

AN OPTIMUM DETECTOR FOR MOVING TARGETS IN HEAVY INTERFERENCE.

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Abstract. A generalized likelihood-ratio test (GLRT) detector is derived for detecting a space-time signal in the presence of unknown subspace interference and unknown target doppler. The near optimality and constant false alarm rate (CFAR) property of the GLRT is shown by the relationship to the uniformly most powerful invariant (UMPI) test using a simple approximation. Examples are presented comparing the detector against the UMPI tests and optimum Gaussian detector using ROC curves. The ROC curves indicate that the GLRT detectors compare favorably to the optimum detectors.

1. INTRODUCTION

The detection of a signal in additive interference is an important and often difficult problem. Two examples include the detection of a sonar echo in clutter and reception of a communication signal over a noisy channel. Although the theory for designing an optimum detector (e.g. using the Neyman-Pearson criterion [1] is well developed and understood, it requires knowledge of the probability density functions and associated parameters for the data under both hypothesis. Unfortunately, pdf's for real data are rarely known. At best the pdf family might be known, e.g., Gaussian, but the associated pdf parameters such as the interference direction of arrival or covariance matrix, signal amplitude, and propagation channel etc. Are usually not. One popular approach is to treat the interference as Gaussian distributed and replace the unknown covariance matrix in the optimum Gaussian detector [1] by an estimate of the interference covariance matrix. This is known as the Sample Matrix Inverse [2].

Disadvantages of the SMI method are the slow rate of adaptation [3,4,5,6] and a requirement for signal-free training data. As an Improvement to the SMI method, KIRSTEINS and TUFTS [] proposed the Principal Component Inverse (PCI) method of reduced-rank interference cancellation. The PCI method exploits the underlying low rank structure of the sample covariance matrix, or equivalently, the data matrix. It is motivated by the observation that the optimum detector for a signal in strong low rank Gaussian interference is a null steered followed by matching filtering.

Although the PCI method performs well, it is not CFAR and signal components within the adaptation set must be weaker than the interference or be absent. This requirement can be restrictive for some applications. For instance, in active sonar, the interference from clutter can be highly non-stationary. Hence, threshold setting is difficult and data outside the detection interval cannot be used reliably for training. It would be desirable to construct a PCI-like test that is CFAR and allows signal presence anywhere within the observed data set. Motivated by the PCI method, it was proposed a new Generalized Likelihood Ratio Test detectors for detecting signals in unknown subspace interference which overcome the forementioned limitation of the PCI method and lead to CFAR tests

that are remarkably close to the UMPI matched subspace detectors of Scharfs et al. [12] and the optimum Gaussian detector.

The objective of this paper are to present the performance of GLRT in the case of a moving target in unknown subspace interference. The rest of the paper is organized as follows: We start by reviewing the signal and subspace interference model followed by UMPI test. Next we derive the GLRT for space time multipath distorted signal from a moving target we present its relationship with UMPI test and we examine the performance degradation due to doppler. Finally numerical examples are presented comparing the performance of the proposed detector against the UMPI test

2. THEORETICAL BACKGROUND

We start by reviewing the subspace interference model. Suppose we have an array of m sensors that are simultaneously sampled at time t_k and the outputs stacked into the vector $x(t_k) = [x_1(t_k) \ x_2(t_k) \ \dots \ x_k(t_k)]^T$. We say the interference is subspace if at any instant of time, it can be represented as

$$x(t_k) = H\Theta_{t_k} = \sum_{n=1}^r h_n \theta_n^{t_k} \quad (1)$$

where H is a $m \times r$ matrix whose columns generate the interference space and is a $r \times 1$ vector of scale factors. That is, $x(t_k)$ is a linear combination of the columns of H . Note that H is fixed, ie., it does not vary as a function of time. The only dependence on time is through. Many type of interference components, eg., clutter, can be represented using a subspace model (see Scharf [7] for an extensive treatise on subspace or reduced-rank modeling and also [8]). For example, if the sensor outputs. $x(t_k) = [g_1 I(t - \tau_1) \ g_2 I(t - \tau_2) \ \dots \ g_m I(t - \tau_m)]^T$ (where $I(t)$ is some interference time series) are time delay steered to align the interference wavefronts (output of the k 'th sensor is delayed by $\tau_k - \tau_1$), then $x(t_k) = I(t_k) [g_1 g_2 \dots g_m]^T$ where g_k is the gain of the k -th sensor. Also narrowband components can be modeled as subspace [7,8]. A useful

aspect of the subspace model is it inherently accounts for array calibration error, eg., gain errors.

2.1 Signal and Interference Model

Now we present the models for the received data. First, the received signal can undergo multipath distortion or time dispersion from the propagation channel. The multipath waveform received by the n 'th sensor is modeled as

$$\bar{s}_n(t) = \sum_{k=1}^{L_n} c_k s\left(\frac{t - \tau_k - \delta\tau_k}{\alpha}\right) \quad (2)$$

where $s(t)$ is the signal replica, τ_k is the propagation time of the signal traveling over the k 'th path to sensor 1, $\delta\tau_k$ is the inter-sensor propagation time delay measured relative to sensor 1, $\alpha = (c+v)/(c-v)$ is the contraction/dilation due to target/platform motion (where c and v are the propagation and relative target velocities, assumed to be the same for each multipath), and c_k is a scalar corresponding to the attenuation of the k 'th path to the n 'th sensor. We point out that the model of [7] is still valid even when there are a continuum of paths, since a continuum of paths can be approximated by a set of sufficiently close discrete paths. The snapshot of sensor outputs at time t_j is then

$$\tilde{s}_j = \sum_{k=1}^L c_k [s(t_j/a - \tau_k/a) \quad s(t_j/a - \tau_k/a - \delta\tau_2/a) \quad \cdots \quad s(t_j/a - \tau_k/a - \delta\tau_m/a)]^T$$

or, after substituting ζ_k^j for the vector of data samples due to the k th path,

$$\tilde{s}_j = \sum_{k=1}^L c_k \zeta_k^j \quad (3)$$

A total of K snapshots of data are collected at times $\{t_1, t_2, \dots, t_K\}$ and stacked into a matrix. The matrix corresponding to the signal component in the data is then

$$D^a = \left[\tilde{\zeta}_1 \mid \tilde{\zeta}_2 \mid \cdots \mid \tilde{\zeta}_K \right] \quad (4)$$

and has the equivalent form

$$D^a = \sum_{k=1}^L c_k D_k^a \quad (5)$$

$$\text{where } D_k^a = \left[\zeta_k^1 \mid \zeta_k^2 \mid \cdots \mid \zeta_k^K \right]. \quad (6)$$

Using the above signal representation, the received space-time data matrices are modeled as

$$H_0 : X = H\theta + N \quad (7)$$

$$H_1 : X = H\theta + \sum_{k=1}^L c_k D_k^a + N \quad (8)$$

under the signal absent and signal present hypotheses respectively where $\theta = [\Theta_1 \Theta_2 \cdots \Theta_k]^T$. The elements of the background noise matrix N are modeled as IID complex Gaussian distributed with zero-mean and variance σ^2 . We now discuss the uniformly most powerful invariant (UMPI) test for this hypothesis testing problem.

2.2 UMPI Test

We want to test the hypothesis that $\sum_k |c_k|^2 = 0$ or $\sum_k |c_k|^2 > 0$ (signal absent or present). We assume that the interference subspace H and doppler a are known, but that the parameters θ , c_k , σ^2 in (7) and (8) are unknown and deterministic, ie. can take on a range of values. In sonar and radar, it is usually difficult or impossible to determine the distributions for the relevant interference and signal parameters since they depend on the physics of the sources and propagation conditions which are often unknown or just

partially known. This type of detection problem is called a composite hypothesis testing problem [1].

It is difficult to find an optimum test when no probability density function is available for the unknown parameters [1,10]. Ideally, we would like to construct an uniformly most powerful (UMP) test [10]. A problem is that UMP tests usually do not exist [8,4]

.In [9] it is argued that principles of invariance should be used to find the UMP test which is invariant to the unknown nuisance parameters (eg., noise variance, signal phase), known as the UMPI test. The motivation is that nuisance parameters are probably responsible for the non-existence of the UMP test in the first place [9]. Also, in many applications the test should be invariant to nuisance parameters such as the background noise level, ie., a CFAR test. However, the UMPI test is still difficult to find and may not exist. An alternative approach frequently used is to form the likelihood-ratio and replace the unknown parameters by their maximum likelihood estimates [1]. This is called the generalized likelihood ratio test (GLRT).

Scharf [12] derived the GLRT for the related problem of detecting a single data snapshot subspace signal in subspace interference (when the subspace is known) and showed that it is the UMPI test. The space-time signal and interference models we have are analogous to the data model used by Scharf [12] if the matrices in (7) and (8) are vectorized (by stacking the matrix columns into a vector). Vectorizing (7) and (8) and applying the results of [12], the UMPI test is easily be shown to be

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$$\frac{\|P_{S'} x'\|_F^2}{\|P_{S'}^\perp x'\|_F^2} \underset{H_0}{\overset{H_1}{>}} \lambda \quad (9)$$

where the projection operators are given by

$$P_H^\perp = I - H(H^H H)^{-1} H^H, \quad P_{S'} = S'(S'^H S')^{-1} S'^H, \quad \text{and} \quad P_{S'}^\perp = I - P_{S'}.$$

The vectors are

$$S' = \left[\text{vec}(P_H^\perp D_1^a) \mid \text{vec}(P_H^\perp D_2^a) \mid \cdots \mid \text{vec}(P_H^\perp D_L^a) \right] \quad \text{and} \quad x' = \text{vec}(P_H^\perp X).$$

The scalar λ is some threshold. The operator $\text{vec}(\cdot)$ takes a matrix and converts it to a vector representation by stacking the columns. The numerator of (9) can be interpreted as the magnitude-squared output of a space-time matched filter (beamforming-matched filter processing) using as the replica the part of the signal(s) which remain after the interference has been nulled. This is then normalized by an estimate of the background noise variance given by the denominator of (9). Test (9) is invariant to scalings of the data matrix and rotations in the column space of H . Thus it is the best possible CFAR detector.

Unfortunately, test (9) usually can not be implemented because the interference subspace matrix H is not known (eg., H is a function of such things as the channel, direction of arrival, array geometry and sensor characteristics which are either unknown or at best, partially known) and also the target multipath structure and doppler are unknown. In previous work KIRSTEINS [11] had proposed a GLRT detector for the above problem assuming doppler is known. The intent here is to extend those results to the case when doppler is not known and determine the effect on performance. In the remainder of the paper we derive the GLRT for the above hypothesis testing problem assuming the interference subspace H and target doppler are unknown and then discuss the relationship of the detector to the UMPI test. Finally, some numerical examples are presented comparing the performances of the GLRT and UMPI tests. We start by deriving the GLRT.

3. GENERALIZED LIKELIHOOD RATIO TEST

It is straight forward to show that a GLRT statistic for choosing between hypotheses (7) and (8) is

$$z_1 = \frac{\min_{H_0, \theta_0} \|X - H_0 \theta_0\|_F^2}{\min_{H_1, \theta_1, a, c_1, \dots, c_L} \left\| X - H_1 \theta_1 - \sum_{k=1}^L c_k D_k^a \right\|_F^2} \quad (10)$$

where H , θ , a , c_k , and σ^2 have been treated as unknown. The GLRT statistic (10) is simply a ratio of fitting errors. The numerator is the error in fitting the matrix X by a rank r matrix and the denominator is the error in jointly fitting X by a rank r matrix and $\sum_{k=1}^L c_k D_k^a$.

The numerator in (10) is easily evaluated using the singular value decomposition (SVD) of X as $\min_{H_0 \theta_0} \|X - H_0 \theta_0\|_F^2 = \sum_{k=r+1}^L \gamma_k^2$ where γ_k^2 are the singular values of X . We need to evaluate the denominator of (10). Unfortunately, a direct solution is not available. We propose an iterative scheme to perform the minimization which is motivated by the criss-cross regressions method of Gabriel [13] for solving the weighted low rank approximation problem. Basically the idea is to linearize the optimization problem, for each hypothesized doppler a , by holding H constant and then minimizing with respect to only θ and c_k . This is a standard linear least-squares fitting problem and is easy to solve. The procedure is then repeated this time replacing θ with its estimate from the previous step and now minimizing with respect to H and the c_k . These steps are continued until convergence and for all hypothesized dopplers. The algorithm steps are summarized below:

a. Initialization. Iteration counter k set to zero $k=0$. Select initial guess H_0 for H .

b. $k=k+1$

c. Holding H_{k-1} fixed, minimize with respect to only θ , the c_k :

$$\theta_k, c_k = \arg \min_{\theta, c_1, \dots, c_L} \left\| X - H_{k-1} \theta - \sum_{k=1}^L c_k D_k^a \right\|_F^2$$

d. Holding θ_k fixed, minimize with respect to only H and c_k :

$$H_k, c_k = \arg \min_{H, c_1, \dots, c_L} \left\| X - H \theta_k - \sum_{k=1}^L c_k D_k^a \right\|_F^2$$

e. Check if converged. If not converged, go back to step b.

The operator \arg here means the solution to the minimization problem.

3.1 Relationship to UMPI TEST

We now discuss the relationship of the proposed GLRT to the optimum UMPI test. It was shown in [11] that when the signal $\sum_{k=1}^L c_k D_k^a$ and background noise N are much weaker than the subspace interference H_0 and doppler a is known, the GLRT has the approximate form

$$z_1 - 1 \approx \frac{\|P_{S_a''} x''\|_F^2}{\|P_{S_a''}^\perp x''\|_F^2} \quad (11)$$

where $P_{S_a''} = S''(S''^H S'')^{-1} S''^H$, $P_{S_a''}^\perp = I - P_{S_a''}$, $P_\theta^\perp = I - \theta^H (\theta \theta^H)^{-1} \theta$,

$x'' = \text{vec}(P_H^\perp X P_\theta^\perp)$, and $S'' = [\text{vec}(P_H^\perp D_1^a P_\theta^\perp) | \text{vec}(P_H^\perp D_2^a P_\theta^\perp) | \dots | \text{vec}(P_H^\perp D_L^a P_\theta^\perp)]$.

The approximation (11) and UMPI test (9) are nearly the same except for a post-multiplication of X and the signal by the projection operator P_θ^\perp . The post-multiplication of X by P_θ^\perp corresponds to an additional temporal nulling of the data (P_θ^\perp projects onto the orthogonal complement of the complex conjugate of the row space of the subspace interference matrix H_0 which corresponds to the time series observed by each sensor due to the interference). This additional temporal nulling can be interpreted as a loss due to estimation. Test (11) is also CFAR since it is invariant to scalings of the data matrix.

The distribution of (11) has been derived in [11] and was shown to be central and non-central F distributed under H_0 and H_1 respectively.

3.2 Performance Degradation

We now discuss the loss in performance when estimating doppler. When the signal is present and not too weak compared to the background noise N , we expect the the test statistic (10) to be almost the same as when doppler is known (at high signal-to-noise ratios doppler is estimated accurately). However, when the signal is not present (noise only case) the value of (10) will clearly increase, resulting in an increased false alarm rate for the same threshold. We now determine the extent of the increase using (11). An exact analysis is difficult since it involves determining order statistics. Here we present an approximate analysis using (11) and assuming the possible range of dopplers is restricted to some small interval (often times we know the feasible target velocities).

Approximation (11) can be rewritten as

$$z_1 \approx \frac{1}{1 - \frac{\|P_{S_a''} x''\|_F^2}{\|x''\|_F^2}} \quad (12)$$

When doppler is being estimated, the above approximation becomes

$$z_1 \approx \max_a \frac{1}{1 - \frac{\|P_{S_a''} x''\|_F^2}{\|x''\|_F^2}} \quad (13)$$

An equivalent test statistic is:

$$z_1' = \max_a \frac{x''^H P_{S_a''} x''}{x''^H x''} \quad (14)$$

Next linearize (14) about a_0 by keeping only up to first-order terms of its Taylor series expansion. The resulting linearization is

$$z_1'(a) \approx \frac{x''^H P_{S_a''} x''}{x''^H x''} + (a - a_0) \left(\frac{\partial}{\partial a} \frac{x''^H P_{S_a''} x''}{x''^H x''} \right) \Bigg|_{a=a_0} \quad (15)$$

If a is restricted to some small interval $[a_0 - \Delta, a_0 + \Delta]$, the maximum of (15) must occur at one of the end points of the interval. Therefore the maximum of (14) is approximately

$$z'_1 \approx \frac{\mathbf{x}''^H \mathbf{P}_{S_a''} \mathbf{x}''}{\mathbf{x}''^H \mathbf{x}''} + \Delta \left| \frac{\partial}{\partial a} \frac{\mathbf{x}''^H \mathbf{P}_{S_a''} \mathbf{x}''}{\mathbf{x}''^H \mathbf{x}''} \right|_{a=a_0} \quad (16)$$

where the first term in (16) is the test when a is known. The last term can be thought of as the perturbation due to estimating doppler.

We now calculate the second moment of the last term in (16) (the first moment is difficult because of the absolute value), that is, the expected value of

$$e = \frac{\Delta^2}{(\mathbf{x}''^H \mathbf{x}'')^2} \left| \frac{\partial}{\partial a} \mathbf{x}''^H \mathbf{P}_{S_a''} \mathbf{x}'' \right|_{a=a_0}^2 \quad (17)$$

Replacing $(\mathbf{x}''^H \mathbf{x}'')^2$ in (17) by its expected value $(\sigma^2)^2((m-r)^2(K-r)^2 + 2(m-r)(K-r))$ and using some results in [14] for the moments of complex Wishart distributed matrices, the expected value of e is found to be

$$E(e) \approx \frac{\Delta^2}{(m-r)^2(K-r)^2 + 2(m-r)(K-r)} \left(\text{trace} \left[\left(\frac{\partial}{\partial a} \tilde{\mathbf{P}}_{S_a''} \Big|_{a=a_0} \right)^2 + \text{trace}^2 \left(\frac{\partial}{\partial a} \tilde{\mathbf{P}}_{S_a''} \Big|_{a=a_0} \right) \right] \right) \quad (18)$$

where $\tilde{\mathbf{P}}_{S_a''}$ is obtained applying the previous formulas using $\tilde{\mathbf{S}}'' = \left[\text{vec}(U_0^H D_1^a V_0) \mid \text{vec}(U_0^H D_2^a V_0) \mid \dots \mid \text{vec}(U_0^H D_L^a V_0) \right]$ in place of \mathbf{S}'' and the orthonormal columns of matrices U_0 and V_0 span the column spaces of \mathbf{P}_H^\perp and \mathbf{P}_θ^\perp respectively.

3.3 Discussion

As expected, to first-order the magnitude of the perturbation (relative to the detector when doppler is known) is related to the doppler resolution of the waveform. To adjust detector thresholds, we can approximately determine the expected value using \sqrt{e} and then (13) to determine the increase of z_1

4. NUMERICAL EXAMPLES

In order to evaluate the performance of the GLRT detector a number of studies were made. We simulate an active sonar system with an array of 10 hydrophones with half-wavelength spacing. The reverberation component was modeled as arising from IID Gaussian point scatterers (Rayleigh distributed amplitudes and uniformly distributed phases) along a line perpendicular to the center of the array. The per sample reverberation power is normalized to unity. The ambient noise component is modeled as white Gaussian with variance 3.125×10^{-3} . The transmitted pulse is a .6 second 400-425 Hz LFM waveform. The received target echo is modeled as Rayleigh fading with variance 1.95×10^{-5} . In all simulations the signal is arriving $\frac{1}{2}$ of beamwidth from broadside, noting that the reverberation is arriving from broadside.

The target velocity was set at 4 m/sec. A total of 200 independent trials were performed. The UMPI test, GLRT when target velocity is known, and GLRT when target doppler is not known (doppler search is restricted to the interval of 0-5 m/sec) were evaluated for each trial using the same realizations of the interference and signal.

The measured ROC curves are plotted in Figure 1. Note that the unknown target doppler GLRT is close in performance to the GLRT using correct target doppler and also the UMPI test. Next the square of (18) (second moment of the increase of the approximate test statistic(14)) vs. Δ is plotted in figure 2 (note that $\Delta=.004$ corresponds to a velocity change of about 30 m/sec). This is compared with experimentally measured second moment. The plots indicate the approximations are accurate over a wide range.

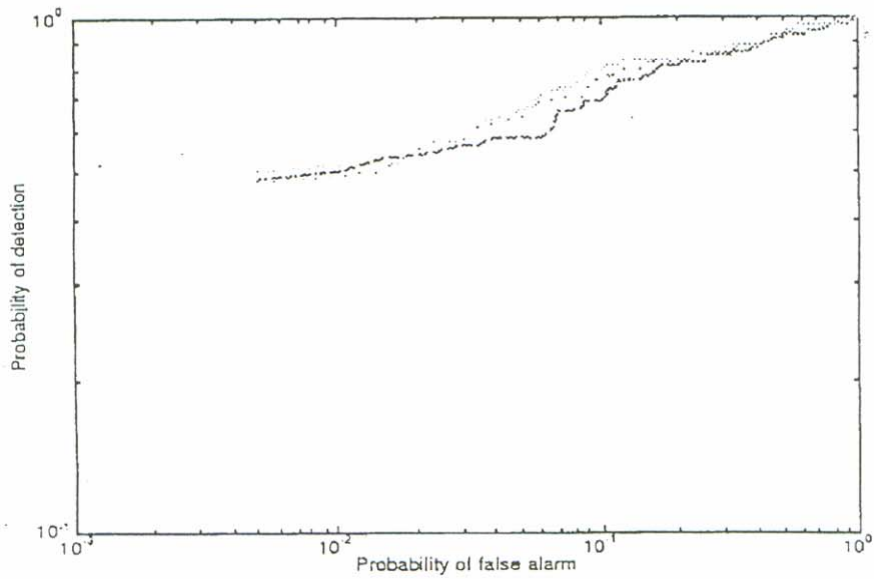


Fig 1: Experimentally measured ROC curves based on 200 trials

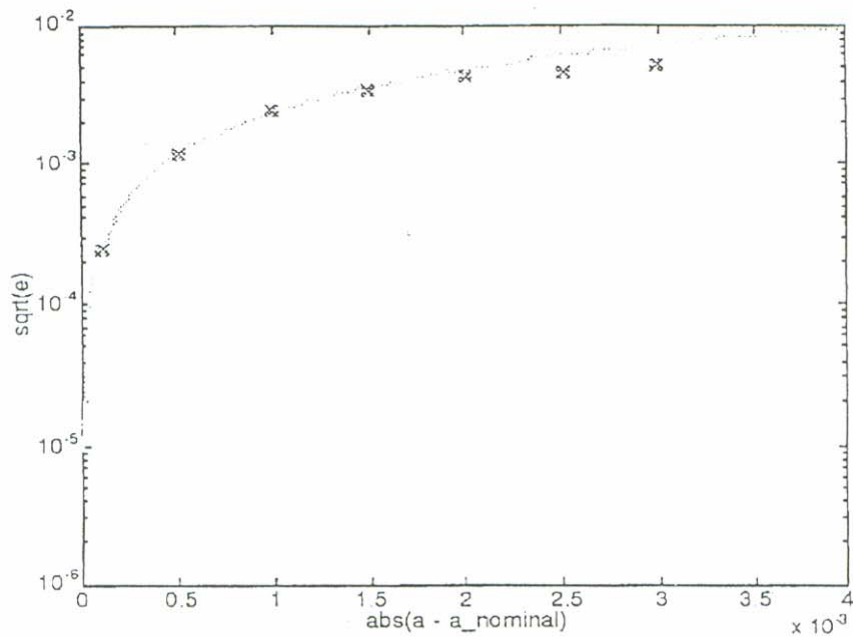


Fig 2: Expected increase of test statistic (14). Solid line is theoretical and * is experimental

5. CONCLUSION

The theoretical and experimental analysis indicates that the proposed GLRT detectors perform well. Furthermore, formulas are provided relating the GLRT's to the UMPI test and allowing the approximate calculation of the expected increase of the test statistic when doppler is estimated.

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