Uniform Distribution of Generalized Polynomials

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Abstract

Generalizied polynomials form a natural family of functions which are obtained from polynomials by adding to the arithmetic operations the operation of taking the greatest integer. Among other results we show that if the coefficients of a generalized polynomial $q(n)$ are sufficiently independent then the sequence $q(n), n = 1, 2, \ldots$, is uniformly distributed (mod 1) (see Theorem 3.1). We also show that for example the sequence $[\alpha n]\beta n, n = 1, 2, \ldots$, is uniformly distributed (mod 1) if and only if either $\alpha^2 \notin \mathbf{Q}$ and β is irrational or $\alpha^2 \in \mathbf{Q}$ and $1, \alpha, \beta$ are rationally independent.

1 Introduction

In [9] H. Weyl proved that for any real-valued polynomial

$$p(x) = a_k x^k + \ldots + a_1 x + a_0, \tag{1}$$

where at least one coefficient $a_i, i \geq 1$, is irrational, the sequence $\{p(n)\}_{n\in\mathbf{N}}$ is uniformly distributed (mod 1). This was a generalization of Hardy and Littlewood's results from 1914 on the fractional part of $n^k\theta$, [3]. Weyl's proof was simpler and it was Weyl who introduced the notion of uniform distribution (mod 1). In [7] van der Corput introduced an even simpler method to prove Weyl's theorem, based on the fact that if the sequence $\{x_{n+h} - x_n\}_{n\in\mathbf{N}}$ is uniformly distributed (mod 1) for all integers $h \geq 1$, then the sequence $\{x_n\}_{n\in\mathbf{N}}$ is also uniformly distributed (mod 1).

A natural extension of the family of real-valued polynomials arises by adding to the arithmetic operations the operation of taking the greatest integer function $[\cdot]$. In this way functions like $q_1(x) = [b_1x^2 + b_2x]b_3x^2 + [b_4x]b_5$ and $q_2(x) = \Big[[b_1x][b_2x^2]b_3\Big]b_4x$ can be obtained. We call such functions generalized polynomials. By restricting the domain to the integers, as we will do, these representations are not unique. However, it is enough to maintain certain conventions in order to treat questions of uniform distribution. $b_1, \ldots, b_5$ are called the coefficients of these concrete representations. The simplest generalized polynomials, next after the polynomials, are those of the form $[p(x)]\beta$, where $p(x)$ is a polynomial and β a real number. The case $[\alpha n]\beta$ is treated in ([5], p.310), and it follows from [8] that $[p(n)]\beta$, where $p(x)$ is a polynomial (1), is well distributed iff either $a_1, \ldots, a_k$ do not lie in a singly generated additive subgroup of the reals and β is irrational, or there exists $\gamma \in \mathbf{R}$ such that $a_i = b_i\gamma$, $b_i \in \mathbf{Q}, i = 1, \ldots, k$, and β is rationally independent of $1, \gamma$.

In this paper we will use van der Corput's method to show that a sequence coming from a generalized polynomial $q(x)$ having coefficients satisfying certain independence conditions, is uniformly distributed (mod 1). However, the proof in this case is far more complicated than for the polynomials.

It is clear that we need some conditions on the coefficients of a generalized polynomial $q(x)$ in order that the sequence $q(n)$ will be uniformly distributed (mod 1). For example, for any real number γ the sequence $\left[[\sqrt{2}n]\sqrt{3}\right]\gamma n - [\sqrt{6}n]\gamma n$ fails to be uniformly distributed because $\left[[\sqrt{2}n]\sqrt{3}\right] = [\sqrt{6}n]$ on a set of n of positive density. A less obvious example of a sequence which is not uniformly distributed (mod 1) is $[\sqrt{2}n][\sqrt{3}n]\sqrt{6}$, see Proposition 5.3. Here $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ are rationally independent, but $\sqrt{2}\sqrt{6} = 2\sqrt{3}$ and $\sqrt{3}\sqrt{6} = 3\sqrt{2}$ are rationally dependent of $1, \sqrt{2}, \sqrt{3}$. However, there are many sequences coming from generalized polynomials having dependent coefficients which are uniformly distributed (mod 1). The van der Corput's method fails to show that the sequences αn, where α is an irrational number, is uniformly distributed (mod 1), because $\alpha(n+h) - \alpha n = \alpha h$ is a constant for each fixed $h \in \mathbf{N}$. In a similar way, the van der Corput's method may fail to work for a uniformly distributed sequence coming from a generalized polynomial having dependent coefficient, as in the case of $[\sqrt{2}n]^2\sqrt{2}$. This sequence can be seen to be uniformly distributed (mod 1) by rewriting it,

$$[\sqrt{2}n]^2\sqrt{2} \equiv -2\sqrt{2}n^2 + \{\sqrt{2}n\}^2\sqrt{2} \pmod 1,$$

and using that $(2\sqrt{2}n^2, \sqrt{2}n)$ is uniformly distributed (mod 1) in $\mathbf{R}^2$. For then we have

$$\begin{aligned} \frac{1}{N}\sum_{n=0}^{N-1} \exp(2\pi i k[\sqrt{2}n]^2\sqrt{2}) &= \frac{1}{N}\sum_{n=0}^{N-1} \exp\Big(2\pi i k(-2\sqrt{2}n^2 + \{\sqrt{2}n\}^2\sqrt{2})\Big) \\ &= \frac{1}{N}\sum_{n=0}^{N-1} f(2\sqrt{2}n^2, \sqrt{2}n) \to \int_0^1\int_0^1 f dxdy = 0 \end{aligned}$$

where $f(x,y) = \exp(2\pi i k(-x + \{y\}^2\sqrt{2}))$ is a Riemann-integrable periodic (mod1) function, so by Weyl's criterion for uniform distribution (see Theorem 2.1 for several equivalent formulations of uniform distribution), $[\sqrt{2}n]^2\sqrt{2}$ is uniformly distributed (mod 1).

On the other hand, $[\sqrt{2}n]\sqrt{2}n$ is not uniformly distributed (mod 1) in the usual sense. This is due to the following observation made by I.Z.Ruzsa, that $[\sqrt{2}n]\sqrt{2}n \equiv 1 - \{\sqrt{2}n\}^2 \pmod 1$. However, $[\sqrt{2}n]\sqrt{2}n$ has $g(x) = \sqrt{1-x}$ as continuous asymptotic distribution function mod 1 (see [5], p.53), so in particular $[\sqrt{2}n]\sqrt{2}n \pmod 1$ is dense in the unit interval.

In [6] Peres shows, using spectral theory, that if a sequence $x(n)$ is uniformly distributed (mod 1) by the van der Corput's method, then the sequence $x([\alpha n])$ is also uniformly distributed (mod 1) for any non-zero $\alpha \in R$. This gives another proof of uniform distribution of $[\sqrt{2}n]^2\sqrt{2}$.

Furstenberg introduced a different way of showing uniform distribution of sequences in [4], where he proved Weyl's theorem by means of ergodic theoretical methods. This approach is also useful for some classes of sequences coming from generalized polynomials. For example, $[\sqrt{2}n^2]\sqrt{2}n$ and any uniformly distributed sequence of form $[\alpha n]\beta n$ can be shown to be uniformly distributed (mod 1) by considering affine transformations on nilmanifolds.

If we let the coefficients $a_1, \ldots, a_k$ of the polynomial (1) be parameters, then a weak version of Weyl's theorem says that polynomials (1) are uniformly distributed (mod 1) for all but countably many k-tuples $(a_1, \ldots, a_k) \in \mathbf{R}^k$. In the same way we can treat the coefficients $b_1, \ldots, b_k$ of a representation of a generalized polynomial $q(x)$ as parameters. We will show in Section 3 that $q(n)$ is uniformly distributed (mod 1) if the k-tuple $(b_1, \ldots, b_k) \in \mathbf{R}^k$ lies outside a set Γ, where Γ is the union of at most countably many hypersurfaces in $\mathbf{R}^k$, see Theorem 3.1.

If the coefficients of $q(x)$ are not in the exceptional set Γ, we say that $q(x)$ has independent coefficients, see Definition 3.1 for a precise definition. In Section 4 we will use the result from Section 3 to show that if the generalized polynomial $\sum_{i=1}^k q_i(x)$ has independent coefficients then $\sum_{i=1}^k [q_i(n)]p_i(n)$ is uniformly distributed (mod 1) for all but countably many $(\gamma_1, \ldots, \gamma_k) \in \mathbf{R}^k$, where γ_i is the leading coefficient of $p_i(n)$, see Theorem 4.1 and its corollary. Note that in contrast to this result, there exist integer sequences a_n with linear rate of growth such that $a_n\alpha$ fails to be uniformly distributed (mod 1) for uncountably many $\alpha \in \mathbf{R}$. In fact, in [1], Boshernitzan shows that for any real sequence b_n there exists a sequence t_n, $t_n \in \{0, 1\}$, such that if $a_n = b_n + t_n$, then the sequence $a_n\alpha$ is not uniformly distributed (mod 1) for uncountably many $\alpha \in \mathbf{R}$.

In Section 5 we give necessary and sufficient conditions for uniform distribution of sequences coming from some classes of generalized polynomials of degree 2.

Remark:
We could sharpen all our results by saying *well-distribution* in place of uniform distribution, because there is a well-distribution version of van der Corput's method ([5], p.240) and in the proofs we reduce everything to linear polynomials which are well-distributed if they are uniformly distributed.

2 Definitions and Lemmas

Denote by [r] the greatest integer less than or equal to r, and {r} the fractional part of r, so that $r = [r] + \{r\}$. We will refer to the greatest integer function on $\mathbf{R}$ as *bracket operation.* If $f_1(x)$ and $f_2(x)$ are real-valued functions, then

$$\begin{aligned} g_1(x) &= [f_1(x)]f_2(x) \\ g_2(x) &= [f_1(x)][f_2(x)] \\ g_3(x) &= \Big[[f_1(x)]f_2(x)\Big] \\ g_4(x) &= \Big[[f_1(x)] + f_2(x)\Big]f_1(x) \end{aligned}$$

are examples of new real-valued functions, obtained from $f_1(x)$ and $f_2(x)$ by the use of bracket operations, products and sums. Note that $g_2(x)$ is a *product* of brackets, but that $g_3(x)$ has *nested* brackets.

Definition 2.1 *A* generalized polynomial $q(x)$ *is a real-valued function on* $\mathbf{R}$ *obtained from a finite number of real-valued polynomials by the use of bracket operations, sums and products.*

Example 1 *Let* $a, b, c, d, x \in \mathbf{R}$*. Then*

- $q_1(x) = [ax](bx^2 + cx)$

- $q_2(x) = \Big[[ax]bx\Big][cx^3+dx]^4$
- $q_3(x) = \Big[[ax]^3bx^2\Big]c$

are generalized polynomials.

We will always write a polynomial in the form $p(x) = a_kx^k + \cdots + a_1x + a_0$, where $a_i \in \mathbf{R}, i = 0,\ldots,k$. With this agreement the symbolic representation of a polynomial is unique. Also, a polynomial is uniquely determined by its values on $\mathbf{N}$. The situation is diffierent for generalized polynomials since two different generalized polynomials can have the same values on $\mathbf{N}$. For example, $[2x]\alpha x^2 \neq 2\alpha x^3$ as functions on $\mathbf{R}$, but $[2n]\alpha n^2 = 2\alpha n^3$ for any $n \in \mathbf{N}$.

Since we are only interested in uniform distribution of sequences coming from generalized polynomials, we will from now on only deal with the sequences and not the generalized polynomials themselves. For simplicity reasons we will call these sequences generalized polynomials. Furthermore, we are not interested in terms which take only integer values. So we will leave them out as long as they are not multiplied by another generalized polynomial.

A generalized polynomial may have many symbolic representations. For example, we have for any generalized polynomial $q(n)$, the identities

$$\Big[[[q(n)]\alpha]\beta\Big] = \Big[[q(n)]\alpha\beta\Big] - 1 \ \text{ if } \ \alpha \text{ is irrational and } \beta < 1$$

and

$$\Big[[q(n)]\tfrac{b}{a_1\cdots a_l}\Big] = \Big[\Big[\cdots\Big[\Big[[q(n)]\tfrac{b}{a_1}\Big]\tfrac{1}{a_2}\Big]\cdots\Big]\tfrac{1}{a_l}\Big] \text{ if } b, a_i \in \mathbf{Z}\setminus\{0\}, i = 1,\ldots,l.$$

Furthermore, we have identities like

$$[\alpha n]\beta n = \beta n^2 + [\alpha_1 n]\beta n \text{ when } \alpha = 1 + \alpha_1\ .$$

However, to prove uniform distribution of a sequence $q(n)$, it is enough to use some representation of $q(n)$. And that is what we will do. We will allow abuse of language, saying that $q(n)$ is a generalized polynomial when we actually mean that $q(n)$ is a fixed representation of the corresponding generalized polynomial.

Nevertheless, we will need to make certain agreements for writing a generalized polynomial because of the way we define coefficients and put conditions on them. It will be done inductively. Let R_0 be the set of sequences of the form

$$p(n) = a_kn^k + \cdots + a_1n + a_0,\ a_i \in \mathbf{R}\setminus\mathbf{Z},$$

and let R_1 be the set of all sequences

$$q(n) = \sum_{i=1}^{l}\prod_{j=1}^{l_i}[q_{ij}(n)]^{k_{ij}}p_i(n) \tag{2}$$

where $q_{ij}(n), p_i(n) \in R_0$ with at least one $q_{ij}(n)$ non-constant and such that

$$\text{if } i_1 \neq i_2, \text{ then } \textstyle\prod_j[q_{i_1j}(n)]^{k_{i_1j}} \neq \prod_j[q_{i_2j}(n)]^{k_{i_2j}}. \tag{3}$$

Suppose R_k is defined for $k < K$ and let R_K be all the sequences of form (2), where $p_i(n) \in R_0$ and $q_{ij}(n) \in R_k$, $k < K$, with at least one $q_{ij}(n) \in R_{K-1}$, and such that (3) holds. Then

$$\mathcal{G} = \bigcup_{i=0}^{\infty} R_i$$

is the family of generalized polynomial sequences, and we agree to write any generalized polynomial in form (2).

Definition 2.2 *If $q(n) \in R_k$ then we will say that $q(n)$ has a sequence of* nested brackets *of length k, i.e., $q(n)$ has a term like $[\cdots[[q_1(n)]q_2(n)]\cdots]q_{k+1}(n)$, and we write $B(q) = k$.*

In Example 1, $B(q_1) = 1$ and $B(q_2) = B(q_3) = 2$.

Definition 2.3 *Define the* set $S(q)$ of coefficients *of (a representation of) a generalized polynomial $q(n)$ inductively. If $q(n) = a_k n^k + \cdots + a_1 n + a_0$ is a polynomial, then as usual, $S(q) = \{a_i \mid a_i \neq 0\}$. If*

$$q(n) = \sum_i \prod_j [q_{ij}(n)]^{k_{ij}} p_i(n) \in R_K,$$

where $q_{ij}(n) \in R_k, k < K, p_i(n) \in R_0$, then

$$S(q) = \bigcup_{i,j} S(q_{ij}) \cup \bigcup_i S(p_i).$$

A real number α is a coefficient *of $q(n)$ if $\alpha \in S(q)$. α is an* outer coefficient *if $\alpha \in \bigcup_i S(p_i)$.*

Definition 2.4 *$q(n)$ is a* simple *generalized polynomial if its representation does not contain any sums.*

In other words, a simple generalized polynomial is a real-valued function on **R** obtained from a finite number of real-valued monomials by using bracket operations and products. For example, $q_3(x)$ in Example 1 is simple, but $q_1(x)$ and $q_2(x)$ are not.

Definition 2.5 *Let $q(n)$ and $q_1(n)$ be simple generalized polynomials.*

$q_1(n)$ is an inner subpolynomial *of $q(n)$ if $[q_1(n)]$ occurs in the representation of $q(n)$.*

$q_1(n)$ is an outer subpolynomial *of $q(n)$ if $q_1(n)$ is the remaining part of $q(n)$ when one or more inner subpolynomials are removed from $q(n)$.*

$q_1(n)$ is an induced inner subpolynomial *of $q(n)$ if $q_1(n)$ is an inner subpolynomial of an outer subpolynomial of $q(n)$.*

Finally, $q_1(n)$ is a subpolynomial *of $q(n)$ if it is an inner, outer or induced inner subpolynomial of $q(n)$.*

If $q(n)$ is a sum of simple generalized polynomials, then we will say that $q_1(n)$ is a subpolynomial of $q(n)$ if $q_1(n)$ is a subpolynomial of a simple term of $q(n)$.

Note that an inner subpolynomial is also an induced inner subpolynomial, and that if $\Big[[q_1(n)]q_2(n)\Big]$ occurs in the representation of $q(n)$, then $q_2(n)$ is an induced inner subpolynomial of $q(n)$ since $q_1(n)$ is an inner subpolynomial which can be removed.

Example 2 *If*

$$q(n) = \Big[[\alpha n]\beta n^2\Big]\gamma + [\delta n][\rho n]\lambda n,$$

then $\alpha n, \delta n, \rho n, [\alpha n]\beta n^2$ *are inner subpolynomials of* $q(n)$*,* βn^2 *is an induced inner subpolynomial of* $q(n)$ *and* $[\beta n^2]\gamma, \gamma, [\delta n]\lambda n, [\rho n]\lambda n, \lambda n$ *are outer subpolynomials of* $q(n)$*.*

Note also that if $x(n)$ is a subpolynomial of $y(n)$ and $y(n)$ is a subpolynomial of $z(n)$, then $x(n)$ is also a subpolynomial of $z(n)$.

Definition 2.6 *The* degree *of a generalized polynomial q(n), denoted by* $\deg(q)$*, is the degree of its underlying polynomial, i.e. the polynomial obtained by disregarding all the brackets.*

Like in the case of polynomials, we would expect a generalized polynomial $q(n)$ of degree 0 to act more or less like a constant. Especially, if $q(n)$ has degree 0, we would not expect the sequence $\{q(n)\}_{n=1}^{\infty}$ to be uniformly distributed (mod 1). However, it will be shown in Section 5 that the sequence $q(n) = [\alpha n]\alpha n - \alpha^2 n^2$, which comes from the generalized polynomial $q(n)$ of degree 0, is uniformly distributed (mod 1) if $1, \alpha, \alpha^2$ are rationally independent.
Uniform distribution (mod 1) of sequences in $\mathbf{R}^l$, $l \geq 1$, is defined in the following way.

Definition 2.7 *A sequence* $(x_1(n), \ldots, x_l(n))$*,* $n = 1, 2, 3, \ldots$*, is* uniformly distributed (mod 1) *in* $\mathbf{R}^l$ *if for any real numbers* $0 \leq a_i < b_i \leq 1$*,* $i = 1, \ldots, l$*,*

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{card} \left(\{n \leq N \mid (\{x_1(n)\}, \ldots, \{x_l(n)\}) \in \prod_{i=1}^{l} [a_i, b_i)\} \right) = \prod_{i=1}^{l} (b_i - a_i).$$

There are many equivalent formulations which we will use freely.

Theorem 2.1 ([5]) *Let* $\mathbf{x}(n) = (x_1(n), \ldots, x_l(n))$ *be a sequence in* $\mathbf{R}^l$*. Then the following statements are equivalent:*

(i) $\mathbf{x}(n)$ *is uniformly distributed (mod 1) in* $\mathbf{R}^l$*.*

(ii) $\sum_{i=1}^{l} k_i x_i(n)$ *is uniformly distributed (mod 1) in* $\mathbf{R}$ *for all l-tuples* $(k_1, \ldots, k_l) \neq (0, \ldots, 0)$ *of integers.*

(iii) Weyl's criterion:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp\Big(2\pi i \sum_{j=1}^{l} k_j x_j(n)\Big) = 0$$

for all l-tuples $(k_1, \ldots, k_l) \neq (0, \ldots, 0)$ *of integers.*

(iv) *For every Riemann-integrable function f on* $[0,1]^l$*,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\{\mathbf{x}(n)\}) = \int_0^1 \cdots \int_0^1 f(\mathbf{x}) dx_1 \cdots dx_l.$$

Let $q(n)$ be a generalized polynomial and β a real number. By Lemma 2.2, below, the new generalized polynomial $[q(n)]\beta$ is uniformly distributed (mod 1) in $\mathbf{R}$ if $(q(n)\beta, q(n))$ is uniformly distributed (mod 1) in $\mathbf{R}^2$. Note that we don't have implication the other way. For if for example $q(n) = n$, then $[q(n)]\beta = n\beta$ is uniformly distributed (mod 1) if β is irrational, but $(q(n)\beta, q(n)) = (n\beta, n)$ is certainly not uniformly distributed (mod 1). However, if $q(n)$ is a polynomial with at least one irrational coefficient other than the constant term, then $[q(n)]\beta$ is uniformly distributed (mod 1) if and only if $(q(n)\beta, q(n))$ is uniformly distributed (mod 1). This follows from Proposition 2.3 below and is also shown in [5, p. 310] for the special case $q(n) = \alpha n$.

If $x(n)$ is a sequence in $\mathbf{R}^k$ and $y(n)$ a sequence in $\mathbf{R}^l$ such that $x(n)$ is uniformly distributed (mod 1) if $y(n)$ is, then we will write

$$x(n)y(n).$$

We have already seen that $[q(n)]\beta(q(n)\beta, q(n))$.

It follows from the above discussion that for certain generalized polynomials we can, by going up in dimension, get rid of a bracket which has only a constant on the outside. In the same way we could remove brackets from the new terms by going further up in dimension. This can be repeated until all the terms are of one of the forms $[q_1(n)][q_2(n)]\beta$ or $[q_1(n)]p(n)$, where $q_1(n)$ and $q_2(n)$ are generalized polynomials of positive degrees and $p(n)$ is a non-constant polynomial.

If $q(n) = [\cdots[[q_1(n)]\lambda_1]\cdots]\lambda_k$ is a simple generalized polynomial where $q_1(n)$ is of one of the above mentioned forms, then we will use the following notation:

$$q^i(n) = [\cdots[[q_1(n)]\lambda_1]\cdots]\lambda_i, \quad 0 \le i < k,$$

$$\overline{q}^i(n) = q_1(n)\lambda_1\cdots\lambda_i, \quad 0 \le i < k$$

and

$$\overline{q}(n) = q_1(n)\lambda_1\cdots\lambda_k.$$

Note that $q^i(n)$ is an inner subpolynomial of $q(n)$. It follows from the next lemma that

$$q(n)\ \ (\overline{q}(n), \overline{q}^{k-1}(n), \ldots, \overline{q}^0(n)).$$

Lemma 2.2 *Let $q_i(n) = [\cdots[[q_{i1}(n)]\lambda_{i1}]\cdots]\lambda_{ik_i}, i = 1, \ldots, s,$ be simple generalized polynomials and $q(n) = \sum_{i=1}^{s} a_i q_i(n)$, $a_i \in \mathbf{Z}$. Let $r_1(n), \ldots, r_m(n)$ be all the distinct elements from the set $\{\overline{q}_i^l(n) \mid l = 0, \ldots, k_i - 1, i = 1, \ldots, s\}$. Then*

$$q(n) \left(\sum_{i=1}^{s} a_i \overline{q}_i(n), r_1(n), \ldots, r_m(n) \right).$$

Proof: We will first prove by induction on k that if

$$q(n) = [\cdots[q_1(n)]\lambda_1\cdots]\lambda_k,$$

where $q_1(n)$ is a simple generalized polynomial such that $\overline{q}_1(n) = q_1(n)$, then

$$q(n) = \overline{q}(n) - \{q^0(n)\}\lambda_1\cdots\lambda_k - \{q^1(n)\}\lambda_2\cdots\lambda_k - \cdots - \{q^{k-1}(n)\}\lambda_k \tag{4}$$

and

$$q^i(n) = \overline{q}^i(n) - \{q^0(n)\}\lambda_1\cdots\lambda_i - \cdots - \{q^{i-1}(n)\}\lambda_i.$$

If $k = 1$, then

$$q(n) = q_1(n)\lambda_1 - \{q_1(n)\}\lambda_1 = \overline{q}(n) - \{q^0(n)\}\lambda_1$$

which proves (4) in this case. Assume now that (4) is true for k, and let $q(n) = [\cdots[q_1(n)]\lambda_1\cdots]\lambda_{k+1}$. Then we have, by using the induction hypothesis,

$$\begin{aligned} q(n) &= [\cdots[q_1(n)]\lambda_1\cdots]\lambda_k\lambda_{k+1} - \{[\cdots[q_1(n)]\lambda_1\cdots]\lambda_k\}\lambda_{k+1} \\ &= \overline{q}(n) - \{q^0(n)\}\lambda_1\cdots\lambda_k\lambda_{k+1} - \cdots - \{q^{k-1}(n)\}\lambda_k\lambda_{k+1} - \{q^k(n)\}\lambda_{k+1} \end{aligned}$$

which we wanted to prove.

Note that this gives us that $q(n)$ is a function of $\overline{q}(n)$ and the $\overline{q}^j(n)$'s. If $q(n) = \sum_{i=1}^s a_i q_i(n)$ and

$$q_i(n) = [\cdots[q_{i1}(n)]\lambda_{i1}\cdots]\lambda_{ik_i}$$

then

$$q(n) = \sum_{i=1}^s a_i\overline{q}_i(n) - \sum_{i=1}^s\sum_{j=1}^{k_i}\{q_i^{j-1}(n)\}\lambda_{ij}\cdots\lambda_{ik_i}.$$

Let $r_i(n), i = 1, \ldots, m$, be all the distinct elements from the set $\{\overline{q}_i^j(n) \mid j = 0, \ldots, k_i - 1\,,\, i = 1, \ldots, s\}$ and for each $b \in \mathbf{Z} \setminus \{0\}$, define a function

$$f_b(x_0, x_1, \ldots, x_m) = \exp\, 2\pi i b\Big(x_0 - \sum_{i=1}^s\sum_{j=1}^{k_i}\{f_{ij}(x_1, \ldots, x_m)\}\lambda_{ij}\cdots\lambda_{ik_i}\Big)$$

where $f_{ij}(x_1, \ldots, x_m)$ is defined such that

$$f_{ij}(r_1(n), \ldots, r_m(n)) = \overline{q}_i^{j-1}(n) - \sum_{l=1}^{j-1}\{q_i^{l-1}(n)\}\lambda_{il}\ldots\lambda_{i,j-1}.$$

It follows that for any b, f_b is a Riemann-integrable, periodic (mod1) function with integral equal 0. Hence, if $(\sum_i a_i\overline{q}_i(n), r_1(n), \ldots, r_m(n))$ is uniformly distributed (mod 1), then

$$\begin{aligned} \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i b q(n)} &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f_b\Big(\sum_i a_i\overline{q}_i(n), r_1(n), \ldots, r_m(n)\Big) \\ &= \int f_b(x_0, \ldots, x_m)dx_0\cdots dx_m = 0 \end{aligned}$$

which shows that $q(n)$ is uniformly distributed (mod 1).

□

If $q(n)$ is as in Lemma 2.2 such that $s = 1$ and $q_{11}(n)$ is a polynomial, then we have the following stronger result.

Proposition 2.3 *If $p(x) = a_l x^l + \cdots + a_1 x + a_0$ is a polynomial with real coefficients, then the following are equivalent:*

(i) $(p(n), [p(n)]\lambda_1, \ldots, [\cdots [p(n)]\lambda_1 \cdots]\lambda_k)$ *is uniformly distributed (mod 1) in* $\mathbf{R}^{k+1}$.

(ii) $(p(n), p(n)\lambda_1, \ldots, p(n)\lambda_1 \cdots \lambda_k)$ *is uniformly distributed (mod 1) in* $\mathbf{R}^{k+1}$.

(iii) *If there exist* $b_i \in \mathbf{Q}$ *and an irrational number* α *such that the coefficients* $a_i = b_i\alpha$, $i = 1, \ldots, l$, *then* $1, \alpha, \alpha\lambda_1, \ldots, \alpha\lambda_1 \cdots \lambda_k$ *are rationally independent.*
Or if there exist coefficients a_i *and* a_j, $i, j \neq 0$, *such that* $a_i/a_j \notin \mathbf{Q}$, *then* $1, \lambda_1, \ldots, \lambda_1 \cdots \lambda_k$ *are rationally independent.*

We will make use of Weyl's theorem ([9]) in the proof.

Theorem 2.4 (Weyl) *If* $p(x)$ *is a real polynomial with at least one coefficient other than the constant term irrational, then the sequence* $p(n)$ *is uniformly distributed (mod 1).*

Proof of Proposition 2.3: We have (ii) $\Leftrightarrow$ (iii), for by Theorem 2.4 and Theorem 2.1, (iii) implies (ii), and since it is clear that a polynomial with only rational coefficients is not uniformly distributed (mod 1), (ii) implies (iii).

(ii) $\Rightarrow$ (i) follows from Lemma 2.2. For by Theorem 2.1, $\left(p(n), [p(n)]\lambda_1, \ldots, [\cdots [p(n)]\lambda_1 \cdots]\lambda_k\right)$ is uniformly distributed (mod 1) in $\mathbf{R}^{k+1}$ if for any $d = (d_0, \ldots, d_k) \in \mathbf{Z}^{k+1} \setminus \{(0, \ldots, 0)\}$, $g_d(n) = d_0 p(n) + \sum_{i=1}^{k} d_i [\cdots [p(n)]\lambda_1 \cdots]\lambda_i$ is uniformly distributed (mod 1). Without loss of generality we may assume $d_k \neq 0$. Then by Lemma 2.2, $g_d(n)$ is uniformly distributed (mod 1) if

$$\left(d_0 p(n) + \sum_{i=1}^{k} d_i p(n)\lambda_1 \cdots \lambda_i,\ p(n),\ p(n)\lambda_1, \ldots,\ p(n)\lambda_1 \cdots \lambda_{k-1}\right)$$

is uniformly distributed (mod 1) in $\mathbf{R}^{k+1}$, hence if

$$(p(n),\ p(n)\lambda_1,\ \ldots,\ p(n)\lambda_1 \cdots \lambda_{k-1},\ p(n)\lambda_1 \cdots \lambda_k)$$

is uniformly distributed (mod 1) in $\mathbf{R}^{k+1}$.

We will use induction on k to show (i) $\Rightarrow$ (iii). If $k = 0$, then it is trivial since (ii)$\Leftrightarrow$(iii). Suppose (i) $\Rightarrow$ (iii) is true for $k - 1$ for some $k \geq 1$. Let us first consider the case when $a_i = c_i\alpha, i = 1, \ldots, l$, where $c_i \in \mathbf{Q}$ and α is an irrational number. Then $p(n) = \alpha p_1(n)$, where $p_1(n)$ is a polynomial with rational coefficients. Let $a \in \mathbf{Z}$ be such that $ap_1(n)$ has integer coefficients. Since we are assuming that

$$\left(p(n), [p(n)]\lambda_1, \ldots, [\cdots [p(n)]\lambda_1 \cdots]\lambda_k\right)$$

is uniformly distributed (mod 1) in $\mathbf{R}^{k+1}$, it follows by the induction hypothesis that $1, \alpha, \alpha\lambda_1, \ldots$, $\alpha\lambda_1 \cdots \lambda_{k-1}$ are rationally independent. Suppose $\alpha\lambda_1 \ldots \lambda_k$ is rationally dependent of $1, \alpha, \alpha\lambda_1$, $\ldots$, $\alpha\lambda_1 \cdots \lambda_{k-1}$ and that

$$b\alpha\lambda_1 \cdots \lambda_k = b_k + b_0\alpha + b_1\alpha\lambda_1 + \cdots + b_{k-1}\alpha\lambda_1 \cdots \lambda_{k-1}$$

for some integers $b \neq 0$ and b_i, $i = 0, \ldots, k$. Let

$$\beta_{k-i} = \frac{b_k + b_0\alpha + b_1\alpha\lambda_1 + \cdots + b_{k-i-1}\alpha\lambda_1 \cdots \lambda_{k-i-1}}{\alpha\lambda_1 \cdots \lambda_{k-i}},\ i = 1, \ldots, k$$

where we let $\lambda_0 = 1$ and $\lambda_{-1} = 0$ so that $\beta_0 = \frac{b_k}{\alpha}$. Then

$$b\lambda_k = \frac{b_k + b_0\alpha + b_1\alpha\lambda_1 + \cdots + b_{k-2}\alpha\lambda_1\cdots\lambda_{k-2}}{\alpha\lambda_1\cdots\lambda_{k-1}} + b_{k-1} = \beta_{k-1} + b_{k-1}$$

and $\lambda_{k-i}\beta_{k-i} = \beta_{k-i-1} + b_{k-i-1},\ i = 1, \ldots, k-1$. Note that β_{k-i} is irrational for any i. We now have

$$\begin{aligned}
&ab[[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-1}]\lambda_k \\
&= a[[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-1}](\beta_{k-1} + b_{k-1}) \\
&\equiv a[[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-1}]\beta_{k-1} \pmod 1 \\
&= a[[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-2}]\lambda_{k-1}\beta_{k-1} - a\{[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-1}\}\beta_{k-1} \\
&\equiv a[[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-2}]\beta_{k-2} - a\{[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-1}\}\beta_{k-1} \pmod 1.
\end{aligned}$$

Since the process of reducing the number of brackets in the first term can be continued until we get down to the term

$$\begin{aligned}
a[p(n)]\beta_0 &= a[p(n)]\frac{b_k}{\alpha} \\
&= ap_1(n)b_k - a\{p(n)\}\frac{b_k}{\alpha} \\
&\equiv -a\{p(n)\}\frac{b_k}{\alpha} \pmod 1,
\end{aligned}$$

it follows that

$$ab[[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-1}]\lambda_k \equiv -a\sum_{i=1}^{k}\{[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-i}\}\beta_{k-i} \pmod 1.$$

Hence,

$$\begin{aligned}
&\frac{1}{N}\sum_{n=0}^{N-1}\exp(2\pi iab[[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-1}]\lambda_k) \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\exp(-2\pi ia\sum_{i=1}^{k}\{[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-i}\}\beta_{k-i} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} f\Big(p(n), [p(n)]\lambda_1, \ldots, [[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-2}]\lambda_{k-1}\Big),
\end{aligned}$$

where

$$f(x_0, x_1, \ldots, x_{k-1}) = \exp(2\pi ia\sum_{i=1}^{k}\{x_{k-i}\}\beta_{k-i})$$

is a Riemann-integrable, periodic (mod1) function and

$$\begin{aligned}
\int f dx_0\cdots dx_{k-1} &= \prod_{i=1}^{k}\int_0^1 \exp(2\pi iax_{k-i}\beta_{k-i})dx_{k-i} \\
&= \prod_{i=1}^{k}\frac{1-\exp(-2\pi ia\beta_{k-i})}{2\pi ia\beta_{k-i}} \neq 0
\end{aligned}$$

since $\beta_{k-i}, i = 1, \ldots, k$, are irrationals. Now,

$$\Big(p(n), [p(n)]\lambda_1, \ldots, [[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-2}]\lambda_{k-1}\Big)$$

is uniformly distributed (mod 1) so by Theorem 2.1,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(p(n), [p(n)]\lambda_1, \ldots, [[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-2}]\lambda_{k-1}) = \int f dx_0\cdots dx_{k-1} \neq 0,$$

which shows that $[[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-1}]\lambda_k$ is not uniformly distributed (mod 1), contradicting our assumption. Hence (i)⇒(iii) in this case.

The proof in the case where there exist i, j such that $\frac{a_i}{a_j} \notin \mathbf{Q}$ is very similar. For by the induction hypothesis, $1, \lambda_1, \ldots, \lambda_1\cdots\lambda_{k-1}$ are rationally independent and if we assume there exist integers $b \neq 0, b_i$ such that

$$b\lambda_1\cdots\lambda_k = b_k + b_1\lambda_1 + \cdots b_{k-1}\lambda_1\cdots\lambda_{k-1}$$

and let

$$\beta_{k-i} = \frac{b_k + b_1\lambda_1 + \cdots + b_{k-i-1}\lambda_1\cdots\lambda_{k-i-1}}{\lambda_1\cdots\lambda_{k-i}}\ ,\ i = 1, \ldots, k-1,$$

then

$$b[\cdots[p(n)]\lambda_1\cdots]\lambda_k \equiv -\sum_{i=1}^{k-1}\{[\cdots[p(n)]\lambda_1\cdots]\lambda_{k-i}\}\beta_{k-i} \pmod 1$$

so that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\exp(2\pi i b[\cdots[p(n)]\lambda_1\cdots]\lambda_k) \neq 0$$

as in the first case. Hence (i)⇒(iii).

□

Let l be a positive integer. Then for any subset C of $[0,1)^l$, we can define an *indicator function* 1_C on $\mathbf{R}^l$ by

$$1_C(x_1, \ldots, x_l) = \begin{cases} 1 & \text{if } (\{x_1\}, \ldots, \{x_l\}) \in C \\ 0 & \text{otherwise.} \end{cases}$$

We will call $x_1, \ldots, x_l$ the *arguments* of the indicator function 1_C. When it is not necessary to specify the arguments of the indicator function 1_C, we will write $1_C(*)$. If x and y are real numbers, then

$$[x+y] = \begin{cases} [\mathrm{x}] + [y] + 1 & \text{if } \{x\} + \{y\} \geq 1 \\ [\mathrm{x}] + [y] & \text{if } \{x\} + \{y\} < 1 \end{cases}$$

So if we let

$$A = \{(x, y) \in [0,1)^2 \mid x + y \geq 1\}$$

then we have

$$[x+y] = [x] + [y] + 1_A(x, y).$$

Lemma 2.5 *Let $x, x_1, \ldots, x_k$ be real numbers and h a positive integer. Then*

(i) $\left[\sum_{i=1}^{k} x_i\right] = \sum_{i=1}^{k} \Big([x_i] + 1_A(x_i, \sum_{j=1}^{k-i} x_{i+j})\Big)$

(ii) $[hx] = h[x] + \sum_{i=0}^{h-1} i\, 1_{[\frac{i}{h}, \frac{i+1}{h})}(x)$.

Proof: (i) will be proved by induction on k. The case $k = 2$ is done above. Suppose (i) is true for k. Then

$$\begin{aligned}\left[\sum_{i=1}^{k+1} x_i\right] &= [x_1] + \left[\sum_{i=2}^{k+1} x_i\right] + 1_A(x_1, \sum_{i=2}^{k+1} x_i) \\ &= [x_1] + \sum_{i=2}^{k+1}\left([x_i] + 1_A(x_i, \sum_{j=1}^{k+1-i} x_{i+j})\right) + 1_A(x_1, \sum_{j=1}^{k} x_{1+j}) \\ &= \sum_{i=1}^{k+1}\left([x_i] + 1_A(x_i, \sum_{j=1}^{k+1-i} x_{i+j})\right).\end{aligned}$$

Hence, by induction, (i) is true for any natural number k.
To prove (ii), observe first that since $h\{x\} = hx - h[x]$ and $\{hx\} = hx - [hx]$, $h\{x\}$ differs from $\{hx\}$ by an integer. Now, if

$$\frac{i}{h} \le \{x\} < \frac{i+1}{h},$$

then $i \le h\{x\} < i + 1$. So $h\{x\} = \{hx\} + i$, and therefore $[hx] = h[x] + i$, which was to be proved.

□

The previous lemma is fundamental for what we will be doing, because it allows us to write generalized polynomials as a sum of simple generalized polynomials together with indicator functions. It also tells us that we won't do much harm by bringing out positive integers from brackets.
Call a subset C of $[0, 1)^l$ *nice* if C is a finite union of subsets of $[0, 1)^l$ where each subset is bounded by finitely many $(l - 1)$-dimensional planes in $\mathbf{R}^l$.

Lemma 2.6 *A sequence*

$$G(n) = q_0(n) + \sum_{i=1}^{L} q_i(n)\, 1_{C_i}\Big(r_{i1}(n), \ldots, r_{il_i}(n)\Big)$$

where $q_i(n)$, $r_{ij}(n)$, $j = 1, \ldots, l_i$, $i = 1, \ldots, L$, are generalized polynomials and C_i is a nice subset of $[0, 1)^{l_i}$, $i = 1, \ldots, L$,
is uniformly distributed (mod 1) if there exist generalized polynomials $t_1(n), \ldots, t_k(n)$ such that each $r_{ij}(n)$ is a linear combination over $\mathbf{Z}$ of $1, t_1(n), \ldots, t_k(n)$ and such that for all subsets V of $\{q_1(n), \ldots, q_L(n)\}$,

$$\Big(q_0(n) + \sum_{q\in V} q(n)\,,\ t_1(n), \ldots, t_k(n)\Big)$$

is uniformly distributed (mod 1) in $\mathbf{R}^{k+1}$.

Proof: We will rewrite $G(n)$ in terms of new indicator functions. For each $i = 1, \ldots, L$, let C_i^c be the complement of C_i in $[0,1)^{l_i}$. Then C_i and C_i^c give a partition of $[0,1)^{l_i}$ consisting of two nice sets, and $E_1 \times \ldots \times E_l$, where E_i is either C_i or C_i^c, give a partition of $[0,1)^l$ consisting of 2^L nice sets, where $l = \sum l_i$. Call the family of these sets $\mathcal{C}$, and let V_C be the subset of $\{q_1(n), \ldots, q_L(n)\}$ that corresponds to $C \in \mathcal{C}$, i.e., if $C = \prod_{i=1}^{L} E_i$, where $E_i = C_i$ or C_i^c, then $V_C = \{q_i(n) \mid E_i = C_i\}$. Then

$$G(n) = \sum_{C \in \mathcal{C}} \Big(q_0(n) + \sum_{q \in V_C} q(n) \Big) 1_C \Big(r_{11}(n), \ldots, r_{1l_1}(n), \ldots, r_{L1}(n), \ldots, r_{Ll_L}(n) \Big).$$

Since each $r_{ij}(n)$ is a linear combination over $\mathbf{Z}$ of $1, t_1(n), \ldots, t_k(n)$, the partition $\mathcal{C}$ of $[0,1)^l$ induces a partition $\mathcal{D}$ of $[0,1)^k$ also consisting of 2^L nice sets, so that

$$G(n) = \sum_{D \in \mathcal{D}} \Big(q_0(n) + \sum_{q \in V_D} q(n) \Big) 1_D(t_1(n), \ldots, t_k(n)).$$

Let $b \in \mathbf{Z} \setminus \{0\}$. Then

$$\begin{aligned}
&\frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i b G(n)} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \exp\Big(2\pi i b \sum_{D \in \mathcal{D}} \Big(q_0(n) + \sum_{q \in V_D} q(n) \Big) 1_D(t_1(n), \ldots, t_k(n)) \Big) \\
&= \sum_{D \in \mathcal{D}} \frac{1}{N} \sum_{n=0}^{N-1} \exp\Big(2\pi i b \Big(q_0(n) + \sum_{q \in V_D} q(n) \Big) \Big) \, 1_D(t_1(n), \ldots, t_k(n)) \\
&= \sum_{D \in \mathcal{D}} \frac{1}{N} \sum_{n=0}^{N-1} f_D \Big(q_0(n) + \sum_{q \in V_D} q(n) \, , \, t_1(n), \ldots, t_k(n) \Big)
\end{aligned}$$

where $f_D(x, y_1, \ldots, y_k) = e^{2\pi i b x} 1_D(y_1, \ldots, y_k)$ is a Riemann-integrable periodic (mod 1) function having integral equal 0. Therefore, since $\Big(q_0(n) + \sum_{q \in V_D} q(n) \, , \, t_1(n), \ldots, t_k(n) \Big)$ is uniformly distributed (mod 1) for all $D \in \mathcal{D}$,

$$\sum_{D \in \mathcal{D}} \frac{1}{N} \sum_{n=0}^{N-1} f_D \Big(q_0(n) + \sum_{q \in V_D} q(n) \, , \, t_1(n), \ldots, t_k(n) \Big) \to 0$$

as $N \to \infty$ by Theorem 2.1. Hence $G(n)$ is uniformly distributed (mod 1).

□

We will not always be able to write the $r_{ij}(n)$'s as a sum of $t_l(n)$'s as nicely as in this lemma. But by using Lemma 2.5 on the $r_{ij}(n)$'s, we can break them down to a linear combination of simple generalized polynomials together with new indicator functions which can be taken care of in a similar manner.

Lemma 2.7 *Let $q(n)$ be a generalized polynomial. Then there exist simple generalized polynomials $q_i(n)$, $i = 1, \dots, L$, such that*

$$q(n) = \sum_{i=1}^{L} q_i(n) + \sum_{i=1}^{L_1} \sum_{j=1}^{l_i} 1_A\Big(r_{1i}(n), r_{2i}(n)\Big)\, t_{ij}(n), \tag{5}$$

where each $t_{ij}(n)$ is an outer subpolynomial of some $q_l(n), 1 \leq l \leq L$, so that $\deg(t_{ij}) < \deg(q)$, *and each $r_{ji}(n)$ is a sum of induced inner subpolynomials of the $q_i(n)$'s and induced inner subpolynomials multiplied by indicator functions of the same form.*

We will call the $q_i(n)$'s the *simple terms* of $q(n)$.

Proof: We will use induction on $B(q)$. Recall that $B(q)$ is the length of the longest sequence of nested brackets in $q(n)$, Definition 2.2. If $B(q) = 0$, then $q(n)$ is a polynomial, and the statement is trivially true. Suppose the lemma is true for all generalized polynomials $q(n)$ with $B(q) = K$ and let $q(n)$ be such that $B(q) = K + 1$. Without loss of generality we may assume $q(n)$ has only one term, i.e., that $q(n) = \prod_{i=1}^{k} [q_i(n)] p(n)$, where $p(n)$ is a monomial. First, let $k = 1$ and let $q_1(n)$ be a generalized polynomial with $B(q_1) = K$. By the induction hypothesis,

$$q_1(n) = \sum_{i=1}^{L} q_{1i}(n) + \sum_{i=1}^{L_1} \sum_{j=1}^{l_i} 1_A(r_{1i}, r_{2i}) t_{ij}(n)$$

with $q_{1i}(n), t_{ij}(n), r_{ji}(n)$ as stated in the lemma. Now by Lemma 2.5, we have

$$\begin{aligned} q(n) &= \Big[\sum_{i=1}^{L} q_{1i}(n) + \sum_{i,j} 1_A(r_{1i}, r_{2i})\, t_{ij}(n)\Big] p(n) \\ &= \sum_{i=1}^{L} [q_{1i}(n)] p(n) + \sum_{i,j} 1_A(r_{1i}, r_{2i}) [t_{ij}(n)] p(n) \\ &\quad + \sum_{i=1}^{L} 1_A\Big(q_{1i}(n), \sum_{j} q_{1,i+j}(n) + \sum_{l,j} 1_A(r_{1l}, r_{2l})\, t_{lj}(n)\Big) p(n) \\ &\quad + \sum_{i,j} 1_A\Big(1_A(r_{1i}, r_{2i}) t_{ij}(n), \sum_{l,k} 1_A(r_{1,i+l}, r_{2,i+l}) t_{i+l,j+k}(n)\Big) p(n) \end{aligned} \tag{6}$$

which proves (5) in this case since $[q_{1i}(n)]p(n)$ is simple for any i and each $r_{ji}(n)$ is an induced inner subpolynomial of some $q_{1l}(n)$, hence of $[q_{1l}(n)]p(n)$. Since $t_{ij}(n)$ is an outer subpolynomial of some $q_{1l}(n)$, $t_{ij}(n)$ is an induced inner subpolynomial of $[q_{1l}(n)]p(n)$ and $[t_{ij}(n)]p(n)$ is an outer subpolynomial of $[q_{1l}(n)]p(n)$.

Now, if $q(n) = \prod_{i=1}^{k} [q_i(n)] p(n), k > 1$, we get a product of expressions like (6), which when multiplied out, is of form (5).

□

Recall the definitions preceding Lemma 2.2 of $\overline{q}(n), \overline{q}^l(n), q^l(n)$ where $q(n)$ is a generalized polynomial.

Corollary 2.8 *Let $q(n)$, $q_i(n), t_{ij}(n), r_{ji}(n)$ be as in Lemma 2.7 and let $v_1(n), \ldots, v_{m_1}(n)$ be all the distinct simple generalized polynomials appearing in the expressions of the $r_{ji}(n)$'s. Then $q(n)$ is uniformly distributed (mod 1) if either*

(i) $\left(\sum_i q_i(n) + \sum_{i,j} \epsilon_i t_{ij}(n), v_1(n), \ldots, v_{m_1}(n)\right)$
is uniformly distributed (mod 1) in $\mathbf{R}^{m_1+1}$ for all $\epsilon_i \in \{0,1\}$, or

(ii) $\left(\sum_i \overline{q}_i(n) + \sum_{i,j} \epsilon_i \bar{t}_{ij}(n), \overline{w}_1(n), \ldots, \overline{w}_{m_2}(n)\right)$
is uniformly distributed (mod 1) in $\mathbf{R}^{m_2+1}$ for all $\epsilon_i \in \{0,1\}$, where $w_1(n), \ldots, w_{m_2}(n)$ are all the distinct simple generalized polynomials from the set $\{v_i(n), v_i^l(n), q_i^l(n), t_{ij}^l(n) \mid i, j, l\}$.

Proof: (i) follows directly from Lemma 2.6.
(ii). By (i) and Theorem 2.1, $q(n)$ is uniformly distributed (mod 1) if for all (m_1+1)-tuples of integers $a = (a_0, \ldots, a_{m_1}) \neq (0, \ldots, 0)$,

$$g_a(n) = a_0 \left(\sum_i q_i(n) + \sum_i \epsilon_i t_{ij}(n) \right) + \sum_{i=1}^{m_1} a_i v_i(n)$$

is uniformly distributed (mod 1). Let $u_1(n), \ldots, u_{m_3}(n)$ be all the distinct elements from the set $\{v_i^l(n), q_i^l(n), t_{ij}^l(n) \mid i, j, l\}$. By Lemma 2.2,

$$g_a(n)\Big(a_0\Big(\sum_i \overline{q}_i(n) + \sum_{i,j} \epsilon_i \bar{t}_{ij}(n)\Big) + \sum_i a_i \overline{v}_i(n), \overline{u}_1(n), \ldots, \overline{u}_{m_3}(n)\Big).$$

So if $w_1(n), \ldots, w_{m_2}(n)$ are as stated in the lemma, $q(n)$ is uniformly distributed (mod 1) if

$$\left(\sum_i \overline{q}_i(n) + \sum_{i,j} \epsilon_i \bar{t}_{ij}(n), \overline{w}_1(n), \ldots, \overline{w}_{m_2}(n) \right)$$

is uniformly distributed (mod 1) in $\mathbf{R}^{m_2+1}$.

□

Definition 2.8 *Let $\mathcal{S}$ be the space of all linear combinations over $\mathbf{Z}$ of simple generalized polynomials. For any $h \in \mathbf{N}$, define a* generalized derivative *$V_h : \mathcal{S} \to \mathcal{S}$ inductively by*

(i) $V_h n^l = lhn^{l-1}$, $l = 0, 1, 2, \ldots$

(ii) $V_h(q_1(n)q_2(n)) = q_1(n)V_h(q_2(n)) + V_h(q_1(n))\, q_2(n)$*, when $q_1(n)$ and $q_2(n)$ are simple generalized polynomials.*

(iii) *If $q(n)$ is a simple generalized polynomial and $V_h q(n) = \sum_i q_i(n)$, $q_i(n)$ simple, then $V_h[q(n)] = \sum_i [q_i(n)]$.*

Also, let $V_0(q(n)) = q(n)$.

Note that if $p(n)$ is a usual polynomial, then $V_1p(n)$ is the usual derivative $p'(n)$, and that for polynomials, (ii) is just the product rule.

Example 3 *If*

$$q(n) = \Big[[p_1(n)]p_2(n)\Big]\Big[p_3(n)\Big]^2 p_4(n),$$

where the $p_i(n)$'s are monomials, then

$$\begin{aligned} V_hq(n) &= \Big[[V_hp_1(n)]p_2(n)\Big]\Big[p_3(n)\Big]^2 p_4(n) + \Big[[p_1(n)]V_hp_2(n)\Big]\Big[p_3(n)\Big]^2 p_4(n) \\ &\quad +2\Big[[p_1(n)]p_2(n)\Big]\Big[p_3(n)\Big]\Big[V_hp_3(n)\Big]p_4(n) + \Big[[p_1(n)]p_2(n)\Big]\Big[p_3(n)\Big]^2 V_hp_4(n). \end{aligned}$$

Note that if $q(n)$ is a simple generalized polynomial having k non-constant polynomials as subpolynomials, then $V_hq(n)$ is a sum of k simple generalized polynomials.

Recall that the degree of a generalized polynomial is the degree of its underlying polynomial. As in the case of usual derivatives of polynomials, the generalized derivative reduces the degree of a generalized polynomial.

Lemma 2.9 *Let $q(n)$ be a generalized polynomial and let $h \in \mathbf{N}$. Then*

$$\deg(V_hq) = \deg(q) - 1.$$

Proof: This is obvious if $q(n)$ is a polynomial. We will show the general case by induction on $\deg(q)$. Note first that if $q(n) = [\cdots[q_1(n)]\lambda_1\cdots]\lambda_k$ and $V_hq_1(n) = \sum v_i(n)$, then $V_h[q(n)] = \sum[\cdots[v_i(n)]\lambda_1\cdots]\lambda_k$, so that $\deg(V_hq) = \max_i \deg(v_i)$. It is therefore enough to consider polynomials and generalized polynomials of the form $q(n) = [q_1(n)]q_2(n)$, where $q_1(n)$ and $q_2(n)$ are generalized polynomials of positive degrees. Hence if $\deg(q)$=1, then we may assume that $q(n)$ is a linear polynomial so that $\deg(V_hq)$=$\deg(q) - 1 = 0$.

Assume the Lemma is true for $q(n)$ where $\deg(q) \leq K$, and let $q(n) = [q_1(n)]q_2(n)$, where $\deg(q_1) > 0$ and $\deg(q_2) > 0$, be a generalized polynomial of degree $K + 1$. Then

$$V_hq(n) = [q_1(n)]V_hq_2(n) + V_h([q_1(n)])q_2(n).$$

Since $\deg(q_2) > 0$, we are done by induction.

□

We will end this section by stating a version of van der Corput's difference theorem. The proof can be found in [5, p.26, see also Cor.2.1, p.251].

Theorem 2.10 (Van der Corput's difference theorem) *Let $x(n), n = 1, 2, \ldots,$ be a sequence of real numbers. If for all but finitely many integers h, the sequence $x(n + h) - x(n)$, $n = 1, 2, \ldots$, is uniformly distributed (mod 1), then $x(n)$ is uniformly distributed (mod 1).*

3 Generalized Polynomials with Independent Coefficients

Recall the definition of the set $S(q)$ of coefficients of a generalized polynomial $q(n)$, Definition 2.3, and let

$$R(q) = \bigcup S(v),$$

where the union is taken over all generalized polynomials $v(n)$ which are obtained from $q(n)$ by removing any number of nested brackets in $q(n)$. So, if for example

$$q(n) = \Big[[\alpha n][\beta n]\lambda n + [\delta n]^3\sigma\Big]\gamma n^2,$$

then $S(q) = \{\alpha, \beta, \lambda, \delta, \sigma, \gamma\}$ and $R(q) = S(q) \cup \{\lambda\gamma, \sigma\gamma, \alpha\lambda\gamma, \beta\lambda\gamma, \delta\sigma\gamma, \alpha\lambda, \beta\lambda, \delta\sigma\}$.

Definition 3.1 *A (representation of a) generalized polynomial $q(n)$ has* independent coefficients *if $R(q) \cup \{1\}$ is rationally independent.*

The goal of this section is to prove the following theorem.

Theorem 3.1 *If (a representation of) a generalized polynomial q(x) has independent coefficients, then $q(n)$ is uniformly distributed (mod 1).*

We will use induction and van der Corput's difference theorem to prove this theorem, similarly to what is done for polynomials, but it is more complicated in this case. The complications arise because of the form $q^h(n) \stackrel{\text{def}}{=} q(n+h) - q(n)$ takes. If $q(n)$ is a polynomial, $q^h(n)$ is a new polynomial of degree $\deg(q) - 1$. However, if $q(n)$ is a generalized polynomial which is not a usual polynomial, we cannot write $q^h(n)$ as a generalized polynomial of degree $\deg(q) - 1$ without obtaining indicator functions that have to be taken care of. Furthermore, $q^h(n)$ has no longer independent coefficients. Therefore, instead of dealing directly with $q(n)$, we will operate with a class of generalized polynomials coming from $q(n)$.

Let $Q(n) = \sum_i b_i Q_i(n)$ be a generalized polynomial where each $b_i \in \mathbf{Z}$ and each $Q_i(n)$ is simple. Suppose also that $R(Q) \subset \{a\alpha \mid a \in \mathbf{N}, \alpha \in R(q)\}$ for some generalized polynomial $q(n)$. By Lemma 2.5, integers can be brought out of brackets so that for each $Q_i(n)$ there exists a corresponding generalized polynomial $u_i(n)$ with $R(u_i) \subset R(q)$ and such that

$$Q(n) = \sum_i b_i a_i u_i(n) + \sum_i 1_{C_i}(s_i) t_i(n)$$

for some $a_i \in \mathbf{N}$, some generalized polynomials $t_i(n)$ and $s_i(n)$ and some subintervals $C_i \subset [0, 1)$. We will say that $u_i(n)$ is *the reduced term* of $Q_i(n)$.

Definition 3.2 *Let $Q(n)$ be as above. We will say that a simple generalized polynomial $r(n)$ is* a reduced subpolynomial of $Q(n)$ *if there exists a subpolynomial $u(n)$ of $Q(n)$ such that $r(n)$ is the reduced term (with respect to some fixed $q(n)$) of $u(n)$.*

Lemma 3.2 *Let $q(n)$ be a finite sum of simple generalized polynomials and $q^h(n) = q(n+h) - q(n)$. Then*

$$q^h(n) = V_h q(n) + \sum_i b_i(h) s_i(n) + \sum_{i,j} 1_{B_i}(*) d_{ij}(h) t_{ij}(n), \tag{7}$$

and by possibly adding more terms involving indicator functions,

$$q^h(n) = \sum_i a_i(h)\, q_i(n) + \sum_i b_i(h)\, s_i(n) + \sum_{i,j} 1_{B_i}(*) d_{ij}(h)\, t_{ij}(n), \tag{8}$$

where $a_i(h), b_i(h), d_{ij}(h) \in \mathbf{Z}$, each $s_i(n)$, $t_{ij}(n)$ is a simple, reduced subpolynomial of $V_{h_l} \circ \cdots \circ V_{h_1} q(n)$ for some l, the $q_i(n)$'s are the simple, reduced terms of $V_h q(n)$, B_i is some nice subset of either $[0,1)$ or $[0,1)^2$ and each argument of the indicator functions is of form

$$\sum_i c_i(h) r_i(n) + \sum_i 1_{B_i}(*) c_{ij}(h) r_{ij}(n) \tag{9}$$

where $c_i(h), c_{ij}(h) \in \mathbf{Z}$ and $r_i(n), r_{ij}(n)$ are induced inner subpolynomials of $q(n), q_i(n), s_i(n)$ and $t_{ij}(n)$.
Moreover, $\deg(s_i) < \deg(V_h q)$, $\deg(r_i)$,$\deg(r_{ji}) \le \deg(q)$ *and* $\deg(t_{ij}) \le \deg(V_h q)$. *If* $\deg(t_{ij}) = \deg(V_h q)$, *then $d_{ij}(h) = 1$ and $t_{ij}(n)$ equals a reduced term of $V_h q(n)$. If $q(n) = \overline{q}(n)$, then* $\deg(r_i)$, $\deg(r_{ij}) \le \deg(V_h q)$.

Proof: Since the operators $q(n) \mapsto q^h(n)$ and V_h are linear, we may assume that $q(n)$ is simple. We will again use induction on $B(q)$.
If $B(q) = 0$, then $q(n) = \alpha n^k$ for some $k \in \mathbf{N}$ and $\alpha \in \mathbf{R}$. So

$$\begin{aligned} q^h(n) &= \alpha(n+h)^k - \alpha n^k \\ &= \alpha k h n^{k-1} + \alpha \sum_{i=2}^{k} \binom{k}{i} h^i n^{k-i} \\ &= V_h q(n) + \sum_{i=2}^{k} \binom{k}{i} h^i \alpha n^{k-i}. \end{aligned}$$

This gives $b_i(h) = \binom{k}{i} h^i \in \mathbf{Z}$ and for each $i = 2, \ldots, k-1$, $s_i(n) = \alpha n^{k-i}$ is the simple, reduced term of $V_{h_i} \circ \cdots \circ V_{h_1} q(n)$. Also, $a(h) = kh$.
Suppose now that the lemma is true for all simple generalized polynomials $q(n)$ with $B(q) \le K$, and let $q(n)$ be a simple generalized polynomial with $B(q) = K+1$. Then $q(n) = \prod_{i=1}^{k} [q_i(n)] p(n)$, where $B(q_i) = K$ for some i, $1 \le i \le k$. We will prove it in the case $k = 1$. The general case follows similarly. So let $q(n) = [q_1(n)] p(n)$ where $B(q_1) = K$ and $p(n)$ is a polynomial. By the induction hypothesis,

$$q_1^h(n) = V_h q_1(n) + \sum_i b_{1i}(h) s_{1i}(n) + \sum_{i,j} 1_{B_i}(*) d_{1ij}(h) t_{1ij}(n)$$

where

$$V_h q_1(n) = \sum_i q_{1i}(n) = \sum_i a_{1i}(h) u_{1i}(n) + \sum_i 1_{B_i}(*) v_{1i}(n),$$

and

$$p^h(n) = V_h p(n) + \sum_i b_{2i}(h) s_{2i}(n) = \sum_i a_{2i}(h) p_i(n) + \sum_i b_{2i}(h) s_{2i}(n)$$

where $q_{1i}(n), s_{ji}(n), t_{1ij}(n), u_{1i}(n), v_{1i}(n)$ and $p_i(n)$ are all simple generalized polynomials. By using these identities and Lemma 2.5, we have

$$\begin{aligned}
& q^h(n) \\
= \; & [q_1(n+h)]\, p(n+h) - [q_1(n)]p(n) \\
= \; & \Big[q_1(n) + \sum_i q_{1i}(n) + \sum_i b_{1i}(h) s_{1i}(n) + \sum_{i,j} 1_{B_i}(*) d_{1ij}(h) t_{1ij}(n)\Big] \Big(p(n) + \\
& \qquad V_h p(n) + \sum_i b_{2i}(h) s_{2i}(n)\Big) - [q_1(n)]p(n) \\
= \; & [q_1(n)] V_h p(n) + \sum_i [q_{1i}(n)] p(n) \\
& + \sum_i b_{2i}(h)[q_1(n)] s_{2i}(n) + \sum_i b_{1i}(h)[s_{1i}(n)] p(n) \qquad (10) \\
& + \sum_{i,j} \Big(a_{1i}(h)[u_{1i}(n)] + b_{1i}(h)[s_{1i}(n)]\Big) \Big(a_{2j}(h) p_j(n) + b_{2j}(h) s_{2j}(n)\Big) \qquad (11) \\
& + \sum_{i,j} 1_{B_i}(*) d_{ij}(h) t_{ij}(n) \qquad (12)
\end{aligned}$$

where

$$\begin{aligned}
\{t_{ij}(n) \mid i,j\} \;=\; & \Big\{ [t_{1ij}(n)]p(n),\ [t_{1ij}(n)]p_i(n),\ [t_{1ij}(n)]s_{2i}(n), \\
& [v_{1i}(n)]p_j(n),\ [v_{1i}(n)]s_{2j}(n),\ p(n),\ p_i(n),\ s_{2i}(n) \mid i,j \Big\}
\end{aligned}$$

and $d_{ij}(h)$ are the corresponding integer coefficients.

Now, $V_h q(n) = [q_1(n)]V_h p(n) + \sum_i [q_{1i}(n)]p(n)$, so we only need to check that the terms in (10), (11) and (12) have the properties stated in the lemma.

Since each $q_1(n)$, $s_{1i}(n)$, $u_{1i}(n)$ is a reduced outer subpolynomial of some $V_{h_l} \circ \cdots \circ V_{h_1} q_1(n)$ and each $p(n)$, $p_i(n)$, $s_{2i}(n)$ is a reduced term of some $V_{h_{l'}} \circ \cdots \circ V_{h_1} p(n)$, each $[q_1(n)]s_{2i}(n)$, $[s_{1i}(n)]p(n)$, $[u_{1i}(n)]p_j(n)$, $[u_{1i}(n)]s_{2i}(n)$, $[s_{1i}(n)]p_j(n)$, $[s_{1i}(n)]s_{2j}(n)$ is a reduced outer subpolynomial of some $V_{h_l} \circ \cdots \circ V_{h_1} q(n)$. The same is also true about the $t_{ij}(n)$'s.

By the induction hypothesis, $\deg(s_{1i}) < \deg(V_h q_1)$ and $\deg(s_{2i}) < \deg(V_h p)$. Therefore
$\deg([s_{1i}]s_{2i}) < \deg([q_1]s_{2i}) < \deg([q_1]V_h p) = \deg(V_h q)$,
$\deg([s_{1i}]p_j) < \deg([s_{1i}]p) < \deg([V_h q_1]p) = \deg(V_h q)$,
$\deg([u_{1i}]s_{2j}) < \deg([u_{1i}]p_j) \le \deg([V_h q_1]V_h p) < \deg(V_h q)$.
Also, $\deg(t_{1ij}) \le \deg(V_h q_1)$ and $\deg(v_{1i}) < \deg(V_h q_1)$, so it follows that $\deg(t_{ij}) \le \deg(V_h q)$. Now if $\deg(t_{ij}) = \deg(V_h q)$, then $t_{ij}(n) = p(n)$ if $\deg(q_1) = 1$, and $t_{ij}(n) = [t_{1ij}(n)]p(n)$ otherwise, where $\deg(t_{1ij}) = \deg(V_h q_1)$. Therefore, by the induction hypothesis, $d_{1ij}(h) = 1$ and $t_{1ij}(n)$ equals a reduced term of $V_h q_1(n)$. Hence $d_{ij}(h) = 1$ and $t_{ij}(n)$ equals a reduced term of $V_h q(n)$.

By repeated use of Lemma 2.5, the arguments of the indicator functions can be written in the form (9). Any term coming from the arguments of $q_1^h(n)$ has the desired properties by the induction hypothesis. Also, any simple reduced term of an argument comes from $q_1(n+h)$, so it is an induced inner subpolynomial of $q(n)$, $q_i(n)$, $s_i(n)$ or $t_{ij}(n)$. If $q(n) = \overline{q}(n)$, then $\deg(p) \geq 1$, so $\deg(r_i), \deg(r_{ij}) \leq \deg(q_1) < \deg(q)$.

□

Lemma 3.3 *Let $q(n)$ be a simple generalized polynomial and $u(n)$ a simple reduced subpolynomial of $V_{h_{l'}} \circ \cdots \circ V_{h_1} q(n)$. Then the following hold:*

(i) *If $x(n)$ is a reduced subpolynomial of $V_{h_l} \circ \cdots \circ V_{h_1} \overline{u}(n)$, then there exists a reduced subpolynomial $\breve{x}(n)$ of $V_{h_l} \circ \cdots \circ V_{h_1} u(n)$ such that $\overline{\breve{x}}(n) = \overline{x}(n)$.*

(ii) *If $x(n)$ is a reduced subpolynomial of $V_{h_l} \circ \cdots \circ V_{h_1} u(n)$, then $x(n)$ is a reduced subpolynomial of $V_{h_{l+l'}} \circ \cdots \circ V_{h_1} q(n)$.*

Proof: (i) Suppose $u(n) = [\cdots[v_1(n)]\lambda_1 \cdots]\lambda_k$. Let $v(n) = \overline{u}(n) = v_1(n)\lambda_1 \cdots \lambda_k$ and $V_{h_l} \circ \cdots \circ V_{h_1} v_1(n) = \sum_i v_{1i}(n)$. Then

$$V_{h_l} \circ \cdots \circ V_{h_1} u(n) = \sum_i [\cdots[v_{1i}(n)]\lambda_1 \cdots]\lambda_k$$

and

$$V_{h_l} \circ \cdots \circ V_{h_1} v(n) = \sum_i v_{1i}(n)\lambda_1 \cdots \lambda_k.$$

$x(n)$ is either a subpolynomial of $v_{1i}(n)$ for some i, in which case $x(n)$ is also a subpolynomial of $[\cdots[v_{1i}(n)]\lambda_1 \cdots]\lambda_k$, or $x(n)$ is an outer subpolynomial of $v_{1i}(n)\lambda_1 \cdots \lambda_k$ for some i, say $x(n) = y(n)\lambda_1 \cdots \lambda_k$. But then $\breve{x}(n) = [\cdots[y(n)]\lambda_1 \cdots]\lambda_k$ is a subpolynomial of $[\cdots[v_{1i}(n)]\lambda_1 \cdots]\lambda_k$ and $\overline{\breve{x}}(n) = \overline{x}(n)$.

(ii) Bringing out integers from the brackets of a generalized polynomial, which we will refer to as reducing the generalized polynomial, changes only the coefficients and not the structure of the generalized polynomial. Therefore, if $v(n)$ is a subpolynomial of $V_{h_{l'}} \circ \cdots \circ V_{h_1} q(n)$, we will obtain the same set of reduced terms if we reduce $v(n)$ first and then take l generalized derivatives and reduce as if we just reduce the terms of $V_{h_l} \circ \cdots \circ V_{h_1} v(n)$. So if $v(n)$ is such that $u(n)$ is the reduced term of $v(n)$, then $x(n)$ is a reduced term of $V_{h_l} \circ \cdots \circ V_{h_1} v(n)$.

Let $y(n)$ be a subpolynomial of $V_{h_l} \circ \cdots \circ V_{h_1} v(n)$ such that $x(n)$ is the reduced term of $y(n)$. Let $Q(n) = V_{h_{l'}} \circ \cdots \circ V_{h_1} q(n)$. Recall that $v(n)$ is a subpolynomial of $Q(n)$. We will prove by induction on l that any subpolynomial $y(n)$ of $V_{h_l} \circ \cdots \circ V_{h_1} v(n)$ is a subpolynomial of $V_{h_l} \circ \cdots \circ V_{h_1} Q(n) = V_{h_l} \circ \cdots \circ V_{h_1}(V_{h_{l'}} \circ \cdots \circ V_{h_1} q(n)) = V_{h_{l+l'}} \circ \cdots \circ V_{h_1} q(n)$. First, let $l = 1$ and let $a(n)$ be the term of $Q(n)$ such that $v(n)$ is a subpolynomial of $a(n)$. If $V_h v(n) = \sum_{i=1}^k v_i(n)$, where $v_i(n)$ is a simple generalized polynomial, $i = 1, \ldots, k$, then $V_h a(n) = \sum_{i=1}^k a_i(n) + a_0(n)$, where $v_i(n)$ is a subpolynomial of $a_i(n)$, $i = 1, \ldots, k$, and $v(n)$ is a subpolynomial of each term of $a_0(n)$. Hence, any subpolynomial of $V_h v(n)$ is a subpolynomial of $V_h Q(n)$. Assume the statement is true for $l-1$. Since $y(n)$ is a subpolynomial of $V_{h_l} \circ \cdots \circ V_{h_1} v(n)$, there exists a subpolynomial $a(n)$ of $V_{h_l} \circ \cdots \circ V_{h_1} v(n)$ such that $y(n)$ is a subpolynomial of $V_{h_l} a(n)$. By the induction hypothesis, $a(n)$ is a subpolynomial of $V_{h_{l-1}} \circ \cdots \circ V_{h_1} Q(n)$ and hence $y(n)$ is a subpolynomial of $V_{h_l} \circ \cdots \circ V_{h_1} Q(n)$.

□

Let $q(n)$ be a generalized polynomial with independent coefficients. We will put a partial order on the set $S(q)$ of coefficients of $q(n)$ by

$\alpha_1 \prec \alpha_2$ if there exist subpolynomials $u_1(n)$ and $u_2(n)$, $u_2(n) \neq [u_2(n)]$, such that α_1 is a coefficient of $u_1(n)$, α_2 is an outer coefficient of $u_2(n)$ and such that $[u_1(n)]u_2(n)$ is either an inner subpolynomial of $q(n)$ or a term of $q(n)$.

Since $q(n)$ has independent coefficients, the relation $\prec$ is a strict ordering. In order to check that $\prec$ is transitive, let $\alpha_1 \prec \alpha_2$ and $\alpha_2 \prec \alpha_3$. Then there exist generalized polynomials $u(n) = [u_1(n)]u_2(n)$ and $v(n) = [v_1(n)]v_2(n)$ where α_1 is a coefficient of $u_1(n)$, α_2 is an outer coefficient of $u_2(n)$ and also a coefficient of $v_1(n)$, and α_3 is an outer coefficient of $v_2(n)$. Since α_2 is a coefficient of $v_1(n)$ in addition to be an outer coefficient of $u(n)$, $u(n)$ is necessarily an inner subpolynomial of $q(n)$ and therefore is a subpolynomial of $v_1(n)$. Hence, $\alpha_1 \prec \alpha_3$.

Example 4 *If*

$$q(n) = \Big[[\alpha n]\,[[\beta n]\gamma n]\lambda + \delta n^2\Big]\sigma,$$

then

$$\alpha \prec \lambda \prec \sigma$$
$$\beta \prec \gamma \prec \lambda \prec \sigma$$
$$\delta \prec \sigma.$$

Note that there are no relations between α and β or between α and γ.

If $q_i(n)$ is a simple term (see Lemma 2.7) of $q(n)$, then each simple, reduced term of $V_h q_i(n)$ has independent coefficients and the coefficients have the same order as in $q(n)$. We will say that V_h preserves the ordering of the coefficients.

If $q(n)$ is a generalized polynomial with independent coefficients and $q_i(n), i = 1, \ldots, m$, are its simple terms, then let $U_l(q_i)$ be the set of all non-constant, simple reduced subpolynomials of $V_{h_l} \circ \cdots \circ V_{h_1} q_i(n)$ and

$$U(q) = \bigcup_{i=1}^{m} \bigcup_{l=0}^{D_i-1} U_l(q_i)$$

where D_i =deg(q_i). Note that if $u(n) \in U(q)$, then $S(u) \subset S(q)$ and that if $\alpha, \beta \in S(u)$ and $\alpha \prec \beta$ as elements of $S(q)$, then $\alpha \prec \beta$ as elements of $S(u)$. The following proposition is important for the proof of Theorem 3.1.

Proposition 3.4 *Let $q(n)$ be a generalized polynomial with independent coefficients. Then for any subset $\{u_1(n), \ldots, u_k(n)\}$ of $U(q)$, where $u_i(n) \neq u_j(n)$ if $i \neq j$, both*

$$u(n) = (u_1(n), \ldots, u_k(n)) \quad \text{and} \quad \overline{u}(n) = (\overline{u}_1(n), \ldots, \overline{u}_k(n))$$

are uniformly distributed (mod 1) in $\mathbf{R}^k$.

Let us illustrate this with an example. Let

$$q(n) = \Big[[\alpha n^2]^3 \beta n^2 + \lambda n^4\Big]^2 \gamma n,$$

where $1, \alpha, \beta, \lambda, \gamma, \alpha\beta, \alpha\beta\gamma, \lambda\gamma$ are rationally independent. The simple terms of $q(n)$ are then

$$\Big[[\alpha n^2]^3\beta n^2\Big]^2\gamma n,\ \ 2\Big[[\alpha n^2]^3\beta n^2\Big][\lambda n^4]\gamma n,\ \ [\lambda n^4]^2\gamma n.$$

There are too many elements of $U(q)$ to write them all up. So consider the subpolynomial $u(n) = [\alpha n^2]^3\beta n^2 \in U(q)$ of $q(n)$, and

$$U_4(u) = \Big\{[\alpha n^2]^2\beta,\ [\alpha n^2]\beta n^2,\ [\alpha n][\alpha n^2]\beta n,\ [\alpha n]^2\beta n^2,\ [\alpha n]^3\beta n,\ [\alpha n^2][\alpha n]^2\beta\Big\} \subset U(q). \quad (13)$$

Then, by Proposition 3.4,

$$\Big([\alpha n^2]^2\beta,\ [\alpha n^2]\beta n^2,\ [\alpha n][\alpha n^2]\beta n,\ [\alpha n]^2\beta n^2,\ [\alpha n]^3\beta n,\ [\alpha n^2][\alpha n]^2\beta\Big)$$

is uniformly distributed (mod 1) in $\mathbf{R}^6$.

We will need the following lemma in the proof of Proposition 3.4.

Lemma 3.5 *If $q(n)$ is a generalized polynomial with independent coefficients, let $u_1(n), \ldots, u_k(n) \in U(q)$ be k distinct generalized polynomials with* $\deg(u_i) = D, i = 1, \ldots, k$. *Let $\{r_i(n) \mid i\}$ be the set of distinct inner subpolynomials of the $\overline{u}_i(n)$'s of degree $D-1$ and let $c_i \in \mathbf{Z}\setminus\{0\}$, $i = 1, \ldots, k$. If $u_{ij}(n)$ are the reduced generalized polynomials such that*

$$V_h\overline{u}_i(n) = \sum_{j=1}^{k_i} a_{ij}(h)u_{ij}(n) + \sum_j 1_{C_{ij}}(*)t_{ij}(n)$$

for some $a_{ij}(h) \in \mathbf{Z}$, let $v_1(n), \ldots, v_l(n)$ be all the distinct elements from the set $\{\overline{u}_{ij}(n), \overline{r}_i(n) \mid i, j\}$ and $d_i(h) \in \mathbf{Q}$. Then

$$\sum_{i=1}^{k}\sum_{j=1}^{k_i} c_i a_{ij}(h)\overline{u}_{ij}(n) + \sum_i d_i(h)\overline{r}_i(n) = \sum_{i=1}^{l} b_i(h)v_i(n) \quad (14)$$

for some $b_i(h) \in \mathbf{Q}$ such that $(b_1(h), \ldots, b_l(h)) \neq (0, \ldots, 0)$ for all but finitely many h.

Note that if $u_1(n), \ldots, u_6(n)$ are the elements of $U_4(u)$ in the order which they are written in (13), and $c_1, \ldots, c_6 \in \mathbf{Z}$, not all equal 0, then (14) becomes

$$\begin{aligned}
& c_1 4h[\alpha n^2][\alpha n]\beta + c_2\Big(2h[\alpha n]\beta n^2 + 2h[\alpha n^2]\beta n\Big) \\
&\quad + c_3\Big([\alpha h][\alpha n^2]\beta n + 2h[\alpha n]^2\beta n + h[\alpha n][\alpha n^2]\beta\Big) + c_4\Big(2[\alpha h][\alpha n]\beta n^2 + 2h[\alpha n]^2\beta n\Big) \\
&\quad + c_5\Big(3[\alpha h][\alpha n]^2\beta n + h[\alpha n]^3\beta\Big) + c_6\Big(2h[\alpha n]^3\beta + 2[\alpha h][\alpha n^2][\alpha n]\beta\Big) \\
= &\ \Big(h(4c_1 + c_3) + 2[\alpha h]c_6\Big)[\alpha n^2][\alpha n]\beta + \Big(2hc_2 + 2[\alpha h]c_4\Big)[\alpha n]\beta n^2 \\
&\quad + \Big(2hc_2 + [\alpha h]c_3\Big)[\alpha n^2]\beta n + \Big(2h(c_3 + c_4) + 3c_5[\alpha h]\Big)[\alpha n]^2\beta n + \Big(h(c_5 + 2c_6)\Big)[\alpha n]^3\beta
\end{aligned}$$

which is non-zero for all but at most finitely many h (see Lemma **??**).

Proof of Lemma 3.5: Suppose first that $\overline{u}_1(n),\ldots,\overline{u}_k(n)$ are polynomials. Then $u_i(n) = \Big[\cdots[\alpha_i n^D]\lambda_{i1}\cdots\Big]\lambda_{il_i}$, $i=1,\ldots,k$, where $\alpha_i,\lambda_{ij}\in S(q)$ and $\overline{u}_i(n) = \alpha_i\prod_{j=1}^{k_i}\lambda_{ij}n^D$. Note that $u_{i_1}(n)\neq u_{i_2}(n)$ for $i_1\neq i_2$ implies that $\alpha_{i_1}\prod_{j=1}^{k_{i_1}}\lambda_{i_1 j}\neq\alpha_{i_2}\prod_{j=1}^{k_{i_2}}\lambda_{i_2 j}$. For $\alpha_{i_j}\prec\lambda_{i_j 1}\prec\cdots\prec\lambda_{i_j l_{i_j}}$, $\alpha_{i_j}\prod_{l=1}^{k_{i_j}}\lambda_{i_j l}\in R(q)$, $j=1,2$, and $R(q)$ is rationally independent. So

$$V_h\Big(\sum_{i=1}^{k} c_i\overline{u}_i(n)\Big) = \sum_{i=1}^{k} c_i\alpha_i\prod_{j=1}^{k_i}\lambda_{ij}hDn^{D-1} = \sum_{i=1}^{k} b_i(h)\overline{u}_i(n)$$

where $b_i(h) = c_iDh\neq 0$, which was to be proved.

Next, suppose that there is at least one i such that $\overline{u}_i(n)$ is not a polynomial. We will first show that if $w_1(n),\ldots,w_m(n)$ are all the distinct elements from the set $\{\overline{u}_{ij}(n)\mid i,j\}$, and

$$\sum_{i,j} c_i a_{ij}(h)\overline{u}_{ij}(n) = \sum_i B_i(h)w_i(n), \tag{15}$$

then $(B_1(h),\ldots,B_m(h))\neq(0,\ldots,0)$ for all but finitely many h.

Let αn^t be an inner subpolynomial of some $\overline{u}_i(n)$, $t\geq 1$, and let

$$T = \min\{t\geq 1 \mid \ \alpha n^t \text{ is an inner subpolynomial of } \overline{u}_j(n) \text{ for some } j\in\{1,\ldots,k\}\ \},$$

$$R_\alpha = \{\overline{u}_j(n)\mid \ \alpha n^T \text{ is an inner subpolynomial of } \overline{u}_j(n)\ \}$$

and

$$S_\alpha = \{s_{ij}(n) = a_{ij}(h)\overline{u}_{ij}(n)\mid \overline{u}_i(n)\in R_\alpha\}.$$

For each $\overline{u}_i(n)\in R_\alpha$, there exists exactly one corresponding $s_{ij}(n)\in S_\alpha$ with αn^{T-1} as an inner subpolynomial if $T>1$ and with $a_{ij}(h) = [\alpha h]$ if $T=1$. With this corresponding $s_{ij}(n) = a_{ij}(h)\overline{u}_{ij}(n)$, let $w_i(n) = \overline{u}_{ij}(n)$. It follows that if $i_1\neq i_2$ and $\overline{u}_{i_1}(n),\overline{u}_{i_2}(n)\in R_\alpha$, then $w_{i_1}(n)\neq w_{i_2}(n)$.

If $T>1$, the only $\overline{u}_{ij}(n)$'s having αn^{T-1} as an inner subpolynomial, arise from elements in R_α. Since all these $w_i(n) = \overline{u}_{ij}(n)$'s are different, there are no cancellations so that the corresponding coefficients $B_i(h)$'s have the property that $B_i(h)\neq 0$ for all h.

If $T=1$, there may exist $i_1\neq i_2$ and j such that $\overline{u}_{i_1}(n)\in R_\alpha$ and $w_{i_1}(n) = \overline{u}_{i_2 j}(n)$. However, we have already seen that we then have $\overline{u}_{i_2}(n)\notin R_\alpha$. Therefore $a_{i_2 j}(h)\neq[\alpha h]$ because the coefficient $[\alpha h]$ can only come from terms having αn as an inner subpolynomial. Hence $B_{i_1}(h) = c_{i_1}[\alpha h] + Bh + \sum_i e_i[\beta_i h]$ for some $B,e_i\in\mathbf{Z}$ and $\beta_i\in S(q),\beta_i\neq\alpha$. Since $c_{i_1}\neq 0$ and $S(q)$ is rationally independent, $B_{i_1}(h)\neq 0$ for all but finitely many h, see Lemma **??**.

It remains to show that none of the subpolynomials $r_i(n)$ can cancel out all the $w_i(n)$'s in (15). We may assume that $d_i(h)\in\mathbf{Z}$. Suppose that

$$\sum_{i=1}^{m} B_i(h)w_i(n) + \sum_i d_i(h)\overline{r}_i(n) = \sum_{i=1}^{l} b_i(h)v_i(n).$$

We need to show that $(b_1(h),\ldots,b_l(h))\neq(0,\ldots,0)$ for all but finitely many h. Since terms with different outer coefficients cannot cancel each other out, and since the above argument shows that for each distinct outer coefficient of the $u_i(n)$'s there is some $B_i(h) \neq 0$, all the outer coefficients of the $u_i(n)$'s are found in the $w_i(n)$'s but possibly multiplied by some inner coefficients. Let γ be a maximal element in $\bigcup_{i=1}^k S(u_i) \subset S(q)$, with respect to the partial order of the set of coefficients of the $u_i(n)$'s, and let $w_j(n)$ be a generalized polynomial where γ is a factor of the outer coefficient and for which the corresponding $B_j(h) \neq 0$. All the $r_i(n)$'s are inner subpolynomials of the $u_i(n)$'s. Therefore if λ is an outer coefficient of $r_i(n)$, then either $\lambda \prec \gamma$ or there is no relation between λ and γ. That means that $\overline{r}_i(n) \neq w_j(n)$ for all i. Hence there is at least one j such that $b_j(h) \neq 0$.

□

Proof of Proposition 3.4: Since the partial ordering of the coefficients is preserved under $V_{h_l} \circ \cdots \circ V_{h_1}$, $\overline{u}_i(n) \neq \overline{u}_j(n)$ if $i \neq j$. There is also at least one i_0 such that $\overline{u}_{i_0}(n) \neq \overline{u}_i^j(n)$ for all i, j. We need to show that for any $a = (a_1, \ldots, a_k) \in \mathbf{Z}^k \setminus \{(0, \ldots, 0)\}$,

$$u(n) = \sum_{i=1}^k a_i u_i(n)$$

is uniformly distributed (mod 1). Without loss of generality we may assume that $a_i \neq 0$ for all i. Let $w_1(n), \ldots, w_{k_1}(n)$ be all the distinct, simple generalized polynomials from $\{\overline{u}_i^j(n) \mid i, j\}$ and let $v_1(n), \ldots, v_{k_2}(n)$ be all the distinct elements from the set $\{\overline{u}_i(n), w_j(n) \mid i, j\}$. Then by Lemma 2.2,

$$u(n) \left(\sum_{i=1}^k a_i \overline{u}_i(n), w_1(n), \ldots, w_{k_1}(n) \right),$$

and since $\overline{u}_{i_0}(n) \neq w_j(n)$ for all j,

$$u(n)(v_1(n), \ldots, v_{k_2}(n)).$$

It is therefore enough to show that $v(n) = (v_1(n), \ldots, v_k(n))$ is uniformly distributed (mod 1) for any subset

$$\{v_1(n), \ldots, v_k(n)\} \subset \overline{U}(q) = \{\overline{u}(n) \mid u(n) \in U(q)\}$$

such that $v_i(n) \neq v_j(n)$ if $i \neq j$, and we will do this by induction on $\deg(v) \stackrel{\text{def}}{=} \max_i \deg(v_i)$.

If $\deg(v) = 1$, $v(n)$ is uniformly distributed (mod 1) since $v_i(n) = \beta_i n$ for some $\beta_i \in R(q)$, and $R(q) \cup \{1\}$ is rationally independent.

Assume now that $v(n)$ is uniformly distributed (mod 1) if $\deg(v) \leq K$, and let $\deg(v) = K+1$. We will show that $v_c(n) = \sum_{i=1}^k c_i v_i(n)$ is uniformly distributed (mod 1) for any k-tuple of integers $c = (c_1, \ldots, c_k) \neq (0, \ldots, 0)$. If $\deg(v_{i_1})$,..., $\deg(v_{i_{m_1}})$ <$\deg(v)$, then by the induction hypothesis, $(v_{i_1}(n), \ldots, v_{i_{m_1}}(n))$ is uniformly distributed (mod 1). So we may assume without loss of generality that $c_i \neq 0, i = 1, \ldots, k$. Let $u_c(n) = \sum_{i=1}^k c_i u_i(n)$, where $u_i(n) \in U$ and $\overline{u}_i(n) = v_i(n)$.

By Lemma 3.2

$$v_c^h(n) = \sum_{i,j} c_i a_{ij}(h) v_{ij}(n) + \sum_i b_i(h) s_i(n) + \sum_{i,j} 1_{B_i}(*) d_{ij}(h) t_{ij}(n)$$

where each $v_{ij}(n), s_i(n), t_{ij}(n)$ is a simple reduced subpolynomial of some $V_{h_l} \circ \cdots \circ V_{h_1} v_c(n)$. Also, $\deg(s_i) < \deg(V_h v_c) = K$ and if $\deg(t_{ij}) = \deg(V_h v_c)$, then $d_{ij}(h) = 1$.

Let $\{r_i(n) \mid i\}$ be the set of all distinct simple, reduced generalized polynomials appearing in the arguments of the indicator functions. By Lemma 3.2, each $r_i(n)$ is an induced inner subpolynomial of one of the generalized polynomials $v_c(n), v_{lj}(n), s_l(n)$ or $t_{lj}(n)$. Therefore each $r_i(n)$ is a reduced subpolynomial of some $V_{h_l} \circ \cdots \circ V_{h_1} v_c(n)$.

Let furthermore $w_1(n), \ldots, w_{k_3}(n)$ be all the distinct terms from the set $\{r_i(n), r_i^l(n),\ v_{ij}^l(n),\ s_i^l(n),\ t_{ij}^l(n) \mid i,j,l\}$. Then by Lemma 2.6 and Lemma 2.2,

$$v_c^h(n)\Big(\sum_{i,j} c_i a_{ij}(h)\overline{v}_{ij}(n) + \sum_i b_i(h)\overline{s}_i(n) + \sum_i \epsilon_i d_{ij}(h)\overline{t}_{ij}(n), \overline{w}_1(n), \ldots, \overline{w}_{k_3}(n)\Big).$$

Note that each $w_i(n)$ is also a reduced subpolynomial of some $V_{h_l} \circ \cdots \circ V_{h_1} v_c(n)$ so that if $Q_1(n), \ldots, Q_{k_4}(n)$ are all the distinct generalized polynomials from the set $\{v_{ij}(n), t_{ij}(n),\ s_i(n),\ w_i(n) \mid i,j\}$, then each $Q_i(n)$ is a reduced subpolynomial of some $V_{h_l} \circ \cdots \circ V_{h_1} v_c(n)$. By Lemma 3.3(i) there exist a generalized polynomial $\check{Q}_i(n)$ such that $\overline{\check{Q}}_i(n) = \overline{Q}_i(n)$ and such that $\check{Q}_i(n)$ is a reduced subpolynomial of $V_{h_l} \circ \cdots \circ V_{h_1} u_c(n)$ for some l, $i = 1, \ldots, k_4$. So by Lemma 3.3(ii), $\check{Q}_i(n) \in U(q)$ and therefore $\overline{Q}_i(n) \in \overline{U}(q)$. Since $\deg(\overline{Q}_i) \leq K$, it follows by the induction hypothesis that $(\overline{Q}_1(n), \ldots, \overline{Q}_{k_4}(n))$ is uniformly distributed (mod 1) in $\mathbf{R}^{k_4}$.

Now, if

$$g(h) = \sum_{i,j} c_i a_{ij}(h)\overline{v}_{ij}(n) + \sum_i b_i(h)\overline{s}_i(n) + \sum_i \epsilon_i d_{ij}(h)\overline{t}_{ij}(n) + \sum_i e_i\overline{w}_i(n) \neq 0 \tag{16}$$

for any k_4-tuple of rational numbers $(e_1, \ldots, e_{k_4}) \neq (0, \ldots, 0)$, then

$$v_c^h(n)\Big(\overline{Q}_1(n), \ldots, \overline{Q}_{k_4}(n)\Big). \tag{17}$$

It is enough, by van der Corput's difference theorem, to show (17) and hence (16) for all but finitely many h. Furthermore, it suffices to consider the terms of degree K. Note that $\deg(\overline{s}_i) < K$ and $\deg(\overline{t}_{ij}) < K$ unless $d_{ij}(h) = 1$, i.e. $d_{ij}(h)$ is independent of h. Since each $a_{ij}(h)$ is an increasing function of h, and it is sufficient that $g(h) \neq 0$ for all but finitely many h, we need only to consider

$$D(h) = \sum_{i,j} c_i a_{ij}(h)\overline{v}_{ij}(n) + \sum_i e_i\overline{w}_i(n).$$

However, $D(h) \neq 0$ for all but finitely many h, by Lemma 3.5. Therefore $g(h) \neq 0$ and $v_c^h(n)(\overline{Q}_1(n), \ldots, \overline{Q}_{k_4}(n))$ for all but finitely many h.

□

Proof of Theorem 3.1: Let $q(n)$ be a generalized polynomial with independent coefficients. By Corollary 2.8(i), $q(n)$ is uniformly distributed (mod 1) if

$$\left(\sum_{i=1}^{k} q_i(n) + \sum_{i,j} \epsilon_i t_{ij}(n), r_1(n), \ldots, r_m(n)\right)$$

is uniformly distributed (mod 1) for any $\epsilon_i \in \{0,1\}$, where $q_i(n), i = 1,\dots,k$, are the simple terms of $q(n)$, the $t_{ij}(n)$'s are outer subpolynomials of $\sum_{i=1}^k q_i(n)$ and $r_1(n),\dots,r_m(n)$ are distinct induced inner subpolynomials of $\sum_{i=1}^k q_i(n)$. So if $t_1(n),\dots,t_{k_1}(n)$ are all the distinct terms among the $t_{ij}(n)$'s, then

$$q(n)\Big(q_1(n),\dots,q_k(n),t_1(n),\dots,t_{k_1}(n),r_1(n),\dots,r_m(n)\Big).$$

Since all $q_i(n), t_i(n), r_i(n)$ are distinct reduced subpolynomials of $\sum_i q_i(n)$, it follows from Proposition 3.4 ($l = 0$), that $q(n)$ is uniformly distributed (mod 1).

□

4 Main Result

Recall that $R(q)$ is the set of coefficients of the generalized polynomial $q(n)$ and of all generalized polynomials obtained from $q(n)$ by removing nested brackets. In our main theorem, which follows below, we require that the subpolynomials have independent coefficients except that their outer coefficient can take the value 1. In this way, the subpolynomials $q_i(n)$ (see below) can take the form $\prod_{j=1}^l [q_{ij}(n)]$. Define therefore a set $R'(q) = R(q) \setminus \{1\}$.

Theorem 4.1 *Let*

$$q(x) = p_0(x) + \sum_{i=1}^k [q_i(x)]p_i(x)$$

be a generalized polynomial such that $q_i(x), i = 1,\dots,k$, are generalized polynomials and $p_i(x), i = 0,\dots,k$, are polynomials. Suppose that there exists i_0 such that $[q_{i_0}(x)]p_{i_0}(x)$ has a simple term $a(x)$ with $\deg(a) = \deg(q)$ *and $\overline{a}(n)$ is not a polynomial, and such that*

$$R \cup R\left([q_{i_0}]\gamma_{i_0}\right)$$

is rationally independent, where $R = \bigcup_{j=1}^k R'(q_j) \cup \{1\}$ and γ_{i_0} is the leading coefficient of $p_{i_0}(x)$. Then $q(n)$ is uniformly distributed (mod 1).

This gives us immediately the following corollary.

Corollary 4.2 *Let $q_i(x)$, $i = 1,\dots,k$, be generalized polynomials such that $R = \bigcup_{i=1}^k R(q_i) \cup \{1\}$ is rationally independent. Then*

$$q(n) = \sum_{i=1}^k [q_i(n)]p_i(n) + p_0(n)$$

is uniformly distributed (mod 1) for all but countably many k-tuples of monomials $(p_1(n),\dots,p_k(n))$, where deg(p_i) ≥ 1 if $\overline{q}_i(n)$ is a polynomial.

Before we prove Theorem 4.1 we need some lemmas.

Lemma 4.3 *If $q(n)$ is a simple generalized polynomial such that $\overline{q}(n)$ is not a polynomial and $f_l(n) = V_{h_l} \circ \cdots \circ V_{h_1} q(n)$, $l \in \{0, \ldots, \deg(q)-1\}$, then $\overline{f}_l(n)$ is not a polynomial either unless* $\deg(f_l) = 1$ *in which case $\overline{f}_l(n)$ has an integer coefficient $[u(h_l)]$, where $u(n)$ is a subpolynomial of $f_l(n)$ of degree* 1.

Proof: The proof goes by induction on l. The case $l = 0$ is trivial. Assume the statement is true for $l-1$. We will show that $f_l(n)$, where $\deg(f_l) \geq 1$, has the desired properties. Let $r(n)$ be a simple term of $f_{l-1}(n)$ such that $\overline{r}(n)$ is not a polynomial. Since $\deg(f_{l-1}) \geq 2$, $r(n) = [u(n)]v(n)$ for some generalized polynomials $u(n)$ and $v(n)$ of positive degrees. Let $V_{h_l} u(n) = \sum_i u_i(n)$. Then

$$\begin{aligned} V_{h_l} r(n) &= V_{h_l}([u(n)])v(n) + [u(n)]V_{h_l} v(n) \\ &= \sum_i [u_i(n)]v(n) + [u(n)]V_{h_l} v(n), \end{aligned}$$

which is a sum of simple terms of $f_l(n)$. If $\deg(r) \geq 3$, then either $\deg(u) > 1$, in which case $\deg(u_i) \geq 1$, or $\deg(v) > 1$, in which case $\deg(V_{h_l} v) \geq 1$. Hence $\overline{V_{h_l} r}(n)$ is not a polynomial if $\deg(r) \geq 3$. If $\deg(r) = 2$, then $\deg(u) =\deg(v) = 1$ and $u(n) = [\cdots[\alpha n]\lambda_1 \cdots]\lambda_k$ for some $\alpha, \lambda_1, \ldots, \lambda_k \in \mathbf{R}$. Therefore $[V_{h_l} u(n)] = [[\cdots[\alpha h_l]\lambda_1 \cdots]\lambda_k] = [u(h_l)]$ is an integer coefficient of $V_{h_l} r(n)$.

□

Proof of Theorem 4.1:
In order to prove that

$$q(n) = p_0(n) + \sum_{i=1}^{k} [q_i(n)]p_i(n)$$

is uniformly distributed (mod 1), we write, by Lemma 2.7,

$$q(n) = \sum_{i=1}^{k_1} Q_i(n) + \sum_{i,j} 1_A(r_{1i}, r_{2i}) t_{ij}(n)$$

where the $Q_i(n)$'s are the simple terms of $q(n)$, each $t_{ij}(n)$ is an outer subpolynomial of $\sum_{i=1}^{k_1} Q_i(n)$ and each $r_{ji}(n)$ is a sum of simple induced inner subpolynomials of $\sum_{i=1}^{k_1} Q_i(n)$, some which may be multiplied by indicator functions. Note that $\deg(t_{ij}) <\deg(q)$.

Let $v_1(n), \ldots, v_{m_1}(n)$ be all the distinct simple generalized polynomials in the expressions for the $r_{ji}(n)$'s, and let $w_1(n), \ldots, w_{m_2}(n)$ be all the distinct simple generalized polynomials from the set $\{v_i(n), v_i^l(n), Q_i^l(n), t_{ij}^l(n) \mid i, j, l\}$. Then by Corollary 2.8, $q(n)$ is uniformly distributed (mod 1) if for all $\epsilon_i \in \{0, 1\}$,

$$\left(\sum_{i=1}^{k_1} \overline{Q}_i(n) + \sum_{i,j} \epsilon_i \bar{t}_{ij}(n), \overline{w}_1(n), \ldots, \overline{w}_{m_2}(n) \right) \tag{18}$$

is uniformly distributed (mod 1) in $\mathbf{R}^{m_2+1}$. Note that all the $w_i(n)$'s are induced inner subpolynomials of $\sum_{i=1}^{k}[q_i(n)]$, and that $\sum_{i=1}^{k}[q_i(n)]$ has independent coefficients except that the outer coefficients equal 1. Proposition 3.4 can be applied to these induced inner subpolynomials, because if we let $\lambda_i \in \mathbf{R}$ so that $R(Q) \cup \{1\}$ is rationally independent, where $Q(n) = \sum_{i=1}^{k} [q_i(n)]\lambda_i$, then $w_i(n) \in U(Q)$ for each i. Hence, $(\overline{w}_1(n), \ldots, \overline{w}_{m_2}(n))$ is uniformly distributed (mod 1). So by Theorem 2.1, (18) is uniformly distributed (mod 1) if for all $a_i \in \mathbf{Z}$, $a_0 \neq 0$,

$$a_0\Big(\sum_i \overline{Q}_i(n) + \sum_{i,j} \epsilon_i \bar{t}_{ij}(n)\Big) + \sum_{i\geq 1} a_i \overline{w}_i(n)$$

is uniformly distributed (mod 1). It is therefore enough to show that for any $a_i \in \mathbf{Z}$, where $a_0 \neq 0$, and for any subpolynomial $u_i(n)$ of $\sum_{j=1}^{k_1} Q_j(n)$, $i = 1, \ldots, k_2$, such that $u_i(n) \neq Q_j(n)$ for all i, j, the sequence

$$a_0 \sum_{i=1}^{k_1} \overline{Q}_i(n) + \sum_{i=1}^{k_2} a_i \overline{u}_i(n) \tag{19}$$

is uniformly distributed (mod 1).

We will prove a more general statement:
Let $A_l(q)$ be the set of $u(n) \in \bigcup_{j=1}^{k_1} U(Q_j)$ such that $\deg(u) \leq \deg(q) - l$ and if $\deg(u) = \deg(q) - l$ then $u(n) \in \bigcup_{j=1}^{k} U'(q_j)$, where $U'(q_j)$ is the subset of $U(q_j)$ which excludes all generalized polynomials having outer coefficient 1. For each $l \in \{0, 1, \ldots, \deg(q) - 2\}$, let F_l be the set of generalized polynomials

$$f(n) = a_0\Big(\sum_{i=1}^{k_1} \overline{V_{h_l} \circ \cdots \circ V_{h_1} Q_i}(n) + \sum_i \overline{V_{h_l} \circ \cdots \circ V_{h_{l_i+1}} t_i}(n)\Big) + \sum_{i\geq 1} a_i \overline{u}_i(n)$$

where $h_i \in \mathbf{N}, i = 1, \ldots, l$, $a_i \in \mathbf{Z}$, $a_0 \neq 0$, $t_i(n)$ is a reduced term of $V_{h_{l_i}} \circ \cdots \circ V_{h_1} Q_j(n)$ for some j, $1 \leq l_i \leq l$, and $u_i(n) \in A_l(q)$. Note that F_0 contains all generalized polynomials of form (19). Observe also that each $f(n) \in F_l$ has degree $\deg(q) - l$, which we will denote by $\deg(F_l)$. We will use induction on $\deg(F_l)$ to show that any $f(n) \in F_l$ is uniformly distributed (mod 1).

First, let $f(n) \in F_l$, where $\deg(F_l) = 2$, i.e., $l = \deg(q) - 2$. By bringing out integer coefficients from the brackets,

$$a_0 \sum_{i=1}^{k_1} \overline{V_{h_l} \circ \cdots \circ V_{h_1} Q_i}(n) + a_0 \sum_i \overline{V_{h_l} \circ \cdots \circ V_{h_{l_i+1}} t_i}(n) \tag{20}$$

can be written as a linear combination, $\sum_{i=1}^{k_2} d_i v_i(n)$, over $\mathbf{Z}$ of distinct simple reduced terms $v_1(n), \ldots, v_{k_2}(n)$, and a sum of indicator functions multiplied by outer subpolynomials of the generalized polynomial (20). When reduced, the outer subpolynomials of (20) are members of $A_l(q)$. The simple reduced terms of the arguments of the indicator functions are also in $A_l(q)$. Hence, $f(n)$ is uniformly distributed (mod 1) if

$$g(n) = c_0 \sum_{i=1}^{k_2} d_i v_i(n) + \sum_i c_i w_i(n)$$

is uniformly distributed (mod 1) for any $w_i(n) \in A_l(q)$ and $c_i \in \mathbf{Z}$, $c_0 \neq 0$. We will use van der Corput's difference theorem to show that $g(n)$ is uniformly distributed (mod 1). By

Lemma 3.2,

$$g^h(n) = g(n+h) - g(n) = c_0 \sum_{i=1}^{k_2} d_i V_h v_i(n) + \sum_{i\geq 1} c_i V_h w_i(n) + \sum_i 1_A(*) s_i(n).$$

Note that if $V_h w_i(n)$ is not a constant, then $R(w_i) \subset R$. Therefore, since the coefficients of the reduced simple terms of the $s_i(n)$'s are independent of h, and the coefficients of the simple reduced terms of the arguments of the indicator functions are in R, it is enough to prove that the coefficient $\sigma(h)$ of $\sum_{i=1}^{k_2} d_i \overline{V_h v_i}(n)$ is rationally independent of R for all but finitely many h. We can write

$$\sigma(h) = h\theta_0 + \sum_{i=1}^{m} [\phi_i(h)]\theta_i,$$

where $\phi_1(n), \dots, \phi_m(n)$ are distinct generalized polynomials of degree 1, and the coefficients of $\overline{\phi_1}(n), \dots, \overline{\phi_m}(n)$ are rationally independent and contained in R. Furthermore, $\theta_i \in R([q_{j_i}]\gamma_{j_i})$ for some j_i such that γ_{j_i} is a factor of θ_i, $i = 1, \dots, m$, where γ_i is the leading coefficient of $p_i(n)$. Since $[q_{i_0}(n)]p_{i_0}(n)$ has a term $a(n)$ with $\deg(a) = \deg(q)$ and so that $\overline{a}(n)$ is not a polynomial, it follows from Lemma 4.3 that γ_{i_0} is factor of some θ_i, say θ_{i_0}. So by Lemma **??**, $\sigma(h)$ is rationally independent of R for all but finitely many h, which was to be proved.

Assume $f(n) \in F_l$ is uniformly distributed (mod 1) if $\deg(F_l) < K$, for some $K \geq 3$, and let $f(n) \in F_l$, where $l = \deg(q) - K$, i.e. $\deg(f) = K$. Let $f^h(n) = f(n+h) - f(n)$. Then by Lemma 3.2,

$$\begin{aligned} f^h(n) &= a_0 \sum_{i=1}^{k_1} V_h \circ \overline{V_{h_l} \circ \cdots \circ V_{h_1} Q_i}(n) + a_0 \sum_i V_h \circ \overline{V_{h_l} \circ \cdots \circ V_{h_{l_i}} t_i}(n) \\ &\quad + \sum_{i\geq 1} a_i e_{ij}(h) u_{ij}(n) + \sum_i b_i(h) s_i(n) + \sum_{i,j} 1_{B_i}(*) d_{ij}(h) t_{ij}(n), \end{aligned}$$

where $b_i(h), d_{ij}(h), e_{ij}(h) \in \mathbf{Z}$, each $u_{ij}(n)$ is a reduced term of $V_h \overline{u}_i(n)$, and $s_i(n), t_{ij}(n)$ are simple, reduced subpolynomials of some $V_{h'_{l'}} \circ \cdots \circ V_{h'_1} \circ \overline{V_{h_l} \circ \cdots \circ V_{h_1} Q_i}(n)$ or $V_{h'_{l'}} \circ \cdots \circ V_{h'_1} \overline{u}_i(n)$. Here we have used that any simple, reduced subpolynomial of $V_{h'_{l'}} \circ \cdots \circ V_{h'_1} \circ \overline{V_{h_l} \circ \cdots \circ V_{h_1} t_i}(n)$ is a simple, reduced subpolynomial of $V_{h'_{l'}} \circ \cdots \circ V_{h'_1} \circ \overline{V_{h_l} \circ \cdots \circ V_{h_1} Q_i}(n)$, by Lemma 3.3. Each simple, reduced generalized polynomial $r_i(n)$ appearing in the argument of an indicator function is a reduced inner subpolynomial of either $V_h \circ \overline{V_{h_l} \circ \cdots \circ V_{h_1} Q_i}(n)$, $u_{ij}(n), s_i(n)$ or $t_{ij}(n)$. Also, all the generalized polynomials involved have degree less than $\deg(f)$ since $f(n) = \overline{f}(n)$. Let $w_1(n), \dots, w_{m_1}(n)$ be all the distinct non-constant terms from the set of all simple, reduced terms of $(V_h \circ \overline{V_{h_l} \circ \cdots \circ V_{h_1} Q_i})^j(n)$ and of $(V_h \circ \overline{V_{h_l} \circ \cdots \circ V_{h_1} t_i})^j(n)$, all j, and the terms $u^j_{i_1 i_2}(n)$, $s^j_i(n)$, $t^j_{i_1 i_2}(n)$, $r_i(n)$ and $r^j_i(n)$. By Proposition 3.4, $(\overline{w}_1(n), \dots, \overline{w}_{m_1}(n))$ is uniformly distributed (mod 1). So by Corollary 2.8(ii) and Theorem 2.1, since $\overline{V_h \overline{q}} = \overline{V_h q}$ for any generalized polynomial $q(n)$, $f^h(n)$ is uniformly distributed (mod 1) if

$$\begin{aligned} & c_0\Big(a_0 \sum_{i=1}^{k_1} \overline{V_h \circ V_{h_l} \circ \cdots \circ V_{h_1} Q_i}(n) + a_0 \sum_i \overline{V_h \circ V_{h_l} \circ \cdots \circ V_{h_{l_i}} t_i}(n) \\ &\quad + \sum_{i,j} a_i e_{ij}(h) \overline{u}_{ij}(n) + \sum_i b_i(h) \overline{s}_i(n) + \sum_{i,j} \epsilon_i d_{ij}(h) \overline{t}_{ij}(n)\Big) + \sum_{i\geq 1} c_i \overline{w}_i(n) \qquad (21) \end{aligned}$$

is uniformly distributed (mod 1) for all $\epsilon_i \in \{0,1\}$ and any $c_i \in \mathbf{Z}$, $c_0 \neq 0$. Now, each $u_{ij}(n)$, $s_i(n)$, $t_{ij}(n)$, $w_i(n)$ is a reduced subpolynomial of some $V_{h'_{l'}} \circ \cdots \circ V_{h'_1} \circ \overline{V_{h_l} \circ \cdots \circ V_{h_1} Q_i}(n)$. By Lemma 3.3, there exist corresponding $\breve{u}_{ij}(n), \breve{s}_i(n), \breve{t}_{ij}(n), \breve{w}_i(n)$ which are reduced subpolynomials of the $V_{h_{l+l'}} \circ \cdots \circ V_{h_1} Q_i(n)$'s such that $\overline{\breve{u}}_{ij}(n) = \overline{u}_{ij}(n)$, $\overline{\breve{s}}_i(n) = \overline{s}_i(n)$, $\overline{\breve{t}}_{ij}(n) = \overline{t}_{ij}(n)$, $\overline{\breve{w}}_i(n) = \overline{w}_i(n)$. Let

$$V = \{\breve{u}_{ij}(n), \breve{s}_i(n), \breve{t}_{ij}(n), \breve{w}_i(n) \mid i, j\}$$

and let V_1 be the set of all generalized polynomials in V which is either an element of $\bigcup_{i=1}^k U(q_i)$ or has degree strictly less than $\deg(f) - 1$, i.e., $V_1 = V \cap A_{l+1}(q)$. Note that all $\breve{u}_{ij}(n), \breve{s}_i(n), \breve{w}_i(n) \in V_1$. Denote by $v_1(n), \ldots, v_{k_2}(n)$ all the distinct elements of V_1.

If $\breve{t}_{ij}(n) \in V \setminus V_1$, then $t_{ij}(n)$ is a reduced subpolynomial of some $V_h \circ \overline{V_{h_l} \circ \cdots \circ V_{h_1} Q_{i_1}}(n)$ and since $\deg(t_{ij}) = \deg(f) - 1$, it follows from Lemma 3.2 that $t_{ij}(n)$ equals a reduced term of $V_h \circ \overline{V_{h_l} \circ \cdots \circ V_{h_1} Q_{i_1}}(n)$ and that $d_{ij}(h) = 1$. Hence we can write (21) as

$$\begin{aligned} g_{d,h}(n) \quad = \quad & a_0 c_0 \Big(\sum_{i=1}^{k_1} \overline{V_h \circ V_{h_l} \circ \cdots \circ V_{h_1} Q_i}(n) + \sum_i \overline{V_h \circ V_{h_l} \circ \cdots \circ V_{h_{l_i}} t_i}(n) \\ & + \sum_{t_{ij} \notin V_1} \epsilon_i \overline{t}_{ij}(n) \Big) + \sum_{i=1}^{k_2} d_i \overline{v}_i(n) \end{aligned}$$

which for each $h \in \mathbf{N}$, each $d = (d_1, \ldots, d_{k_2}) \in \mathbf{Z}^{k_2}$ and any $\epsilon_i \in \{0,1\}$ lies in F_{l+1}, and hence is uniformly distributed (mod 1) by the induction hypothesis. Therefore $f(n)$ is uniformly distributed (mod 1) by the van der Corput's difference theorem.

□

5 Some special results for generalized polynomial with dependent coefficients

The results in the previous sections concern only generalized polynomials having independent inner coefficients. No good method has been found yet to show uniform distribution of generalized polynomials having relations between their coefficients, except to treat each dependence relation separately. In this section we confine ourselves to (sums of) generalized polynomials of degree 2 and we show that many generalized polynomials having dependent coefficients are uniformly distributed (mod 1), but there are also some which are not.

Lemma 5.1 *Let $1, \alpha_1, \ldots, \alpha_k$ be rationally independent and*

$$q(n) = \sum_{i=1}^{k} [\alpha_i n] \gamma_i n + \alpha_0 n^2 + \beta n.$$

Then $q(n)$ is uniformly distributed (mod 1) if either $2\alpha_0 + \sum_{i=1}^{k} \alpha_i \gamma_i$ or some γ_i, is rationally independent of $1, \alpha_1, \ldots, \alpha_k$.

Proof: Since

$$\begin{aligned} q^h(n) & \Big(\sum_{i=1}^{k}[\alpha_i n]\gamma_i h + [\alpha_i h]\gamma_i n + 2\alpha_0 hn + \beta h, \alpha_1 n, \ldots, \alpha_k n\Big) \\ & \Big(\sum_{i=1}^{k}[\alpha_i h]\gamma_i n + (\sum_{i=1}^{k}\alpha_i\gamma_i + 2\alpha_0)hn,\ \alpha_1 n, \ldots, \alpha_k n\Big), \end{aligned}$$

it follows from the van der Corput's difference theorem that $q(n)$ is uniformly distributed (mod 1) if $\sum_{i=1}^{k}[\alpha_i h]\gamma_i + \left(\sum_{i=1}^{k}\alpha_i\gamma_i + 2\alpha_0\right)h$ is rationally independent of $1, \alpha_1, \ldots, \alpha_k$ for all but finitely many h. By Lemma **??**, this is so if either $2\alpha_0 + \sum_{i=1}^{k}\alpha_i\gamma_i$ or some γ_i, is rationally independent of $1, \alpha_1, \ldots, \alpha_k$.

□

The identity

$$[a]b + [b]a = ab + [a][b] - \{a\}\{b\}, \tag{22}$$

where a, b are real numbers, will be used to prove the following.

Proposition 5.2 *Let $1, \alpha_1, \ldots, \alpha_k$ be rationally independent. Then*

$$q(n) = \sum_{i=1}^{k}[\alpha_i n]\beta_i n + \alpha_0 n^2$$

is uniformly distributed (mod 1) if and only if one of the following conditions holds:

(i) *There exists i such that β_i is rationally independent of $1, \alpha_1, \ldots, \alpha_k$.*

(ii) $\beta_i = a_{i0} + \sum_{j=1}^{k} a_{ij}\alpha_j,\ a_{ij} \in \mathbf{Q},\ i = 1, \ldots, k,$ *and there exist i, j such that $a_{ij} \neq a_{ji}$.*

(iii) $\beta_i = a_{i0} + \sum_{j=1}^{k} a_{ij}\alpha_j,\ a_{ij} \in \mathbf{Q},\ i = 1, \ldots, k,\ a_{ij} = a_{ji}$ *for all i, j and* $\sum_{i=1}^{k}\sum_{j=1}^{i-1} a_{ij}\alpha_i\alpha_j + \frac{1}{2}\sum_{i=1}^{k} a_{ii}\alpha_i^2 + \alpha_0 \notin \mathbf{Q}$.

Proof: It follows from Lemma 5.1 that $q(n)$ is uniformly distributed (mod 1) if (i) holds. Suppose that $\beta_i = \sum_{j=1}^{k} a_{ij}\alpha_j,\ i = 1, \ldots, k$. Let $b \in \mathbf{N}$ be such that $ba_{ij} \in \mathbf{Z}$ and $ba_{ii} \in 2\mathbf{Z}$ for all i, j. Then by using equation (22) we have

$$\begin{aligned} bq(n) &= \sum_{i=1}^{k}\sum_{j=1}^{k} ba_{ij}[\alpha_i n]\alpha_j n + b\alpha_0 n^2 \\ &\equiv \sum_{i<j} ba_{ij}[\alpha_i n]\alpha_j n - \sum_{i>j} ba_{ij}[\alpha_j n]\alpha_i n + \sum_{i=1}^{k} ba_{ii}[\alpha_i n]\alpha_i n \end{aligned}$$

$$+\sum_{i>j} ba_{ij}\alpha_i\alpha_j n^2 - \sum_{i>j} ba_{ij}\{\alpha_i n\}\{\alpha_j n\} + b\alpha_0 n^2 \pmod 1$$

$$\equiv \sum_{i<j} b(a_{ij}-a_{ji})[\alpha_i n]\alpha_j n + \sum_{i>j} ba_{ij}\alpha_i\alpha_j n^2 + \frac{1}{2}b\sum_{i=1}^{k} a_{ii}\alpha_i^2 n^2 + b\alpha_0 n^2$$

$$-\sum_{i>j} ba_{ij}\{\alpha_i n\}\{\alpha_j n\} - \frac{1}{2}b\sum_{i=1}^{k} a_{ii}\{\alpha_i n\}^2 \pmod 1. \tag{23}$$

The generalized polynomial $bq(n)$ is therefore uniformly distributed (mod 1) by Lemma 5.1 if there exist i,j such that $a_{ij}\neq a_{ji}$. If that is not the case, then $bq(n)$ behaves like the polynomial $b\Big(\sum\limits_{i>j} a_{ij}\alpha_i\alpha_j + \frac{1}{2}\sum\limits_{i=1}^{k} a_{ii}\alpha_i^2 + \alpha_0\Big)n^2$ which is uniformly distributed (mod 1) if its coefficient is irrational. Note that $b^2q(n)$ has these same properties. In order to show that $q(n)$ is uniformly distributed (mod 1) if (ii) or (iii) is satisfied, write for each n, $n=mb+r$ where $0\le r<b$. By Lemma 2.5 we have

$$[\alpha_i(bm+r)] = b[\alpha_i m] + [\alpha_i r] + 1_A(b\alpha_i m, \alpha_i r) + \sum_{l=0}^{b-1} l 1_{[\frac{l}{b},\frac{l+1}{b})}(\alpha_i m)$$

so that

$$q_r(m) = q(bm+r) = \sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}b^2[\alpha_i m]\alpha_j m + \alpha_0 b^2 m^2 + \phi_r(m) + I_r(m)$$

where $\phi_r(m)$ is a generalized polynomial of degree 1 and $I_r(m)$ is a sum of generalized polynomials multiplied by indicator functions whose arguments are constants and generalized polynomials of degree 1. It follows from Corollary 2.8 that $q_r(m)$ is uniformly distributed (mod 1) if for certain linear polynomials $p_r(m)$, $(b^2q(m)+p_r(m), \alpha_1 m, \dots, \alpha_k m)$ is uniformly distributed (mod 1). Since $(\alpha_1 m,\dots,\alpha_k m)$ is uniformly distributed (mod 1) and we have already shown by using Lemma 5.1 that $b^2q(m)$ is uniformly distributed (mod 1), it follows from the same lemma that $q_r(m)$ is uniformly distributed (mod 1) for each r. Hence, by letting $M=[N/b]$, we have for each Riemann-integrable function $f(x)$ on $[0,1]$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(q(n)) = \lim_{N\to\infty}\sum_{r=0}^{b-1}\frac{[N/b]}{N}\lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M-1} f\Big(q_r(m)\Big) = \sum_{r=0}^{b-1}\frac{1}{b}\int_0^1 f(x)dx = \int_0^1 f(x)dx$$

which shows that $q(n)$ is uniformly distributed (mod 1).

If neither (i), (ii) nor (iii) holds, then $\beta_i = \sum\limits_{j=1}^{k} a_{ij}\alpha_j$, $a_{ij}\in\mathbf{Q}$, $i=1,\dots,k$, $a_{ij}=a_{ji}$ for all i,j and $\sigma = \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{i-1} a_{ij}\alpha_i\alpha_j + \frac{1}{2}\sum\limits_{i=1}^{k} a_{ii}\alpha_i^2 + \alpha_0 \in \mathbf{Q}$. Let $b\in\mathbf{N}$ be such that $ba_{ij}\in\mathbf{Z}$ and $b\sigma\in\mathbf{Z}$. Then it follows from (23) that

$$2bq(n) \equiv -\Big(2\sum_{i>j} ba_{ij}\{\alpha_i n\}\{\alpha_j n\} + \sum_{i=1}^{k} ba_{ii}\{\alpha_i n\}^2\Big) \pmod 1$$

which is not uniformly distributed (mod 1). By Weyl's criterion for uniform distribution (see Theorem 2.1), $2bq(n)$ is uniformly distributed (mod 1) if $q(n)$ is uniformly distributed (mod 1). Hence, $q(n)$ is not uniformly distributed (mod 1).

□

This gives us for example that

$$[\alpha n]\alpha n - \alpha^2 n^2$$

is uniformly distributed (mod 1) iff $\alpha^2 \notin \mathbf{Q}$. Note however, that it follows from Proposition 5.2 that $2[\alpha n]\alpha n - \alpha^2 n^2$ is not uniformly distributed (mod 1) for any α.

By using the identity (22) and Proposition 5.2 the following proposition follows.

Proposition 5.3 *We have the following table for simple generalized polynomials of degree 2:*

Generalized polynomial $q(n)$	$q(n)$ *is uniformly distributed (mod 1) if and only if one of the following condition(s) hold*
γn^2	γ *is irrational*
$[\alpha n]\gamma n$	(i) $\alpha^2 \notin \mathbf{Q}$ *and* γ *is irrational* (ii) $\alpha^2 \in \mathbf{Q}$ *and* γ *is rationally independent of* $1, \alpha$.
$[\alpha n][\beta n]\gamma$	(i) $\frac{\alpha}{\beta} \neq \sqrt{c}$ *for all* $c \in \mathbf{Q}^+$ *and* γ *is irrational* (ii) $\frac{\alpha}{\beta} = \sqrt{c}$ *for some* $c \in \mathbf{Q}^+$ *and* γ *is rationally independent of* $1, \sqrt{c}$.

One could give similar tables for simple generalized polynomials of higher degrees. It turns out that less restrictions on the coefficients are necessary for some of them. For example, we have shown in [2] that if $\gamma, \alpha_1, \ldots, \alpha_k \in \mathbf{R}$, $k \geq 3$, then

$$[\alpha_1 n][\alpha_2 n] \cdots [\alpha_k n]\gamma$$

is uniformly distributed (mod 1) if and only if γ is irrational. The difference between the cases $k \geq 3$ and $k = 2$ (table) should be noticed.

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