Fuzzy Stability of an $n$-Dimensional Quadratic and Additive Type Functional Equation

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Abstract

In this paper, we investigate a fuzzy version of stability for the functional equation

$$f \left( \sum_{j=1}^{n} x_j \right) + (n - 2) \sum_{j=1}^{n} f(x_j) - \sum_{1 \leq i < j \leq n} f(x_i + x_j) = 0$$

in the sense of A. K. Mirmostafaee and M. S. Moslehian.

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1 Introduction and preliminaries

A classical question in the theory of functional equations is “when is it true that a mapping, which approximately satisfies a functional equation, must be somehow close to a solution of the equation?”. Such a problem, called a stability problem of the functional equation, was formulated by S. M. Ulam [27] in 1940. In the next year, D. H. Hyers [6] gave a partial solution of Ulam’s problem for the case of approximate additive mappings. Subsequently, his result was generalized by T. Aoki [1] for additive mappings, and by Th. M. Rassias [25] for linear mappings, to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [4], [5], [13], [15], [17]-[21], [26].

In this paper, we consider the fuzzy version stability problem in the fuzzy normed linear space setting. The concept of fuzzy norm on a linear space was introduced by A. K. Katsaras [14] in 1984, which was later on studied, following Cheng and Mordeson [3], to give a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [16]. In 2008, A. K. Mirmostafaee and M. S. Moslehian [23,24] used the definition of a fuzzy norm in [2] to obtain a fuzzy version of stability for the Cauchy functional equation:

\[ f(x + y) - f(x) - f(y) = 0 \]  \hspace{1cm} (1)

and the quadratic functional equation:

\[ f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0. \]  \hspace{1cm} (2)

We call a solution of (1) an additive mapping and a solution of (2) is called a quadratic mapping. Now we consider the following \(n\)-dimensional quadratic and additive type functional equation:

\[ f \left( \sum_{j=1}^n x_j \right) + (n - 2) \sum_{j=1}^n f(x_j) - \sum_{1 \leq i < j \leq n} f(x_i + x_j) = 0. \]  \hspace{1cm} (3)


In this paper, we get a general stability result of the \(n\)-dimensional quadratic and additive type functional equation (3) in the fuzzy normed linear space. To do it, we introduce a Cauchy sequence \( \{J_n f(x)\} \) starting from a given mapping \( f \), which converges to the desired mapping \( F \) in the fuzzy sense([7]-[11], [22]).

We use the definition of a fuzzy normed space given in [2] to exhibit a reasonable fuzzy version of stability for the \(n\)-dimensional quadratic and additive type functional equation (3) in the fuzzy normed linear space.
Definition 1.1 ([2]) Let $X$ be a real linear space. A function $N : X \times R \rightarrow [0, 1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on $X$ if for all $x, y \in X$ and all $s, t \in R$,

(N1) $N(x, c) = 0$ for $c \leq 0$;

(N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;

(N3) $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$;

(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;

(N5) $N(x, \cdot)$ is a non-decreasing function on $R$ and $\lim_{t \to \infty} N(x, t) = 1$.

The pair $(X, N)$ is called a fuzzy normed linear space. Let $(X, N)$ be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n \to \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n \to \infty} x_n = x$. A sequence $\{x_n\}$ in $X$ is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists $n_0$ such that for all $n \geq n_0$ and all $p > 0$ we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$. It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

2 Fuzzy stability of (3) for the case $n$ is even

Let $(X, N)$ be a fuzzy normed space and $(Y, N')$ a fuzzy Banach space. And let $n$ be a fixed even number greater than 2. For a given mapping $f : X \rightarrow Y$, we use the abbreviations

\[
Df(x_1, x_2, \ldots, x_n) := f \left( \sum_{j=1}^{2^n} x_j \right) + (n - 2) \sum_{j=1}^{n} f(x_j) - \sum_{1 \leq i < j \leq n} f(x_i + x_j),
\]

\[
\Delta(x) := (\underbrace{x, \ldots, x}_n, -x, \ldots, -x)
\]

for all $x, x_1, x_2, \ldots, x_n \in X$. For given $q > 0$, the mapping $f$ is called a fuzzy $q$-almost quadratic-additive mapping if

\[
N'(Df(x_1, \ldots, x_n), t_1 + \cdots + t_n) \geq \min\{N(x_1, t_1^q), \ldots, N(x_n, t_n^q)\} \quad (4)
\]

for all $x_1, x_2, \ldots, x_n \in X$ and all $t_1, t_2, \ldots, t_n \in [0, \infty)$. The following result gives a fuzzy version of the stability of the $n$-dimensional quadratic and additive type functional equation (3).

Theorem 2.1 Let $q$ be a positive real number with $q \neq \frac{1}{2}, 1$. And let $f$ be a fuzzy $q$-almost quadratic-additive mapping from a fuzzy normed space $(X, N)$
into a fuzzy Banach space \((Y, N')\). Then there is a unique quadratic-additive mapping \(F : X \to Y\) such that

\[
N'(F(x) - f(x), t) \geq \begin{cases} 
\sup_{t' < t} N(x, (n-2)^q(2-2^q)n^{-q}t'^q) & \text{if } q > 1, \\
\sup_{t' < t} N\left(x, \frac{(n-2)^q t^q}{(1-2^q) + n^{-q}t^{-q}}\right) & \text{if } q < 1
\end{cases}
\]  

(5)

for each \(x \in X\) and \(t > 0\), where \(p = 1/q\).

**Proof.** It follows from (N2), (N3), (N4) and (4) that

\[
N'(f(0), t) = N'\left(Df(0, \ldots, 0), \frac{(n-1)(n-2)t}{2}\right) \\
\geq \min \left\{ N\left(0, \left(\frac{(n-1)(n-2)t}{2n}\right)^q\right) \right\} = 1
\]

for all \(t > 0\). So by (N2) we know that \(f(0) = 0\). We will prove the theorem in three cases, \(q > 1\), \(\frac{1}{2} < q < 1\), and \(0 < q < \frac{1}{2}\).

**Case 1.** Let \(q > 1\) and let \(J_m f : X \to Y\) be a mapping defined by

\[
J_m f(x) = 2^{-2m-1}(f(2^m x) + f(-2^m x)) + 2^{-m-1}(f(2^m x) - f(-2^m x))
\]

for all \(x \in X\) and \(m \in N \cup \{0\}\). Then \(J_0 f(x) = f(x)\) and

\[
J_j f(x) - J_{j+1} f(x) = \frac{Df(\Delta(2^j x))}{2 \cdot 4^n(n-2)} + \frac{Df(\Delta(-2^j x))}{2 \cdot 4^n(n-2)} \\
+ \frac{Df(\Delta(2^j x))}{2^j + 2(n-2)} - \frac{Df(\Delta(-2^j x))}{2^j + 2(n-2)}
\]  

(6)

for all \(x \in X\) and \(j \geq 0\). Together with (N3), (N4) and (4), this equation implies that if \(m' + m > m \geq 0\) then

\[
N'(J_m f(x) - J_{m'} f(x), \sum_{j=m}^{m'-m} n \left(\frac{2^p}{2}\right) \frac{j t_p}{n-2}) \\
\geq N'\left(\sum_{j=m}^{m'-m} (J_j f(x) - J_{j+1} f(x)), \sum_{j=m}^{m'-m} n \left(\frac{2^p}{2}\right) \frac{j t_p}{n-2} \right) \\
\geq \min \bigcup_{j=m}^{m'-m} \left\{ N'\left(J_j f(x) - J_{j+1} f(x), n \left(\frac{2^p}{2}\right) \frac{j t_p}{n-2} \right) \right\} \\
\geq \min \bigcup_{j=m}^{m'-m} \left\{ \min \left\{ N'\left(\frac{(2 + 2j n)Df(\Delta(2^j x))}{4^n+1 n(n-2)}, \frac{(2 + 2j n)2^{j p} t_p}{4^n+1 n(n-2)}\right)\right), \\
N'\left(\frac{(2 - 2j n)Df(\Delta(-2^j x))}{4^n+1 n(n-2)}, \frac{(2j n - 2)2^{j p} t_p}{4^n+1 n(n-2)}\right) \right\} \\
\geq \min \bigcup_{j=m}^{m'-m} \left\{ N(2^j x, 2^j t) \right\} = N(x, t)
\]  

(7)
for all $x \in X$ and $t > 0$. Let $\varepsilon > 0$ be given. Since $\lim_{t \to \infty} N(x, t) = 1$, there is $t_0 > 0$ such that

$$N(x, t_0) \geq 1 - \varepsilon.$$ 

We observe that for some $\tilde{t} > t_0$, the series $\sum_{j=0}^{\infty} \frac{n}{2} \left(\frac{2p}{2}\right)^j \frac{\tilde{t}^p}{n-2}$ converges for $p = \frac{1}{q} < 1$. It guarantees that, for an arbitrary given $c > 0$, there exists $m_0 \geq 0$ such that

$$\sum_{j=m}^{m'+m-1} \frac{n}{2} \left(\frac{2p}{2}\right)^j \frac{\tilde{t}^p}{n-2} < c$$

for each $m \geq m_0$ and $m' > 0$. By (N5) and (7), we have

$$N'(J_m f(x) - J_{m'+m} f(x), c) \geq N'(J_m f(x) - J_{m'+m} f(x), \sum_{j=m}^{m'+m-1} \frac{n}{2} \left(\frac{2p}{2}\right)^j \frac{\tilde{t}^p}{n-2}) \geq N(x, \tilde{t}) \geq N(x, t_0) \geq 1 - \varepsilon$$

for all $x \in X$. Hence $\{J_m f(x)\}$ is a Cauchy sequence in the fuzzy Banach space $(Y, N')$, and so we can define a mapping $F : X \to Y$ by

$$F(x) := N' - \lim_{m \to \infty} J_m f(x)$$

for all $x \in X$. Moreover, if we put $m = 0$ in (7), we have

$$N'(f(x) - J_{m'} f(x), t) \geq N\left(x, \frac{(n-2)^q t^q}{\sum_{j=0}^{m'-1} \frac{n}{2} \left(\frac{2p}{2}\right)^j \frac{\tilde{t}^p}{n-2}}\right)$$

for all $x \in X$. Next we will show that $F$ is the desired quadratic additive mapping. Using (N4), we have

$$N'(DF (x_1, \ldots, x_n), t)$$

\begin{align*}
&\geq \min \left\{ N'(F \left(\sum_{j=1}^{n} x_j\right) - J_m f \left(\sum_{j=1}^{n} x_j\right), \frac{t}{4} \right), \\
&\quad \min \bigcup_{j=1}^{n} \left\{ N' \left((n-2) F(x_j) - (n-2) J_m f(x_j), \frac{t}{4n}\right) \right\}, \\
&\quad \min \bigcup_{1 \leq i < j \leq n} \left\{ N' \left(F(x_i + x_j) - J_m f(x_i + x_j), \frac{t}{2n(n-1)}\right) \right\}, \\
&\quad N' \left( DJ_m f(x_1, x_2, \ldots, x_n), \frac{t}{4} \right) \right\} \quad (9)
\end{align*}
for all \( x_1, \ldots, x_n \in X \) and \( m \in N \). The first three terms on the right hand side of (9) tend to 1 as \( m \to \infty \) by the definition of \( F \) and (N2), and the last term holds

\[
N'(DJ_m f(x_1, x_2, \ldots, x_n), t) \geq \min\left\{ N'(Df(2^m x_1, \ldots, 2^m x_n), t), N'(Df(-2^m x_1, \ldots, -2^m x_n), t) \right\}
\]

for all \( x_1, x_2, \ldots, x_n \in X \) and \( t > 0 \). By (N3) and (4), we obtain

\[
N'(Df(\pm 2^m x_1, \ldots, \pm 2^m x_n), t) \geq \min\left\{ N(x_1, 2^{(2q-1)m-3q}n^{-q}t^q), \ldots, N(x_n, 2^{(2q-1)m-3q}n^{-q}t^q) \right\}
\]

and

\[
N'(Df(\pm 2^m x_1, \ldots, \pm 2^m x_n), t) \geq \min\left\{ N(x_1, 2^{(q-1)m-3q}n^{-q}t^q), \ldots, N(x_n, 2^{(q-1)m-3q}n^{-q}t^q) \right\}
\]

for all \( x_1, \ldots, x_n \in X \) and \( m \in N \). Since \( q > 1 \), together with (N5), we can deduce that the last term of (9) also tends to 1 as \( m \to \infty \). It follows from (9) that

\[
N'(DF(x_1, x_2, \ldots, x_n), t) = 1
\]

for all \( x_1, x_2, \ldots, x_n \in X \) and \( t > 0 \). By (N2), this means that \( DF(x_1, x_2, \ldots, x_n) = 0 \) for all \( x_1, x_2, \ldots, x_n \in X \). Now we approximate the difference between \( f \) and \( F \) in a fuzzy sense. For an arbitrary fixed \( x \in X \) and \( t > 0 \), choose \( 0 < \varepsilon < 1 \) and \( 0 < t' < t \). Since \( F \) is the limit of \( \{J_m f(x)\} \), there is \( m \in N \) such that

\[
N'(F(x) - J_m f(x), t - t') \geq 1 - \varepsilon.
\]

By (8), we have

\[
N'(F(x) - f(x), t) \geq \min\left\{ N'(F(x) - J_m f(x), t - t'), N'(J_m f(x) - f(x), t') \right\}
\]

\[
\geq \min\left\{ 1 - \varepsilon, N\left(x, \frac{(n - 2)^q t^q}{\sum_{j=0}^{n-2} \left( \frac{2^p}{2} \right)^j} \right) \right\}
\]

\[
\geq \min\left\{ 1 - \varepsilon, N\left(x, (n - 2)^q(2 - 2^p)q^{-q}t^q \right) \right\}.
\]
Because $0 < \varepsilon < 1$ is arbitrary, we get the inequality (5) in this case. Finally, to prove the uniqueness of $F$, let $F' : X \to Y$ be another quadratic-additive mapping satisfying (5). Then by (6), we get

$$
\begin{align*}
\begin{cases}
F(x) - J_m F(x) = \sum_{j=0}^{m-1} (J_j F(x) - J_{j+1} F(x)) = 0 \\
F'(x) - J_m F'(x) = \sum_{j=0}^{m-1} (J_j F'(x) - J_{j+1} F'(x)) = 0
\end{cases}
\end{align*}
$$

(10)

for all $x \in X$ and $m \in \mathbb{N}$. Together with (N4) and (5), this implies that

$$
N'(F(x) - F'(x), t) = N'(J_m F(x) - J_m F'(x), t) \geq \min \left\{ N'(\frac{(F - f)(2^m x)}{2 \cdot 4^m}, \frac{t}{8}), N'\left(\frac{(f - F')(2^m x)}{2 \cdot 4^m}, \frac{t}{8}\right) \right\}
$$

$$
\geq \min \left\{ N'\left(\frac{(F - f)(2^m x)}{2 \cdot 4^m}, \frac{t}{8}\right), N'\left(\frac{(f - F')(2^m x)}{2 \cdot 4^m}, \frac{t}{8}\right) \right\}
$$

$$
\geq \sup_{t' < t} N(x, 2^{(q-1)m-2q(n-2)}(2 - 2^q)^n t'^q)
$$

for all $x \in X$ and $m \in \mathbb{N}$. Observe that, for $q = \frac{1}{p}$, the last term of the above inequality tends to 1 as $m \to \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$ and so we get

$$
F(x) = F'(x)
$$

for all $x \in X$ by (N2).

**Case 2.** Let $\frac{1}{2} < q < 1$ and let $J_m f : X \to Y$ be a mapping defined by

$$
J_m f(x) = 2^{-2m-1} (f(2^m x) + f(-2^m x)) + 2^{m-1} \left( f\left(\frac{x}{2^m}\right) - f\left(-\frac{x}{2^m}\right) \right)
$$

for all $x \in X$. Then we have $J_0 f(x) = f(x)$ and

$$
J_j f(x) - J_{j+1} f(x) = \frac{Df(\Delta(2^j x))}{2 \cdot \mathcal{P}(n-2)} + \frac{Df(-\Delta(2^j x))}{2 \cdot \mathcal{P}(n-2)}
$$

$$
- \frac{2^{j-1}Df(\Delta(\frac{x}{2^{j+1}}))}{n-2} + \frac{2^{j-1}Df(-\Delta(\frac{x}{2^{j+1}}))}{n-2}
$$
for all $x \in X$ and $j \geq 0$. If $m' + m > m \geq 0$, then

$$N' \left( J_m f(x) - J_{m'} f(x) \right), \sum_{j=m}^{m'+m-1} \left( \left( \frac{2^p}{4} \right)^j + \frac{n}{2^p} \left( \frac{2}{2^p} \right)^j \right) \frac{4^p}{(n-2)}$$

$$\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ \min \left\{ N' \left( \frac{D f(\Delta(2^j x))}{2 \cdot 4^j n(n-2)}, \frac{2^j p}{2 \cdot 4^j n(n-2)} \right), \right. \right.$$ 

$$N' \left( \frac{D f(-\Delta(2^j x))}{2 \cdot 4^j n(n-2)}, \frac{2^j p}{2 \cdot 4^j n(n-2)} \right), \right.$$ 

$$N' \left( -\frac{2^{j-1} D f(\Delta(\frac{p}{2^{j-2}}))}{n-2}, \frac{2^j \cdot 1-i n p}{2^{j-1} n p(n-2)} \right), \right.$$ 

$$\left. \left. \bigg\{ \right. \bigg\} \right\}$$

$$\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ N(2^j x, 2^j t), N \left( \frac{x}{2^{j+1}}, \frac{t}{2^{j+1}} \right) \right\} \quad (11)$$

for all $x \in X$ and $t > 0$. In the similar argument following (7) of the previous case, we can define the limit $F(x) := N' - \lim_{m \to \infty} \left( J_m f(x) \right)$ of the Cauchy sequence $\{ J_m f(x) \}$ in the Banach fuzzy space $Y$. Moreover, putting $m = 0$ in the above inequality, we have

$$N'(f(x) - J_m f(x), t) \geq N \left( x, \frac{(n-2)^q t^q}{\left( \sum_{j=0}^{m'-1} \left( \frac{2^p}{4} \right)^j + \frac{n}{2^p} \left( \frac{2}{2^p} \right)^j \right)^q} \right)$$

for each $x \in X$. To prove that $F$ is a quadratic additive mapping, we have enough to show that the last term of (9) in Case 1 tends to 1 as $m \to \infty$. By (N3) and (4), we get

$$N' \left( D J_m f(x_1, x_2, \ldots, x_n), \frac{t}{4} \right)$$

$$\geq \min \left\{ N' \left( \frac{D f(2^m x_1, \ldots, 2^m x_n)}{2 \cdot 4^m}, \frac{t}{16} \right), N' \left( \frac{D f(-2^m x_1, \ldots, -2^m x_n)}{2 \cdot 4^m}, \frac{t}{16} \right), \right.$$ 

$$N' \left( 2^{m-1} D f \left( \frac{x_1}{2^m}, \ldots, \frac{x_n}{2^m} \right), \frac{t}{16} \right), \right.$$ 

$$\left. \left. \bigg\{ \right. \bigg\} \right\}$$

$$\geq \min \left\{ N(x_1, 2^{(2q-1)m-3q_n - q t q}), \ldots, N(x_n, 2^{(2q-1)m-3q_n - q t q}), \right.$$ 

$$N(x_1, 2^{(1-q)m-3q_n - q t q}), \ldots, N(x_n, 2^{(1-q)m-3q_n - q t q}) \right\} \right.$$
for all \( x_1, x_2, \ldots, x_n \in X \) and \( t > 0 \). Observe that all the terms on the right hand side of the above inequality tend to 1 as \( m \to \infty \), since \( \frac{1}{2} < q < 1 \). Hence, together with the similar argument after (9), we can say that \( DF(x_1, x_2, \ldots, x_n) = 0 \) for all \( x_1, x_2, \ldots, x_n \in X \). Recall, in Case 1, the inequality (5) follows from (8). By the same reasoning, we get (5) from (11) in this case. Now to prove the uniqueness of \( F \), let \( F' \) be another quadratic additive mapping satisfying (5). Then, together with (N4), (5), and (10), we have

\[
N'(F(x) - F'(x), t) 
\geq \min \left\{ N'(J_mF(x) - J_mf(x), \frac{t}{2}^2), N'(J_mf(x) - J_mF(x), \frac{t}{2}^2) \right\} 
\geq \min \left\{ N'(\frac{(F - f)(2^m x)}{2 \cdot 4^m}, \frac{t}{8}), N'(\frac{(f - F')(2^m x)}{2 \cdot 4^m}, \frac{t}{8}) \right\} 
\geq \min \left\{ \sup_{t' < t} N\left( x, \frac{2^{(2q-1)m-2q}(n-2)^q}{(4 - 2p)q} \right), \sup_{t' < t} N\left( x, \frac{2^{(1-q)m-2q}(n-2)^q}{(4 - 2p)q} \right) \right\} 
\geq \min \left\{ \sup_{t' < t} N\left( x, \frac{2^{(2q-1)m-2q}(n-2)^q}{(4 - 2p)q} \right), \sup_{t' < t} N\left( x, \frac{2^{(1-q)m-2q}(n-2)^q}{(4 - 2p)q} \right) \right\} 
\]

for all \( x \in X \) and \( m \in N \). Since \( \lim_{m \to \infty} 2^{(2q-1)m-2q} = \lim_{m \to \infty} 2^{(1-q)m-2q} = \infty \) in this case, both terms on the right hand side of the above inequality tend to 1 as \( m \to \infty \) by (N5). This implies that \( N'(F(x) - F'(x), t) = 1 \) and so \( F(x) = F'(x) \) for all \( x \in X \) by (N2).

**Case 3.** Finally, we take \( 0 < q < \frac{1}{2} \) and define \( J_mf : X \to Y \) by

\[
J_mf(x) = \frac{4^m(f(2^{-m}x) + f(-2^{-m}x)) + 2^m(f(2^{-m}x) - f(-2^{-m}x))}{2}
\]

for all \( x \in X \). Then we have \( J_0f(x) = f(x) \) and

\[
J_jf(x) - J_{j+1}f(x) = \frac{2 \cdot 4^jDf(\Delta(x_{2^{-j+1}}))}{n(n - 2)} + \frac{2 \cdot 4^jDf(\Delta(-x_{2^{-j+1}}))}{n(n - 2)} - \frac{2^{j-1}Df(\Delta(x_{2^{-j+1}}))}{n - 2} + \frac{2^{j-1}Df(\Delta(-x_{2^{-j+1}}))}{n - 2}
\]
which implies that if $m' + m > m \geq 0$ then

$$
N' \left( J_m f(x) - J_{m'} f(x), \sum_{j=m}^{m'+m-1} \left( \frac{4}{2^p} \right)^{j+1} + \frac{n}{2^p} \left( \frac{2}{2^p} \right)^j \frac{t^p}{n-2} \right)
$$

$$
\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ N' \left( \frac{2 \cdot 4^j Df(\Delta(\frac{x}{2^{j+1}}))}{n(n-2)}, \frac{2 \cdot 4^j t^p}{(n-2)2^{(j+1)p}} \right), \frac{N' \left( 2 \cdot 4^j Df(\Delta(-\frac{x}{2^{j+1}}))}{n(n-2)}, \frac{2 \cdot 4^j t^p}{(n-2)2^{(j+1)p}} \right), \right. \\
\left. \frac{N' \left( -2^{j-1} Df(\Delta(\frac{x}{2^{j+1}}))}{n(n-2)}, \frac{2^{j-1} n t^p}{2^{(j+1)p}(n-2)} \right), \frac{2^{j-1} n t^p}{2^{(j+1)p}(n-2)} \right\}
$$

$$
\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ N \left( \frac{x}{2^{j+1}}, \frac{t}{2^{j+1}} \right) \right\}
= N(x,t)
$$

for all $x \in X$ and $t > 0$. Similar to the previous cases, it leads us to define the mapping $F : X \to Y$ by $F(x) := N' - \lim_{m \to \infty} J_m f(x)$. Putting $m = 0$ in the above inequality, we have

$$
N'(f(x) - J_0 f(x), t) \geq N \left( x, \frac{(n-2)^q t^q}{\left( \sum_{j=0}^{m'-1} \left( \frac{4}{2^p} \right)^{j+1} + \frac{n}{2^p} \left( \frac{2}{2^p} \right)^j \right)^q} \right)
$$

(12)

for all $x \in X$ and $t > 0$. Notice that

$$
N' \left( DJ_m f(x_1, x_2, \cdots, x_n), \frac{t}{4} \right)
$$

$$
\geq \min \left\{ N' \left( \frac{4m Df(x_1/2, x_2/2, \cdots, x_n/2)}{2}, \frac{t}{16} \right), N' \left( \frac{4m Df(x_1/2, x_2/2, \cdots, x_n/2)}{2}, \frac{t}{16} \right), N' \left( \frac{2m-1 Df(x_1/2, x_2/2, \cdots, x_n/2)}{2}, \frac{t}{16} \right), N' \left( \frac{-2m-1 Df(-x_1/2, -x_2/2, \cdots, -x_n/2)}{2}, \frac{t}{16} \right) \right\}
$$

$$
\geq \min \left\{ N \left( x_1, 2^{(1-q)m-3q-1} t^q \right), \cdots, N \left( x_n, 2^{(1-q)m-3q-1} t^q \right), N \left( x_1, 2^{(1-q)m-3q-1} t^q \right), \cdots, N \left( x_n, 2^{(1-q)m-3q-1} t^q \right) \right\}
$$

for all $x_1, x_2, \cdots, x_n \in X$ and $t > 0$. Since $0 < q < \frac{1}{2}$, all terms on the right hand side tend to 1 as $m \to \infty$, which implies that the last term of (9) tends
function satisfying (5). Then by (10), we get
\[ Y \]

To prove the uniqueness of this case, we follow the similar argument after (9) in Case 1, we get the inequality (5) from (12) in this case.

To prove the uniqueness of \( F \), let \( F' : X \to Y \) be another quadratic additive function satisfying (5). Then by (10), we get
\[
N'(F(x) - F'(x), t) \geq \min \left\{ N'(J_mF(x) - J_mF'(x), \frac{t}{2}), N'(J_mf(x) - J_mF'(x), \frac{t}{2}) \right\} \\
\geq \min \left\{ N'(\frac{4m}{2}(F - f)\left(\frac{x}{2^m}\right), \frac{t}{8}), N'(\frac{4m}{2}(f - F')\left(\frac{x}{2^m}\right), \frac{t}{8}) \right\} \\
N'(2^{m-1}(F - f)\left(\frac{x}{2^m}\right), \frac{t}{8}), N'(2^{m-1}(f - F')\left(\frac{x}{2^m}\right), \frac{t}{8}) \right\} \\
N'(2^{m-1}(F - f)\left(\frac{x}{2^m}\right), \frac{t}{8}), N'(2^{m-1}(f - F')\left(\frac{x}{2^m}\right), \frac{t}{8}) \right\} \\
\geq \sup_{t < 1} N' \left( x, 2^{(1-2q)m-2q(n-2)}t^q \left(\frac{4}{2^{q-4}} + \frac{n}{2^{q-2}}\right)^q \right)
\]
for all \( x \in X \) and \( m \in N \). Observe that, for \( 0 < q < \frac{1}{2} \), the last term tends to 1 as \( m \to \infty \) by (N5). This implies that \( N'(F(x) - F'(x), t) = 1 \) and \( F(x) = F'(x) \) for all \( x \in X \) by (N2). We have completed the proof of Theorem 2.1.

We can use Theorem 2.1 to get a classical result in the framework of normed spaces. Let \( (X, \| \cdot \|) \) be a normed linear space. Then we can define a fuzzy norm \( N_X \) on \( X \) by following
\[
N_X(x, t) = \begin{cases} 
0, & t \leq \|x\|, \\
1, & t > \|x\|,
\end{cases}
\]
where \( x \in X \) and \( t \in R \), see [17]. Suppose that \( f : X \to Y \) is a mapping into a Banach space \( (Y, ||\cdot||) \) such that
\[
|||Df(x_1, x_2, \ldots, x_n)||| \leq ||x_1||^p + ||x_2||^p + \cdots + ||x_n||^p
\]
for all \( x_1, x_2, \ldots, x_n \in X \), where \( p > 0 \) and \( p \neq 1, 2 \). Let \( N_Y \) be a fuzzy norm on \( Y \). Then we get
\[
N_Y(Df(x_1, \ldots, x_n), t_1 + \cdots + t_n) = \begin{cases} 
0, & t_1 + \cdots + t_n \leq |||Df(x_1, \ldots, x_n)||| \\
1, & t_1 + \cdots + t_n > |||Df(x_1, \ldots, x_n)|||
\end{cases}
\]
for all \(x_1, x_2, \ldots, x_n \in X\) and \(t_1, \ldots, t_n \in R\). Consider the case

\[N_Y(Df(x_1, \ldots, x_n), t_1 + \cdots + t_n) = 0\]

which implies that

\[
\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p \geq \|Df(x_1, \ldots, x_n)\| \geq t_1 + \cdots + t_n
\]

and so one of \(i\) satisfies \(\|x_i\|^p \geq t_i\) in this case. Hence, for \(q = \frac{1}{p}\), we have

\[
\min\{N_X(x_1, t_1^q), \ldots, N_X(x_n, t_n^q)\} = 0
\]

for all \(x_1, x_2, \ldots, x_n \in X\) and \(t_1, \ldots, t_n > 0\). Therefore, in every case, the inequality

\[N_Y(Df(x_1, \ldots, x_n), t_1 + \cdots + t_n) \geq \min\{N_X(x_1, t_1^q), \ldots, N_X(x_n, t_n^q)\}\]

holds. It means that \(f\) is a fuzzy \(q\)-almost quadratic additive mapping, and by Theorem 2.1, we get the following stability result.

**Corollary 2.2** Let \((X, \|\cdot\|)\) be a normed linear space and let \((Y, |||\cdot|||)\) be a Banach space. If \(n\) is an even number greater than 3 and \(f : X \to Y\) satisfies

\[|||Df(x_1, x_2, \ldots, x_n)||| \leq \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p\]

for all \(x_1, x_2, \ldots, x_n \in X\), where \(p > 0\) and \(p \neq 1, 2\), then there is a unique quadratic-additive mapping \(F : X \to Y\) such that

\[|||F(x) - f(x)||| \leq \begin{cases} \left(\frac{4}{|2^p-4|} + \frac{n}{2^p-2}\right) \frac{\|x\|^p}{n-2} & \text{if } p > 1, \\ \frac{n\|x\|^p}{(2-2^p)(n-2)} & \text{if } p < 1 \end{cases}\]

for all \(x \in X\).

### 3 Fuzzy stability of (3) for the case \(n\) is odd.

Let \(n\) be a fixed odd number greater than 2. We use the following abbreviation:

\[\tilde{\Delta}(x) := (\underbrace{x, \ldots, x}_{\frac{n+1}{2}}, \underbrace{-x, \ldots, -x}_{\frac{n-1}{2}})\]

for all \(x \in X\).
Theorem 3.1 Let \(q\) be a positive real number with \(q \neq \frac{1}{2}, 1\) and let \(f\) be a fuzzy \(q\)-almost quadratic-additive mapping from a fuzzy normed space \((X, N)\) into a fuzzy Banach space \((Y, N')\). Then there is a unique quadratic-additive mapping \(F : X \to Y\) such that

\[
N'(F(x) - f(x), t) \geq \begin{cases} 
\sup_{t' \leq t} N \left( x, (n-1)^{q}(2n)^{-q}(2-2^{p})^{q}t'^{q} \right) & \text{if } q > 1, \\
\sup_{t' \leq t} N \left( x, \frac{(n-1)^{q}(2n)^{-q}(2-2^{p})^{q}t'^{q}}{(\frac{2n}{2^{p}})^{q}} \right) & \text{if } q < 1 
\end{cases}
\]

(13)

for all \(x \in X\) and \(t > 0\), where \(p = 1/q\).

Proof. We easily know that \(f(0) = 0\) from (4) as in the proof of Theorem 2.1. We will prove the theorem in three cases, \(q > 1, \frac{1}{2} < q < 1\), and \(0 < q < \frac{1}{2}\).

Case 1. Let \(q > 1\) and let \(J_{m}f : X \to Y\) be a mapping defined by

\[
J_{m}f(x) = 2^{-2m-1}(f(2^{m}x) + f(-2^{m}x)) + 2^{-m-1}(f(2^{m}x) - f(-2^{m}x))
\]

for all \(x \in X\). Then \(J_{0}f(x) = f(x)\) and

\[
J_{j}f(x) - J_{j+1}f(x) = \frac{Df(\tilde{\Delta}(2jx))}{2 \cdot 4^{j}(n-1)^{2}} + \frac{Df(\tilde{\Delta}(-2jx))}{2 \cdot 4^{j}(n-1)^{2}}
\]

\[
+ \frac{Df(\Delta(2jx))}{2^{j+1}(n-1)} - \frac{Df(\Delta(-2jx))}{2^{j+1}(n-1)}
\]

for all \(x \in X\) and \(j \geq 0\). Together (N3), (N4) and (4), this equation implies that if \(m' + m > m \geq 0\) then

\[
N' \left( J_{m}f(x) - J_{m'+m}f(x), \sum_{j=m}^{m'+m-1} \frac{2^{j+p}nt^{p}}{2^{j}(n-1)} \right)
\]

\[
\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ N' \left( \frac{(2^{j}(n-1) + 1)Df(\tilde{\Delta}(2jx))}{2 \cdot 4^{j}(n-1)^{2}}, \frac{(2^{j}(n-1) + 1)2^{j+p}nt^{p}}{2 \cdot 4^{j}(n-1)^{2}} \right), N' \left( \frac{(1 - 2^{j}(n-1))Df(\tilde{\Delta}(-2jx))}{2 \cdot 4^{j}(n-1)^{2}}, \frac{(2^{j}(n-1) - 1)2^{j+p}nt^{p}}{2 \cdot 4^{j}(n-1)^{2}} \right) \right\}
\]

\[
\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ N(2^{j}x, 2^{j}t) \right\}
\]

\[= N(x, t) \]

for all \(x \in X\) and \(t > 0\). Therefore the Cauchy sequence \(\{J_{m}f(x)\}\) has the limit \(F(x) := N' - \lim_{m \to \infty} J_{m}f(x)\) and

\[
N'(f(x) - J_{m}f(x), t) \geq N \left( x, \frac{(n-1)^{q}t^{q}}{n \sum_{j=0}^{m-1} \left( \frac{2^{p}}{2} \right)^{j} \left( \frac{q}{q} \right)} \right).
\]

(14)
By the same reasoning as in the proof of Case 1 in Theorem 2.1, the inequality (13) follows from (14) and the rest of the proof is same with that of Case 1 in Theorem 2.1.

**Case 2.** Let $\frac{1}{2} < q < 1$ and let $J_m f : X \to Y$ be a mapping defined by

$$J_m f(x) = 2^{-2m-1}f(2^m x) + f(-2^m x) + 2^{m-1}(f(2^{-m} x) - f(-2^{-m} x))$$

for all $x \in X$. Then we have $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \frac{D f(\Delta(2^j x))}{2 \cdot 4^j (n-1)^2} + \frac{D f(\Delta(-2^j x))}{2 \cdot 4^j (n-1)^2}$$

$$- \frac{2^j D f(\Delta(x/n^{j+1}))}{n-1} + \frac{2^j D f(\Delta(-x/n^{j+1}))}{n-1}$$

for all $x \in X$ and $j \geq 0$. If $m' + m > m \geq 0$, then

$$N'(J_m f(x) - J_{m'+m} f(x), \sum_{j=m}^{m'+m-1} \left(\frac{2^m}{4} + (n-1) \left(\frac{2}{2^p}\right)^{j+1}\right) \frac{nt^p}{(n-1)^2})$$

$$\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ \min \left\{ N'(\frac{D f(\Delta(2^j x))}{2 \cdot 4^j (n-1)^2}, \frac{2^{j+1} nt^p}{2 \cdot 4^j (n-1)^2})ight\}ight.$$

$$N'(\frac{D f(\Delta(x/n^{j+1}))}{n-1}, \frac{2^j nt^p}{2^{(j+1)p}(n-1)})$$

$$N'(\frac{D f(\Delta(-x/n^{j+1}))}{n-1}, \frac{2^j nt^p}{2^{(j+1)p}(n-1)}) \right\} \right\}$$

$$\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ N(2^j x, 2^j t), N \left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}}\right) \right\}$$

$$= N(x, t)$$

for all $x \in X$ and $t > 0$. Therefore the Cauchy sequence $\{J_m f(x)\}$ has the limit $F(x) := N' - \lim_{m \to \infty} J_m f(x)$ and

$$N'(f(x) - J_m f(x), t) \geq N \left(x, \frac{(n-1)^2 t^q}{n \sum_{j=0}^{m-1} \left(\left(\frac{2^m}{4} + (n-1) \left(\frac{2}{2^p}\right)^{j+1}\right)^q}\right)$$

(15)

for each $x \in X$. By the same reasoning as in the proof of Case 2 in Theorem 2.1, the inequality (13) follows from (15) and the rest of the proof is same with that of Case 2 in Theorem 2.1.
Case 3. Finally, we take $0 < q < \frac{1}{2}$ and define $J_m f : X \to Y$ by

$$J_m f(x) = 2^{2m-1}(f(2^{-m}x) + f(-2^{-m}x)) + 2^{m-1}(f(2^{-m}x) - f(-2^{-m}x))$$

for all $x \in X$. Then we have $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \frac{-2 \cdot 4^j Df(\Delta(\frac{x}{2^{j+1}}))}{(n-1)^2} + \frac{-2 \cdot 4^j Df(\Delta(-\frac{x}{2^{j+1}}))}{(n-1)^2}$$

$$- \frac{2^j Df(\Delta(\frac{x}{2^{j+1}}))}{n-1} + \frac{2^j Df(\Delta(-\frac{x}{2^{j+1}}))}{n-1}$$

for all $x \in X$ and $j \geq 0$. If $m' + m > m \geq 0$, then

$$N'(J_m f(x) - J_{m'+m} f(x), \sum_{j=m}^{m'+m-1} \left( \left( \frac{4}{2^p} \right)^{j+1} + (n-1) \left( \frac{2}{2^p} \right)^{j+1} \right) \frac{n t^p}{(n-1)^2})$$

$$\geq \min \bigg\{ \min \left\{ N'(\frac{-2 \cdot 4^j Df(\Delta(\frac{x}{2^{j+1}}))}{(n-1)^2}, \frac{2 \cdot 4^j n t^p}{(n-1)^2 2^{(j+1)p}}), \right. \right.$$  

$$\left. N'(\frac{2 \cdot 4^j Df(\Delta(-\frac{x}{2^{j+1}}))}{(n-1)^2}, \frac{2 \cdot 4^j n t^p}{(n-1)^2 2^{(j+1)p}}), \right. \right.$$  

$$\left. N'(\frac{-2^j Df(\Delta(\frac{x}{2^{j+1}}))}{n-1}, \frac{2^j n t^p}{2^{(j+1)p}(n-1)}), \right. \right.$$  

$$\left. N'(\frac{2^j Df(\Delta(-\frac{x}{2^{j+1}}))}{n-1}, \frac{2^j n t^p}{2^{(j+1)p}(n-1)}) \} \bigg\}$$

$$\geq \min \bigg\{ \min \left\{ N\left( \frac{x}{2^{j+1}}, \frac{t}{2^{j+1}} \right) \right. \right.$$  

$$= N(x, t)$$

for all $x \in X$ and $t > 0$. Therefore the Cauchy sequence $\{J_m f(x)\}$ has the limit $F(x) := N' - \lim_{m \to \infty} J_m f(x)$ and

$$N'(f(x) - J_m f(x), t) \geq N\left( x, \frac{(n-1)^2 q t^q}{\left( n \sum_{j=0}^{m-1} \left( \left( \frac{4}{2^p} \right)^{j+1} + (n-1) \left( \frac{2}{2^p} \right)^{j+1} \right) \right)^q} \right)$$

(16)

for all $x \in X$. By the same reasoning as in the proof of Case 3 in Theorem 2.1, the inequality (13) follows from (16) and the rest of the proof is same with that of Case 3 in Theorem 2.1.

Theorem 3.1 can be regarded as a generalization of the classical stability result in the framework of normed spaces by following. We can show it by the same argument before Corollary 2.3.
Corollary 3.2 Let $f$ be a mapping from a normed space $(X, \| \cdot \|)$ into a Banach space $(Y, || \cdot ||)$ and $p$ a nonnegative real number with $p \neq 1, 2$. If $n$ is a fixed odd number greater than 3 and $f$ satisfies the inequality
\[ |||Df(x_1, x_2, \cdots, x_n)||| \leq ||x_1||^p + ||x_2||^p + \cdots + ||x_n||^p \]
for all $x_1, x_2, \cdots, x_n \in X$. Then there is a unique quadratic-additive function $F : X \to Y$ such that
\[ |||F(x) - f(x)||| \leq \begin{cases} \frac{2n||x||^p}{(2-2p)(n-1)} & \text{if } p > 1, \\ \left(\frac{4}{||x||^2} + \frac{2(n-1)}{||x||^2} \right) \frac{n||x||^p}{(n-1)^2} & \text{if } p < 1 \end{cases} \]
for all $x \in X$.

Remark 3.3 Let $n$ be an arbitrary fixed natural number greater than 2. Consider a mapping $f : X \to Y$ satisfying (4) for all $x_1, x_2, \cdots, x_n \in X$ and a real number $q < 0$. Take any $t > 0$. If we choose a real number $s$ with $0 < ns < t$, then we have
\[ N'(Df(x_1, \cdots, x_n), t) \geq N'(Df(x_1, \cdots, x_n), ns) \geq \min \{N(x_1, s^q), \cdots, N(x_n, s^q)\} \]
for all $x_1, x_2, \cdots, x_n \in X$. Since $q < 0$, we have $\lim_{s \to 0^+} s^q = \infty$. This implies that
\[ \lim_{s \to 0^+} N(x_1, s^q) = \cdots = \lim_{s \to 0^+} N(x_n, s^q) = 1 \]
and so
\[ N'(Df(x_1, \cdots, x_n), t) = 1 \]
for all $x_1, \cdots, x_n \in X$ and $t > 0$. By (N2), it allows us to get $Df(x_1, \cdots, x_n) = 0$ for all $x_1, \cdots, x_n \in X$. In other words, $f$ is itself a quadratic additive mapping if $f$ is a fuzzy $q$-almost quadratic-additive mapping for the case $q < 0$.

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