if \( A \land \bigwedge_{j \in \{i_1, \ldots, i_p\}} (t|_j \downarrow = X_j), \quad C \land \bigwedge_{j \in \{i_1, \ldots, i_p\}} H_C(t|_j) \) are satisfiable and

\[\text{POS} = \{p_1, \ldots, p_k-1\}, S = [0, p_{k+1}, \ldots, p_n] \] if \( \exists k \in [2..n] : p_1, \ldots, p_{k-1} \neq 0 \) and \( p_k = 0 \)

\[\text{POS} = \{p_1, \ldots, p_n\}, S = [] \] if \( p_1, \ldots, p_n \neq 0 \) or \( [p_1, \ldots, p_n] = [] \)

\[\text{POS} = \emptyset, S = [p_1, \ldots, p_n] \] if \( p_1 = 0 \)

Narrow:

\[
\begin{align*}
\left\{ t[0, p_1, \ldots, p_n] \right\}, &\quad A, \quad C \\
\{ u^S \}, &\quad A', \quad C
\end{align*}
\]

where \( u = u_i, S = LS(top(u_i)), A' = A \land \sigma_i \) if \( t \sim^e \sigma_i u_i \) and \( A \land \sigma_i \) is satisfiable

or \( u^S = t[p_1, \ldots, p_n], A' = A \land \bigwedge_{i=1}^l \sigma_i, \quad \text{and} \quad \sigma_i, i \in [1..l] \) are all nar. subst. as above

or \( u^S = t[p_1, \ldots, p_n], A' = A \)

if \( t \) is not narrowable at the top position

or \( \forall \sigma \) nar. subst. of \( t \) at the top position, \( A \land \sigma \) is not satisfiable

Stop:

\[
\begin{align*}
\left\{ t[p_1, \ldots, p_n] \right\}, &\quad A, \quad C \\
\emptyset, &\quad A \land H_A(t), \quad C \land H_C(t)
\end{align*}
\]

if \( A \land H_A(t), \quad C \land H_C(t) \) are satisfiable

and \( H_A(t) = \begin{cases} 
\top & \text{if } [p_1, \ldots, p_n] = [] \\
& \text{or any ground instance of } t \\
& \text{is in normal form} \\
t|_j \downarrow = X & \text{otherwise.} 
\end{cases} \)

\( H_C(t) = \begin{cases} 
\top & \text{if } [p_1, \ldots, p_n] = [] \\
& \text{or } \text{TERMIN(Local-Strat, } t) \\
t_{\text{ref}} > t & \text{otherwise.} 
\end{cases} \)
the form \((\emptyset, A, C)\).

8.3 Examples

Example 8.3.1. Let us recall the rules of the example given in the introduction.

\[
\begin{align*}
f(i(x)) & \rightarrow \text{ite}(\text{zero}(x), g(x), f(h(x))) \\
\text{zero}(0) & \rightarrow \text{true} \\
\text{zero}(s(x)) & \rightarrow \text{false} \\
\text{ite}(\text{true}, x, y) & \rightarrow x \\
\text{ite}(\text{false}, x, y) & \rightarrow y \\
h(0) & \rightarrow i(0) \\
h(x) & \rightarrow s(i(x))
\end{align*}
\]

The LS-strategy is the following:

- \(\text{LS}(\text{ite}) = [1; 0]\).
- \(\text{LS}(f) = \text{LS}(\text{zero}) = \text{LS}(h) = [1; 0]\) and 
- \(\text{LS}(g) = \text{LS}(i) = [1]\).

Let us prove the termination of this system on the signature 
\(F = \{f : 1, \text{zero} : 1, \text{ite} : 3, h : 1, s : 1, i : 1, g : 1, 0 : 0\}\).

Applying the inference rules on \(f(x_1)\), we get:

\[
f(x_1)^{[1, 0]} \\
A = \top, C = \top
\]

\[
\text{Abstract}
\]

\[
f(X_1)^{[0]} \\
A = (x_1 = X_1) \\
C = (f(x_1) > x_1)
\]

\[
\text{Narrow} \quad \sigma = (X_1 = i(X_2)) \quad \text{Narrow}
\]

\[
\text{ite}(\text{zero}(X_2), g(X_2), f(h(X_2)))^{[1, 0]} \\
A = (x_1 = i(X_2)) \\
C = (f(x_1) > x_1)
\]

\[
\text{Abstract}
\]

\[
f(X_1)^{[1]} \\
A = (x_1 = X_1) \land (X_1 \neq i(X_2)) \\
C = (f(x_1) > x_1)
\]

\[
\text{Stop}
\]

\[
\text{ite}(X_3, g(X_2), f(h(X_2)))^{[0]} \\
A = (x_1 = i(X_2) \land \text{zero}(X_2)) = X_3 \\
C = (f(x_1) > x_1)
\]

\[
\text{Abstract}
\]

\[
f(X_1)^{[0]} \\
A = (x_1 = X_1) \land (X_1 \neq i(X_2)) \\
C = (f(x_1) > x_1)
\]

\[
\emptyset
\]

\[
\text{Stop}
\]

\[
A = (x_1 = X_1) \land (X_1 \neq i(X_2)) \\
C = (f(x_1) > x_1)
\]

**Abstract** applies on \(f(x_1)\), since \(C\) is satisfiable by any ordering having the subterm property. \(A\) is satisfiable with any instantiation \(\theta\) such that \(\theta x_1 = \theta X_1 = 0\).

**Narrow** expresses the fact that \(\sigma f(X_1)\) is reducible if \(\sigma\) is such that \(\sigma x_1 = i(X_2)\), and that the other instances \((\sigma' f(X_1))\) with \(\sigma' X_1 \neq i(X_2)\) cannot be reduced.

The renaming of \(x_2\) into \(X_2\) in \(\sigma_A\) comes from the fact that \(x_2\) occurs in \(i(x_2)\) at an LS-position in \(\sigma = (X_1 = i(x_2))\).
Then, the constraint formula $A$ on the left branch is satisfiable by any instantiation $\theta$ such that $\theta X_2 = 0$ and $\theta x_1 = i(0)$. The constraint formula on the complementary branch is satisfied by any instantiation $\theta$ such that $\theta x_1 = \theta X_1 = \theta X_2 = 0$.

**Abstract** applies here on the first branch, since $\text{zero}(X_2)$ can be abstracted, thanks to a version of Proposition 6.2.1 adapted to local strategies [Fissore et al. 2001]. Indeed, $\mathcal{U}(\text{zero}(X_2)) = \{\text{zero}(0) \rightarrow \text{true}, \text{zero}(s(x)) \rightarrow \text{false}\}$, and both rules can be oriented by a LPO $\succ$ with the precedence $\text{zero} \succ \text{true}$ and $\text{zero} \succ \text{false}$. Then we have $\text{TERMIN}(\text{Local-strat}, \text{zero}(X_2))$.

The next constraint formula $A$ is satisfiable with any instantiation $\theta$ such that $\theta X_2 = 0$, $\theta X_3 = \text{true}$ and $\theta x_1 = i(0)$.

Then, **Narrow** applies on the left branch:

\[
\begin{align*}
\text{Abstract} & \quad \text{Abstract} \\
\quad g(X_2) & \quad f(h(X_2)) \\
A = (x_1 \downarrow = i(X_2) \wedge & \quad A = (x_1 \downarrow = i(X_2) \wedge \\
\quad \text{zero}(X_2) \downarrow = \text{true}) & \quad \text{zero}(X_2) \downarrow = \text{false} \wedge h(X_2) \downarrow = X_4 \\
C = (f(x_1) > x_1) & \quad C = (f(x_1) > x_1) \\
\end{align*}
\]

**Stop**

\[
\begin{align*}
A = (x_1 \downarrow = i(X_2) \wedge & \quad A = (x_1 \downarrow = i(X_2) \wedge \\
\quad \text{zero}(X_2) \downarrow = \text{true}) & \quad \text{zero}(X_2) \downarrow = \text{false} \wedge h(X_2) \downarrow = X_4 \\
C = (f(x_1) > x_1) & \quad C = (f(x_1) > x_1) \\
\end{align*}
\]

The first constraint formula $A$ is satisfiable by any instantiation $\theta$ such that $\theta X_2 = 0$ and $\theta x_1 = i(0)$. The second one is satisfiable by any instantiation $\theta$ such that $\theta X_2 = s(0)$ and $\theta x_1 = i(s(0))$. The third one (see below) is satisfiable by any instantiation $\theta$ such that $\theta X_3 = \text{zero}(i(0))$, $\theta X_2 = i(0)$ and $\theta x_1 = i(i(0))$.

**Abstract** trivially applies on $g(X_2)$: since $X_2$ is an abstraction variable, there is no need to abstract it.

The second **Abstract** applies on $f(h(X_2))$, thanks to the previous adaptation of Proposition 6.2.1 to local strategies. Indeed, $\mathcal{U}(h(X_2)) = \{h(0) \rightarrow i(0), h(x) \rightarrow s(i(x))\}$, and both rules can be oriented by the same LPO as previously with the
additional precedence $h \succcurlyeq i$ and $h \succcurlyeq s$. Then we have $\text{TERMIN}(\text{Local-strat},\ h(X_2))$.

The constraint formula associated to $f(X_4)^0$ is satisfiable by any instantiation $\theta$ such that $\theta X_4 = s(i(s(0)))$, $\theta X_2 = s(0)$ and $\theta x_1 = i(s(0))$.

One could have tried to narrow $f(X_4)$, by using the first rule and the narrowing substitution $\sigma_A = (X_4 = i(X_5))$. But then $A \land \sigma_A$ would lead to $(x_1 \downarrow = i(X_2) \land \text{zero}(X_2) \downarrow = \text{false} \land h(X_2) \downarrow = i(X_5))$. For any $\theta$ satisfying $A \land \sigma_A$, $\theta$ must be such that $\theta h(X_2) \downarrow = h(\theta X_2) \downarrow = i(\theta X_5)$. If $\theta X_2 \downarrow \neq 0$, then, according to $R$, $h(\theta X_2) \downarrow \rightarrow s(i(\theta X_2) \downarrow)$, where $s$ is a constructor. Then we cannot have $h(\theta X_2) \downarrow = i(\theta X_5)$, so $\theta$ must be such that $\theta X_2 \downarrow = 0$. But then $\theta \text{zero}(X_2) \downarrow = \text{true}$, which makes $A \land \sigma_A$ unsatisfiable. Therefore there is no narrowing.

For the third branch, we have:

\[
\text{ite}(X_3, g(X_2), f(h(X_2)))
\]

\[
A = (x_1 \downarrow = i(X_2) \land \\
\text{zero}(X_2) \downarrow = X_3 \land \\
X_3 \neq \text{true} \land X_3 \neq \text{false})
\]

\[
C = (f(x_1) > x_1)
\]

\[
\text{Stop}
\]

\[
\emptyset
\]

\[
A = (x_1 \downarrow = i(X_2) \land \\
\text{zero}(X_2) \downarrow = X_3 \land \\
X_3 \neq \text{true} \land X_3 \neq \text{false})
\]

\[
C = (f(x_1) > x_1)
\]

Like for the defined symbols $\text{ite}, \text{zero}, h$, the inference rules apply successfully through one $\text{Abstract}, \text{Narrow}, \text{Abstract}$ with no abstraction position, $\text{Narrow}$ and $\text{Stop}$ application. Therefore $R$ is LS-terminating.

Let us now give an example that cannot be handled with the context-sensitive approach.

**Example 8.3.2.** Let $R$ be the following TRS

\[
f(a, g(x)) \rightarrow f(a, h(x))
\]

\[
h(x) \rightarrow g(x)
\]

with the LS-strategy: $LS(f) = [0; 1; 2]$, $LS(h) = [0]$ and $LS(g) = [1]$.

The context-sensitive strategy would allow to permute the reducible arguments of $f$, so that we also could evaluate terms with $LS(f) = [1; 2; 0]$. We let the user check that, with this strategy, $R$ does not terminate.

Applying the rules on $f(x_1, x_2)$, we get:
Applying the rules on $h(x_1)$, we get:

\[
\begin{align*}
h(x_1) &\quad A = T, \quad C = T \\
\sigma = \text{Id} &\quad \text{Narrow} \\
g(x_1) &\quad A = T, \quad C = T \\
\text{Abstract} &\quad g(X_1) \\
A &\quad A = (x_1 \models X_1), \quad C = (h(x_1) > x_1) \\
\text{Stop} &\quad \emptyset
\end{align*}
\]
9. CONCLUSION

The generic termination proof method presented in this paper is based on the simple ideas of schematizing and observing the derivation trees of ground terms and of using an induction ordering to stop derivations as soon as termination is ensured by induction. The method makes clear the schematization power of narrowing, abstraction and constraints. Constraints are heavily used on one hand to gather conditions that the induction ordering must satisfy, on the other hand to represent the set of ground instances of generic terms.

We now have an implementation of our technique, in a system named CARIBOO [Fissore et al. 2002a; Fissore 2003; Fissore et al. 2004a], providing a termination proof tool for the innermost, the outermost, and the local strategies 1. CARIBOO consists of two main parts:

(1) The proof procedure, written in ELAN, which is a direct translation of the inference rules. It generates the proof trees, dealing with the ordering and the abstraction constraints. It is worth emphasizing the reflexive aspect of this proof procedure, written in a rule-based language, to allow termination of rule-based programs.

(2) A graphical user interface (GUI), written in Java. It provides an edition tool to define specifications of TRSs which are then transformed into an ELAN specification used by the proof procedure. It also displays the detailed results of the proof process: which defined symbols have already been treated and, for each of them, the proof tree together with the detail of each state. Trace files can be generated in different formats (HTML, ps, pdf...)

To deal with the generated constraints, the proof process of CARIBOO can use integrated features, like the computation of usable rules, the use of the subterm ordering or the Lexicographic Path ordering to satisfy ordering constraints, and the test of sufficient conditions of Section 4.4 for detecting unsatisfiability of A.

It can also delegate features, as solving the ordering constraints or orienting the usable rules when the LPO fails, proving termination of a term, or testing satisfiability of A. Delegation is either proposed to the user, or automatically ensured by the ordering constraint solver Cäne2.

CARIBOO provides several automation modes for dealing with constraints. Dealing with unsatisfiability of A allows a complete automatic mode, providing a termination proof for a large class of examples (a library is available with the distribution of CARIBOO).

It is interesting to note that thanks to the power of induction, and to the help of usable rules, the generated ordering constraints are often simple, and are easily satisfied by the subterm ordering or an LPO.

Finally, the techniques presented here have also been applied to weak termination in [Fissore et al. 2004b].

As our proof process is very closed to the rewriting mechanism, it could easily be extended to conditional, equational and typed rewriting, by simply adapting the narrowing definition. Our approach is also promising to tackle inductive proofs of other term properties like confluence or ground reducibility.

1Available at http://protheo.loria.fr/softwares/cariboo/
APPENDIX

A. THE LIFTING LEMMA

The lifting lemma for standard narrowing [Middeldorp and Hamoen 1994] can be locally adapted to S-rewriting with non-normalized substitutions provided they fulfill some constraints on the positions of rewriting. To do so, we need the following two propositions (the first one is obvious).

**Proposition A.1.** Let \( t \in T(F, X) \) and \( \sigma \) a substitution of \( T(F, X) \). Then \( \text{Var}(\sigma t) = (\text{Var}(t) - \text{Dom}(\sigma)) \cup \text{Ran}(\sigma \text{Var}(t)) \).

**Proposition A.2.** Suppose we have substitutions \( \sigma, \mu, \nu \) and sets \( A, B \) of variables such that \((B - \text{Dom}(\sigma)) \cup \text{Ran}(\sigma) \subseteq A \). If \( \mu = \nu[A] \) then \( \mu \sigma = \nu \sigma[B] \).

**Proof.** Let us consider \((\mu \sigma)_B\), which can be divided as follows: \((\mu \sigma)_B = (\mu \sigma)_{B \cap \text{Dom}(\sigma)} \cup (\mu \sigma)_{B - \text{Dom}(\sigma)}\).

For \( x \in B \cap \text{Dom}(\sigma) \), we have \( \text{Var}(\sigma x) \subseteq \text{Ran}(\sigma) \), and then \((\mu \sigma)x = \mu(\sigma x) = \mu_{\text{Ran}(\sigma)}(\sigma x) = (\mu_{\text{Ran}(\sigma)})_x\).

Therefore \((\mu \sigma)_{B \cap \text{Dom}(\sigma)} = (\mu_{\text{Ran}(\sigma)} \sigma)_{B \cap \text{Dom}(\sigma)}\).

For \( x \in B - \text{Dom}(\sigma) \), we have \( \sigma x = x \), and then \((\mu \sigma)x = \mu(\sigma x) = \mu x\). Therefore we have \((\mu \sigma)_{B - \text{Dom}(\sigma)} = \mu_{B - \text{Dom}(\sigma)}\).

Henceforth we get \((\mu \sigma)_B = (\mu_{\text{Ran}(\sigma)} \sigma)_{B \cap \text{Dom}(\sigma)} \cup \mu_{B - \text{Dom}(\sigma)}\).

By a similar reasoning, we get \((\nu \sigma)_B = (\nu_{\text{Ran}(\sigma)} \sigma)_{B \cap \text{Dom}(\sigma)} \cup \nu_{B - \text{Dom}(\sigma)}\).

By hypothesis, we have \( \text{Ran}(\sigma) \subseteq A \) and \( \mu = \nu[A] \). Then \( \mu_{\text{Ran}(\sigma)} = \nu_{\text{Ran}(\sigma)}\). Likewise, since \( B - \text{Dom}(\sigma) \subseteq A \), we have \( \mu_{B - \text{Dom}(\sigma)} = \nu_{B - \text{Dom}(\sigma)}\).

Then we have \((\mu \sigma)_B = (\mu_{\text{Ran}(\sigma)} \sigma)_{B \cap \text{Dom}(\sigma)} \cup \mu_B - \text{Dom}(\sigma) = (\nu_{\text{Ran}(\sigma)} \sigma)_{B \cap \text{Dom}(\sigma)} \cup \nu_B - \text{Dom}(\sigma) = (\nu \sigma)_B\). Therefore \((\mu \sigma) = (\nu \sigma)[B]\).

**Lemma 4.3.1 (S-lifting Lemma).** Let \( R \) be a TRS. Let \( s \in T(F, X) \), \( \alpha \) a ground substitution such that \( as \) is \( S \)-reducible at a non variable position \( p \) of \( s \), and \( Y \subseteq X \) a set of variables such that \( \text{Var}(s) \cup \text{Dom}(\alpha) \subseteq Y \). If \( as \overset{S_{p \leftarrow l \leftarrow r}}{\rightarrow} t' \), then there exist a term \( s' \in T(F, X) \) and substitutions \( \beta, \sigma = \sigma_0 \land \bigwedge_{j \in [1..k]} \overline{\sigma_j} \) such that:

1. \( s \overset{S_{p \leftarrow l \leftarrow r \leftarrow \sigma}}{\rightarrow} s' \),
2. \( \beta s' = t' \),
3. \( \beta \sigma_0 = \alpha[Y] \),
4. \( \beta \) satisfies \( \bigwedge_{j \in [1..k]} \overline{\sigma_j} \).

where \( \sigma_0 \) is the most general unifier of \( s |_p \) and \( l \) and \( \sigma_j, j \in [1..k] \) are all most general unifiers of \( \sigma_0 |_{p'} \) and a left-hand side \( \ell' \) of a rule of \( R \), for all position \( p' \) which are \( S \)-better positions than \( p \) in \( s \).

**Proof.** In the following, we assume that \( Y \cap \text{Var}(l) = \emptyset \) for every \( l \rightarrow r \in R \).

If \( as \overset{S_{p \leftarrow l \leftarrow r}}{\rightarrow} t' \), then there exists a substitution \( \tau \) such that \( \text{Dom}(\tau) \subseteq \text{Var}(l) \) and \((\alpha s)|_p = \tau l\). Moreover, since \( p \) is a non variable position of \( s \), we have \((\alpha s)|_p = \alpha(s)|_p\). Denoting \( \mu = \alpha \tau \), we have:

\[
\begin{align*}
\mu(s)|_p &= \alpha(s)|_p \quad \text{for } \text{Dom}(\tau) \subseteq \text{Var}(l) \text{ and } \text{Var}(l) \cap \text{Var}(s) = \emptyset \\
&= \tau l \quad \text{by definition of } \tau \\
&= \mu l \quad \text{for } \text{Dom}(\alpha) \subseteq Y \text{ and } Y \cap \text{Var}(l) = \emptyset,
\end{align*}
\]
and therefore $s|_p$ and $l$ are unifiable. Let us note $\sigma_0$ the most general unifier of $s|_p$ and $l$, and $s' = \sigma_0(s|_p)$.

Since $\sigma_0$ is more general than $\mu$, there exists a substitution $\rho$ such that $\rho\sigma_0 = \mu$. Let $Y_1 = (Y - Dom(\sigma_0)) \cup Ran(\sigma_0)$. We define $\beta = \rho\gamma$. Clearly $Dom(\beta) \subseteq Y_1$.

We now show that $Var(s') \subseteq Y_1$, by the following reasoning:

— since $s' = \sigma_0(s|_p)$, we have $Var(s') = Var(\sigma_0(s|_p))$;
— the rule $l \rightarrow r$ is such that $Var(r) \subseteq Var(l)$, therefore we have $Var(\sigma_0(s|_p)) \subseteq Var(\sigma_0(s|_p[l|_p]))$; and therefore, thanks to the previous point, $Var(s') \subseteq Var(\sigma_0(s|_p[l|_p]))$.
— since $\sigma_0(s[l|_p]) = \sigma_0(s|_p[l|_p])$ and since $\sigma_0$ unifies $l$ and $s|_p$, we get $\sigma_0(s[l|_p]) = (\sigma_0 s)[\sigma_0(s)]_p = \sigma_0(s|_p[l|_p])$ and, thanks to the previous point: $Var(s') \subseteq Var(\sigma_0(s))$;
— according to Proposition A.1, we have $Var(\sigma_0(s)) = (Var(s) - Dom(\sigma_0)) \cup Ran(\sigma_0\gamma)$ (by hypothesis, $Var(s) \subseteq Y$). Moreover, since $Ran(\sigma_0\gamma) \subseteq Ran(\sigma_0)$, we have $Var(\sigma_0(s)) \subseteq (Y - Dom(\sigma_0)) \cup Ran(\sigma_0)$, that is $Var(\sigma_0(s)) \subseteq Y_1$. Therefore, with the previous point, we get $Var(s') \subseteq Y_1$.

From $Dom(\beta) \subseteq Y_1$ and $Var(s') \subseteq Y_1$, we infer $Dom(\beta) \cup Var(s') \subseteq Y_1$.

Let us now prove that $\beta s' = t'$. Since $\beta = \rho\gamma$, we have $\beta = \rho[\gamma]$. Since $Var(s') \subseteq Y_1$, we get $\beta s' = \rho s'$. Since $s' = \sigma_0(s|_p)$, we have $\beta s' = \rho\sigma_0(s|_p) = \mu(s|_p) = \mu s|_p$. Then $\beta s' = \mu s|_p$.

We have $\mu \tau \subseteq \sigma_0(\tau)$, so we get $\mu = \alpha[\gamma]$. Since $Var(s) \subseteq Y$, we get $\mu s = \alpha s$.

Likewise, by hypothesis we have $\mu \tau \subseteq \sigma_0(\tau)$, $\mu \tau \subseteq \sigma_0(\tau)$, and $\mu \tau \subseteq \sigma_0(\tau)$, so we get $\mu \tau \subseteq \sigma_0(\tau)$. Finally, as $\beta s' = \mu s|_p$, we get $\beta s' = t'$.

Next let us prove that $\beta s_0 = \alpha[\gamma]$. Remembering that $Y_1 = (Y - Dom(\sigma_0)) \cup Ran(\sigma_0)$, Proposition A.2 (with the notations $A$ for $Y_1$, $B$ for $Y$, $\nu$ for $\rho$ and $\sigma$ for $\sigma_0$) yields $\beta s_0 = \rho s_0[\gamma]$. We already noticed that $\mu = \alpha[\gamma]$. Linking these two equalities via the equation $\rho s_0 = \mu s_0 = \alpha[\gamma]$.

Let us now suppose that there exist a rule $l' \rightarrow r' \in R$, a position $p'$ better than $p$ and a substitution $\sigma_1$ such that $\sigma_1(\sigma_0(s|_p)) = \sigma_1 t'$.

Let us now suppose that $\beta$ does not satisfy $\bigwedge_{i \in [1..k]}[\sigma_i]$. There exists $i \in [1..k]$ such that $\beta$ satisfies $\sigma_i = \bigwedge_{i \in [1..n]}(x_i = u_i)$. So $\beta$ is such that $\bigwedge_{i \in [1..n]}(\beta x_i = \beta u_i)$.

Thus, on $Dom(\beta) \cap Dom(\sigma_1) \subseteq \{x_i, i \in [1..n]\}$, we have $\beta x_i = \beta u_i$, so $\beta \sigma_1 = \beta$. Moreover, as $\beta$ is a ground substitution, $\sigma_1 \beta = \beta$. Thus, $\beta \sigma_1 = \sigma_1 \beta$.

On $Dom(\beta) \cup Dom(\sigma_1) - (Dom(\beta) \cap Dom(\sigma_1))$, either $\beta = Id$, or $\sigma_1 = Id$, so $\beta \sigma_1 = \sigma_1 \beta$.

As a consequence, $\alpha(s) = \sigma_1 \alpha(s) = \sigma_1 \beta \sigma_0(s) = \beta \sigma_1 \sigma_0(s)$ is reducible at position $p'$ with the rule $l'$, which is impossible by definition of $S$-reducibility of $\alpha(s)$ at position $p$. So the ground substitution $\beta$ satisfies $\bigwedge_{i \in [1..k]}[\sigma_i]$ for all most general unifiers $\sigma_1$ of $\sigma_0 s$ and a left-hand side of rule of $R$ at $S$-better positions of $p$.
Therefore, denoting \( \sigma = \sigma_0 \land \bigwedge_{i \in [1..k]} \overline{\sigma_i} \), from the beginning of the proof, we get \( s \sim_{[p,t\rightarrow r,s]}^S s' \), and then the point (1) of the current lemma holds. \( \square \)

B. PROOF OF THE GENERIC TERMINATION RESULT

Let us remind that \( \text{SUCCESS}(g, \succ) \) means that the application of \( \text{Strat-Rules}(S) \) on \((g(x_1, \ldots , x_m))\), \( \top, \top \) gives a finite proof tree, whose sets \( C \) of ordering constraints are satisfied by a same ordering \( \succ \), and whose leaves are either states of the form \((\emptyset, A, C)\) or states whose set of constraints \( A \) is unsatisfiable.

**Theorem 5.4.1.** Let \( R \) be a TRS on a set \( F \) of symbols containing at least a constructor constant. If there exists an \( F \)-stable ordering \( \succ \) having the subterm property, such that for each symbol \( g \in \text{Def} \), we have \( \text{SUCCESS}(g, \succ) \), then every term of \( T(F) \) terminates with respect to the strategy \( S \).

**Proof.** We use an emptiness lemma, an abstraction lemma, a narrowing lemma, and a stopping lemma, which are given after this main proof.

We prove by induction on \( T(F) \) that any ground instance \( \theta f(x_1, \ldots , x_m) \) of any term \( f(x_1, \ldots , x_m) \in T(F, A) \) S-terminates. The induction ordering is constrained along the proof. At the beginning, it has at least to be \( F \)-stable and to have the subterm property, which ensures its noetherianity. Such an ordering always exists on \( T(F) \) (for instance the embedding relation). Let us denote it \( \succ \).

If \( f \) is a constructor, then \( \theta f(x_1, \ldots , x_m) \downarrow = f(\theta x_1, \ldots , \theta x_m) \downarrow \in [\theta x_1, \ldots , \theta x_m] \downarrow \), where \( \{i_1, \ldots , i_p\} \in [1..m] \) are the highest positions in \( f(\theta x_1, \ldots , \theta x_m) \), where subterms can be normalized, according to the strategy \( S \). (More specifically, \( \{i_1, \ldots , i_p\} = [1..m] \) if \( S = \text{Innermost or S = Outermost} \), \( \{i_1, \ldots , i_p\} = \{j \mid j \in \{p_1, \ldots , p_n\}, j \neq 0 \} \) where \( \{p_1, \ldots , p_n\} = \text{LS}(f) \) if \( S = \text{Local-Strat.} \).

By subterm property of \( \succ \), we have \( \theta f(x_1, \ldots , x_m) \downarrow = f(\theta x_1, \ldots , \theta x_m) \uparrow \succ \theta x_i_1, \ldots , \theta x_i_p \). Then, by induction hypothesis, we suppose that \( \theta x_1, \ldots , \theta x_i_p \) S-terminate, and so their respective normal forms \( \theta x_1, \ldots , \theta x_i_p \downarrow \) exist and \( f(\theta x_1, \ldots , \theta x_m) \downarrow \) is in normal form. We may thus restrict our attention to terms headed by a defined symbol.

If \( f \) is not a constructor, let us denote it \( g \) and prove that \( g(\theta x_1, \ldots , \theta x_m) \) S-terminates for any \( \theta \) satisfying \( A = \top \) if we have \( \text{SUCCESS} \sim S (h, \succ) \) for every defined symbol \( h \). Let us denote \( g(x_1, \ldots , x_m) \) by \( t_{ref} \) in the sequel of the proof.

To each state \( s \) of the proof tree of \( g \), characterized by a current term \( t \) and the set of constraints \( A \), we associate the set of ground terms \( G = \{ \alpha t \mid \alpha \) satisfies \( A \} \), that is the set of ground instances represented by \( s \).

Inference rule **Abstract** (resp. **Narrow**) transforms \((\{t\}, A)\) into \((\{t'\}, A')\) to which is associated \( G' = \{ \beta t' \mid \beta \) satisfies \( A' \} \) (resp. into \((\{t'\}, A'_i)\), \( i \in [1..l] \) to which are associated \( G'_i = \{ \beta_i t'_i \mid \beta_i \) satisfies \( A'_i \}) \).

By abstraction (resp. narrowing) Lemma, applying **Abstract** (resp. **Narrow**), for each \( \alpha t \) in \( G \), there exists a \( \beta t' \) (resp. \( \beta_i t'_i \)) in \( G' \) and such that S-termination of \( \beta t' \) (resp. of the \( \beta_i t'_i \)) implies S-termination of \( \alpha t \).

When the inference rule **Stop** applies on \( (\{t\}, A, C) \):
either A is satisfiable, in which case, by stopping lemma, every term of $G = \{ \alpha t \mid \alpha \text{ satisfies } A \}$ is S-terminating,

—or A is unsatisfiable. In this case, G is empty. By emptyness lemma, all previous states on the branch correspond to empty sets $G_i$, until an ancestor state $\{t_p\}, A_p, C_p)$, where $A_p$ is satisfiable. Then every term $\alpha t$ of $G_p$ is irreducible, otherwise, by Abstraction and Narrowing lemmas, $G_{p+1}$ would not be empty.

Therefore, S-termination is ensured for all terms in all sets $G$ of the proof tree.

As the process is initialized with $\{t_{ref}\}$ and a constraint problem satisfiable by any ground substitution, we get that $g(\theta x_1, \ldots, \theta x_m)$ is S-terminating, for any $t_{ref} = g(x_1, \ldots, x_m)$, and any ground instance $\theta$. ∎

**Lemma (Emptyness Lemma).** Let $\{\{t\}, A, C\}$ be a state of any proof tree, giving $\{\{t’\}, A’, C’\}$ by application of Abstract or Narrow. If $A$ is unsatisfiable, then so is $A’$.

**Proof.** If Abstract is applied, then if $A$ is unsatisfiable, $A’ = A \land t|_{i_1} = X_{i_1} \ldots \land t|_{i_p} = X_{i_p}$ is also unsatisfiable.

If Narrow is applied, then if $A$ is unsatisfiable (which does not occur for local strategies), $A’ = A \land \sigma$ in the innermost case, and $A’ = R(t) \land A \land \sigma$ in the outermost case are also unsatisfiable. ∎

**Lemma (Abstraction Lemma).** Let $\{\{t\}, A, C\}$ be a state of any proof tree, giving the state $\{\{t’ = t[X_j]_{j\in\{i_1, \ldots, i_p\}}\}, A’, C’\}$ by application of Abstract.

For any ground substitution $\alpha$ satisfying $A$, if $\alpha t$ is reducible, there exists $\beta$ such that S-termination of $\beta t’$ implies S-termination of $\alpha t$. Moreover, $\beta$ satisfies $A’$.

**Proof.** We prove that $\alpha t \rightarrow^{S} \beta t’$, where $\beta = \alpha \cup \bigcup_{j\in\{i_1, \ldots, i_p\}} X_j = \alpha t|_{j}$.

First, whatever the strategy $S$, the abstraction positions in $t$ are chosen so that the $\alpha t|_{j}$ can be supposed terminating w.r.t. $S$. Indeed, each term $t|_{i_j}$ is such that:

—either $\text{TERMIN}(S, t|_{j})$ is true, and then by definition of the predicate $\text{TERMIN}$, $\alpha t|_{j}$ S-terminates;

—or $t_{ref} > t|_{j}$ is satisfiable by $\succ$, and then, by induction hypothesis, $\alpha t|_{j}$ S-terminates.

So the $\alpha t|_{j}$ exist.

Then, let us consider the different choices of abstraction positions w.r.t the strategy $S$:

—Either $S = \text{Innermost}$ and whatever the positions $i_1, \ldots, i_p$ in $t$, $\alpha t \rightarrow^{S \text{Inn}} \alpha t[\alpha t|_{i_1} t|_{i_1}, \ldots, [\alpha t|_{i_p} t|_{i_p}] = \beta t’$.

—Or $S = \text{Outermost}$ and $t$ is abstracted at positions $i_1, \ldots, i_p$ if $t[X_j]_{j\in\{i_1, \ldots, i_p\}}$ is not outermost narrowable at prefix positions of $i_1, \ldots, i_p$, which warrants that the only redex positions of $\alpha t$ are suffixes of the $j$, and then that $\alpha t \rightarrow^{S \text{Outermost}} \alpha t[\alpha t|_{i_1} t|_{i_1}, \ldots, [\alpha t|_{i_p} t|_{i_p}] = \beta t’$. 


—Either \( S = \text{Local–Strat} \) and \( \text{top}(t) = f \) with \( LS(f) = [p_1, \ldots, p_n] \). The term \( t \) is abstracted at positions \( i_1, \ldots, i_p \in \{p_1, \ldots, p_k-1\} \), where \( p_k = 0 \), or at positions \( i_1, \ldots, i_p \in \{p_1, \ldots, p_n\} \) if \( p_1, \ldots, p_n \neq 0 \). According to the definition of local strategies, \( \alpha \xrightarrow{\text{Local–Strat}} \alpha t \downarrow_{i_1} \ldots \downarrow_{i_p} = \beta t' \). If \( LS(f) = [] \) or \( LS(f) = [0, p_2, \ldots, p_n] \), then \( t = t' \) and \( A = A' \), so \( \alpha t = \beta t' \).

So \( \alpha t \xrightarrow{\text{S}} \beta t' \) for any normal form \( \alpha t \downarrow_j \) of \( \alpha t \), for \( j \in \{i_1, \ldots, i_p\} \). Then, S-termination of \( \beta t' \) implies S-termination of \( \alpha t \).

Clearly in all cases, \( \beta \) satisfies \( A' = A \land t_{i_1} = X_{i_1} \ldots \land t_{i_p} = X_{i_p} \), provided the \( X_i \) are not in \( \text{Dom}(\alpha) \), which is true since the \( X_i \) are fresh variables not appearing in \( A \).

\[
\square
\]

**Lemma (narrowing lemma).** Let \((\{t\}, A, C)\) be a state of any proof tree, giving the states \((\{v_i\}, A', C'), i \in [1..l] \), by application of Narrow. For any ground substitution \( \alpha \) satisfying \( A \), if \( \alpha t \) is reducible, then, for each \( i \in [1..l] \), there exist \( \beta_i \) such that S-termination of the \( \beta_i v_i, i \in [1..l] \), implies S-termination of \( \alpha t \). Moreover, \( \beta_i \) satisfies \( A'_i \) for each \( i \in [1..l] \).

**Proof.** We reason by case on the different strategies.

—Either \( S = \text{Innermost} \), and By lifting lemma, there is a term \( v \) and substitutions \( \beta \) and \( \sigma = \sigma_0 \land \bigwedge_{j \in [1..k]} \sigma_j \), corresponding to each rewriting step \( \alpha f(u_1, \ldots, u_m) \xrightarrow{\text{inner}}_{p, l-r, \sigma} v \), such that:

1. \( t = f(u_1, \ldots, u_m) \xrightarrow{\text{inner}}_{p, l-r, \sigma} v \),
2. \( \beta v = t' \),
3. \( \beta \sigma_0 = \alpha \bigwedge_{j \in [1..k]} \sigma_j \).

where \( \sigma_0 \) is the most general unifier of \( l_p \) and \( l \) and \( \sigma_j, j \in [1..k] \) are all most general unifiers of \( \sigma_0 l_{p'} \) and a left-hand side \( l' \) of a rule of \( \mathcal{R} \), for all position \( p' \) which are suffix positions of \( p \) in \( t \).

These narrowing steps are effectively produced by the rule Narrow, applied in all possible ways on \( f(u_1, \ldots, u_m) \). So a term \( \beta v \) is produced for every innermost rewriting branch starting from \( \alpha t \). Then innermost termination of the \( \beta v \) implies innermost termination of \( \alpha t \).

Let us prove that \( \beta \) satisfies \( A' = A \land \sigma_0 \land \bigwedge_{j \in [1..k]} \sigma_j \).

By lifting lemma, we have \( \alpha = \beta \sigma_0 \) on \( \mathcal{Y} \). As we can take \( \mathcal{Y} \supseteq \text{Var}(A) \), we have \( \alpha = \beta \sigma_0 \) on \( \text{Var}(A) \).

More precisely, on \( \text{Ran}(\sigma_0) \), \( \beta \) is such that \( \beta \sigma_0 = \alpha \) and on \( \text{Var}(A) \setminus \text{Ran}(\sigma_0) \), \( \beta = \alpha \). As \( \text{Ran}(\sigma_0) \) only contains fresh variables, we have \( \text{Var}(A) \cap \text{Ran}(\sigma_0) \neq \emptyset \), so \( \text{Var}(A) \setminus \text{Ran}(\sigma_0) = \text{Var}(A) \). So \( \beta = \alpha \) on \( \text{Var}(A) \) and then, \( \beta \) satisfies \( A \). Moreover, as \( \beta \sigma_0 = \alpha \) on \( \text{Dom}(\sigma_0) \), \( \beta \) satisfies \( \sigma_0 \).

So \( \beta \) satisfies \( A \land \sigma_0 \). Finally, with the point 4. of the lifting lemma, we conclude that \( \beta \) satisfies \( A' = A \land \sigma_0 \land \bigwedge_{j \in [1..k]} \sigma_j \).
—or $S = \text{Local–Strat}$, and Narrow is applied on \{t = f(u_1, \ldots, u_m)\} with $l = [0, p_1, \ldots, p_n]$. For any $\alpha$ satisfying $A$,—either $\alpha f(u_1, \ldots, u_m)$ is irreducible at the top position, but may be reduced at the positions $p_1, \ldots, p_n$. In this case, either $f(u_1, \ldots, u_m)$ is not narowable at the top position, or $f(u_1, \ldots, u_m) \sim_{\epsilon, \sigma_i} v_i$ for $i \in [1..l]$ and $A \land \sigma_i$ is unsatisfiable for each $i$, or there exists $i \in [1..l]$ such that $f(u_1, \ldots, u_m) \sim_{\epsilon, \sigma_i} v_i$ and $A \land \sigma_i$ is satisfiable.

In the first two cases, Narrow produces the state $(\{t[p_1, \ldots, p_n]\}, A, C)$, and setting $\beta = \alpha$, we have termination of $\beta t[p_1, \ldots, p_n]$ implies termination of $\alpha t[0, p_1, \ldots, p_n]$, and $\beta$ satisfies $A' = A$.

In the third case, Narrow produces the state $(\{t[p_1, \ldots, p_n]\}, A \land (\bigwedge_{i=1}^l \overline{p_i}), C)$, and setting $\beta = \alpha$, we have termination of $\beta t[p_1, \ldots, p_n]$ implies termination of $\alpha t[0, p_1, \ldots, p_n]$. Moreover, as $\alpha t$ is reducible at the top position, $\alpha$ satisfies $(\bigwedge_{i=1}^l \overline{p_i})$. Thus, as $\alpha$ satisfies $A$, $\beta$ satisfies $A' = A \land (\bigwedge_{i=1}^l \overline{p_i})$.

—or $\alpha f(u_1, \ldots, u_m)$ is reducible at the top position, and by lifting lemma, there is a term $v$ and substitutions $\beta$ and $\sigma_0$ corresponding to each rewriting step $\alpha f(u_1, \ldots, u_m) \rightarrow_{\epsilon, l \rightarrow r} v$, such that:

1. $t = f(u_1, \ldots, u_m) \sim_{l \rightarrow r, \sigma_0} v$,
2. $\beta v = t'$,
3. $\beta \sigma_0 = \alpha[Y]$.

where $\sigma_0$ is the most general unifier of $t$ and $l$.

These narrowing steps are effectively produced by Narrow, which is applied in all possible ways on $f(u_1, \ldots, u_m)$ at the top position. So a term $\beta v$ is produced for every LS-rewriting step applying on $\alpha t$ at the top position. Then termination of the $\beta v$ implies termination of $\alpha t$ for the given LS-strategy.

We prove that $\beta$ satisfies $A \land \sigma_0$ like in the innermost case, except that there is no negation of substitution here.
By definition of $A_0$, the $\beta_0$ are the $\alpha$ verifying the reduction formula $f(u_1, \ldots, u_m) \rightsquigarrow^* f(x_1, \ldots, x_n)^0$, with $p = \langle x_1 \rightsquigarrow^* x'_1 \ldots (x_k \rightsquigarrow^* x'_k) \rangle$. We have $\text{Dom}(\alpha) = \text{Var}(A) \cup \{x_1, \ldots, x_k\}$. The domain of $\beta_0$ is $\text{Dom}(\alpha) \cup \{x'_1, \ldots, x'_k\}$.

Then $\beta_0 = 0 [\text{Dom}(\alpha)]$ and by definition of the reduction formula, the $\beta_0 x'_i$ are such that $t[\beta_0 x'_i]_{p_1} \ldots [\beta_0 x'_k]_{p_k}$ is the first reduced form of $f(u_1, \ldots, u_n)$ in any outermost rewriting chain starting from $\alpha f(u_1, \ldots, u_n)$, having an outermost rewriting position at a non variable position of $f(u_1, \ldots, u_n)$.

Then, by definition of the outermost strategy, the $\beta_0 t_0$ represent any possible outermost reduced form of $\alpha t$ just before the reduction occurs at a non variable occurrence of $f(u_1, \ldots, u_n)$. Thus, outermost termination of the $\beta_0 t_0$ implies outermost termination of the $\alpha t$.

Then $t_0$ is narrowed in all possible ways into terms $v_i$ at positions $p_i$ with substitutions $\sigma_i$, provided $p_i$ and $\sigma_i$ satisfy the outermost narrowing requirements, as defined in Definition 4.3.2. We now show that if $\beta_0 t_0$ is reducible, then there exist $\beta_i$ satisfying $A'$ such that outermost termination of the $\beta_i v_i$ implies outermost termination of $\beta_0 t_0$.

We have $\beta_0 t_0 \rightsquigarrow^{\text{Out}}_{p_i \leftarrow r, \sigma} t'$ and $p \in \overline{r}(t_0)$ since $t_0 = t^0$.

By lifting lemma, there is a term $v$ and substitutions $\beta$ and $\sigma = \sigma_0 \land \bigwedge_{j \in \Gamma \setminus \{1 \ldots k\}} \overline{s_j}$, corresponding to each rewriting step $\beta_0 t_0 \rightsquigarrow^{\text{Out}}_{p_i \leftarrow r, \sigma} t'$, such that:

1. $t_0 \rightsquigarrow^{\text{Out}}_{p_i \leftarrow r, \sigma} v$,  
2. $\beta v = t'$,  
3. $\beta \sigma_0 = \beta_0 [\gamma]$,  
4. $\beta$ satisfies $\bigwedge_{j \in \Gamma \setminus \{1 \ldots k\}} \overline{s_j}$.

where $\sigma_0$ is the most general unifier of $t_0|_p$ and $l$ and $\sigma_j, j \in \Gamma \setminus \{1 \ldots k\}$ are all most general unifiers of $\sigma_0 t_0|_p$ and a left-hand side $v'$ of a rule of $R$, for all position $p'$ which are prefix positions of $p$ in $t_0$.

These narrowing steps are effectively produced by the rule $\text{Narrow}$, applied in all possible ways. So a term $\beta v$ is produced for every outermost rewriting branch starting from $\beta_0 t_0$. Then outermost termination of the $\beta v$ implies outermost termination of $\beta_0 t_0$.

We prove that $\beta$ satisfies $A' = A_0 \land \bigwedge_{j \in \Gamma \setminus \{1 \ldots k\}} \overline{s_j}$ like in the innermost case.

\[ \square \]

**Lemma (Stopping Lemma).** Let $\{ t \}, A, C$ be a state of any proof tree, with $A$ satisfiable, and giving the state $(\emptyset, A', C')$ by application of an inference rule. Then for any ground substitution $\alpha$ satisfying $A$, at $S$-terminates.

**Proof.** The only rule giving the state $(\emptyset, A', C')$ is $\text{Stop}$. When $\text{Stop}$ is applied, then

1. $\text{TERMIN}(S, t)$ and then at $S$-terminates for any ground substitution $\alpha$,
2. $(t_{\text{ref}} > t)$ is satisfiable. Then, for any ground substitution $\alpha$ satisfying $A$, at $t_{\text{ref}} \succ \alpha$. By induction hypothesis, at $S$-terminates.

\[ \square \]
C. THE USABLE RULES

To prove Lemma 6.2.1, we need the next three lemmas. The first two ones are pretty obvious from the definition of the usable rules.

**Lemma C.1.** Let $\mathcal{R}$ be a TRS on a set $\mathcal{F}$ of symbols and $t \in T(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A)$. Then, every symbol $f \in \mathcal{F}$ occurring in $t$ is such that $\text{Rls}(f) \subseteq U(t)$.

**Proof.** We proceed by structural induction on $t$.

— If $t \in \mathcal{X} \cup \mathcal{X}_A$, the property is trivially true;

— If $t$ is a constant $a$, $U(t = a) = \text{Rls}(a) \cup \cup_{i \rightarrow r \in \text{Rls}(a)} U(r)$; the only symbol of $t$ is $a$, and we have $\text{Rls}(a) \subseteq U(t)$.

Let us consider a non-constant and non-variable term $t \in T(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A)$, of the form $f(u_1, \ldots, u_n)$. Then, by definition of $U(t)$, we have $U(t) = \text{Rls}(f) \cup \cup_{i=1}^n U(u_i) \cup \cup_{i \rightarrow r \in \text{Rls}(f)} U(r)$. Then, whatever $g$ symbol of $t$, either $g = f$ and then $\text{Rls}(g) \subseteq U(t)$, or $g$ is a symbol occurring in some $u_i$ and by induction hypothesis on $u_i$, $\text{Rls}(g) \subseteq U(u_i)$, with $U(u_i) \subseteq U(t)$. □

**Lemma C.2.** Let $\mathcal{R}$ be a TRS on a set $\mathcal{F}$ of symbols and $t \in T(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A)$. Then $l \rightarrow r \in U(t) \Rightarrow U(r) \subseteq U(t)$.

**Proof.** According to the definition of the usable rules, if a term $t$ is such that $\text{Var}(t) \cap \mathcal{X} \neq \emptyset$, then $U(t) = \mathcal{R}$, and then the property is trivially true. We will then suppose in the following that $t$ does not contain any variable of $\mathcal{X}$.

Let $l \rightarrow r \in U(t)$. By definition of $U(t)$, since $\text{Var}(t) \cap \mathcal{X} = \emptyset$, among all recursive applications of the definition of $U$ in $U(t)$, there is an application $U(t')$ of $U$ to some term $t'$ such that $U(t') = \text{Rls}(g) \cup U(t'_1) \cup \cup_{i \rightarrow r \in \text{Rls}(g)} U(r)$, with $U(t') \subseteq U(t)$, and $l \rightarrow r \in \text{Rls}(g)$, with $g = \text{top}(l)$.

Since $l \rightarrow r \in \text{Rls}(g)$, by definition of $U(t')$, we have $U(r) \subseteq \cup_{i \rightarrow r' \in \text{Rls}(g)} U(r')$, and then $U(r) \subseteq U(t') \subseteq U(t)$. □

**Lemma C.3.** Let $\mathcal{R}$ be a TRS on a set $\mathcal{F}$ of symbols and $t \in T(\mathcal{F}, \mathcal{X} \cup \mathcal{X}_A)$. Whatever $\alpha$ ground normalized substitution and $at \rightarrow_{p_1, l_1 \rightarrow r_1} t_1 \rightarrow_{p_2, l_2 \rightarrow r_2} t_2 \rightarrow \ldots \rightarrow_{p_n, l_n \rightarrow r_n} t_n$ rewrite chain starting from $at$, the defined symbol of $t_k, 1 \leq k \leq n$ at a redex position of $t_k$ is either a symbol of $t$ or one of the $r_i, i \in [1..k]$.

**Proof.** We proceed by induction on the length of the derivation. The property is obviously true for an empty derivation i.e. on $at$.

Let us show the property for the first rewriting step $at \rightarrow_{p_1, l_1 \rightarrow r_1} t_1$. By definition of rewriting, $\exists \sigma : \sigma t_1 = \alpha t|_{p_1}$ and $t_1 = \alpha t[\sigma r_1]|_{p_1}$. Let $f$ be the redex symbol of $t_1$ at a position $p$, and let us show that $f$ comes either from $t$ or from $r_1$.

Since $t_1 = \alpha t[\sigma r_1]|_{p_1}$, either $p$ is a position of the context $\alpha t|_{p_1}$, which does not change by rewriting, so we already have $f$ as redex symbol of $at$ at position $p$. As $\alpha$ is normalized, $p$ is a position of $t$, so $f$ is a symbol of $t$.

Or $p$ corresponds in $t_1$ to a non variable position of $r_1$, so $f$ is a symbol of $r_1$.

Or $p$ corresponds in $t_1$ to a position $r$ in $\sigma x$, for a variable $x \in \text{Var}(r_1)$ at position $q$ in $r_1$; we have $p = p_1 q r$. In this case, since $\text{Var}(r_1) \subseteq \text{Var}(l_1)$, we have $x \in \text{Var}(l_1)$, so $\sigma x$ is also a subterm of $at$, and $f$ occurs in $at$ at position $p' = p_1 q r$, where $q'$ is a position of $x$ in $l_1$.
Moreover, as $p$ is a redex position in $t_1$, then by definition of the innermost strategy, there is no suffix redex position of $p$ in $t_1$. As $t_1|_p = \alpha t|_p$, then similarly $p'$ is a redex position in $\alpha t$. As $\alpha$ is normalized, $p'$ is a position of $t$, so $f$ is a symbol of $t$.

Then, let us suppose the property true for any term of the rewrite chain $\alpha t \rightarrow_{p_1,i_1} t_1 \rightarrow \ldots \rightarrow_{p_k,i_k} t_k$, i.e. any redex symbol $f$ of $t_k$ is also a symbol of $t$, or a symbol of one of the $r_i, i \in [1..k]$, and let us consider $t_k \rightarrow_{p_{k+1},i_{k+1}} t_{k+1}$.

By a similar reasoning than previously, we establish that any redex symbol $f$ of $t_{k+1}$ is also a symbol of $t_k$, or a symbol of $r_{k+1}$. We then conclude with the previous induction hypothesis. 

We are now able to prove Lemma 6.2.1.

**Lemma 6.2.1.** Let $R$ be a TRS on a set $F$ of symbols and $t \in T(F, X \cup X_A)$. Whatever at ground instance of $t$ and at $\rightarrow_{p_1,i_1} t_1 \rightarrow_{p_2,i_2} t_2 \rightarrow \ldots \rightarrow_{p_n,i_n} t_n$ rewrite chain starting from $at$, then $l_i \rightarrow r_i \in U(t)$, $\forall i \in [1..n]$.

**Proof.** If a variable $x \in X$ occurs in $t$, then $U(t) = R$ and the property is trivially true. We then consider in the following that $t \in T(F, X_A)$, and then that $\alpha$ is a (ground) normalized substitution.

We proceed by induction on $T(F, X_A)$ and on the length of the derivation.

The property is trivially true if $\alpha t$ is in normal form. For any $\alpha \rightarrow_{p_1,i_1} t_1$, since $\alpha$ is normalized, $p_1$ corresponds in $at$ to a non-variable position of $t$. Let $f$ be the symbol at position $p_1$ in $t$. Since $f$ is the symbol at the redex position $p_1$ of $\alpha t$ with the rule $l_1 \rightarrow r_1$, then $l_1 \rightarrow r_1 \in Rls(f)$. Moreover, thanks to Lemma C.1, $Rls(f) \subseteq U(t)$. Therefore, $l_1 \rightarrow r_1 \in U(t)$.

Let us now suppose the property is true for any derivation chain starting from $at$ whose length is less or equal to $k$, and consider the chain: $\alpha \rightarrow_{p_1,i_1} t_1 \rightarrow_{p_2,i_2} t_2 \rightarrow \ldots \rightarrow_{p_k,i_k} t_k \rightarrow_{p_{k+1},i_{k+1}} t_{k+1}$. Let $f$ be the symbol at position $p_{k+1}$ in $t_k$. Since $p_{k+1}$ is a redex position of $t_k$ with the rule $l_{k+1} \rightarrow r_{k+1}$, then $l_{k+1} \rightarrow r_{k+1} \in Rls(f)$.

By Lemma C.3 with a derivation of length $k$, we have two cases:

—either the symbol $f$ at position $p_{k+1}$ in $t_k$ is a symbol of $t$; then, thanks to Lemma C.1 on $t$, we get $Rls(f) \subseteq U(t)$; henceforth $l_{k+1} \rightarrow r_{k+1} \in U(t)$;

—or the symbol $f$ at position $p_{k+1}$ in $t_k$ is a symbol of a $r_i, i \in [1..k]$; then, thanks to Lemma C.1 on $r_i$, we get $Rls(f) \subseteq U(r_i)$; henceforth $l_{k+1} \rightarrow r_{k+1} \in U(r_i)$; by induction hypothesis we have $l_i \rightarrow r_i \in U(t)$ and, thanks to Lemma C.2, we have $U(r_i) \subseteq U(t)$. Henceforth $l_{k+1} \rightarrow r_{k+1} \in U(t)$.

Proposition 6.2.1. Let $R$ be a TRS on a set $F$ of symbols, and $t$ a term of $T(F, X \cup X_A)$. If there exists a simplification ordering $\succ$ such that $\forall l \rightarrow r \in U(t) : l \succ r$, then any ground instance of $t$ is terminating.

**Proof.** As $\succ$ orients the rules used in any reduction chain starting from $\alpha t$ for any ground substitution $\alpha$, by properties of the simplification orderings, $\succ$ also orients the reduction chains, which are then finite.
D. A LEMMA SPECIFIC TO THE OUTERMOST CASE

**Lemma 7.3.1.** Let \(\{t_i\}, A_i, C_i\) be the \(i^{th}\) state of any branch of the derivation tree obtained by applying the strategy \(S\) on \((\{t_{ref}\}, \top, \top)\), and \(\succ\) an \(F\)-stable ordering having the subterm property. If every reduction formula in \(A_i\) can be reduced to a formula \(\bigwedge_j x_j = x'_j\), then we have:

for all variable \(x\) of \(t_i\) in \(X\): \((t_{ref} \succ x)/A_i\) is satisfiable by \(\succ\).

**Proof.** The proof is made by induction on the number \(i\) of applications of the inference rules from \((\{t_{ref}\}, \top, \top)\) to the state \((\{t_i\}, A_i, C_i)\).

Let us prove that the property holds for \(i = 0\). We have \(t_0 = t_{ref}\) and then \(\text{Var}(t_0) = \text{Var}(t_{ref})\). Consequently, for every \(x \in \text{Var}(t_0)\), whatever the ground substitution \(\alpha\) such that \(\text{Var}(t_{ref}) \subseteq \text{Dom}(\alpha)\), \(\alpha x\) is a subterm of \(\alpha t_{ref}\). The induction ordering \(\succ\) satisfying the conditions of the rules before the application of these rules can be any \(F\)-stable ordering having the subterm property. We then have \(\alpha t_{ref} \succ \alpha z\).

We now prove that if the property holds for \(i - 1\), it also holds for \(i\).

If the rule used at the \(i^{th}\) step is **Stop**, then \(\text{Var}(t_i) = \emptyset\), and then, the property is trivially verified.

If the rule used at the \(i^{th}\) step is **Abstract**, as the rule **Abstract** replaces subterms in \(t_{i-1}\) by new variables of \(X_A\), then \((\text{Var}(t_i) \cap X) \subseteq (\text{Var}(t_{i-1} \cap X))\), so the property still holds.

If the rule used at the \(i^{th}\) step is **Narrow** then, by hypothesis, the reduction renaming applied to \(t_{i-1}\) and giving a term \(t'_{i-1}\) just consists in a mere renaming of the variables of \(t_{i-1}\). Let \(t_i\) be a term obtained by narrowing \(t'_{i-1}\) with the substitution \(\sigma\).

Let \(z \in \text{Var}(t_i)\), and \(\alpha\) a substitution satisfying \(A_i\). We show that \(\alpha t_{ref} \succ \alpha z\).

We have two cases:

1. either \(z\) is a fresh variable introduced by the narrowing step. Let \(x' \in \text{Var}(t'_{i-1})\) such that \(z \in \text{Var}(\sigma x')\), and \(x \in \text{Var}(t_{i-1})\) such that \(x'\) is a renaming of \(x\). By hypothesis, every reduction formula in \(A_i\) can be reduced to a formula \(\bigwedge_j x_j = x'_j\). This is then the same for \(A_{i-1}\). Moreover, since \(\alpha\) satisfies \(A_i\), then it satisfies in particular \(A_{i-1}\). Then, by induction hypothesis, \(\alpha t_{ref} \succ \alpha x\) and, since \(\alpha\) satisfies \(x = x'\), we also have \(\alpha t_{ref} \succ \alpha x'\).

By hypothesis, \(\sigma\) contains the equality \(x' = C[z]\), with \(C[z]\) a (possibly empty) context of \(z\). Moreover, by definition of the rule **Narrow**, \(A_i = A_{i-1} \land \text{R}(t_{i-1}) \land \sigma\). So \(A_i\) contains the equality \(x' = C[z]\).

Then, as \(\alpha\) satisfies \(A_i\), \(\alpha\) is such that \(\alpha x' = \alpha C[z]\). Since \(\alpha t_{ref} \succ \alpha x'\), we have \(\alpha t_{ref} \succ \alpha C[z]\) and then, by subterm property, \(\alpha t_{ref} \succ \alpha z\).

2. or \(z \in \text{Var}(t'_{i-1})\); by the same reasoning as in the previous point for \(x'\), we have \(\alpha t_{ref} \succ \alpha z\).
REFERENCES


