

Coordinate Descent with Online Adaptation of Coordinate Frequencies

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Abstract

Coordinate descent (CD) algorithms have become the method of choice for solving a number of optimization problems in machine learning. They are particularly popular for training linear models, including linear support vector machine classification, LASSO regression, and logistic regression. We consider general CD with non-uniform selection of coordinates. Instead of fixing selection frequencies beforehand we propose an online adaptation mechanism for this important parameter, called the adaptive coordinate frequencies (ACF) method. This mechanism removes the need to estimate optimal coordinate frequencies beforehand, and it automatically reacts to changing requirements during an optimization run. We demonstrate the usefulness of our ACF-CD approach for a variety of optimization problems arising in machine learning contexts. Our algorithm offers significant speed-ups over state-of-the-art training methods.

1 Introduction

Coordinate Descent (CD) algorithms are becoming increasingly important for solving machine learning tasks. They have superseded other gradient-based approaches such as stochastic gradient descent (SGD) for solving certain types of problems, such as training of linear support vector machines (SVMs) as well as LASSO regression and other L_1 regularized learning problems [8, 15, 6]. There is growing interest in machine learning applications of CD in the field of optimization, see e.g. [24, 27].

Natural competitors for solving large scale convex problems are (trust region/pseudo) Newton methods and (stochastic) gradient descent. In contrast

to these approaches, CD needs only a single component of the full gradient per iteration and is thus particularly efficient if such a partial derivative is much faster to compute than the gradient vector. This is often the case in machine learning problems. The difference in computational effort per step can be huge, differing by a factor as big as the number of data points. Stochastic gradient descent (SGD) has the very same advantage over (plain) gradient descent. An ubiquitous problem of SGD is the need to set a learning rate parameter, possibly equipped with a cooling schedule. This is a cumbersome task, and success of a learning method—at least within reasonable computational limits—can well depend on this choice.

CD algorithms do also have parameters. The most generic such parameter is the frequency of choosing a coordinate for descent, e.g., in randomized CD algorithms. This parameter is not obvious from a machine learning perspective because uniform coordinate selection is apparently dominant in all kinds of applications of CD. This is different in the optimization literature on CD where non-uniform distributions have been considered. This literature also offers a few criteria for choosing selection probabilities (see e.g. [24, 28] and refer to the more detailed discussion in section 2). Interestingly these recommendations are all static in nature, i.e., the selection probabilities are set before the start of the optimization run and are then kept constant. This proceeding may be suitable for simple (i.e., quadratic) objectives, however, it is difficult to propose good settings of the parameters for realistic optimization scenarios.

In contrast, for SGD there have been a number of proposals for adapting such parameters online during the optimization run for optimal progress (see [30] and references therein). This does not only effectively remove the need to adjust parameters to the problem instance before the run, which is anyway often difficult due to missing information. It also allows to react to changing requirements during the optimization run. Similarly, trust region methods and many other optimization strategies take online information into account for adapting their parameters to the local characteristics of the problem instance they are facing.

The present paper proposes an online adaptation technique for the coordinate selection probability distribution of CD algorithms. We refer to this technique as *Adaptive Coordinate Frequencies* (ACF), and to the resulting coordinate descent scheme as ACF-CD. We have first proposed this algorithm in [10, 9]; the present paper broadens and extends this work.

Our approach is inspired by previous work. First of all, the formulation of our method is most natural in the context of random coordinate descent as discussed by [24]. It is closest in spirit to the Adaptive Coordinate Descent algorithm by [22] that adapts a coordinate system of descent directions online with the goal to make steps independent. This algorithm maintains a number of state variables (directions) that are subject to online adaptation. However, this algorithm is deemed to be inefficient unless arbitrary directional derivatives can be computed cheaply, which most often is not the case. Online adaptation in general turns out to be a technique applied in many different optimization strategies (refer to section 4 for a detailed discussion).

The remainder of this paper is organized as follows. First we review the basic coordinate descent algorithm with a focus on coordinate selection techniques and summarize its use for the solution of various machine learning problems. We then review online parameter adaptation techniques applied in different (machine learning relevant) optimization methods. Then we present our online parameter adaptation algorithm for coordinate selection probabilities, followed by a Markov chain analysis of its convergence behavior. The new algorithm is thoroughly evaluated on a diverse set of problems against state-of-the-art CD solvers.

2 Coordinate Descent

We consider convex optimization with variable $w = (w_1, \dots, w_n)$. In the simplest case each $w_i \in \mathbb{R}$ is a real value, however, in general we want to allow for a decomposition of the search space \mathbb{R}^N into n sub-spaces $w_i \in \mathbb{R}^{N_i}$ with $N = \sum_{i=1}^n N_i$. For simplicity we refer to each component w_i as a coordinate in the following. The generalization to subspaces of more than one dimension is implied. We denote the set of coordinate indices by $I = \{1, \dots, n\}$.

Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ denote the objective function to be minimized. Constraints may of course be present; they are handled implicitly in this general presentation since the exact constraint handling technique is problem specific. The basic CD scheme for solving this problem iteratively is presented in algorithm 1.

Algorithm 1 Coordinate Descent (CD) algorithm.

input: $w^{(0)} \in \mathbb{R}^n$

$t \leftarrow 1$

repeat

 select active coordinate $i^{(t)} \in I$

 solve the optimization problem with additional constraints

$$w_j^{(t)} = w_j^{(t-1)} \text{ for all } j \in I \setminus \{i^{(t)}\}$$

$t \leftarrow t + 1$

until stopping criterion is met

CD methods have advantages over other gradient-based optimization schemes if the partial derivatives¹ $\frac{\partial f}{\partial w_i}(w)$ are significantly faster to compute than the complete gradient $\nabla_w f(w)$. In many machine learning problems this difference is of order $\Theta(n)$, i.e., the computation of the full gradient is about n times more expensive than the computation of a partial derivative. Based on a partial derivative a coordinate descent solver performs a step on only the i -th coordinate by solving the (often one-dimensional) sub-problem either optimally or approximately, e.g., with a gradient descent step, a Newton step, line search, or with a problem specific (possibly iterative) strategy.

¹We refer to $\frac{\partial f}{\partial w_i}(w)$ as a partial derivative of f . It is understood that in the case of subspace descent it consists of a vector of N_i partial derivatives.

Convergence properties of CD iterates and their values have been established, e.g. in [23, 34], and runtime analysis results in [28]. A recent analysis in a machine learning context can be found in [31].

2.1 Coordinate Selection

CD can come in a number of variations, e.g., differing in how the sub-problem in each iteration t is solved. Here we want to highlight the selection of coordinates.

A first difference is between deterministic and randomized choices of coordinates $i^{(t)}$. The most prominent deterministic scheme is the simple cyclic rule $i^{(t)} = t \bmod n$. It is often implemented as an outer (epoch) loop and an inner loop sweeping over all coordinates in the set I . This means that all coordinates are visited equally often and always in their natural order. This basic scheme is sometimes randomized by permuting the indices in the inner loop randomly in each epoch. The predefined but often arbitrary order of the coordinates is thus avoided, but this method still sticks to paying equal attention to each coordinate. Both approaches can be viewed as uniform coordinate selection with different dependency structures between variables.

Moving away from the epoch-based approach it is most natural to pick $i^{(t)}$ i.i.d. at random from some distribution π on I . We denote the probability of selecting coordinate i with π_i . The simplest choice for π is the uniform distribution. This choice seems to be distinguished since it is the most obvious and in fact the only unbiased one. Also, it allows to sample an index in constant time (based on a random number generator that samples from the uniform distribution on the unit interval). Nesterov [24] proposes an algorithm for drawing a sample from a non-uniform distribution π in $\log(n)$ time, which is often tolerable.

Better performance (in terms of progress per iteration) can be expected if the best coordinate is chosen in each iterations, e.g., in a greedy manner. This requires knowledge of the full gradient, a prerequisite that usually renders CD methods inefficient. However, there are notable exceptions. The standard solver for training non-linear (kernelized) SVMs is based on the SMO algorithm [25] but with highly developed working set selection heuristics [7, 11]. When dropping the bias term from the SVM model these methods reduce to CD with (approximately) greedy coordinate selection (refer to [32] for an extensive study). The reason for the efficiency of CD in this case is that computation of the full gradient takes $\mathcal{O}(n^2)$ operations, but after a CD step the new gradient can be obtained from the old one in only $\mathcal{O}(n)$ operations. A similar technique has been applied in [16] for a sparse matrix factorization problem. However, even linear time complexity is prohibitive in many applications of CD algorithms. Then greedy selection is not feasible and coordinate selection needs to revert to sampling from a coordinate distribution π .

2.2 Non-uniform Distributions

Despite the seemingly distinguished properties of the uniform distribution it is in general implausible that selecting all coordinates equally often should be optimal. In a machine learning problem a coordinate often corresponds either to a training example or to a feature, and it is understood that some data points (and some features) are more important than others. Such important coordinates should be chosen much more frequently than others.

However, knowing that a non-uniform distribution is advantageous does not tell us in which direction to deviate from uniformity. The question for the relative importance of coordinates for optimization is often about as hard to answer as solving the optimization problem itself.

In the literature on CD the problem of finding good or even optimal probabilities π_i has been addressed mostly in terms of upper runtime bounds, and under the additional assumptions that all partial derivatives are Lipschitz continuous and that upper bounds on the corresponding Lipschitz constants are known. Nesterov derives a runtime bound (equation (2.12) in [24]) that can be minimized given upper bounds on the Lipschitz constants, but the conclusions drawn from his analysis instead consider the achievable *worst case* convergence rate as compared to other approaches. A more direct approach is proposed by Richtárik and Takáč in [28] where minimization of a runtime bound for a fixed problem instance is proposed as an optimization strategy (see section 4 in [28]).

These approaches offer invaluable theoretical insights but turn out to be problematic in practice. If coordinates correspond to data points or features then the due to symmetrical treatment of all coordinates in the machine learning optimization problem all a-priori upper bounds on the Lipschitz constants of coordinate-wise derivatives coincide, resulting in uniform coordinate selection. Data-dependent bounds can be tighter and promise to be non-uniform. However, their computation may be too costly to be practical. Even worse, for the procedure to be effective these bounds would need to be continuously updated since the relative importance of variables can change drastically during an optimization run.

It may be for the reasons outlined above or just for simplicity of concepts and implementations that standard algorithms in statistics and machine learning rely nearly exclusively on uniform coordinate selection. Actually, the only exception we are aware of is the shrinking technique for linear SVM optimization (see section 3.2).

3 Coordinate Descent in Machine Learning

CD methods have been popularized in statistics and machine learning especially for certain regularized empirical risk minimization problems. CD methods are particularly well suited for problems with sparse solutions. One advantage is that they can quickly set single coordinates to exact zero. This is in contrast to (stochastic) gradient descent, which is often the most natural competitor.

Hence intermediate solutions are often sparse, which can greatly speed up computations. A follow up advantage is that sparsity can be taken into account by coordinate selection algorithms. This insight is at the heart of shrinking techniques for SVM training [19, 7, 6].

In machine learning, sparsity is often a result of regularization, most prominently with an L_1 penalty on the weight vector of a linear model, which is the case in least absolute shrinkage and selection operator (LASSO) models [8]. Logistic regression with L_1 regularization is another prominent example [37, 38]. Alternatively, sparsity (of the dual solution) can result from the empirical risk term, e.g., in a support vector machine with hinge loss. CD training of linear SVMs has been demonstrated to outperform competing methods [15].

In the following we present four prototypical supervised machine learning problems that are commonly solved with CD algorithms. They will serve as testbeds throughout this paper. Of course there exist many more application areas such as sparse matrix factorization [16], stochastic variational inference [14], and others. We start with data $\{(x_1, y_1), \dots, (x_\ell, y_\ell)\}$ composed of inputs $x_i \in \mathbb{R}^d$ and labels $y_i \in Y$. Let $L(h(x), y)$ denote a loss function comparing model outputs $h(x)$ and ground truth labels y . The (primal, unconstrained) regularized empirical risk minimization training problem of the linear predictor $h_w(x) = \langle w, x \rangle$ amounts to

$$\min_{w \in \mathbb{R}^d} f(w) = \frac{\lambda}{p} \|w\|_p^p + \frac{1}{\ell} \sum_{i=1}^{\ell} L(\langle w, x_i \rangle, y_i) \quad (1)$$

where p is typically either 1 or 2 and $\lambda > 0$ is a complexity control parameter.

3.1 The LASSO

The LASSO problem is an instance of this problem with $p = 1$. In its simplest form it is applied to a regression problem with $Y = \mathbb{R}$ and $L(h(x), y) = \frac{1}{2}(h(x) - y)^2$. Friedman et al. [8] propose to solve this problem with CD with a simple cyclic coordinate selection rule.

With all coordinates except i fixed the resulting one-dimensional problem is piecewise quadratic. The empirical risk term is a quadratic term, and the regularizer restricted to coordinate w_i reduces to $\lambda|w_i|$. Given the partial derivative $\frac{\partial f(w)}{\partial w_i}$ this problem can be solved in constant time: a gradient step equals a Newton step since the second derivative equals one, and a case distinction needs to be made for whether the component w_i after the Newton step is optimal, changed sign, or ends up at exact zero.

The most costly step is the computation of the derivative. It takes $O(n_{\text{nz}})$ operations with n_{nz} denoting the number of non-zeros in the i -th *column* of the data matrix X composed of the inputs x_1, \dots, x_ℓ , i.e., the number of inputs with non-zero i -th component $(x_j)_i$. We will find in the following that this situation is rather typical.

3.2 Linear SVMs

With binary classification labels $Y = \{-1, +1\}$, hinge loss $L(h(x), y) = \max\{0, 1 - yh(x)\}$ and $p = 2$ we obtain the linear soft margin SVM from equation (1). Hsieh et al. [15] solve the corresponding dual problem

$$\begin{aligned} \min_{\alpha \in \mathbb{R}^\ell} \quad & f(\alpha) = \frac{1}{2} \sum_{i,j=1}^{\ell} \alpha_i \alpha_j y_i y_j \langle x_i, x_j \rangle - \sum_{i=1}^{\ell} \alpha_i \\ \text{s.t.} \quad & 0 \leq \alpha_i \leq C = \frac{1}{\lambda} \end{aligned} \quad (2)$$

with CD. The key technique for making CD iterations fast is to keep track of the model vector $w = \sum_{i=1}^{\ell} \alpha_i y_i x_i \in \mathbb{R}^d$ during optimization. A CD step in this box-constrained quadratic program amounts to a one-dimensional, interval-constrained Newton step. The first derivative of the objective function w.r.t. α_i is $y_i \langle w, x_i \rangle - 1$, the second derivative is $\langle x_i, x_i \rangle$, which can be precomputed. The resulting CD step reads

$$\alpha_i^{(t)} = \left[\alpha_i^{(t-1)} - \frac{1 - y_i \langle w, x_i \rangle}{\langle x_i, x_i \rangle} \right]_0^C,$$

where $[x]_a^b = \max\{a, \min\{b, x\}\}$ denotes truncation of the argument x to the interval $[a, b]$. With densely represented $w \in \mathbb{R}^d$ and sparse data x_i the complexity of a step is not only independent of the data set size ℓ , but even as low as the number of non-zero entries in x_i (and therefore often much lower than the data dimension d). Again we arrive at a complexity of $\mathcal{O}(n_{\text{nz}})$, where in this case n_{nz} denotes the number of non-zeros in the i -th *row* of the data matrix X .

The liblinear algorithm [15, 6] applies a shrinking heuristic that removes bounded variables from the problem during optimization. This technique was originally proposed for non-linear SVM optimization [19], and it can give considerable speed-ups. From a CD perspective this technique sets the probabilities π_i of removed coordinates to zero while normalizing the remaining probabilities to a uniform distribution on the active variables. As such it is the only technique in common use that actively adapts coordinate selection probabilities π_i online during the optimization run. It should be noted that the decision which coordinates to remove is based on a heuristic. No matter how robust this heuristic is designed, it is subject to infrequent failure, resulting in costly convergence to a sub-optimal point, followed by a warm-start.

Online adaptation of π is well justified for SVM training. This is because the relative importance of coordinates changes significantly over the course of an optimization run. Consider a variable α_i corresponding to an outlier (x_i, y_i) . At first, starting at $\alpha_i = 0$, this coordinate is extremely important since it needs to move by the maximal possible amount of C . Once it arrives at the upper bound the constraint $\alpha_i \leq C$ becomes active, essentially fixing the variable at its current value. Thus its importance for further optimization drops to zero. Of course the constraint may become inactive later on. Hence, any a-priori

estimation of relative importance of coordinates (e.g., based on upper bounds on Lipschitz constants) is not helpful in this case and online adaptation of π is of uttermost importance.

3.3 Multi-class SVMs

A non-trivial extension to this problem is multi-class SVM classification. Different extensions to the binary problem exist. The arguably simplest and most generic one is the one-versus-all approach that reduces the multi-class problem to a set of two-class problems. Other approaches attempt to generalize the margin concept to multiple classes. There is the classic approach by Weston and Watkins [36] that turns out to be equivalent to a similar proposals in [35] and [4]. A popular alternative was proposed by Crammer and Singer [5], and a considerably different approach by Lee et al. [20]. All of these approaches differ only in the type of (piecewise linear) large-margin loss function as a multi-class replacement for the hinge loss.

Although these methods were originally designed for non-linear SVMs they can be applied for linear SVM training. From an optimization perspective a decisive difference from binary classification is that the corresponding dual problems contain $\mathcal{O}(K)$ variables per training example, where K denotes the number of classes. This calls for a solution with proper subspace descent with $N_i \in \mathcal{O}(K) > 1$. Sub-problems can be solved either with a general purpose QP solver or with a SMO-style technique [25].

We consider the “default” multi-class SVM proposed by Weston and Watkins. Its dual problem is a box-constrained quadratic program that could well be solved with a standard CD approach. Treating it as a subspace descent problem is computationally attractive: once the partial derivative is computed the (non-trivial) sub-problem can be solved to arbitrary precision – without a need for further derivative computations in each sub-step.

3.4 Logistic Regression

Logistic regression is closely related to the linear SVM problem, but with smooth loss function $L(h(x), y) = \log(1 + \exp(-yh(x)))$ replacing the non-smooth hinge loss. Its dual problem

$$\begin{aligned} \min_{\alpha \in \mathbb{R}^\ell} f(\alpha) &= \frac{1}{2} \sum_{i,j=1}^{\ell} \alpha_i \alpha_j y_i y_j \langle x_i, x_j \rangle + \sum_{i=1}^{\ell} \alpha_i \log(\alpha_i) + (C - \alpha_i) \log(C - \alpha_i) \\ \text{s.t.} \quad 0 &\leq \alpha_i \leq C = \frac{1}{\lambda} \end{aligned} \tag{3}$$

can be solved efficiently with CD methods. The dual variables are connected through a quadratic term, while the more difficult to handle logarithmic terms as well as the constraints are separable. This problem shares many properties with the dual linear SVM problem, however, the logarithmic terms do not allow for an exact solution of the one-dimensional CD sub-problem. Instead an

iterative solver with second order steps is employed in [37] and implemented in liblinear [6]. Also, the solution is dense which means that shrinking techniques are not applicable. Thus this problem is solved with uniform coordinate probabilities (coordinate sweeps in random order).

4 Online Parameter Adaptation

The coordinate sampling distribution π is represented as an n -dimensional vector $\pi \in \mathbb{R}^n$, subject to simplex constraints $\pi_i \geq 0$ and $\sum_{i=1}^n \pi_i = 1$. This vector is a parameter of the CD algorithm, and as such it can be subject to online tuning. Before we propose such a procedure in the following section we review existing parameter adaptation techniques.

Online adaptation of the values of algorithm parameters is found in many different types of algorithms. Here we restrict ourselves to optimization techniques. Driving this to the extreme one may consider any non-trivial aspect of the state of an optimization algorithm that goes beyond the current search point (and its properties, such as derivatives) as parameters that may or may not be subject to online adaptation.

Sometimes parameters can be set to robust default values, making online adaptation essentially superfluous. However, it turns out that in many cases performance can be increased when tuning such parameters to the problem *instance* at hand, which is often most efficiently done in an online fashion. In the most extreme case an algorithm can break down completely if online adaptation of a parameter is switched off. The types of parameters present in different optimization algorithms differ vastly. We revisit a few prototypical examples in the following.

4.1 Stochastic Gradient Descent

Stochastic Gradient Descent (SGD) algorithms are widespread in machine learning. They are appreciated for their ability to deliver a usable (although often poor) model even before the first sweep over the data set is finished. In a machine learning context stochasticity is introduced artificially into the gradient descent procedure by approximating the empirical risk term

$$\frac{1}{\ell} \sum_{i=1}^{\ell} L(h(x_i), y_i)$$

with the loss $L(h(x_i), y_i)$ for a single pattern. This estimate is unbiased, and this property carries over to its gradient $\nabla_w L(h_w(x_i), y_i)$. For the sake of simplicity we assume minimization of the empirical risk in the following. Then an SGD iteration performs the update step

$$w^{(t)} = w^{(t-1)} - \eta^{(t)} \cdot \nabla_w L(h_w(x_i), y_i)$$

where $\eta^{(t)} > 0$ is a learning rate. Cooling schedules such as $\eta^{(t)} \sim 1/t$ result in strong convergence guarantees [3], however, in some cases they turn out to be inefficient in practice. This problem has been solved by online adaptation, e.g., by Schaul et al. [30]. In their approach the objective function is modeled as a quadratic function. The model is estimated online from the available information. The learning rate is then adjusted in a way that is optimal given the current model. This scheme was demonstrated to outperform plain SGD as well as number of alternative methods on the task of training deep neural networks (see [30] and references therein).

4.2 Resilient Propagation

The Resilient Propagation (Rprop) algorithm was originally designed for back-propagation training of neural networks [29]. However, it constitutes a general optimization technique. The method maintains coordinate-wise step sizes $\gamma_i > 0$. In each iteration the algorithm roughly follows the gradient of the objective function by evaluating only the signs of the partial derivatives:

$$w_i^{(t)} = w_i^{(t-1)} - \text{sign} \left(\frac{\partial f(w^{(t-1)})}{\partial w_i} \right) \cdot \gamma_i$$

The coordinate-wise step sizes are adjusted by multiplication with a constant $\eta^+ > 1$ or $\eta^- < 1$ if the signs of derivatives in consecutive iterations agree or disagree, respectively. This simple scheme turns out to be highly efficient for many problems. It has been further refined and evaluated in [18].

It is understood that this algorithm would show poor performance without online adaptation of γ_i . First of all it would be restricted to a fixed grid of values. Even worse, it would be unable to adjust its initial step size settings to the characteristics of a problem instance. Thus it would most probably be deemed to making either too large or too small steps.

4.3 Evolution Strategies

Evolution Strategies (ES) are a class of randomized zeroth order (direct) optimization methods. In the last 15 years these evolutionary algorithms have evolved into highly efficient optimizers. In each iteration (called generation in the respective literature) the algorithm samples one or more search points (called offspring individuals) from a Gaussian search distribution $\mathcal{N}(\mu, \Sigma)$. The mean μ is centered either on the best sample so far or on a weighted mean of recent best samples. In simple ES the covariance matrix is restricted to the form $\Sigma = \sigma^2 \mathbf{I}$ (with \mathbf{I} denoting the unit matrix in \mathbb{R}^n). The “step size” parameter $\sigma > 0$ turns out to be crucial. Any fixed choice results in extremely poor search performance. This is an example of a parameter that cannot be fixed beforehand. Instead it needs to decay and remain roughly proportional to the distance to the optimum. Various schemes exist for its online adaptation, e.g., the classic 1/5-rule [26]. Modern ES treat the whole covariance matrix

$\Sigma \in \mathbb{R}^{n \times n}$ as a free parameter. It is adapted essentially by low-pass filtered weighted maximum likelihood estimation of the distribution that generates the most successful recent search points [13, 33, 12, 2]. Online parameter adaptation is an essential integral building block of these algorithms. On the other hand, and despite their practical success, some of these mechanisms lack satisfactory theoretic backup.

It is understood that our discussion of online parameter adaptation techniques remains incomplete. For example, we did not cover trust region Newton methods such as the Levenberg-Marquardt algorithm. The various examples in this section should in any case suffice to demonstrate that online parameter adaptation is a powerful and sometimes critical technique for optimization performance. This naturally raises the question why it has to date not been applied to the coordinate selection distribution π of the CD algorithm.

5 Online Adaptation of Coordinate Frequencies

In this section we develop an online adaptation method for the CD coordinate selection distribution π . As a first step towards a practical algorithm we ask the following questions:

1. What is the goal of adaptation?
2. Which quantity should trigger adaptation, i.e., which compact statistics of the optimization history indicates that adaptation is beneficial, and into which direction to adapt?

The answer to the first question seems clear: we'd like to minimize the runtime of the CD algorithm, or nearly equivalently, to maximize the pace of convergence to the optimum. Since the optimum is of course unknown this condition is hard to verify. However, experimentation with a controlled family of CD problems as done in [9] (and which was redone in a cleaner fashion, see section 6) reveals that this property seems to coincide with an easy-to-measure statistics: maximization of the rate of convergence on unconstrained quadratic problems is highly correlated with the fact that on average the relative progress

$$\frac{f(w^{(t)}) - f(w^{(t-1)})}{f(w^{(t)}) - f^*} \tag{4}$$

(where f^* denotes the optimal objective value) becomes independent of the coordinate doing the step.² This observation provides us with a powerful tool, namely with the working assumption that

- (a) maximizing the convergence rate and

²This statement and several related ones in this section are on an intuitive level; they will be made rigorous in the next section.

- (b) making average relative progress equal for all coordinates³

are equivalent. We decide for (b) as our answer to the first question in the following.

This results in a straightforward quantity to monitor, namely average relative progress per coordinate, or more precisely, their differences. A similar and closely related observation is that increasing π_i for some coordinate i decreases relative progress, on average. This makes intuitive sense since progress in that coordinate is split over more frequent and hence smaller steps. Assuming a roughly monotonically decreasing relation we should increase π_i as soon as relative progress with coordinate i is above average and the other way round. This answers the second question above, namely how to do the adaptation.

Next we turn these concepts into an actual algorithm. Monitoring relative progress is impossible without knowledge of the optimum. However, CD algorithm often make relatively little progress per step so that the denominator in equation (4) can be assumed to remain nearly constant over a considerable number of iterations. Thus we may well replace relative progress with absolute progress $\Delta f = f(w^{(t)}) - f(w^{(t-1)})$, which is just the numerator of equation (4).

Formally speaking we have added an assumption that goes beyond the standard CD algorithm at this point: we assume that the progress Δf can be computed efficiently. It turns out that in many cases including all examples given in section 3 the computation of Δf is a cheap (constant time) by-product of the CD step.

It holds $\pi_i \leq 1/n$ for at least one coordinate i (and usually for the majority of them), so that for large n only very few progress samples can be acquired per coordinate, and we should avoid relying on too old samples. Therefore we do not perform any averaging of coordinate-wise progress. Instead we maintain an exponentially fading record of overall average progress, denoted by \bar{r} , and compare each single progress sample against this baseline in order to judge whether progress in coordinate i is better or worse than average.

The difference $\Delta f - \bar{r} = [f(w^{(t)}) - f(w^{(t-1)})] - \bar{r}$ triggers a change of π_i . The exact quantitative form of this update is rather arbitrary, and many possible forms should work just fine as long as the change is into the right direction and the order of magnitude of the change is reasonable. For efficiency reasons we do not represent π_i in the algorithm directly, instead we adapt unnormalized preferences p_i , track their sum $p_{\text{sum}} = \sum_{i=1}^n p_i$, and define $\pi_i = p_i/p_{\text{sum}}$. Then we manipulate p_i according to the update rule

$$p_i \leftarrow \left[\exp \left(c \cdot \left(\frac{\Delta f}{\bar{r}} - 1 \right) \right) \cdot p_i \right]_{p_{\min}}^{p_{\max}}$$

where p_{\min} and p_{\max} are lower and upper bounds and $[t]_a^b = \min\{\max\{t, a\}, b\}$ denotes clipping of t to the interval $[a, b]$. Given a coordinate i and its single

³Note that we need to exclude trivial solutions such as putting all probability mass on one coordinate, which makes progress vanish in all coordinates. For technical details refer to the next section.

step progress Δf this update step is made formal in algorithm 2. We call it the *Adaptive Coordinate Frequencies* (ACF) method. Its parameters are the lower and upper bounds p_{\min} and p_{\max} and the learning rates c for preference adaptation and the exponential fading record \bar{r} of average progress. Default values for these parameters are given in table 1. The algorithm state consists of π and \bar{r} . The former can be initialized to the uniform distribution unless a more informed setting is available, the latter can be initialized to the average progress over a brief warm-up phase (without adaptation), i.e., a single sweep over the coordinates.

Algorithm 2 Adaptive Coordinate Frequencies (ACF) Update

$$\begin{aligned}
 p_{\text{new}} &\leftarrow \left[\exp(c \cdot (\Delta f / \bar{r} - 1)) \cdot p_i \right]_{p_{\min}}^{p_{\max}} \\
 p_{\text{sum}} &\leftarrow p_{\text{sum}} + p_{\text{new}} - p_i \\
 p_i &\leftarrow p_{\text{new}} \\
 \bar{r} &\leftarrow (1 - \eta) \cdot \bar{r} + \eta \cdot \Delta f
 \end{aligned}$$

parameter	value
c	1/5
p_{\min}	1/20
p_{\max}	20
η	1/n

Table 1: Default parameter values for the ACF algorithm. These values were set rather ad-hoc; in particular they did not undergo extensive tuning. The algorithm was found to be rather insensitive to these settings.

Until now we did not specify how samples are drawn from π . I.i.d. coordinate selection is the simplest possibility, however, it requires $\Theta(\log(n))$ time per sample [24]. This is in contrast to uniform selection, the time complexity of which is independent of n . Ideally we would like to achieve the same for an arbitrary distribution π . This can be done by relaxing the i.i.d. assumption and instead selecting coordinates in blocks of size $\Theta(n)$. Here we present a deterministic variant for drawing $\Theta(n)$ samples from π in $\Theta(n)$ operations, hence with amortized constant time complexity per CD iteration. Despite drawing a finite set of indices algorithm 3 respects the exact distribution π over time with the help of accumulator variables $a = (a_1, \dots, a_n)$.

The algorithm outputs a sequence of on average n and at most $2 \cdot n$ coordinates at a time at a cost of $\Theta(n)$ operations while guaranteeing that each coordinate has a waiting time of at most

$$\lceil 1/(n \cdot p_i) \rceil \leq \lceil 1/(n \cdot p_{\min}) \rceil = \tau < \infty$$

Algorithm 3 Creation of a sequence J of coordinates according to π

```

 $J \leftarrow \{\}$ 
for  $i \in I$  do
   $a_i \leftarrow a_i + n \cdot p_i / p_{\text{sum}}$ 
   $\lfloor a_i \rfloor$  times: append index  $i$  to list  $J$ 
   $a_i \leftarrow a_i - \lfloor a_i \rfloor$ 
end for
shuffle list  $J$ 

```

sweeps for its next inclusion. This property guarantees convergence of the resulting CD algorithm with the same arguments as in the proof of theorem 1 by [15], which is based on theorem 2.1 in [23]. Alternatively, in the terminology of Tseng [34] algorithm 3 realizes an *essentially cyclic rule* for coordinate selection. Thus, our ACF-CD algorithm enjoys the same convergence guarantees as other CD schemes with fixed, e.g., cyclic coordinate selection.

6 Randomized CD as a Markov Chain

In this section we analyze the qualitative behavior of the ACF algorithm. We formalize most of the intuition presented in the previous section in terms of (properties of) Markov chains.

In a first step we capture the behavior of the CD algorithm for a fixed distribution π . Then we formalize a central conjecture in mathematical terms and show—based on this assumption—that in expectation the ACF method drives this distribution into the vicinity of the optimal distribution.

We perform this analysis for an unconstrained quadratic problem

$$\min_{w \in \mathbb{R}^n} f(w) = \frac{1}{2} w^T Q w$$

with strictly positive definite, symmetric Hessian matrix $Q \in \mathbb{R}^{n \times n}$.

A particularly simple (e.g., diagonal) structure of Q allows to locate the optimum exactly after finitely many iterations (e.g., after exactly n iteration with a cyclic coordinate selection rule). In this case an exact runtime analysis is trivial, but this case does not play any role in practice. In the remainder of this section we consider the more relevant case of an infinite chain: we assume $P(w^{(t)} = 0) = 0$ for all $t \in \mathbb{N}$. This will allow us to “divide by $f(w)$ ”.

The unconstrained quadratic problem is relevant for the understanding of the convergence speed of CD on a large class of optimization problems. For this sake every twice continuously differentiable objective function f can be well approximated in the vicinity of the optimum by its second order Taylor polynomial, and under mild technical assumptions it can be assume that after some iterations t_0 all constraints either remain active or inactive so that the problem can be treated essentially as an unconstrained problem on the free

variables. In the context of SVM optimization such an argument is found e.g. in [7] (based on an earlier result by Lin [21]).

6.1 The CD Markov Chain for fixed π

We start with the analysis for fixed π , i.e., without ACF. In each iteration $t \in \mathbb{N}$ the algorithm picks an index $i^{(t)} \in I$ according to a predefined distribution π on I and then solves the one-dimensional sub-problem in $w_{i^{(t)}}$ optimally⁴ with a one-dimensional Newton step

$$w_{i^{(t)}}^{(t)} = w_{i^{(t)}}^{(t-1)} - \frac{Q_{i^{(t)}}^T w^{(t-1)}}{Q_{i^{(t)}, i^{(t)}}},$$

where Q_i denotes the i -th column of Q . This iteration scheme is expressed equivalently in vector notation as $w^{(t)} = T_{i^{(t)}} w^{(t-1)}$ with

$$T_i = \begin{pmatrix} 1 & & \dots & & 0 \\ \vdots & \ddots & & & \vdots \\ -\frac{Q_{i1}}{Q_{ii}} & \dots & 1 - \frac{Q_{ii}}{Q_{ii}} = 0 & \dots & -\frac{Q_{in}}{Q_{ii}} \\ \vdots & & & \ddots & \vdots \\ 0 & & \dots & & 1 \end{pmatrix}.$$

The matrix T_i fulfills $T_i^2 = T_i$; it defines a projection onto the hyperplane $H_i = \{w \in \mathbb{R}^n \mid Q_i^T w = 0\}$. The transition operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $T(w) = T_i w$ with $i \sim \pi$ performs one iteration of the randomized CD algorithm.⁵

For convenience, the distribution π is represented as an element of the probability simplex

$$\Delta = \left\{ p \in \mathbb{R}^n \mid p_i \geq 0 \text{ and } \sum_{i=1}^n p_i = 1 \right\}.$$

With $\overset{\circ}{\Delta}$ we denote the interior of the simplex. Thus $\pi \in \overset{\circ}{\Delta}$ is equivalent to $\pi_i > 0$ for all $i \in I$.

The problem instance Q and the distribution π define a time homogeneous Markov chain $w^{(t)} \in \mathbb{R}^n$ with random transition operator T . We start out by collecting elementary properties of this chain.

Assume the optimization works as expected then the chain $w^{(t)}$ converges to the optimum. Thus the only stationary limit distribution should be a Dirac peak over the optimum. This distribution does not provide any insights into the actual optimization process. One way of describing the regularity of the process

⁴Optimality of course refers to single-step behavior, i.e., the algorithm solves the one-dimensional sub-problem in a greedy manner.

⁵The square matrices T_i are linear operators on states, not on probability distributions. They are not to be confused with transition matrices of Markov chains on finite state spaces, even if their role is similar.

is by considering the distribution of *directions* from which the optimum is approached. This property can be captured by a scale invariant state description. In the following we construct a scale-invariant Markov chain with a non-trivial limit distribution.

Lemma 1. *The Markov chain is scale invariant, i.e., the transition operator commutes with scaling by any factor $\alpha \neq 0$.*

Proof. We have to show that $T(\alpha \cdot w) = \alpha \cdot T(w)$. This is a trivial consequence of the linearity of T_i , since application of T amounts to the application of a random T_i , all of which are linear operators. \square \square

Scaling the initial solution $w^{(0)}$ by a scalar factor $\alpha \neq 0$ results in the chain $\alpha \cdot w^{(t)}$. Hence the projection of the chain onto the projective space $\mathbb{P}(\mathbb{R}^n)$ is well-defined. The projective space is the “space of lines”, i.e., the space of equivalence classes of the relation $w \sim w' \Leftrightarrow w = \alpha \cdot w'$ for some $\alpha \neq 0$ on $\mathbb{R}^n \setminus \{0\}$. Equivalently, the projective space is obtained by identifying antipodal points on the sphere; it is thus compact. We denote the corresponding chain of equivalence classes (lines) by $z^{(t)} = \kappa(w^{(t)})$. Here $\kappa : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{P}(\mathbb{R}^n)$, $\kappa(w) = (\mathbb{R} \setminus \{0\}) \cdot w$ denotes the canonical projection.

Any CD step with coordinate $i \in I$ ends on the hyperplane H_i : $T_i w \in H_i$ for all $i \in I$ and $w \in \mathbb{R}^n$, and hence $w^{(t)} \in H_{i^{(t)}}$. Let $\mu^{(t)}$ denote the distribution of $w^{(t)}$. The support of the distribution $\mu^{(t)}$, $t \in \mathbb{N}$, is restricted to the union $H = \bigcup_{i=1}^n H_i$ of the hyperplanes H_i . The distribution can hence be written as a superposition $\mu^{(t)} = \sum_{i=1}^n \pi_i \mu_i^{(t)}$, where each $\mu_i^{(t)}$ is a distribution on H_i .

Lemma 2. *Consider $\pi \in \hat{\Delta}$. For each $w \in \mathbb{R}^n \setminus \{0\}$ we define the expected one-step progress rate $r(\pi, w) = \mathbb{E}[f(T(w))]/f(w)$. Then there exists a constant $U_\pi < 1$ such that it holds $r(\pi, w) \leq U_\pi$ for all $w \in \mathbb{R}^n$. In other words the expected distance to the optimal value $\mathbb{E}[f(w^{(t)})]$ converges to zero at least at a linear rate of U_π .*

Proof. For each w progress can be made in at least one coordinate $i \in I$ and because of $\pi_i > 0$ it holds $r(\pi, w) < 1$. The function $r : \hat{\Delta} \times (\mathbb{R}^n \setminus \{0\}) \rightarrow [0, 1)$ depends continuously on w and on π . Furthermore we have $r(\pi, w) = r(\pi, \alpha \cdot w)$ for all $\alpha \neq 0$ by scale invariance, which means that $r(\pi, \cdot)$ can be lifted to $\mathbb{P}(\mathbb{R}^n)$, the compactness of which implies that the supremum

$$U_\pi = \sup_{w \in \mathbb{R}^n \setminus \{0\}} \left\{ r(\pi, w) \right\}$$

is attained. It follows $U_\pi < 1$. \square \square

We are interested in the dependency of the rate of convergence on the distribution π since we aim to eventually improve or even maximize the progress rate of the CD algorithm.

For all $t \in \mathbb{N}$ we define the mixture distributions $\nu^{(t)} = \sum_{i=1}^n \pi_i \nu_i^{(t)}$ on $\mathbb{P}(\mathbb{R}^n)$, defined by $\nu_i^{(t)}(E) = \mu_i^{(t)}(\kappa^{-1}(E))$ for all measurable $E \subset \mathbb{P}(\mathbb{R}^n)$. The

support of $\nu_i^{(t)}$ is restricted to $\kappa(H_i) \subset \mathbb{P}(\mathbb{R}^n)$. By definition it holds $z^{(t)} \sim \nu^{(t)}$. Our further analysis is based on the fact that the scale invariant component $z^{(t)}$ inherits the Markov property.

Lemma 3. *The scale-invariant variables $z^{(t)}$ form a time-homogeneous Markov chain on the compact space $\mathbb{P}(\mathbb{R}^n)$.*

Proof. We show that the transition operator T lifted to $\mathbb{P}(\mathbb{R}^n)$ is well-defined, i.e., that it holds $\kappa(w) = \kappa(w') \Rightarrow \kappa(T(w)) = \kappa(T(w'))$. This is a trivial consequence of scale invariance: $\kappa(w) = \kappa(w')$ implies the existence of $\alpha \neq 0$ such that $w' = \alpha \cdot w$, and hence

$$\begin{aligned} \kappa(T(w')) &= \kappa(T(\alpha \cdot w)) \\ &= \kappa(\alpha \cdot T(w)) \\ &= \kappa(T(w)) . \end{aligned}$$

Let T' denote the now well-defined lift of the transition operator, and T'_i the corresponding step with coordinate index i . Then $z^{(t)} = T'(z^{(t-1)})$ depends on the chain's history only through its predecessor state. Time-homogeneity of $z^{(t)}$ is a direct consequence of time-homogeneity of $w^{(t)}$. \square \square

Under weak technical assumptions $\nu^{(t)}$ converges to a stationary distribution ν^∞ , e.g., by excluding exact cycles of $z^{(t)}$.⁶ It is important to note that the stationary distribution is *not independent* of the initial state $z^{(0)}$.

Lemma 4. *The distribution ν^∞ is of the form $\nu^\infty = \sum_{i=1}^n \pi_i \nu_i^\infty$, with the support of ν_i^∞ restricted to $\kappa(H_i)$.*

Proof. The form of ν^∞ is an elementary consequence of the homogeneous forms of $\nu^{(t)} = \sum_{i=1}^n \pi_i \nu_i^{(t)}$ with the same coefficients π_i for all $t \in \mathbb{N}$. \square \square

The quantity of ultimate interest is the progress of the chain $w^{(t)}$ towards the optimum while the projection $z^{(t)}$ converges to its stationary distribution. This progress rate is captured as follows. We define the coefficients

$$\rho_{ij} = \mathbb{E}_{z \sim \nu_i^\infty} \left[\log(f(w)) - \log(f(T_j w)) \right]$$

measuring average progress of transitions from H_i to H_j . Note that $\log(f(w)) - \log(f(T_j w))$ is invariant under scaling of w and thus well-defined given $z = \kappa(w)$. The aggregations

$$\begin{aligned} \rho_i &= \mathbb{E}_{j \sim \pi} [\rho_{ij}] = \sum_{j=1}^n \pi_j \rho_{ij} \\ \rho &= \mathbb{E}_{i,j \sim \pi} [\rho_{ij}] = \sum_{i,j=1}^n \pi_i \pi_j \rho_{ij} \end{aligned} \tag{5}$$

⁶This is another minor technical prerequisite on the problem instance Q ; it essentially excludes a zero set of instances.

measure average progress of steps with coordinate $i \in I$ and overall average progress, respectively.

Lemma 5. *It holds*

$$\begin{aligned} \rho &= \lim_{t \rightarrow \infty} \frac{1}{t} \cdot \mathbb{E} \left[\log(f(w^{(0)})) - \log(f(w^{(t)})) \right] \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \cdot \left[\log(f(w^{(0)})) - \log(f(w^{(t)})) \right]. \end{aligned}$$

where the last equality holds almost everywhere.

Proof. The iterates take the form $w^{(t)} = T^t w^{(0)}$ with T being random matrices from the set $\{T_1, \dots, T_n\}$, distributed according to π . The multiplicative ergodic theorem by Oseledec (theorem 1.6 and corollary 1.7 in [1]) guarantees that the limits of the sequences $\frac{1}{t} \log(\|T^t\|)$ (with $\|\cdot\|$ denoting a sub-multiplicative matrix norm) and $\frac{1}{t} \log(\|T^t w^{(0)}\|)$ exist a.e. An application with norm $\|w\|_Q = \sqrt{w^T Q w} = \sqrt{f(w)}$ (and the induced matrix norm) gives the second equality. The first equality is an immediate consequence of the definition of ρ and the fact that the chain $z^{(t)}$ converges to its equilibrium distribution. $\square \quad \square$

The relation $\exp(-\rho) \leq U_\pi < 1$ is obvious, where U_π is the constant defined in lemma 2. The asymptotic convergence rate is given by $\exp(-\rho)$, while U_π is a (non-asymptotic) upper bound.

6.2 Optimal Coordinate Distribution and ACF

The goal of coordinate frequency adaptation is to maximize the pace of convergence, or equivalently to maximize ρ . In this context we understand $\rho = \rho(\pi)$ as a function of π with an implicit dependency on the (fixed) problem instance Q . We aim for an adaptation rule that drives π towards a maximizer of ρ .

The CD algorithm converges to the optimum (with full probability) for all interior points $\pi \in \overset{\circ}{\Delta}$ (since all coordinates are selected arbitrarily often), and it converges to a sub-optimal point for boundary points $\pi \in \partial\Delta$ (since at least one coordinate remains fixed). We conclude that boundary points cannot be maximizers of ρ . Hence continuity of ρ implies the existence of a maximizer $\pi^* \in \arg \max_\pi \{\rho(\pi)\} \subset \overset{\circ}{\Delta}$ in the interior of the simplex.

The identification of the problem-dependent distribution π^* is thus the goal of ACF online adaptation. This could be attempted by maximization of equation (5), which requires a descent understanding of the Markov chain $z^{(t)}$ and its stationary distribution $\nu^{(\infty)}$.

It turns out that most standard tools for the analysis of continuous state space Markov chains are not applicable in this case. For example, the chain is not ϕ -irreducible⁷ and thus we cannot hope that its stationary distribution ν^∞ is independent of the starting state $z^{(0)}$ —however, we conjecture that the

⁷This is easy to see from the fact that only countably many points are reachable from each initial point.

resulting progress rate ρ is. The exact functional dependency of ρ on π (and on Q) turns out to be complicated, and its detailed analysis is beyond the scope of this paper. This situation excludes direct maximization of equation (5). Instead we propose an indirect way of identifying π^* by means of the following conjecture, which is a formalization of the empirical observation presented in section 5:

Conjecture 1. *The maximizer π^* of the progress rate $\rho(\pi)$ is the only distribution in $\hat{\Delta}$ that fulfills the equilibrium condition $\rho_i(\pi^*) = \rho(\pi^*)$ for all $i \in I$.*

This conjecture gives a relatively easy to test condition for the identification of π^* without the need for a complete understanding of the underlying Markov chain.

We proceed by testing the conjecture numerically. For this purpose we simulate the Markov chain $z^{(t)}$ over extended periods of time. This allows for accurate measurement of ρ . The coordinate-wise components ρ_i can be measured accordingly. However, for high dimensions n the accuracy of these measurements becomes poor because for at least one $i \in I$ the number of samples available for the estimation of ρ_i is at least n times lower than for ρ . Thus a numerical test of the conjecture is feasible only for small n .

We have performed experiments with random matrix instances Q in dimensions $n \in \{4, 5, 6, 7\}$. Random problem instances Q were created as follows: A set of n points $x_i \in \mathbb{R}^2$ was drawn i.i.d. from a standard normal distribution. The matrix Q was then defined as the kernel Gram matrix of these points w.r.t. the Gaussian RBF kernel function

$$Q_{ij} = k(x_i, x_j) = \exp\left(-\frac{\|x_i - x_j\|^2}{2\sigma^2}\right)$$

for $\sigma = 3$. This model problem is related to learning and optimization problems arising in kernel-based machine learning. Other choices of Q , e.g., as a product $Q = A^T A$ with standard normally distributed entries A_{ij} gave similar results.

Starting from a uniform distribution we have adjusted π so as to balance the coordinate-wise progress rates ρ_i . This was achieved by adaptively increasing π_i if $\rho_i > \rho$ and decreasing π_i if $\rho_i < \rho$ with an Rprop-style algorithm [29]. We denote the resulting distribution by $\tilde{\pi}$. Then we have systematically varied this distribution along n curves $\gamma_{\tilde{\pi},i}(t)$ through the probability simplex, defined as

$$\begin{aligned} \tilde{\gamma}_{\pi,i}(t) &= \pi + (2^t - 1)\pi_i e_i \\ \gamma_{\pi,i}(t) &= \frac{1}{\|\tilde{\gamma}_{\pi,i}(t)\|} \cdot \tilde{\gamma}_{\pi,i}(t) \ , \end{aligned}$$

where e_i is the i -th unit vector. Values $t \in \{-1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{1}{4}, \frac{1}{2}, 1\}$ were chosen for evaluation. The progress rate $\rho(\pi)$ was estimated for each of these distributions numerically by simulating the Markov chain until an estimate of the standard deviation of ρ fell below a threshold of $10^{-4} \cdot \rho$. The resulting one-dimensional performance curves are displayed in figure 1. It turns out that

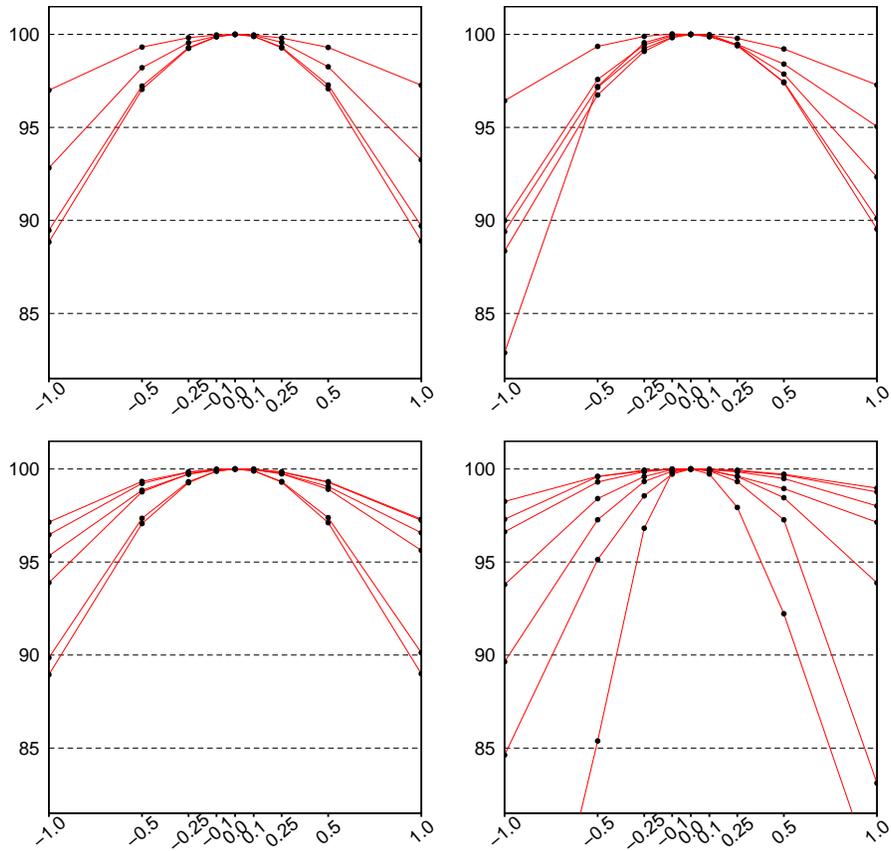


Figure 1: Curves $t \mapsto \rho(\gamma_{\bar{\pi},i}(t))/\rho(\bar{\pi})$, $i \in I = \{1, \dots, n\}$, for random problem instances in dimensions $n = 4$ (top left), $n = 5$ (top right), $n = 6$ (bottom left), and $n = 7$ (bottom right). The numerical optimum is located at $t = 0$, corresponding to the distribution $\bar{\pi} \approx \pi^*$.

the maximum is attained at position $t = 0$, corresponding to $\gamma_{\bar{\pi},i}(t) = \bar{\pi}$. All curves are uni-modal with a single maximum, clearly hinting at $\bar{\pi} \approx \pi^*$.

These experiments indicate that conjecture 1 may indeed hold true. In any case we argue that the empirical evidence is sufficient to justify the design of a heuristic online adaptation strategy for π that is based on equalizing all ρ_i . Moreover, the uni-modality of the performance curves indicates that the identification of the global optimum may be possible with iterative methods starting from any initial configuration.

The ACF algorithm performs a similar type of adaptation as described above for obtaining an estimate of $\bar{\pi}$, with the decisive difference that this adaptation is performed online and without knowledge of the optimum (which would be necessary for the computation of ρ and ρ_i). It aims at maximizing ρ by driving the coordinate-wise progress deviations $\rho_i - \rho$ towards zero.

The intuition behind ACF's adaptation mechanism is as follows: first of all, increasing the i -th coordinate's probability π_i results in a decrease of its progress rate. This is because the progress in direction i is spread over more CD steps. In the extreme case of performing two consecutive steps with the same coordinate the second step does not make any further progress (provided that one-dimensional sub-problems are solved optimally). The second insight is that there is no need to compute $\log(f(w))$ (or more generally $\log(f(w) - f^*)$ where f^* denotes the unknown optimal objective value) in order to compare coordinate-wise progress. Instead it is sufficient to compare the step-wise gains $f(w^{(t-1)}) - f(w^{(t)})$ for different coordinates, or equivalently, to compare them to the moving average \bar{r} in algorithm 2. Adjusting π_i so that all coordinate-wise gains become equal should be about the same as equalizing coordinate-wise progress rates. This intuition is made explicit in the following theorem. It ensures that under a number of conditions the ACF algorithm indeed adapts the expected coordinate distribution π so as to maximize the progress rate.

Theorem 6. *Assume the preconditions*

1. *the Markov chain $z^{(t)}$ is in its stationary distribution,*
2. *the progress rate ρ is infinitesimal, or in other words, the first order Taylor approximation*

$$\log(f(w)) - \log(f(T_i(w))) \approx \frac{f(w) - f(T_i(w))}{f(w)}$$

in w becomes exact,

3. *the estimate \bar{r} of average progress in algorithm 2 is exact, i.e., $\bar{r} = \mathbb{E}[f(w) - f(T(w))]$.*
4. *$\rho_i(\gamma_{\pi,i}(t))$ is strictly monotonically decreasing for all $\pi \in \hat{\Delta}$ and $i \in I$.*

Let $\pi^{(t)}$ denote the sequence of distributions generated by the ACF algorithm with learning rate $0 < \eta \ll 1$ and bounds $p_{\min} = 0$ and $p_{\max} = \infty$. Then $\mathbb{E}[\pi^{(t)}]$ fulfills the equilibrium condition $\rho_i = \rho$ for all $i \in I$.

Proof. From the prerequisites we obtain

$$\begin{aligned} \rho_i &= \mathbb{E}[\log(f(w)) - \log(f(T_i(w)))] \\ &= \mathbb{E}\left[\frac{f(w) - f(T_i(w))}{f(w)}\right] \\ &= c \cdot \mathbb{E}[f(w) - f(T_i(w))] \end{aligned}$$

with constant of proportionality $c = 1/f(w)$, which is quasi constant due to prerequisite 2, and

$$\rho = \mathbb{E}[\rho_i] = c \cdot \mathbb{E}[f(w) - f(T(w))] = c \cdot \bar{r} .$$

We conclude that in expectation the deviation of additive progress $f(w) - f(T_i(w))$ from its mean \bar{r} is proportional to the deviation of coordinate-wise progress $\rho_i(\pi)$ from its mean $\rho(\pi)$:

$$\mathbb{E}\left[\left(f(w) - f(T_i(w))\right) - \bar{r}\right] \propto \rho_i(\pi) - \rho(\pi) .$$

In the ACF algorithm the term inside the expectation on the left hand side drives the adaptation of π , which again impacts the right hand side according to prerequisite 4. Thus, in expectation and for a small enough learning rate η the ACF rule drives the distribution towards the equilibrium distribution π^* . Any stationary point of this process fulfills $(f(w) - f(T_i(w))) = \bar{r}$ for all $i \in I$, which is equivalent to $\rho_i = \rho$ for all $i \in I$. \square \square

None of the prerequisites of theorem 6 is a strong assumption, indeed, all of them should be fulfilled in practice in good approximation: CD algorithms usually perform extremely many cheap update steps so the chain has enough time to approach its stationary distribution, with the same arguments gains per single step are small, and the monotonicity property 4 can easily be validated in the very same experiments that test the conjecture. It is also clear that some prerequisites are only fulfilled approximately. Thus the ACF algorithm's stationary distribution is only a proxy of the ideal distribution π^* . However, the deviation of the resulting progress rates is usually small since the function ρ is rather flat in the vicinity of its optimizer π^* (see figure 1).

A reasonably careful interpretation of theorem 6 is that provided conjecture 1 holds the ACF algorithm adjusts the distribution π in a close to optimal way, on average. Note that the statement gives no guarantees on the variance of $\pi^{(t)}$. For large dimensions n this variance can be significant since only few samples are available for the estimation of ρ_i . This is why in rare cases ACF may even have a deteriorating effect on performance in practice as can be seen from some of the experimental results in the next section.

7 Empirical Evaluation

In this section we investigate the performance of the ACF method. We have run algorithm 1 in a number of variants reflecting the state-of-the-art in the

respective fields against the ACF-CD algorithm for solving a number of instances of problem (1), namely the four problems discussed in section 3.

We have implemented the ACF algorithm directly into the software liblinear [6] for binary SVM and logistic regression training of linear models. An efficient C implementation was created for the LASSO problem with quadratic loss. We have implemented a CD solver for linear multi-class training into the Shark machine learning library [17].

The stopping criteria for all algorithms were set in analogy to the standard stopping criterion for linear and non-linear SVM training as found in libsvm [7], liblinear [6], and Shark [17]. In the case of dual SVM training the algorithm is stopped as soon as all Karush-Kuhn-Tucker (KKT) violations drop below a threshold ε . The default value for non-linear SVM training has been established as $\varepsilon = 0.001$, and less tight values such as $\varepsilon = 0.01$ and even $\varepsilon = 0.1$ are commonly applied for fast training of linear models. For SVMs (without bias term) this measure simply computes the largest absolute component of the gradient of the dual objective that is not blocked by an active constraint. For (essentially) unconstrained problems such as LASSO and logistic regression the stopping criterion checks whether all components of the dual gradient have dropped below ε .

The number of CD iterations is a straightforward performance indicator. For sparse data this indicator is not always a reliable indicator of computational effort, since not all coordinates correspond to roughly equal numbers of non-zeros. In fact, in the LASSO experiments the cost of the derivative computation that dominates the cost of a CD iteration varies widely and cannot be assumed to be roughly constant. Wall clock optimization time is a more relevant measure. We measure optimization time only for exactly comparable implementations and otherwise resort to the number of multiplications and additions required to compute the derivatives, hereafter referred to as the *number of operations*. This quantity is a very good predictor of the actual runtime, with the advantage of being independent of implementation, CPU, memory bandwidth, and all kinds of inaccuracies associated with runtime measurements.

Comparing training times in a fair way is non-trivial. This is because the selection of a good value of the regularization parameter (λ or C) requires several runs with different settings, often performed in a cross validation manner. The computational cost of finding a good value can easily exceed that of training the final model, and even a good range is often hard to guess without prior knowledge. The focus of the present study is on optimization. Therefore we don't fix a specific model selection procedure and instead report training times over reasonable ranges of values.

The various data sets used for evaluation are available from the libsvm data website

<http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/> .

The complete source code of our experiments is available at

<http://www.ini.rub.de/PEOPLE/glasmtbl/code/acf-cd/> .

Problem	Instances (ℓ)	Features ($n = d$)
news 20	19,996	1,355,191
rcv1	20,242	47,236
E2006-tfidf	16,087	150,360

Table 2: Benchmark problems for the LASSO experiments.

7.1 LASSO Regression

To demonstrate the versatility of our approach we furthermore compared ACF-CD to the LASSO solver proposed by [8]. This is a straightforward deterministic CD algorithm, iterating over all coordinates in order. We have used the data sets listed in table 2, and for each of these problems we have varied the parameter λ in a range so that the resulting number of non-zero features varies between very few (less than 10) and many (more than 10,000), covering the complete range of interest. This gives a rather complete picture of the relative performance of both algorithms over a wide range of relevant optimization problems. The results are summarized in table 3.

The ACF-CD algorithm is never significantly slower than uniform CD and in some cases faster by one to two orders of magnitude, while obtaining solutions of equal quality (as indicated by the objective function value). This marks a significant speed-up of ACF-CD over uniform CD.

7.2 Linear SVM Training

We compared ACF-CD in an extensive experimental study to the liblinear SVM solver [6, 15]. This is an extremely strong and widely used baseline. The liblinear CD solver sweeps over random permutations of coordinates in epochs. In addition it applies a shrinking heuristic that removes bounded variables from the problem. In other words the solver performs a simple type of online adaptation of coordinate frequencies that is closely tied to the structure of the SVM optimization problem, while ACF-CD applies its general-purpose adaptation rule.

Our evaluation was based on six data sets listed in table 4. They range from medium sized to extremely large.

Both algorithms return accurate solutions to the SVM training problem. The test errors coincide exactly. The algorithms don't differ in the quality of the solution (dual objective values are extremely close; often they coincide to 10 significant digits), but only in the time it takes to compute this solution. Training times of both algorithms are comparable since we have implemented ACF-CD directly into the liblinear code.⁸

⁸An arbitrary outer loop iteration limit of 1000 is hard-coded into liblinear version 1.9.2. We have removed this limit for the sake of a meaningful comparison.

problem	λ	uniform		ACF		speed-up	
		iterations	operations	iterations	operations	iter.	oper.
rcv1	0.001	$7.06 \cdot 10^8$	$2.24 \cdot 10^{10}$	$7.50 \cdot 10^7$	$4.63 \cdot 10^9$	9.4	4.8
	0.01	$9.21 \cdot 10^7$	$2.92 \cdot 10^9$	$1.86 \cdot 10^7$	$1.36 \cdot 10^9$	5.0	2.1
	0.1	$4.95 \cdot 10^7$	$1.57 \cdot 10^9$	$4.14 \cdot 10^6$	$4.43 \cdot 10^8$	12.0	3.5
	1	$2.36 \cdot 10^7$	$7.48 \cdot 10^8$	$1.53 \cdot 10^6$	$2.24 \cdot 10^8$	15.4	3.3
	10	$5.38 \cdot 10^6$	$1.71 \cdot 10^8$	$1.21 \cdot 10^6$	$1.79 \cdot 10^8$	4.4	1.0
	100	$4.25 \cdot 10^5$	$1.35 \cdot 10^7$	$2.36 \cdot 10^5$	$8.20 \cdot 10^6$	1.8	1.6
news 20	0.1	$2.64 \cdot 10^9$	$1.78 \cdot 10^{10}$	$3.88 \cdot 10^7$	$1.49 \cdot 10^9$	68.0	11.9
	1	$1.47 \cdot 10^9$	$9.89 \cdot 10^9$	$3.19 \cdot 10^7$	$7.50 \cdot 10^8$	46.1	13.2
	10	$3.78 \cdot 10^8$	$2.54 \cdot 10^9$	$2.30 \cdot 10^7$	$1.98 \cdot 10^8$	16.4	12.8
	100	$6.78 \cdot 10^6$	$4.55 \cdot 10^7$	$9.49 \cdot 10^6$	$6.42 \cdot 10^7$	0.7	0.7
E2006-tfidf	0.001	$2.38 \cdot 10^9$	$3.16 \cdot 10^{11}$	$4.08 \cdot 10^7$	$2.57 \cdot 10^{10}$	58.3	12.3
	0.01	$3.40 \cdot 10^8$	$4.51 \cdot 10^{10}$	$8.37 \cdot 10^6$	$4.02 \cdot 10^8$	40.6	112.2
	0.1	$2.59 \cdot 10^7$	$3.44 \cdot 10^9$	$5.70 \cdot 10^6$	$1.38 \cdot 10^9$	4.5	2.5
	1	$2.56 \cdot 10^6$	$3.40 \cdot 10^8$	$2.71 \cdot 10^6$	$3.75 \cdot 10^8$	0.9	0.9

Table 3: Performance of uniform CD (baseline) and the ACF-CD algorithm for LASSO training. The table lists numbers of iterations and operations, as well as the “speed-up” factor by which ACF-CD outperforms the uniform baseline (higher is better, values larger than one are speed-ups). The regularization parameter λ so as to give the full range in between extremely sparse models with less than 10 non-zeros and quite rich models with up to 10^4 non-zero coefficients.

Problem	Instances ($n = \ell$)	Features (d)
cover type	581,012	54
kkd-a	8,407,752	20,216,830
kkd-b	19,264,097	29,890,095
news 20	19,996	1,355,191
rcv1	20,242	47,236
url	2,396,130	3,231,961

Table 4: Benchmark problems for linear SVM training.

The results are reported compactly in figure 2. The figure includes three-fold cross validation performance which gives an indication of which C values are most relevant. The best value is contained in the interior of the tested range in all cases. For completeness, all timings and iteration numbers are listed tables 5 and 6.

In most cases the ACF-CD algorithm is faster than liblinear. For large values of C it can outperform the baseline by more than an order of magnitude (note

the logarithmic scale in figure 2).

The cover type problem is an exception. This problem is special for its low feature dimensionality of only 54 features, which means that the 581,012 dual variables are highly redundant. This implies that optimal solution can be represented with many possible subsets of variables α_i which makes adaptation of coordinate frequencies superfluous. In this case the overhead of coordinate adaptation causes a considerable slowdown. It is actually well known that this problem can be solved more efficiently in the primal, e.g., with liblinear’s trust region method.

Overall (e.g., summing over all experiments) the ACF-CD method clearly outperforms the liblinear algorithm, which is a strong baseline.

Data Set	Solver	$C = 0.01$	$C = 0.1$	$C = 1$	$C = 10$	$C = 100$	$C = 1000$
cover type	liblinear	1.29	2.73	12.5	69.5	533	4,450
	ACF	<small>$3.31 \cdot 10^6$</small>	<small>$7.41 \cdot 10^6$</small>	<small>$3.38 \cdot 10^7$</small>	<small>$1.80 \cdot 10^8$</small>	<small>$1.37 \cdot 10^9$</small>	<small>$1.14 \cdot 10^{10}$</small>
		4.50	6.43	16.3	121	676	8280
kkd-a	liblinear	429	2,340	31,200	138,000	345,000	—
	ACF	<small>$3.07 \cdot 10^8$</small>	<small>$1.57 \cdot 10^9$</small>	<small>$1.88 \cdot 10^{10}$</small>	<small>$8.77 \cdot 10^{10}$</small>	<small>$2.35 \cdot 10^{11}$</small>	—
		357	705	1,980	7,990	62,700	—
kkd-b	liblinear	1,150	5,140	53,300	612,000	—	—
	ACF	<small>$6.92 \cdot 10^8$</small>	<small>$2.86 \cdot 10^9$</small>	<small>$3.11 \cdot 10^{10}$</small>	<small>$3.42 \cdot 10^{11}$</small>	—	—
		828	2,050	4,280	9,350	107,000	—
news 20	liblinear	0.56	0.60	2.30	3.56	7.39	100
	ACF	<small>$8.03 \cdot 10^4$</small>	<small>$1.22 \cdot 10^5$</small>	<small>$4.04 \cdot 10^5$</small>	<small>$6.38 \cdot 10^5$</small>	<small>$1.38 \cdot 10^6$</small>	<small>$2.47 \cdot 10^7$</small>
		0.63	1.05	1.71	2.33	2.62	3.78
rcv1	liblinear	0.09	0.13	0.46	1.76	4.27	14.1
	ACF	<small>$9.36 \cdot 10^4$</small>	<small>$1.46 \cdot 10^5$</small>	<small>$4.77 \cdot 10^5$</small>	<small>$1.70 \cdot 10^6$</small>	<small>$4.19 \cdot 10^6$</small>	<small>$1.43 \cdot 10^7$</small>
		0.15	0.24	0.38	0.70	0.94	1.32
url	liblinear	67.9	353	4,140	22,100	121,000	469,000
	ACF	<small>$4.05 \cdot 10^7$</small>	<small>$1.93 \cdot 10^8$</small>	<small>$2.22 \cdot 10^9$</small>	<small>$1.45 \cdot 10^{10}$</small>	<small>$8.04 \cdot 10^{10}$</small>	<small>$2.74 \cdot 10^{11}$</small>
		86.7	192	614	1,810	5,910	22,800
		<small>$6.24 \cdot 10^7$</small>	<small>$1.30 \cdot 10^8$</small>	<small>$4.24 \cdot 10^8$</small>	<small>$1.16 \cdot 10^9$</small>	<small>$4.34 \cdot 10^9$</small>	<small>$1.73 \cdot 10^{10}$</small>

Table 5: Results of linear SVM training with low accuracy $\varepsilon = 0.01$. The table lists runtime in seconds and (small font below) the number of CD iterations. Runs marked with “—” did not finish after several weeks of training.

Problem	Solver	$C = 0.01$	$C = 0.1$	$C = 1$	$C = 10$	$C = 100$	$C = 1000$
cover type	liblinear	1.28	2.75	12.5	69.5	597	4,750
	ACF	<small>$3.31 \cdot 10^6$</small> 4.50	<small>$7.41 \cdot 10^6$</small> 8.74	<small>$3.38 \cdot 10^7$</small> 32.4	<small>$1.80 \cdot 10^8$</small> 220	<small>$1.78 \cdot 10^9$</small> 2.140	<small>$1.44 \cdot 10^{10}$</small> 17,300
kdd-a	liblinear	817	9,660	239,000	4,410,000	—	—
	ACF	<small>$1.11 \cdot 10^9$</small> 725	<small>$9.16 \cdot 10^9$</small> 1,580	<small>$1.59 \cdot 10^{11}$</small> 5,080	<small>$1.66 \cdot 10^{12}$</small> 48,800	430,000	—
kdd-b	liblinear	2,610	20,500	459,000	—	—	—
	ACF	<small>$1.94 \cdot 10^9$</small> 2,090	<small>$1.17 \cdot 10^{10}$</small> 3,330	<small>$2.73 \cdot 10^{11}$</small> 10,600	<small>$4.28 \cdot 10^{10}$</small> 69,500	—	—
news 20	liblinear	0.56	0.78	8.54	9.84	11.9	103
	ACF	<small>$8.03 \cdot 10^4$</small> 0.68	<small>$1.54 \cdot 10^5$</small> 0.97	<small>$1.55 \cdot 10^6$</small> 1.65	<small>$1.87 \cdot 10^6$</small> 2.03	<small>$2.90 \cdot 10^6$</small> 2.59	<small>$2.50 \cdot 10^7$</small> 4.49
rcv1	liblinear	0.09	0.17	2.74	2.85	4.73	18.4
	ACF	<small>$9.40 \cdot 10^4$</small> 0.22	<small>$1.93 \cdot 10^5$</small> 0.53	<small>$3.36 \cdot 10^6$</small> 0.60	<small>$3.36 \cdot 10^6$</small> 0.74	<small>$5.63 \cdot 10^6$</small> 1.05	<small>$2.14 \cdot 10^7$</small> 1.32
url	liblinear	139	2,100	22,100	135,000	402,000	703,000
	ACF	<small>$8.27 \cdot 10^7$</small> 152	<small>$1.18 \cdot 10^9$</small> 978	<small>$1.46 \cdot 10^{10}$</small> 3,390	<small>$7.61 \cdot 10^{10}$</small> 17,700	<small>$2.35 \cdot 10^{11}$</small> 32,100	<small>$3.78 \cdot 10^{11}$</small> 36,600

Table 6: Results of linear SVM training with high accuracy $\varepsilon = 0.001$. The table lists runtime in seconds and (small font below) the number of CD iterations. Runs marked with “—” did not finish after several weeks of training.

7.3 Multi-class SVM Training with Subspace Descent

We evaluate a learning problem that naturally corresponds to a subspace descent optimization problem, namely multi-class SVM training. The WW multi-class SVM extension was implemented into the Shark [17] machine learning library, version 3.0 (beta). The K -dimensional sub-problems were solved with up to $10 \cdot K$ iterations of an inner CD solver picking the largest derivative component for descent (here K denotes the number of classes).

The data sets for evaluation are listed in table 7. A separate test set was used to estimate a reasonable range for the regularization parameter C . The parameter was varied on a grid of the form $C = 10^k$, and result are reported for a grid of size 5 around the best value.

As discussed above, the liblinear algorithm applies a shrinking technique to reduce the problem size during the optimization run. This technique does not carry over in a one-to-one fashion to the multi-class problem. For comparison we have implemented a similar shrinking heuristic into the multi-class SVM

solver. However, for this problem the heuristic did not perform significantly better than the uniform baseline and sometimes lead to considerably longer optimization times due to wrong shrinking decisions. This is different from the binary SVM case where shrinking works well in most cases. Therefore we have dropped shrinking and instead compare ACF-CD against the better performing uniform baseline.

Problem	Instances ($n = \ell$)	Features (d)	Classes (K)
iris	105	4	3
soybean	214	35	19
news-20	15,935	62,061	20
rcv1	15,564	47,236	53

Table 7: Benchmark problems for multi-class SVM (subspace descent) experiments.

The experimental results are presented in table 8. The ACF algorithm clearly outperforms the uniform coordinate selection baseline. It is noteworthy that ACF does not only perform better but also scales much more gracefully to hard optimization problems, corresponding to large values of C .

7.4 Logistic Regression

We have implemented ACF-CD into the liblinear logistic regression solver [6, 37], analog to the linear SVM solver. There are two major differences to the linear SVM case. First, the dual logistic regression solution is not sparse and thus shrinking is not applicable. Hence, liblinear applies uniform coordinate selection. Second, the one-dimensional sub-problems cannot be solved analytically. Instead a series of Newton steps is applied.

The data sets news 20, rcv1, and url from table 4 were used for comparison. We have tuned the regularization parameter C on the grid 10^k based on three-fold cross-validation (CV). In table 9 we report results for a problem specific range of five settings centered on the best three-fold CV performance.

The results exhibit the usual pattern. In some highly regularized cases ACF-CD is a bit slower, but in most cases it improves performance. Saving are most significant where they are most relevant, namely for parameter configurations that result in long training times. On the logistic regression problem ACF-CD is up to two orders of magnitude faster than the liblinear solver.

7.5 Discussion

Our results show that the ACF algorithm is superior to uniform CD. Of course the algorithm can unfold its full potential only if it performs sufficiently many sweeps over the coordinates. Highly regularized machine learning problems tend to be simple in the sense that the stopping criterion can be met already after

problem	C	test	uniform		ACF		speed-up	
		accuracy	iterations	seconds	iterations	seconds	iter.	time
iris	10^{-2}	60.0%	4,095	0.003	2,625	0.002	1.6	1.5
	10^{-1}	60.0%	42,735	0.023	11,130	0.007	3.8	3.3
	10^0	100.0%	238,140	0.116	27,300	0.009	8.7	12.8
	10^1	95.6%	5,007,870	2.16	410,445	0.279	12.2	7.7
	10^2	95.6%	2,095,065	0.959	267,855	0.194	7.8	4.9
soybean	10^{-2}	69.9%	20,972	0.0434	12,412	0.027	1.7	1.6
	10^{-1}	88.2%	78,752	0.150	42,800	0.101	1.8	1.5
	10^0	91.4%	377,282	0.664	93,732	0.218	4.0	3.0
	10^1	86.0%	607,974	1.04	113,206	0.271	5.3	3.8
	10^2	81.7%	7,038,032	11.8	1,346,916	2.79	5.2	4.2
news 20	10^{-4}	76.7%	270,895	2.62	334,635	3.22	0.8	0.8
	10^{-3}	81.9%	2,230,900	23.7	318,700	3.23	7.0	7.3
	10^{-2}	83.4%	1,290,735	14.4	462,115	5.04	2.8	2.9
	10^{-1}	81.5%	6,023,430	63.3	780,815	8.77	7.7	7.2
	10^0	79.2%	60,234,300	632	1,481,955	18.9	40.6	33.4
rcv1	10^{-2}	81.6%	513,612	15.5	295,716	9.13	1.7	1.7
	10^{-1}	87.8%	2,241,216	76.4	513,612	18.2	4.4	4.1
	10^0	88.8%	4,264,536	153	606,996	23.1	7.0	6.6
	10^1	88.2%	12,746,916	468	793,764	32.6	16.0	14.3
	10^2	87.8%	10,287,804	381	996,096	40.0	10.3	9.5

Table 8: Number of iterations and training times in seconds for multi-class SVM training with subspace descent, as well as corresponding speed-up factors (higher is better). Test errors indicate that the parameter C is varied within a reasonable range.

very few sweeps. In this case uniform coordinate selection is usually a good strategy, and the computational overhead of adaptation can be saved. This is why ACF does not beat the uniform baseline in all cases, in particular for small values of the regularization parameter C . However, these optimization runs are anyway extremely fast and do not pose a computational challenge. As soon as the problem becomes more involved ACF starts to pay off, often saving 90% of the training time and sometimes even more. This is the dominating effect, e.g., when performing grid search or a parameter study. The ACF algorithm does not only outperform the uniform selection baseline but also the standard SVM training algorithm with its shrinking technique. The only exception is the cover type data set. Shrinking is a strong competitor since it is a domain-specific technique designed explicitly to take a-priori knowledge about the dual SVM solution into account. In contrast, ACF is a generic CD speedup technique applicable to all problem types. We argue that outperforming a problem specific technique such as shrinking with a general purpose method such as ACF is a

problem	C	3-fold CV	liblinear		ACF		speed-up	
			iterations	seconds	iterations	seconds	iter.	time
news 20	10^2	96.3%	1,459,708	4.22	660,500	1.84	2.2	2.3
	10^3	96.4%	14,317,136	41.7	560,268	1.77	25.6	23.6
	10^4	96.5%	136,972,600	406	2,019,200	5.54	67.8	73.3
	10^5	96.4%	1,292,781,392	3,720	10,355,564	27.7	124.8	134.3
	10^6	96.5%	3,929,561,008	26,834	158,291,120	382	24.8	70.3
rcv1	10^0	96.1%	121,452	0.081	545,462	0.252	0.2	0.3
	10^1	96.7%	263,146	0.153	345,099	0.188	0.8	0.8
	10^2	96.7%	1,497,908	0.819	788,883	0.341	1.9	2.4
	10^3	96.5%	14,007,464	7.46	2,389,345	1.03	5.9	7.2
	10^4	96.3%	136,005,998	72.9	11,274,174	4.85	12.1	15.0
url	10^0	99.17%	251,593,650	565	217,914,274	445	1.2	1.3
	10^1	99.42%	2,317,057,710	4,265	723,665,786	1,478	3.2	2.9
	10^2	99.44%	665,404,720	51,734	1,940,658,131	3,992	0.3	13.0
	10^3	99.42%	—	—	16,933,597,906	44,115	—	—
	10^4	—	—	—	56,218,463,548	139,041	—	—

Table 9: Number of iterations and training times in seconds for logistic regression, as well as corresponding speed-up factors (higher is better). Three-fold cross-validation performance indicates that the parameter C is varied within a reasonable range. The *url* runs for $C \geq 1,000$ with liblinear are marked with “—”. They had to be stopped without finishing after five days of training.

strong result.

8 Conclusion

We have introduced the Adaptive Coordinate Frequencies (ACF) algorithm. It adapts the relative frequencies of coordinates in coordinate descent (CD) optimization *online* to the problem at hand. The aim of the adaptation is to maximize convergence speed and thus to minimize the time complexity of various machine training problems.

This technique allows to efficiently solve CD problems that greatly profit from non-uniform coordinate selection probabilities. The need for non-uniformity is obvious at least in a machine learning context where coordinates correspond to data points or features, some of which are known to be more important than others, only it is hard to say beforehand which ones are how important. Our method allows to start the CD algorithm with an uninformed guess—the uniform distribution—or a more informed choice if available, and to adapt the coordinate selection distribution online for optimal progress. This is particularly helpful for problems with *changing* importance of coordinates, e.g., when

constraints become active or inactive.

We have presented a first analysis of the ACF method based on a Markov chain perspective. We conjecture that the coordinate selection distribution that maximizes the convergence rate is characterized by equal progress in each coordinate. This property, granted that it holds true in sufficient generality, provides an explanation of why the ACF algorithm works so well. We show that in expectation and under certain simplifying assumptions the ACF algorithm drives the coordinate selection distribution towards this equilibrium. Extending our understanding of this process and the underlying Markov chains resulting from coordinate descent is a primary research goal for future investigations.

It turns out that many successful applications of CD algorithms in machine learning rely on uniform coordinate selection. The only notable exception is linear SVM training where a shrinking heuristic can set a coordinate probability to exact zero. We compared ACF to state-of-the-art machine training implementations for four different problems. Overall the new algorithm shows impressive performance. It systematically outperforms the established algorithm, sometimes by an order of magnitude and more, and falls behind only in rare special cases. We therefore recommend online adaptation of coordinate frequencies as a general tool for coordinate descent optimization, in particular in the domain of machine learning.

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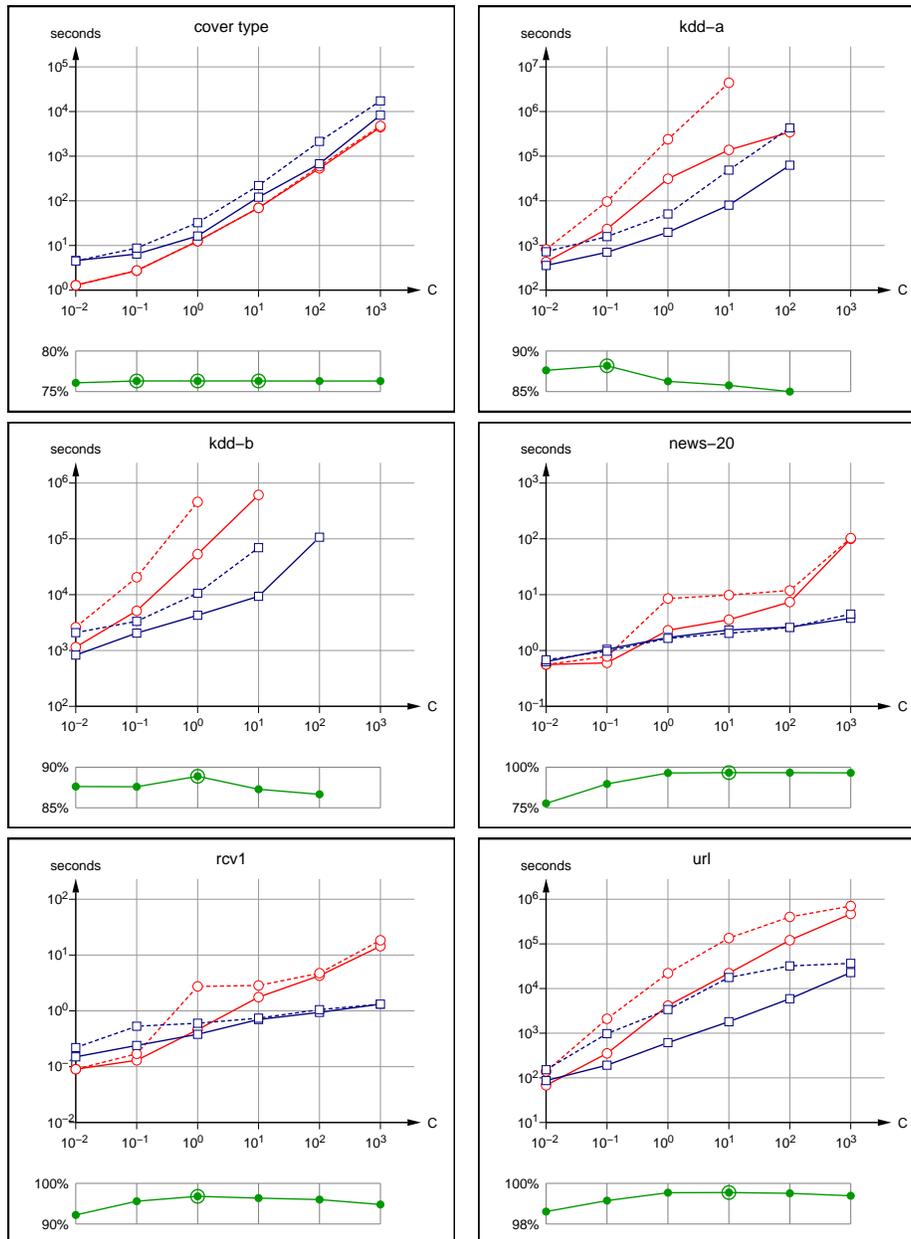


Figure 2: Training times with the original liblinear algorithm (red circles) and with ACF-CD (blue squares) as a function of the regularization parameter C . The target accuracy is $\epsilon = 0.01$ for the solid curve and $\epsilon = 0.001$ for the dashed curves. For reference, three-fold cross validation performance (percent correct) is plotted below the curves in green, with best configurations circled. In all cases the best value(s) are contained in the interior of the chosen parameter range.