The Reasoning Mechanism of Fuzzy Cognitive Maps

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Abstract: Fuzzy Cognitive Maps (FCMs) can represent and reason causal knowledge with stronger semantics. And the causal knowledge widely exists in Knowledge Grid. To provide information services with stronger semantics in Knowledge Grid, we need to know the reasoning mechanism and the characteristics of FCMs. In this paper, we have proved that the reasoning process of FCMs is a discrete topological semi-dynamic system. This theory can help us analyze the reasoning process and find the new characteristics of FCMs, which can guide us using FCMs to provide intelligent information services flexibly in Knowledge Grid.

1 Introduction

It is difficult to represent and reason causal knowledge with stronger semantics in Knowledge Grid. Fuzzy Cognitive Maps (FCMs) are the combination of fuzzy logic and neural networks, which are widely used to represent and reason casual knowledge with stronger semantics [1] [2]. In Knowledge Grid, the casual knowledge can be represented and reasoned by FCMs. For using FCMs to provide information services with stronger semantics in Knowledge Grid, we need to know the reasoning mechanism and the characteristics of FCMs.

FCMs were proposed by Kosko more than twenty years ago [2]. Many extended FCMs were put forward, such as contextual fuzzy cognitive maps [3], rule based fuzzy cognitive maps [4], and dynamic cognitive networks [5] and so on. Fuzzy cognitive maps as well as their extended kinds emphasize on the reasoning process of causality. But the reasoning mathematical mechanism of fuzzy cognitive maps is...
unknown [6]. We only know the reasoning results may be achieved on “fixed point”, “limited cycle” or “chaos” [6]. Y. Miao et al. [5] have proved that the problem of finding whether a state is reachable in the FCM is nondeterministic polynomial (NP) hard. It is difficult to find new characteristics of FCMs because there is no theory to guide or help researchers to analyze the reasoning process of FCMs. In this paper, we mainly discuss the reasoning mathematical mechanism of FCMs.

FCMs have been widely used in economics, fault analysis [7], information systems [3] [8], tacit knowledge management [9], industry control [10] [11], virtual world [6] [12] and electronic commerce [13] and so on.

2 Reasoning Mechanism of Fuzzy Cognitive Maps

2.1 Reasoning process of fuzzy cognitive maps

The reasoning process of fuzzy cognitive maps mainly uses the adjacency matrix $W$ and the set of concepts’ state values $V(t)$. For instance, a FCM has eight concepts that initial state values are represented by $V(0) = \{v_1(0), v_2(0), ..., v_8(0)\}$. The adjacency matrix of the FCM is shown as follows:

$$W = \begin{pmatrix}
0 & w_{21} & w_{31} & w_{41} & w_{51} & w_{61} & w_{71} & w_{81} \\
0 & 0 & w_{32} & \cdots & \cdots & \cdots & \cdots & w_{82} \\
w_{13} & 0 & w_{23} & \cdots & \cdots & \cdots & \cdots & w_{83} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
w_{16} & \cdots & \cdots & 0 & w_{76} & w_{86} \\
w_{17} & \cdots & \cdots & \cdots & 0 & w_{77} \\
w_{18} & w_{28} & w_{38} & w_{48} & w_{58} & w_{68} & w_{78} & 0
\end{pmatrix}$$

where $w_{ij}$ is the weight of the arc which connects from $C_i$ to $C_j$. Then the reasoning process of the FCM as follows:

$$(v_1(t+1), v_2(t+1), ..., v_8(t+1)) = (v_1(t), v_2(t), ..., v_8(t)) * W .$$

$$(v_1(t+1), v_2(t+1), ..., v_8(t+1)) = (f(v_1(t+1)), f(v_2(t+1)), ..., f(v_8(t+1)))$$

where $f(x)$ is the threshold function of FCM’s concepts, usually it is a sigmoid function. Each concept of the FCM’s reasoning can be represented as:

$$v_1(t+1) = g(v_1(t+1)w_{12} + v_2(t+1)w_{13} + v_3(t+1)w_{14} + ..., v_8(t+1)w_{18}) .$$

For instance, in FCM1 (Fig.1), the state values are in the interval $[0, 1]$, the
weights between concepts are in the interval \([-1, 1]\), and \(W_1\) is the adjacency matrix of FCM1. FCM1 will create a path in the seven dimensions state space of causality when the state values of FCM1’s concepts are changed [6]. If the concepts’ initial state values of FCM1 are \([0.2, 0.3, 0.4, 0.7, 0.6, 0.8, 0.75]\), then the reasoning process of FCM1 is shown in Tab.1.

![Fig.1](image)

**Fig.1** The Fuzzy Cognitive Map in [14] denoted as FCM1

\[
W_1 = \begin{bmatrix}
0 & 0 & 0.6 & 0.9 & 0 & 0 & 0 \\
0.1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.7 & 0 & 0.9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.9 \\
-0.3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.8 & 0 \\
\end{bmatrix}
\]

**Tab.1** the reasoning results of FCM1

<table>
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<tr>
<th>t=0</th>
<th>V_1</th>
<th>V_2</th>
<th>V_3</th>
<th>V_4</th>
<th>V_5</th>
<th>V_6</th>
<th>V_7</th>
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<td>0.4000</td>
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<td>0.6000</td>
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<td>0.5222</td>
<td>0.5332</td>
<td>0.5352</td>
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<td>0.5715</td>
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<td>0.4982</td>
<td>0.5716</td>
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<td>0.5274</td>
<td>0.5222</td>
<td>0.5332</td>
<td>0.5352</td>
<td>0.4982</td>
<td>0.5716</td>
<td></td>
</tr>
</tbody>
</table>
2.2 Reasoning mechanism of fuzzy cognitive maps

Topological dynamics has the following definition:

If X is compact metric space, and exists the continuous mapping sequence in X:

\[ \{ p^0, p^1, p^2, \ldots, p^n \} \]

then \( \{p^0, p^1, p^2, \ldots, p^n\} \) is called “the discrete topological semi-dynamics system generated by iteration of continuous mapping p in X”.

**Proposition:** The reasoning process of fuzzy cognitive maps is a discrete topological semi-dynamics system.

**Proof:** If we want to prove the reasoning process of fuzzy cognitive maps is a discrete topological semi-dynamics system, we only need to prove that the reasoning process of fuzzy cognitive maps accords with the following conditions:

- The metric space of all the concepts’ state values of fuzzy cognitive maps is a compact metric space.
- Reasoning process of fuzzy cognitive maps is a continuous mapping sequence.

(1) The metric space of all the concepts’ state values of fuzzy cognitive maps is a compact metric space.

According to the property of fuzzy cognitive maps, the concepts’ state values of fuzzy cognitive maps are in the interval \([0, 1]\), so all the concepts’ state values belong to real space that have bourn.

According to the Heine-Borel theorem, we know that \([0, 1]\) is compact. Suppose the number of a fuzzy cognitive map’s concepts is \(n\), and the metric space \(X=\{0,1\}^n\) of all the concepts’ state values of FCM can be regarded as the \(n\) time product of \([0, 1]\), and then according to the property of multiplicative space, we know that \(X\) is a compact metric space, herein we regard \(X\) as an induced subspace of \(R^n\). Therefore, the metric space of all the concepts’ state values of fuzzy cognitive map is a compact metric space.

(2) Reasoning process of FCMs is a continuous mapping sequence.

The state values of FCM’s concepts at time \(t\) are \(V(t) = (v_1(t), v_2(t), \ldots, v_n(t))\), the reasoning process of FCMs can be regarded as following:

\[ V(t+1) = F(V(t)) = G(V(t) \times W), \]

where \(W\) is adjacency matrix, and \(G\) is a mapping: \(R^n \rightarrow R^n\); \(G(x_1, x_2, \ldots, x_n) = (f(x_1), f(x_2), \ldots, f(x_n))\), where \(f\) is a continuous function on the real domain \(R\), usually
\[ f(x) = \frac{1}{1 + e^{-x}}. \]

Then it will be shown that \( F \) is continuous on \([0, 1]^n\). To do this, we should first give a norm \( v \) on \( R^n \), here we give \( v \) as: \( v(X) = \|X\| = \max_i |x_i| \), where \( X = (x_1, x_2, \ldots, x_n) \).

To show that \( F \) is continuous at \( X \in [0, 1]^n \) \(, \) we must show that \( \forall \epsilon > 0, \exists \delta > 0, \forall Y \in [0, 1]^n \) and satisfying \( v(X - Y) < \delta \), then \( v(F(X) - F(Y)) < \epsilon \).

\[
F(X) - F(Y) = G(X \times W) - G(Y \times W)
\]

\[
= G(X \times W_1, X \times W_2, \ldots, X \times W_n) - G(Y \times W_1, Y \times W_2, \ldots, Y \times W_n)
\]

\[
= (f(X \times W_1), f(X \times W_2), \ldots, f(X \times W_n)) - (f(Y \times W_1), f(Y \times W_2), \ldots, f(Y \times W_n))
\]

\[
= (f(X \times W_1) - f(Y \times W_1), f(X \times W_2) - f(Y \times W_2), \ldots, f(X \times W_n) - f(Y \times W_n)),
\]

where \( W = (W_1, W_2, \ldots, W_n) \).

\[
v(F(X) - F(Y)) = \|F(X) - F(Y)\|_v.
\]

\[
= \max_j |(F(X) - F(Y))_j|
\]

\[
= |f(X \times W_j) - f(Y \times W_j)|, \text{for some } j, 1 \leq j \leq n.
\]

Because \( f(x) \) is a continuous function on \( R \), then \( \exists \delta_1 > 0, \forall Y \in [0, 1]^n \) and satisfying \( |X \times W_j - Y \times W_j| < \delta_1 \), then \( f(X \times W_j) - f(Y \times W_j) < \epsilon \), that is \( v(F(X) - F(Y)) < \epsilon \).

\[
|X \times W_j - Y \times W_j| = \|(X - Y) \times W\|
\]

\[
= |\sum_{k=1}^{n} (x_k - y_k) \times W_{jk}|
\]

\[
\leq \sum_{k=1}^{n} |(x_k - y_k) \times W_{jk}|
\]

\[
\leq \sum_{k=1}^{n} \text{max}_k |(x_k - y_k) \times \text{max}_{j,k} |W_{jk}| |
\]

\[
= \sum_{k=1}^{n} \|X - Y\|_{\infty} \times \text{max}_{j,k} |W_{jk}|
\]

\[
v(X - Y) \times n \times \text{max}_{j,k} |W_{jk}|.
\]

where \( \text{max}_{j,k} |W_{jk}| > 0 \), because the adjacency matrix \( W \) is not a zero matrix.

If we let \( \delta = \frac{\delta_1}{n \times \text{max}_{j,k} |W_{jk}|} \), then \( v(X - Y) < \delta \Rightarrow v(X - Y) \times n \times \text{max}_{j,k} |W_{jk}| < \delta_1 \).

then \( |X \times W_j - Y \times W_j| \leq v(X - Y) \times n \times \text{max}_{j,k} |W_{jk}| < \delta_1 \Rightarrow v(F(X) - F(Y)) < \epsilon \).

Then \( v(X - Y) < \delta \Rightarrow v(F(X) - F(Y)) < \epsilon \). So \( \forall \epsilon > 0, \exists \delta > 0, \forall Y \in [0, 1]^n \) and satisfying \( v(X - Y) < \delta \), then \( v(F(X) - F(Y)) < \epsilon \).
Then we have proved that $F$ is continuous at any $X \in [0, 1]^n$, so $F$ is continuous on $[0, 1]^n$.

According to the definition of the discrete topological semi-dynamics system and the proof of (1) and (2) we know that the reasoning process of fuzzy cognitive maps is a discrete topological semi-dynamics system.

3 The characteristics of FCMs

In this section, we first give the definition of fixed point and then the theorem of Brouwer’s fixed point is also given; finally one characteristic of FCMs is achieved, which is difficult to find if we have no the guiding of the reasoning mechanism of FCMs.

**Fixed point of the state space of FCMs:** For any state value $V(t)$ of FCMs, if $V(t+1) = F(V(t)) = V(t)$, then $V(t)$ is called a fixed point of the state space of FCMs.

**Brouwer’s fixed-point theorem:** For a bounded closed and convex set $D$ in the $n$-dimensional real domain $\mathbb{R}^n$, if there is a continuous mapping $F: D \rightarrow D$ into itself, then there exists at least one point $X \in D$ such that $F(X) = X$.

**Characteristic:** $F$ has at least one fixed point on $[0, 1]^n$.

**Proof:** It is obvious that $[0, 1]^n$ is a bounded closed and convex set in the $n$-dimensional real domain $\mathbb{R}^n$. The reasoning mechanism of FCMs has proved that $F$ is a continuous mapping on $[0, 1]^n$. So from the Brouwer's fixed-point theorem, $F$ has at least one fixed point on $[0, 1]^n$.

4 Conclusions

In this paper, we have proved that the reasoning process of FCMs is a discrete topological semi-dynamic system. This theory can help us to analyze the reasoning process and find the new characteristics of FCMs, which can guide us using FCMs to represent and reason the causal knowledge that widely exist in Knowledge Grid and provide information services with stronger semantics flexibly in Knowledge Grid.
References