Unknown Input Observer for Vehicle Lateral Dynamics Based on a Takagi-Sugeno Model with Unmeasurable Premise Variables

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Abstract—In this paper, an observer is proposed for estimating vehicle states which are not available for measurement. A new representation of a nonlinear vehicle model by exact Takagi-Sugeno structure is proposed based on sector nonlinearity approach. It leads to a multiple model with unmeasurable premise variables which constitutes the first contribution of the present paper. As a second contribution, a state estimation by an Unknown Input Observer (UIO) with unmeasurable premise variables is considered. It takes into account the variations of the longitudinal velocity in order to reflect realistic driving conditions. The convergence conditions of this observer are formulated as an optimization problem subjected to constraints established using Lyapunov theory and expressed under Linear Matrix Inequalities. Finally, some simulations are performed on the nonlinear model of the vehicle including the variations of the longitudinal velocity.

I. INTRODUCTION

Driving assistance and active safety systems have become these last years unavoidable equipments for driver’s safety and comfort enhancement. Integrated into the vehicle, they have proven their efficiency, such as the ESP, ABS, ACC. Their implementation on the vehicle needs some necessary informations to be available and known, as the side slipslip, lateral velocity etc. However, these variables are often not available for measurement, either for technological or economical reasons. The estimation of these unavailable parameters is then a widespread research that had been conducted to overcome the lack of information needed to the vehicle assistance systems.

Systems are often subject to known inputs that represent the system control inputs, and unknown inputs (UI) that can be identified as disturbances, measurement noise, modeling uncertainties etc. These unknown inputs affect the normal behavior of the process and their estimation may be used to design control systems to minimize their effects. A vehicle is a typical case of such systems, because it is subject to unknown inputs that affect its behavior (crosswinds, road curvature, friction coefficient...). Facing the unknown input problem, many studies on unknown input observers are initiated and developed for estimating the state of the system despite the presence of unknown inputs [2]. The idea is to decouple it and make the estimation error insensitive to these last [12], [7], [19]. This kind of approach highlights the simplicity of implementation, but strong structural conditions on the model matrices are required. Furthermore, with noisy measurements, the estimation of the unknown input is always corrupted due to the high sensitivity of the time derivative of the estimated states to measurement noises.

Observers that estimate the unknown input at the same time as the state system have been proposed thereafter [20]. This category of observer is no longer trying to mask the effect of the unknown inputs, but rather to refine the state estimation using an integral action. This observer is the Proportional-Integral (PI) that provides a simultaneous estimation of the state system and its unknown inputs. Such observers provide a good estimation of the unknown inputs, even in the presence of noisy measurements, [6].

Many works based on PI observers have been undertaken in the linear case, especially for the SISO case, [20], [15] and then extended to the MIMO one, [13]. The assumption of constant unknown inputs is required for the theoretical part of the proof concerning the convergence of the estimation error to zero, using the second Lyapunov method. This limits the class of signals that can be estimated by the PI. However, in practice it is quite possible to estimate the signals with slow dynamics, by increasing the observer gain, [14]. The work cited above, concern mainly the design of unknown input observer PI for linear time invariant systems. Many of them have been extended to the nonlinear case, but only for some restricted classes, where many physical systems can’t be represented [17], [5] and [9].

The Takagi-Sugeno (T-S) modeling framework provides a useful tool for representing with good precision a large class of nonlinear systems. Furthermore, by using the sector transformation, the nonlinear model can be exactly represented in a T-S form, that is a convex sum of linear models corresponding to different regions of the state space, via weighting functions. The main advantage of T-S structure is its simplicity that comes from the interpolation of linear systems. Thus, analysis and design methods developed for linear systems can be easily generalized to nonlinear systems.

In recent years, the topic of state estimation for T-S models has been widely investigated (see [18] and references therein). However, most of the work undertaken concerns the case of measurable variables ([2], [16]), which is not always the case, particularly when using the sector transformation that leads inevitably to a T-S model with unmeasurable premise variables. As a matter of fact, the estimation under unmeasurable premise variables is harder to address than the one with measurable variables, but the first formulation gives a more precise and realistic representation and may include a larger class of nonlinear systems [4], [21], [11].
In the present paper, we propose to develop an observer that estimates accurately the lateral dynamics (sideslip angle and lateral velocity for which a sensor would be too expensive for car industry) of a vehicle and the unknown input (road curvature) by using a lateral vehicle model related to the road. The model is described by a quasi-exact T-S structure, obtained by sector nonlinearity transformation that should be able to describe the whole state space, including the linear and the saturated behavior of the lateral tire-road forces occurring for higher tire sideslip angles. The considered premise variables depend on the state of the system which are not available and thus lead to a T-S model with unmeasurable premise variables. The convergence conditions of the estimation error are given under LMI formulation, obtained via the Lyapunov theory. Most of the work dedicated to estimate vehicle lateral dynamics supposes constant longitudinal velocity which is not very realistic. We propose to extend the PI observer for the vehicle nonlinear system described by T-S model.

The paper starts with a short description of the nonlinear vehicle lateral dynamics related to the road. A T-S model is then derived from sector nonlinearity transformation which constitutes a first contribution of the proposed work since it allows describing exactly the nonlinear behavior. Using the sector nonlinearity approach to obtain the T-S model, leads to a T-S model with state dependent weighting functions which are not measurable, and is more difficult to study compared to the case where all the premise variables are known (measured). This is the second distinctive theoretical point of the proposed work. Finally, simulations are performed, to prove the efficiency of the proposed T-S observer.

II. VEHICLE LATERAL DYNAMICS MODEL AND POSITIONING RELATIVE TO ROAD

This section is dedicated to the presentation of the vehicle model used for the observer synthesis. First, a nonlinear yaw-drift model is considered. Then, the positioning of the vehicle according to the road section is established. The overall model including the lateral dynamics and the position relative to the road is then given as a four state model with the road curvature as an unknown input.

A. Lateral-drift model

Most of the work that has dealt with lateral dynamics of the vehicle uses a linear models. They are based on simplification assumptions, which limit the evolution domain to the linear region only, see [8]. Instead, we propose to use a nonlinear yaw-drift model to overcome this limitation. From [8] and [1], the lateral model is described by

$$
\begin{align*}
\dot{\beta} &= \frac{m}{I_z} (F_f + F_r - mv\dot{\psi}) \\
I_z \dot{\psi} &= a_f F_f - a_r F_r
\end{align*}
$$

Where $\beta$ is the sideslip angle, $\psi$ is the yaw rate, $v$ is the longitudinal velocity, $m$ is the mass of the vehicle, $I_z$ is the yaw moment of inertia, $a_f$ and $a_r$ are respectively the distances from the front and rear axle to the center of gravity, $F_f$ and $F_r$ are the lateral front and rear forces respectively, expressed by the Pacejka’s magic formula [3]

$$
F_i = D_i \sin \left( C_i \tan^{-1} \left( (B_i (1 - E_i) \alpha_i + E_i \tan^{-1} (B_i \alpha_i)) \right) \right)
$$

where $i \in \{f, r\}$, $D_i$, $B_i$, $E_i$, and $C_i$ are parameters that depend on the tire characteristics. $\alpha_f$ and $\alpha_r$ are the front and rear sideslip angles of the tires given by:

$$
\begin{align*}
\alpha_f &= \delta_f - \beta - \tan^{-1} \left( \frac{a_f}{\psi} \cos(\beta) \right) \\
\alpha_r &= -\beta + \tan^{-1} \left( \frac{a_r}{\psi} \cos(\beta) \right)
\end{align*}
$$

III. T-S FORMULATION OF THE LATERAL-DRIFT MODEL RELATED TO THE ROAD

In this section, we aim to re-write the nonlinear model expressed in (6) in a T-S formulation without simplification, using the sector nonlinearity approach [18]. First, let us re-write the expressions of Pacejka’s forces, since the model nonlinearities are located there. The general form is

$$
F = D \sin(C \tan^{-1}((B(1 - E)\alpha + E \tan^{-1} (B \alpha))))
$$

with simple mathematical manipulations, it is easy to write

$$
F = f(\alpha) \alpha
$$

where

$$
f(\alpha) = A \frac{\sin(S_3) \tan^{-1}(S_2)}{S_3} \frac{\tan^{-1}(S_1)}{S_2} + B \frac{\sin(S_3) \tan^{-1}(S_2) \tan^{-1}(S_1)}{S_3} \frac{\tan^{-1}(S_1)}{S_2} \frac{\tan^{-1}(S_1)}{S_1}
$$

It is known that $\frac{\sin(x)}{x}$ is defined on $\mathbb{R}$ and when $x \to 0 \frac{\sin(x)}{x} \to 1$. It is the case also for the function $\tan^{-1}(x)$. It

$$
S_1 = B \alpha, \quad S_2 = B(1 - E)\alpha + E \tan^{-1} (S_1) \quad S_3 = C \tan^{-1}(S_1), \quad A = BCD(1 - E), \quad B = BCD,$$

$$
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$$
is obvious that \( \lim_{x \to 0} \left( \frac{\tan^{-1}(x)}{x} \right) = 1 \). Knowing that \( \alpha \) is a bounded angle, the function \( \frac{\tan^{-1}(x)}{x} \) is also bounded. The function \( f(\alpha) \) is bounded \( \forall \alpha: f_{\min} \leq f(\alpha) \leq f_{\max} \). Let us define the \( f(\alpha) \) as a premise variable, it follows

\[
\mu_1(\alpha) = \frac{f(\alpha) - f_{\min}}{f_{\max} - f_{\min}}, \quad \mu_2(\alpha) = \frac{f_{\max} - f(\alpha)}{f_{\max} - f_{\min}} \tag{10}
\]

Thus, the exact T-S model is given by \( F = \sum_{i=1}^{2} \mu_i(\alpha)M_i \), where the parameters \( M_i, i = 1,2 \) are defined by \( M_1 = f_{\max} \) and \( M_2 = f_{\min} \). Next, both \( F_f \) and \( F_r \) will be expressed in exact T-S formulations. Rewriting the Pacejka’s lateral forces under the T-S formulation using the above formulas, the new expressions of these forces are as follows:

\[
\begin{align*}
F_f &= \sum_{i=1}^{2} \mu_{fi} \left( \begin{array}{ccc}
-M_{fi} & -\frac{a_{fi}}{v} M_{fi} & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) x + M_{fi} \delta_f \\
F_r &= \sum_{j=1}^{2} \mu_{rj} \left( \begin{array}{ccc}
-M_{rj} & \frac{a_{rj}}{v} M_{rj} & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) x
\end{align*}
\]

where \( x^T(t) = [\beta \ \dot{\psi} \ \psi_L \ y_L]^T \) is the state vector, the functions \( \mu_{fi} \) and \( \mu_{rj} \) satisfy the convex sum property i.e. \( \sum_{i=1}^{2} \mu_{fi} = 1, \sum_{j=1}^{2} \mu_{rj} \leq 1 \). Using (11) in the vehicle lateral-drift model related to the road, its dynamics can be expressed in a T-S formulation

\[
\dot{x}(t) = \sum_{i=1}^{4} h_i(x(t)) (A_i x(t) + B_i u(t)) \tag{12}
\]

where \( 0 \leq h_i \leq 1, i = 1,4 \), and \( \sum_{i=1}^{4} h_i = 1 \). The matrices of the model are defined by \( A_1 = \hat{A}_{11}, A_2 = A_{12}, A_3 = A_{21} \) and \( A_4 = A_{22} \) where

\[
A_{ij} = \begin{pmatrix}
a_{ij1} & a_{ij2} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & v & 1 & 0 \\
v & 0 & 0 & 0
\end{pmatrix}, \quad i,j = 1,2
\]

\[
a_{11} = \frac{\delta}{v} (M_f + M_r), \quad a_{12} = \frac{\delta}{v} (\frac{M_f}{v} M_f - \frac{a_{rj}}{v} M_r + 1) \\
a_{21} = \frac{\delta}{v} (\frac{a_{fi}}{v} M_f + a_{fi} M_r), \quad a_{22} = \frac{\delta}{v} (M_f - \frac{a_{fi}}{v} M_r)
\]

\[
B_1 = \begin{pmatrix}
M_f \\
0 \\
0 \\
0
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
\frac{M_r}{v} \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[
B_3 = B_1, \quad B_4 = B_2
\]

And the weighting functions \( h_i \) are defined by:

\[
h_1 = \mu_{f1} \times \mu_{r1}, h_2 = \mu_{f2} \times \mu_{r1}, h_3 = \mu_{f1} \times \mu_{r2}, h_4 = \mu_{f2} \times \mu_{r2}
\]

The input of the system is defined by \( u^T(t) = [\delta_f \ \rho]^T \) and the output \( y(t) = C^T x(t) \) is expressed with the observation matrix constructed as follows:

\[
C = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

The global T-S model of the vehicle considered in this work is then given in (13), where the weighting function depend on the system state variables which are not totally measurable, leading to a T-S model with unmeasurable premise variables:

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{4} h_i(x(t)) (A_i x(t) + B_i u(t)) \\
y(t) &= C x(t)
\end{align*} \tag{13}
\]

Next, the obtained T-S model of the vehicle is used to design the PI observer to estimate the road curvature and the sideslip angle needed to evaluate the lateral velocity.

IV. OBSERVER SYNTHESIS

An observer estimating the state \( x(t) \) of the system and the unknown input \( \rho(t) \) (road curvature) is proposed in this section. For that purpose, let us consider the T-S model

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{4} h_i(x(t)) (A_i x(t) + B_i u(t)) + B_{\rho} \rho(t) \\
y(t) &= C x(t)
\end{align*} \tag{14}
\]

where \( B_{\rho} \) are matrices containing respectively the first columns of the matrices \( B_i \) and the matrix \( B_\rho \) contains the second columns of \( B_i \) which are the same for all \( i \). Considering the model (14), the proposed observer is given by the equations (15), under the following assumptions [10]

**Assumption 1. Assume that**

- A1. The state \( x(t) \) of the system (14) is bounded.
- A2. \( \delta_f(t) \) and \( \rho(t) \) are bounded.
- A3. \( \dot{\rho}(t) = 0 \).

These assumptions are not always restrictive. Furthermore, the last one is an assumption for the theory setting part that can be easily overstepped by increasing the observer gain.

The structure of the observer is then given by

\[
\begin{align*}
\dot{\hat{x}}(t) &= \sum_{i=1}^{4} h_i(\hat{x}(t)) \left( A_i \hat{x}(t) + B_i \delta_f(t) + B_\rho \rho(t) \right) + K_P (y(t) - \hat{y}(t)) \\
\dot{\hat{\rho}}(t) &= \sum_{i=1}^{4} h_i(\hat{x}(t)) K_F (y(t) - \hat{y}(t)) \\
\hat{y}(t) &= C \hat{x}(t)
\end{align*} \tag{15}
\]

\( \dot{x}(t) \) and \( \dot{\rho} \) are the estimates of \( x(t) \) and \( \rho(t) \). This observer takes the form of a proportional integral observer (PI) (see [10]). The equation \( \dot{\hat{\rho}} \) allows to estimate the unknown input (road curvature).

So by comparing the observer (15) with the model (14), this last can be re-written as a disturbed system, where the weighting functions depend on the estimate state, as follows

\[
\dot{x}(t) = \sum_{i=1}^{4} h_i(\hat{x}(t)) (A_i x(t) + B_i \delta_f(t) + B_\rho \rho(t) + \Delta(t)) \tag{16}
\]

where

\[
\begin{align*}
\Delta(t) &= \sum_{i=1}^{4} (h_i(x(t)) - h_i(\hat{x}(t))) (A_i x(t) + B_i \delta_f(t) + B_\rho \rho(t)) \\
&\quad + B_{\rho} \rho(t)
\end{align*} \tag{17}
\]
Since \( B_\rho \) is a constant matrix, and knowing that \( \sum_{i=1}^{4} (h_i(x(t)) - h_i(\hat{x}(t))) = 0 \), the term \( \Lambda(t) \) is reduced to

\[
\Lambda(t) = \sum_{i=1}^{4} (h_i(x(t)) - h_i(\hat{x}(t))) (A_i x(t) + B_{ui} \delta_f(t))
\]  (18)

This last term is seen as a bounded vanishing disturbance to be minimized, according to the two first assumptions cited above and the definition of the weighting functions. Besides, we have \( \Lambda(t) \to 0 \) if \( \epsilon(t) \to 0 \) where \( \epsilon(t) = x(t) - \hat{x}(t) \). Knowing that \( \sum_{i=1}^{4} (h_i(x(t)) - h_i(\hat{x}(t))) = 0 \) and some elements of the matrices \( A_i \) and \( B_{ui} \) are constant or zero, in addition, the first and the second components of the state vector \( x(t) \) are bounded. It follows that the disturbance term \( \Lambda(t) \) is bounded. Let us consider the augmented state vector \( x^e_a(t) = [x^e(t) \; \rho(t)]^T \), the augmented system is

\[
\begin{align*}
\dot{x}_a(t) &= \sum_{i=1}^{4} h_i(\hat{x}(t)) \left( \hat{A}_i x_a(t) + B_{ui} \delta_f + \Gamma \Lambda(t) \right) \\
y(t) &= \hat{C} x_a(t)
\end{align*}
\]  (19)

where

\[
\hat{A}_i = \begin{pmatrix} A_i & B_\rho \\ 0 & 0 \end{pmatrix}, \quad \hat{B}_{ui} = \begin{pmatrix} B_{ui} \\ 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} I \\ 0 \end{pmatrix}
\]

\( \hat{C} = \begin{pmatrix} C \\ 0 \end{pmatrix} \)

The same manipulation is performed on the observer (15)

\[
\begin{align*}
\dot{\hat{x}}_a(t) &= \sum_{i=1}^{4} h_i(\hat{x}(t)) \left( \hat{A}_i \hat{x}_a(t) + \hat{B}_{ui} \delta_f + K_i(y(t) - \hat{y}(t)) \right) \\
\hat{y}(t) &= \hat{C} \hat{x}_a(t)
\end{align*}
\]  (20)

where

\[
K_i = \begin{pmatrix} K_{Pi} \\ K_{Ii} \end{pmatrix}
\]  (21)

For the augmented system, the state estimation error \( e_a(t) = x_a(t) - \hat{x}_a(t) \) obeys to the differential equation

\[
e_a(t) = \sum_{i=1}^{4} h_i(\hat{x}(t)) \left( \left( \hat{A}_i - K_i \hat{C} \right) e_a(t) + \Gamma \Lambda(t) \right)
\]  (22)

The convergence of the state estimation error towards a small region around the origin and the boundedness of this error is ensured if the conditions in the theorem 1 hold.

**Theorem 1.** Under the Assumption 1 and for a given parameter \( \sigma \in [0, 1] \), if there exists a symmetric and positive definite matrix \( P \), gain matrices \( \mathcal{K}_i \) and positive scalars \( \gamma \) and \( \alpha \) solutions to the following optimization problem, \( i = 1, \ldots, 4 \)

\[
\min_{P, \mathcal{K}_i, \gamma, \alpha} \left( \sigma \alpha + (1 - \sigma) \gamma \right)
\]

\[
\left( \begin{array}{cc}
A^T P + P A_i - K_i \hat{C} C^T K_i^T & \gamma I \\
-\frac{\gamma I}{P} & \frac{1}{\gamma I} \end{array} \right) < 0
\]

and

\[
\left( \begin{array}{cc}
-\frac{\gamma I}{P} & \frac{1}{\gamma I} \\
0 & 0
\end{array} \right) \\
P \geq I
\]

then the error dynamics (22) is ISS with respect to \( \Lambda(t) \) and \( e_a(t) \) satisfies the following inequality

\[
\| e_a(t) \| \leq \sqrt{\frac{\gamma \alpha}{\Lambda_{\max}(P)}} \left( e^{-\gamma \alpha t} \| e_a(0) \| + \gamma \Lambda_{\infty} \right)
\]  (26)

The gains of the observer are computed by \( \mathcal{M}_i = P^{-1} \mathcal{K}_i \). The attenuation level of the transfer from \( \Lambda(t) \) to the estimation error \( e_a(t) \) is \( \gamma = \sqrt{\alpha} \) and the \( \Lambda_{\max}(P) \leq \alpha \) where \( \alpha = \sqrt{\gamma} \). Hence, the convergence set is given by the quantity

\[
\sqrt{\frac{\Lambda_{\max}(P)}{\Lambda_{\min}(P)}} \gamma \Lambda_{\infty}
\]

where \( \Lambda_{\infty} \) is the upper bound of the norm \( \| \Lambda(t) \| \).

**Proof:** Assume that the LMIs (23) are feasible. Let us consider the vector \( \xi(t) = \left[ e_a^T(t) \; A^T(t) \right] \) T. Pre- and post multiplying (23) by \( \xi(t) \) and \( \xi^T(t) \) respectively and with \( \gamma = \sqrt{\alpha} \) and \( \Phi_i = A_i - K_i \hat{C} \), the following is obtained

\[
e_a^T(t) \left( \Phi_i^T P + P \Phi_i \right) e_a(t) + e_a^T(t) P \Lambda(t) + \Lambda^T(t) P e_a(t) + e_a^T(t) e_a(t) - \gamma^2 \Lambda^T(t) \Lambda(t) < 0
\]

Multiplying (27) by \( \sum_{i=1}^{4} \mu_i(\hat{x}(t)) \), one obtains

\[
\sum_{i=1}^{4} \mu_i(\hat{x}(t)) \left( \Phi_i^T P + P \Phi_i \right) e_a(t) + e_a^T(t) P \Lambda(t) + \Lambda^T(t) P e_a(t) + e_a^T(t) e_a(t) - \gamma^2 \Lambda^T(t) \Lambda(t) < 0
\]

where

\[
V(t) = e_a^T(t) P e_a(t), \quad P = P^T > 0
\]

From (30), it obviously follows that

\[
\Lambda_{\min}(P) \| e_a(t) \|^2 \leq V(t) \leq \Lambda_{\max}(P) \| e_a(t) \|^2
\]

Consequently, if (23) holds, the time derivative of \( V(t) \) is bounded as follows

\[
\dot{V}(t) \leq - \frac{1}{\Lambda_{\max}(P)} V(t) + \gamma^2 \| \Lambda(t) \|^2
\]

and thus

\[
V(t) \leq V(0) e^{-\gamma \alpha t} + \gamma^2 \int_0^t e^{-\gamma \alpha s} \| \Lambda(s) \|^2 ds
\]

Defining \( \Lambda_{\infty} \) the upper bound of the Euclidean norm of \( \Lambda(t) \) (i.e. \( \| \Lambda(t) \| \leq \Lambda_{\infty} \), \( t \)), it follows

\[
V(t) \leq V(0) e^{-\gamma \alpha t} + \gamma^2 \Lambda_{\infty}^2 \Lambda_{\max}(P)
\]

Finally, using (31) with the square root, one obtains the inequality (26). From this inequality, we conclude that if \( \Lambda(t) \equiv 0 \) then \( e_a(t) \to 0 \) when \( t \to \infty \). Moreover, in the presence of the perturbation \( \Lambda(t) \), the error \( \| e_a(t) \| \) is bounded by \( \sqrt{\Lambda_{\max}(P)} \gamma \Lambda_{\infty} \) when \( t \to \infty \). The inequality (26) establishes the ISS of (22).

Note that the size of the convergence set depends on the selected matrix \( P \) and the parameter \( \gamma \). This set should be made as small as possible to ensure a good accuracy of convergence. The choice of \( \gamma \) and \( P \) providing a small set of convergence is not obvious because the problem is not convex. In the next, a technique is proposed to transform the non convex problem to a convex one under LMI constraints.

Let us consider the following inequality

\[
\sqrt{\frac{\Lambda_{\max}(P)}{\Lambda_{\min}(P)}} \leq \sqrt{\alpha}
\]  (35)
where $\alpha$ is a positive scalar to minimize. Since $\lambda_{\text{max}}(P) > \lambda_{\text{min}}(P)$, the minimal value of $\sqrt{\frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)}}$ which can be obtained is equal to 1. From this, one can impose $P \succeq I$ which leads to $\lambda_{\text{min}}(P) \geq 1$. It follows

$$\lambda_{\text{max}}(P) \leq \alpha \iff P^T P - \alpha^2 I \leq 0 \quad (36)$$

Using Schur’s complement lemma

$$\begin{pmatrix} -\alpha^2 I & P \\ P & -I \end{pmatrix} \leq 0 \quad (37)$$

defining $\alpha = \alpha^2$ and considering the objective function

$$\min \sigma \hat{\alpha} + (1 - \sigma) \hat{\gamma} \quad (38)$$

where $\sigma \in [0, 1]$, the optimization problem (38) under LMI constraints (23)-(25) is then obtained which ends the proof.

In the next section, simulations are performed to show the effectiveness of the precision of the T-S model developed in this present work. Afterward, simulations are carried out to estimate the unmeasurable states, the unknown input and the lateral velocity by $v_{\text{lat}} = \beta \times v$, where $v$ is the longitudinal velocity, considered to be varying.

V. SIMULATION RESULTS

The developed T-S model is compared to the nonlinear one. Figure 1 shows the obtained simulation results. The plots are perfectly identical, which reflects the high accuracy by the weighting functions.

The estimated states and unknown input in this case are those of the real states. This is due to the cost function minimization and the integral action that allows a good estimation of the states and UI.

![Fig. 2. Evolution of the bound $\sqrt{\frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)}} \gamma$ with respect to $\sigma$](image)

![Fig. 3. State estimation with the proposed PI observer](image)

![Fig. 4. Lateral velocity and its estimate (top) Road curvature and its estimate (bottom)](image)

A. Time-varying longitudinal velocity

As mentioned before, the PI observer is constructed assuming a constant longitudinal velocity $v = 30\text{m/s}$ as a nominal value. Now the observer is tested for the nonlinear model of the vehicle with time-varying longitudinal velocity. The estimation of the lateral velocity is obtained by $v_{\text{lat}} = \beta \times v$. The longitudinal velocity $v$ is considered time-varying $v = 30 + 5\sin(0.5t)$, then $v$ varied in the range $[25, 35]\text{m/s}$. The estimated states and unknown input in this case are illustrated in figure 5. The estimated lateral velocity with
time-varying longitudinal velocity and the road curvature estimation are displayed in figure 6. One can conclude that the observer provides satisfactory estimations.

![Fig. 5. State estimation in time-varying longitudinal velocity case](image)

![Fig. 6. Lateral velocity estimation (top) curvature estimation (bottom)](image)

VI. CONCLUSION

In this paper, an observer for estimating the lateral velocity of the vehicle and the road curvature is proposed. The nonlinear model is first transformed into a T-S model using the sector nonlinearity approach. The weighting functions of the obtained model depend on unmeasurable premise variables (state variables). The PI observer is constructed using a nominal value of the road curvature and applied directly to the nonlinear one. The convergence conditions of the observer are established by using Lyapunov theory. The weighting functions of the vehicle and the road curvature is proposed. The state variables, unknown inputs and lateral velocity with their obtained model depend on unmeasurable premise variables: an uncertain system approach. Fuzzy Sets and Systems, 160(12):1738–1748, 2009.

The presented PI observer, allows to estimate simultaneously the states and the unknown input. However, the limitation known to the PI is the condition omitted on the UI that should be constant. In practice, the PI observer is able to estimate slowly time-varying UI. In future works, it would be interesting to extend the present work to a Proportional Multi Integral observer to overcome the limits of the PI for the sake of developing efficient driving assistance. Furthermore, it would be interesting to consider the nonlinearities in the slide angles. Finally, in future papers, experimental results will be provided.

REFERENCES