Existence of solutions for fourth order differential equation with four-point boundary conditions

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Abstract

In this paper we investigate the existence of solutions of a class of four-point boundary value problems for a fourth order ordinary differential equation. Our analysis relies on a nonlinear alternative of Leray–Schauder type.

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1. Introduction and preliminaries

As is pointed out in [2,12], boundary value problems for second and higher order differential equations play a very important role in both theory and applications. Recently an increasing interest in studying the existence of solutions and positive solutions to boundary value problems for second and higher order differential equations is observed; see for example [1,4,6–11].

Very recently, Chen, Ni and Wang [5] investigated the fourth order nonlinear ordinary differential equation

\[ u^{(4)}(t) = f(t, u(t)), \quad 0 < t < 1, \]  

with the four-point boundary conditions

\[
\begin{aligned}
u(0) &= u(1) = 0, \\
u''(\xi_1) - bu'''(\xi_1) &= 0, \\
cu''(\xi_2) + du'''(\xi_2) &= 0,
\end{aligned}
\]

where \(0 \leq \xi_1 < \xi_2 \leq 1\). They proved the following lemma (a key lemma):

\textbf{Lemma} (See [5], Lemma 2.2). Suppose \(a, b, c, d, \xi_1, \xi_2\) are nonnegative constants satisfying \(0 \leq \xi_1 < \xi_2 \leq 1\), \(b - a\xi_1 \geq 0, d - c + c\xi_2 \geq 0\) and \(\delta = ad + bc + ac(\xi_2 - \xi_1) \neq 0\). If \(u(t) \in C^4[0, 1]\) satisfies

\[ u^{(4)}(t) \geq 0, \quad t \in (0, 1), \]

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by using a nonlinear alternative of Leray–Schauder type.

Unfortunately this lemma is wrong. We now give a counterexample to illustrate it.

Counterexample. Let \( u(t) = \frac{1}{4}t^4 + \frac{1}{3}t^3 - 2t^2 + \frac{7}{6}t \in C^1[0, 1], \xi_1 = \frac{1}{8}, \xi_2 = \frac{1}{6}, a, b, c, d \) are nonnegative constants satisfying \( b \geq \frac{8}{9}a = a\xi_1, d = \frac{2}{3}c = (1 - \xi_2)c \) and \( \delta = ad + bc + \frac{1}{2}ac \neq 0 \). Then we have

\[
\begin{align*}
\mathbf{u}^{(4)}(t) &= 12 > 0, \quad t \in (0, 1), \\
u(0) &= 0, \quad \mathbf{u}(1) = 0, \\
u''(\xi_1) - bu''(\xi_1) &= 2\left[6t^2 + 2t - 4\right]_{t=1/6} - b\left[12t + 2\right]_{t=1/6} \\
&= -3\frac{21}{32}a - 3\frac{1}{2}b \leq 0,
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{c}u''(\xi_2) + du''(\xi_2) &= c\left[6t^2 + 2t - 4\right]_{t=1/6} + d\left[12t + 2\right]_{t=1/6} \\
&= -\frac{7}{2}c + 4d = -\frac{7}{2}c + 4 \cdot \frac{5}{6}c = -\frac{1}{6}c \leq 0.
\end{align*}
\]

But

\[
\mathbf{u}\left(\frac{11}{12}\right) = -0.0013 < 0,
\]

that is, Lemma 2.2 in [5] is incorrect.

So the conclusions of [5] should be reconsidered. The aim of this paper is concerned with the existence of solutions to the BVP (1.1) and (1.2) by using a nonlinear alternative of Leray–Schauder type.

2. Main result

First, we give some lemmas which are needed in our discussion of the main results.

Lemma 2.1. If \( \delta = ad + bc + ac(\xi_2 - \xi_1) \neq 0 \) and \( h \in C[0, 1] \), then the boundary value problem

\[
\begin{align*}
v''(t) &= h(t), \quad t \in [0, 1], \\
av(\xi_1) - bu'(\xi_1) &= 0, \quad cv(\xi_2) + dv'(\xi_2) = 0, \tag{2.1}
\end{align*}
\]

has a unique solution

\[
v(t) = \int_{0}^{\xi_1} (s - t)h(s)\,ds + \frac{1}{\delta} \int_{\xi_1}^{\xi_2} (a(\xi_1 - t) - b)(c(\xi_2 - s) + d)h(s)\,ds + \int_{0}^{t} (t - s)h(s)\,ds, \tag{2.3}
\]

where \( \delta = ad + bc + ac(\xi_2 - \xi_1) \).

Proof. By (2.1), it is easy to know that

\[
v(t) = C_1 + C_2t + \int_{0}^{t} (t - s)h(s)\,ds, \tag{2.4}
\]

where \( C_1, C_2 \) are any two constants. Substituting the boundary conditions (2.2) into (2.4), by a routine calculation, we get

\[
C_1 = \int_{0}^{\xi_1} sh(s)\,ds + \frac{1}{\delta}(a\xi_1 - b) \int_{\xi_1}^{\xi_2} (c(\xi_2 - s) + d)h(s)\,ds, \tag{2.5}
\]

and
Lemma 2.1. Let \( \xi_1 = 0, \xi_2 = 1 \), then (2.3) reduces to the following
\[
v(t) = -\int_0^1 G(t, s) h(s) \, ds,
\]
where
\[
G(t, s) = \frac{1}{\delta} \begin{cases} 
( (as + b)(d + c(1-t)) ) , & 0 \leq s \leq t \leq 1 , \\
( (at + b)(d + c(1-s)) ) , & 0 \leq t < s \leq 1 .
\end{cases}
\]

Remark 2.2. For
\[
u(t) = -\int_0^1 G(t, s) h(s) \, ds,
\]
where
\[
G(t, s) = \frac{1}{\delta} \begin{cases} 
( (as + b)(d + c(1-t)) ) , & 0 \leq s \leq t \leq 1 , \\
( (at + b)(d + c(1-s)) ) , & 0 \leq t < s \leq 1 .
\end{cases}
\]

In [5] they actually have
\[
G_1(t, s) = \begin{cases} 
\frac{s}{\delta} (1-t) , & 0 \leq s \leq t \leq 1 , \\
\frac{t}{\delta} (1-s) , & 0 \leq t < s \leq 1 .
\end{cases}
\]

But (2.7) is wrong. Indeed, by Lemma 2.1, (2.7) should be replaced by the following:
\[
u(t) = -\int_0^1 \int_0^1 G_1(t, \xi) R(\xi) \, d\xi + \int_0^1 G_1(t, \xi) \int_{\xi_1}^{\xi_2} G_2(\xi, s) h(s) \, ds \, d\xi,
\]
where
\[
R(t) = \frac{1}{\delta} ((a(t - \xi_1) + b)x_3 + (c(\xi_2 - t) + d)x_2).
\]

Remark 2.3. In Theorem 3.1 [5], the operator \( A : C[0, 1] \to C[0, 1] \) is defined as follows:
\[
Au(t) = \int_0^1 G_1(t, \eta) \int_{\xi_1}^{\xi_2} G_2(\eta, s) g(s, u(s)) \, ds \, d\eta,
\]
where \( G_1(t, s) \) and \( G_2(t, s) \) are as in Remark 2.2. By Lemma 2.1 and Remark 2.2, the definition of \( A \) is incorrect. In fact, the operator \( A \) should be defined as follows:
\[
Au(t) = \int_0^1 G_1(t, \eta) \left( \int_{\xi_1}^{\eta} (s - \eta)g(s, u(s))ds \right) d\eta \\
+ \frac{1}{\delta} \int_0^1 G_1(t, \eta) \left( \int_{\xi_1}^{\xi_2} (b - a(\xi_1 - \eta))(c(\xi_2 - s) + d)g(s, u(s))ds \right) d\eta,
\]

where \(g\) is as in Theorem 3.1 [5].

The following is a fixed point result of nonlinear alternative of Leray–Schauder type which will be needed in this paper.

**Lemma 2.2** (3]. Let \(E\) be a Banach space with \(C \subset E\) closed and convex. Assume \(U\) is a relatively open subset of \(C\) with \(0 \in U\) and \(A : U \rightarrow C\) is a continuous, compact map. Then either

(1) \(A\) has a fixed point in \(U\); or

(2) there exists \(u \in \partial \bar{U}\) and \(\lambda \in (0, 1)\) with \(u = \lambda Au\).

We are now in a position to present and prove our main result.

**Theorem 2.3.** Suppose the following conditions are satisfied:

\((H_1)\) \(a, b, c, d, \xi_1, \xi_2\) are nonnegative constants satisfying \(0 \leq \xi_1 < \xi_2 \leq 1\), \(b - a\xi_1 \geq 0\) and \(\delta = ad + bc + ac(\xi_2 - \xi_1) \neq 0\).

\((H_2)\) \(f \in C([0, 1] \times \mathbb{R}, \mathbb{R})\). Moreover, there exist \(\alpha \in C([0, 1], [0, \infty))\) and a continuous, nondecreasing function \(g : [0, \infty) \rightarrow (0, \infty)\) with \(|f(t, w)| \leq \alpha(t)g(w)\) for a.e. \(t \in [0, 1]\) and all \(w \geq 0\)

and

\[
\sup_{c \in (0, \infty)} \frac{c}{g(c)} > k, \tag{2.9}
\]

where

\[
k = \frac{1}{12} \left[ \int_0^{\xi_1} s^3(2 - s)\alpha(s)ds + \int_0^{1} (1 - s)^3(1 + s)\alpha(s)ds \right. \\
+ \left. \frac{2(b - a\xi_1) + a}{\delta} \int_{\xi_1}^{\xi_2} (c(\xi_2 - s) + d)\alpha(s)ds \right]. \tag{2.10}
\]

Then BVP (1.1) and (1.2) has a solution \(u \in C[0, 1]\).

**Proof.** Define the operator \(T : C[0, 1] \rightarrow C[0, 1]\) by

\[
Tu(t) := \int_0^1 G_1(t, \eta) \left( \int_{\xi_1}^{\eta} (s - \eta)f(s, u(s))ds \right) d\eta \\
+ \frac{1}{\delta} \int_0^1 G_1(t, \eta) \left( \int_{\xi_1}^{\xi_2} (b - a(\xi_1 - \eta))(c(\xi_2 - s) + d)f(s, u(s))ds \right) d\eta,
\]

where \(G_1(t, s)\) is as in (2.8). Equivalently,

\[
Tu(t) = -\int_0^{\xi_1} G_1(t, \eta) \left( \int_{\xi_1}^{\eta} f(s, u(s))ds \right) d\eta - \int_0^{1} G_1(t, \eta) \left( \int_{\xi_1}^{\eta} (s - \eta)f(s, u(s))ds \right) d\eta \\
+ \frac{1}{\delta} \int_0^1 G_1(t, \eta) \left( \int_{\xi_1}^{\xi_2} (b - a(\xi_1 - \eta))(c(\xi_2 - s) + d)f(s, u(s))ds \right) d\eta.
\]

Solving BVP (1.1) and (1.2) is equivalent to finding a fixed point of the operator \(T\) (By Remark 2.3). It is easy to see that \(T\) is continuous in \(C[0, 1]\). Now let \(B \subset C[0, 1]\) be a bounded subset of \(C[0, 1]\), and \(M_1 > 0\) be a constant such that \(\|u\| \leq M_1\) for all \(u \in B\). Thus, there exists a constant \(M_2 > 0\) such that

\[
|f(t, u)| \leq M_2, \quad \text{on } [0, 1] \times [0, M_1]. \tag{2.13}
\]
since $f$ is continuous on $[0, 1] \times [0, M_1]$. Therefore, we have by (2.12) and (H1) that
\[
\|Tu\| \leq M_2 \left[ \int_0^{\xi_1} G_1(\eta, \eta) \int_\eta^{\xi_1} (s - \eta) ds d\eta + \int_1^\eta G_1(\eta, \eta) \int_\eta^{\xi_1} (\eta - s) ds d\eta \\
+ \frac{1}{\delta} \int_0^1 G_1(\eta, \eta) \int_\xi^{\xi_1} (b - a(\xi_1 - \eta))(c(\xi_2 - s) + d) ds d\eta \right],
\]
which implies the boundedness of $T(B)$. Furthermore, for $u \in B$ we get by (2.11) that
\[
(Tu)'(t) = -\int_0^t \eta \int_\xi^{\xi_1} (s - \eta) f(s, u(s)) ds d\eta + \int_1^\eta (1 - \eta) \int_\xi^{\xi_1} (s - \eta) f(s, u(s)) ds d\eta \\
+ \frac{1}{\delta} \left[ -\int_0^t \eta \int_\xi^{\xi_1} (b - a(\xi_1 - \eta))(c(\xi_2 - s) + d) f(s, u(s)) ds d\eta \\
+ \int_1^t (1 - \eta) \int_\xi^{\xi_1} (b - a(\xi_1 - \eta))(c(\xi_2 - s) + d) f(s, u(s)) ds d\eta \right].
\]
These and (2.13) imply $||(Tu)'|| < M$ for some positive constant $M$. Hence, $T(B)$ is equicontinuous. By the Arzela–Ascoli theorem, we know that the operator $T$ is completely continuous.

We will apply Lemma 2.2 with
\[
U := \{x \in C[0, 1] : |x| < R \} \quad \text{and} \quad C = E = C[0, 1],
\]
where $R > 0$ satisfies
\[
\frac{R}{g(R)} > k. \quad (2.14)
\]
Here, $k$ is as in (2.10). Now let $u \in C[0, 1]$ be any solution of $u = \lambda Tu$ for $0 < \lambda < 1$. Then for $t \in [0, 1]$, we have by (H1), (H2) and (2.12) that
\[
|u(t)| = |\lambda Tu(t)| \leq \int_0^{\xi_1} G_1(t, \eta) \int_\eta^{\xi_1} (s - \eta) \alpha(s) g(u(s)) ds d\eta + \int_1^\eta G_1(t, \eta) \int_\eta^{\xi_1} (\eta - s) \alpha(s) g(u(s)) ds d\eta \\
+ \frac{1}{\delta} \int_0^1 G_1(t, \eta) \int_\xi^{\xi_1} (b - a(\xi_1 - \eta))(c(\xi_2 - s) + d) \alpha(s) g(u(s)) ds d\eta \\
\leq \int_0^{\xi_1} G_1(\eta, \eta) \int_\eta^{\xi_1} (s - \eta) \alpha(s) g(||u||) ds d\eta + \int_1^\eta G_1(\eta, \eta) \int_\eta^{\xi_1} (\eta - s) \alpha(s) g(||u||) ds d\eta \\
+ \frac{1}{\delta} \int_0^1 G_1(\eta, \eta) \int_\xi^{\xi_1} (b - a(\xi_1 - \eta))(c(\xi_2 - s) + d) \alpha(s) g(||u||) ds d\eta \\
= g(||u||) \left[ \int_0^{\xi_1} \eta(1 - \eta) \int_\eta^{\xi_1} (s - \eta) \alpha(s) ds d\eta + \int_1^\eta (1 - \eta) \int_\eta^{\xi_1} (\eta - s) \alpha(s) ds d\eta \\
+ \frac{1}{\delta} \int_0^1 \eta(1 - \eta) \int_\xi^{\xi_1} (b - a(\xi_1 - \eta))(c(\xi_2 - s) + d) \alpha(s) ds d\eta \right] \\
= \frac{1}{12} g(||u||) \left[ \int_0^{\xi_1} s^3(2 - s) \alpha(s) ds + \int_1^{\xi_1} (1 - s)^3(1 + s) \alpha(s) ds \\
+ \frac{2(b - a\xi_1) + a}{\delta} \int_\xi^{\xi_1} (c(\xi_2 - s) + d) \alpha(s) ds \right].
Consequently
\[
\frac{\|u\|}{g(\|u\|)} \leq k. \tag{2.15}
\]
Now (2.14) and (2.15) imply \(\|u\| \neq R\). Lemma 2.2 now guarantees that BVP (1.1) and (1.2) has a solution \(u \in C[0, 1]\). \(\square\)

**Example 2.1.** Consider the boundary value problem
\[
\begin{aligned}
&u^{(4)}(t) = 12\sqrt{t} \sin t \cdot e^{u(t)}, &0 < t < 1, \\
&u(0) = u(1) = 0, &\\
&u''(1/3) - u'''(1/3) = 0, &u''(2/3) + u'''(2/3) = 0.
\end{aligned} \tag{2.16}
\]

To show (2.16) has a solution we apply Theorem 2.3 with \(f(t, u) = 12\sqrt{t} \sin t \cdot e^{u}, a = b = c = d = 1, \xi_1 = 1/3\) and \(\xi_2 = 2/3\). Clearly (H1) is satisfied. Since
\[
|f(t, u)| \leq 12\sqrt{te^u}, \quad t \in (0, 1), \quad u \geq 0,
\]
we have
\[
\alpha(t) = 12\sqrt{t}, \quad g(u) = e^u.
\]

Notice
\[
k = \int_0^{1/3} s^{7/2}(2-s)ds + \int_{1/3}^1 (1-s)^3(1+s)\sqrt{s}ds + \int_{1/3}^{2/3} \left(\frac{5}{3} - s\right)\sqrt{s}ds
\]
\[
= 0.3238.
\]

In addition
\[
\sup_{c \in (0, \infty)} \frac{c}{g(c)} = \frac{1}{e} = 0.3679 > 0.3238 = k.
\]
So (H2) is satisfied. Thus, Theorem 2.3 now guarantees that BVP (2.16) has a solution \(u \in C[0, 1]\).

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