Black hole entropy from Quantum Geometry

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Abstract

Quantum Geometry provides microscopic degrees of freedom that account for the black-hole entropy. However, the procedure for state counting used in the literature contains an error and the number of the relevant horizon states is underestimated. In our paper a correct method of counting is presented. Our results lead to a revision of the literature of the subject. It turns out that the contribution of spins greater then 1/2 to the entropy is not negligible. Hence, the value of the Barbero-Immirzi parameter involved in the spectra of all the geometric and physical operators in this theory is different than previously derived.

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I. INTRODUCTION AND RESULTS

Quantum Geometry [1–3] provides microscopic degrees of freedom that account for the black-hole entropy. The issue was raised in works by Krasnov [4] and Rovelli [5]. The final quantum Isolated Horizon framework was introduced by Ashtekar, Baez, Corichi and Krasnov [8]. The framework describes a black-hole in equilibrium surrounded by a quite arbitrary space-time. In the canonical formulation of the theory, the classical phase space is restricted to the set of space-times containing a black-hole of a fixed area a. After quantization, the kinematic Hilbert space is the tensor product of the horizon Hilbert space $\mathcal{H}_S$ and the bulk Hilbert space $\mathcal{H}_\Sigma$. Elements of $\mathcal{H}_S$ are identified with the black-hole kinematic quantum states. They represent quantum excitations of the $U(1)$ spin connection defined on the horizon. Mathematically, they are described by the quantum Chern-Simons theories on a punctured sphere, where all possible sets of punctures are admitted. Therefore, they are labeled by certain sequences of numbers (referred to as spins) assigned to the punctures. The bulk states, that is elements of $\mathcal{H}_\Sigma$, on the other hand, are described by Quantum Geometry, and define the quantum area of the horizon [6,7]. The quantum constraints commute with the quantum horizon area operator. Most importantly, all the solutions whose quantum areas fall into any given finite interval $[a-\delta a, a+\delta a]$ can be labeled by a finite number of the black-hole states (and by other labels corresponding to the bulk states). That number defines the Ashtekar-Baez-Corichi-Krasnov black-hole entropy. However, the procedure introduced in [8] for state counting contains a spurious constraint on admissible sequences of the labels, and the number of the relevant horizon states is underestimated. The goal of our work[1] is to point out this error, to formulate a correct counting procedure, and to study the consequences. Several conclusions of [8] have to be re-examined:

- the statement that the leading contribution to the entropy comes entirely from the simplest states (characterized by punctures with labels $\pm 1/2$),
- the confirmation of the Bekenstein and Hawking formula [9],
- the determination of the value the free parameter γ introduced in Quantum Geometry by Barbero and Immirzi, as

$$\gamma_{\mathrm{ABCK}} = \frac{\ln 2}{\pi\sqrt{3}}. \tag{1}$$

The Barbero-Immirzi parameter labeles different sectors of Quantum Geometry. The spectra of all the geometric and physical operators depend on γ. For example the quantized area is proportional to γ.

Therefore, in our paper we go back to the issue of calculating the entropy within the Quantum Geometry and the Ashtekar-Baez-Corichi-Krasnov framework. We formulate an exact combinatoric problem whose solution is the entropy: we carefully define a set of sequences freely labeling the black-hole quantum states which correspond to eigenvalues of the quantum area operator equal or less than a. The entropy is $\ln N(a)$, where $N(a)$ is the

[1] This project was a part of the diploma thesis of one of the authors [13].

number of elements of this set. Next, we address the issue of the proportionality of the entropy to the area. We consider the ratio

$$\frac{\ln N(a)}{a} \tag{2}$$

which is expected to be constant in the limit as $a \to \infty$. We find that for large a,

$$\frac{\ln 2}{4\pi\gamma\ell_{\rm Pl}^2}\, a \ \leq\ \ln N(a) \leq\ \frac{\ln 3}{4\pi\gamma\ell_{\rm Pl}^2}\, a\,. \tag{3}$$

As before (in [8]), the Barbero-Immirzi parameter γ can be fixed by assuming the agreement with the Bekenstein-Hawking entropy (the proportionality of the entropy to the area in the leading order is not proved yet, at this point, but see below). Then, (3) provides lower and upper bounds for γ,

$$\frac{\ln 2}{\pi} \leq \gamma \leq \frac{\ln 3}{\pi}. \tag{4}$$

These results lead to a revision of the literature of the subject. They show, that the first of the statements of [8], itemized above, is not true. The horizon states labeled by higher spins do contribute to the leading term of the entropy. Secondly, a value of the parameter γ can not be that of (1).

We were not able to verify the proportionality of the horizon entropy to the area ourselves. However, the combinatoric formulation of the procedure of calculating the entropy presented in this paper has been recently solved in an exact way by Krzysztof Meissner in the the accompanying paper [10]. He calculated the number $\ln N(a)$ rigorously, and showed that

$$\ln N(a) \ =\ \frac{\gamma_M}{\gamma}\frac{a}{4\ell_{\rm Pl}^2} - \frac{1}{2}\ln a + O(1) \tag{5}$$

in the limit of large horizon area a, where the value of the parameter γ_M can be calculated at arbitrary accuracy. In conclusion, taking into account the results of [10], the proportionality of the entropy to the area in the leading order for large a is confirmed, and even the sub-leading terms are calculated. The physical meaning of these new results will be discussed in a joint paper [11].

II. THE HORIZON QUANTUM STATES

Our departure point is the quantum Isolated Horizon theory introduced in [8]. A classical (weakly) isolated horizon is a null, non-expanding world-surface of a space-like 2-sphere S, equipped with a null vector field of a constant self-acceleration. This idea gave rise to a new quasi-local theory of black-hole in equilibrium interesting also from purely classical point of view [12]. The local degrees of freedom of space-time geometry out of the horizon are not restricted. An additional assumption satisfied by the isolated horizon geometry considered in [8] is the spherical symmetry (see [3] for a discussion of the general case). The classical phase space introduced in the Ashtekar-Baez-Corichi-Krasnov framework consists of all the gravitational fields which can exist in a neighborhood of an isolated horizon.

The location and area a of the horizon are fixed. In the corresponding $3+1$ Hamiltonian framework the horizon is represented by a 2-sphere S, whereas the remaining part of space under consideration (bulk) is the exterior region, a 3-manifold Σ bounded by S. The only kinematic degrees of freedom of the horizon S are given by a U(1) connection defined on a spin bundle over S. The gravitational field data is defined on Σ. The bulk and the horizon data are subject to a consistency condition ensuring that the world-surface of S is indeed an isolated horizon. In the quantum theory of this system, the bulk gravitational field is described by Quantum Geometry. The quantum consistency condition between the bulk and the horizon implies that the quantum degrees of freedom of the horizon should be described by a union of the Chern-Simons theories on arbitrarily punctured S.

Below, we outline the elements of the quantum theory which play a role in the entropy calculation. The kinematic Hilbert space of the quantum Isolated Horizon theory is *contained* in the tensor product $\mathcal{H}_S \otimes \mathcal{H}_\Sigma$, where $\mathcal{H}_S$ is the Hilbert space of the quantum horizon degrees of freedom whereas $\mathcal{H}_\Sigma$ is the Hilbert space of the quantum geometry defined in the bulk Σ.

To characterize the bulk quantum geometry Hilbert space $\mathcal{H}_\Sigma$, it is convenient to use a certain orthogonal decomposition adapted to the boundary S. Consider a finite set of points,

$$\mathcal{P} \; = \{p_1, ..., p_n\} \subset S, \tag{6}$$

and a labeling of elements of $\mathcal{P}$ by numbers $j = (j_1, ..., j_n)$, and $m = m_1, ..., m_n)$ where

$$j_i \in \frac{1}{2}\mathbb{N}, \qquad m_i \in \{-j_1, ..., j_i\} \tag{7}$$

for $i = 1, ..., n$. The j_i labels are assumed not to vanish,

$$j_i \neq 0\,. \tag{8}$$

The space $\mathcal{H}_\Sigma$ is the orthogonal sum

$$\mathcal{H}_\Sigma \;=\; \bigoplus_{(\mathcal{P},j,m)} \mathcal{H}_\Sigma^{\mathcal{P},j,m}, \tag{9}$$

where $\mathcal{P}$ runs through all the finite subsets of S, (j, m) through all the finite labellings (7,8), and the empty set $\mathcal{P} = \emptyset$ is also associated a (non-trivial) single Hilbert space $\mathcal{H}_\Sigma^\emptyset$. Each of the subspaces $\mathcal{H}_\Sigma^{\mathcal{P},j,m}$ is a certain (infinitely dimensional) Hilbert space. To understand the (quantum) geometric meaning of these subspaces, consider any piece S' of the horizon S (more precisely, S' is an arbitrary 2-sub-manifold in the 2-sphere S). The quantum area operator $\hat{A}_{S'}$ is defined in $\mathcal{H}_\Sigma$. Another operator of the quantum geometry in Σ assigned to S' is the flux $\hat{E}_{S'}$ of the vector field normal to S.[2] Every subspace $\mathcal{H}_\Sigma^{\mathcal{P},j,m}$ is the eigenspace of the operators. The corresponding eigenvalues are:

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[2]There are two subtleties, which may be intriguing for the reader. The first one is that classically the flux of a vector field normal to a surface is the same as the area of the surface. In our quantum case however, the vector field is a quantum operator itself, in fact it is one of the SU(2) spin operators. The area, on the other hand, is given by the total SU(2) spin operator. The second subtlety is, that S' contained in the *horizon* is assigned the area and flux by the *bulk* geometry.

$$a^{\mathcal{P},j}_{S'} \; = \; 8\pi\gamma\ell^2_{\rm Pl} \sum_{p_i\in\mathcal{P}\cap S'} \sqrt{j_i(j_i+1)}, \tag{10}$$

$$e^{\mathcal{P},m}_{S'} \; = \; 8\pi\gamma\ell^2_{\rm Pl} \sum_{p_i\in\mathcal{P}\cap S'} m_i \tag{11}$$

where only those points $p_i \in \mathcal{P}$ contribute which are contained also in S', and $\gamma > 0$ is a free parameter of Quantum Geometry known as Barbero-Immirzi parameter. In particular, if we take for S' the whole horizon S, then the corresponding eigenvalue of the horizon area operator is (we drop $\mathcal{P}$ because only the set of the values of j matters)

$$a^j_S \; = \; 8\pi\gamma\ell^2_{\rm Pl} \sum_i \sqrt{j_i(j_i+1)}. \tag{12}$$

Of course the quantum area and flux operators are defined also for 2-dimensional submanifolds of Σ which do not intersect S, and there are also other interesting operators of Quantum Geometry acting in $\mathcal{H}_\Sigma$ but they are not used in the Isolated Horizon framework.

We turn now to the horizon Hilbert space $\mathcal{H}_S$. Recall, that the ABCK framework requires fixing an arbitrary value for the classical horizon area a. In order to quantize consistently the horizon degrees of freedom, it is assumed in [8] that the fixed classical area a is quantized in the following way,

$$a \; = \; 4\pi\gamma\ell^2_{\rm Pl}k, \quad k \in \mathbb{N}, \tag{13}$$

where k is arbitrary. The horizon Hilbert space can also be orthogonally decomposed in a way analogous to (9). In this case the labeling set consists of all the finite sequences $\vec{\mathcal{P}}$ of points in S, labeled by arbitrary non-zero integers defined modulo k, and whose sum is zero modulo k,

$$\vec{\mathcal{P}} \; = \; (p_1, ..., p_n) \quad b \; = \; (b_1, ..., b_n)\,, \tag{14}$$

$$b_i \; \in \; \mathbb{Z}_k, \quad i = 1, ..., n, \quad \sum_{i=1}^{n} b_i = 0\,. \tag{15}$$

To every labeled sequence $(\vec{\mathcal{P}}, b)$ there is assigned a 1-dimensional Hilbert space $\mathcal{H}^{\vec{\mathcal{P}},b}_S$. The points p_i are often called punctures, and for every p_i, the $U(1)$ element $e^{i\frac{2\pi b_i}{k}}$ is the eigenvalue of the holonomy transport along a circle containing p_i. The full horizon Hilbert space can be written as the orthogonal sum

$$\mathcal{H}_S = \bigoplus_{(\vec{\mathcal{P}},b)} \mathcal{H}^{\vec{\mathcal{P}},b}_S, \tag{16}$$

where to the labeling set of pairs $(\vec{\mathcal{P}}, b)$ we add the empty sequence which labels a single, 1-dimensional space $\mathcal{H}^{\vec{\emptyset}}_S$.

Now, the quantum condition that S be a section of a spherically symmetric isolated horizon is introduced in the product Hilbert space

$$\mathcal{H}_S \otimes \mathcal{H}_\Sigma \; = \; \bigoplus_{(\vec{\mathcal{P}},b)} \bigoplus_{(\mathcal{P}',j,m)} \left(\mathcal{H}^{\vec{\mathcal{P}},b}_S \otimes \mathcal{H}^{\mathcal{P}',j,m}_\Sigma\right). \tag{17}$$

It involves the holonomy operators acting in $\mathcal{H}_S^{\vec{\mathcal{P}},b}$ on the one hand and the flux operators acting in $\mathcal{H}_\Sigma^{\mathcal{P}',j,m}$ on the other hand (see [8] or [3] for a summary). The set of solutions is the orthogonal sum of products $\mathcal{H}_S^{\vec{\mathcal{P}},b} \otimes \mathcal{H}_\Sigma^{\mathcal{P},j,m}$ such that the points coincide modulo the ordering

$$\vec{\mathcal{P}} = (p_1, ..., p_n), \quad \mathcal{P} \; = \; \{p_1, ..., p_n\} \tag{18}$$

and the labellings agree with each other in the following way

$$b_i = -2m_i \mod k. \tag{19}$$

In conclusion, the kinematic Hilbert space of states in the ABCK quantum Isolated Horizon framework is the orthogonal sum with respect to all the labeled sequences of points in S which satisfy (18,19),

$$\mathcal{H}_{\rm kin} \; = \; \bigoplus_{\vec{\mathcal{P}},j,m} \mathcal{H}_S^{\vec{\mathcal{P}},b(m)} \otimes \mathcal{H}_\Sigma^{\mathcal{P},j,m}\,, \tag{20}$$

where we emphasized the dependence of each sequence $b = (b_1, ..., b_n)$ on a sequence $m = (m_1, ..., m_n)$ via (19).

The final step of the ABCK framework is to take into account the quantum Einstein constraints [1–3] adjusted to the Isolated Horizon framework [8]. The vector constraints generate diffeomorphisms of Σ which preserve the boundary S. The diffeomorphisms act naturally in $\mathcal{H}_{\rm kin}$ and this action is unitary. In the Quantum Geometry framework, the quantum vector constraints come down to the diffeomorphism invariance. Whereas there is only one normalizable diffeomorphism invariant state in $\mathcal{H}_{\rm kin}$, the invariance condition is imposed in a certain larger space (dual to an appropriate subspace of $\mathcal{H}_{\rm kin}$). A Hilbert space of solutions is constructed via the diffeomorphism averaging procedure whose Quantum Geometry version is well described in [1,3]. It is particularly easy to explain the result of the analogous diffeomorphism averaging suitable for the space $\mathcal{H}_S$ itself. Assign to every finite sequence $b = (b_1, ..., b_n)$ defined in (15) a 1-dimensional Hilbert space $\mathcal{H}_S^b$, and define

$$\mathcal{H}_{S,{\rm phys}} \; = \; \bigoplus_b \mathcal{H}_S^b\,. \tag{21}$$

The averaging applied to the full space $\mathcal{H}_{\rm kin}$ provides a Hilbert space of the following structure

$$\mathcal{H} \; = \; \bigoplus_{j,m} \tilde{\mathcal{H}}^{b(m),j,m}, \tag{22}$$

where $b = (b_1, ..., b_n)$, $j = (j_1, ...j_n)$ and $m = (m_1, ..., m_n)$ are arbitrary triples of finite sequences such that (7,15,19) holds. The remaining constraints are the scalar and the Gauss constraint. The Gauss constraint amounting to the invariance with respect to the local SU(2) rotations of the spin frames is already satisfied at S due to (19). The scalar constraint, on the other hand, has been solved at the horizon already on the classical level. Since the quantum numbers (j, m) used in (23) characterize the quantum geometry of bulk Σ right at the boundary S, the plausible, but non-trivial assumption made in [8] is that the bulk scalar constraints with laps functions vanishing on S do not affect (j, m), and that for every choice of (j, m) there is a solution of the bulk scalar constraints.

We are lead to a conclusion, that the final physical Hilbert $\mathcal{H}_{\rm phys}$ of the quantum states can be characterized as

$$\mathcal{H}_{\rm phys} \ = \ \bigoplus_{j,m} \mathcal{H}^{b(m),j,m}, \tag{23}$$

where $b = (b_1, ..., b_n)$, $j = (j_1, ...j_n)$ and $m = (m_1, ..., m_n)$ are arbitrary triples of finite sequences such that (7,15,19) is satisfied, and none of the spaces $\mathcal{H}^{b(m),j,m}$ is empty. Since the horizon area operator $\hat{A}_S$ commutes with all the constraints in this framework, it passes to the physical Hilbert space. It also follows from the details of the diffeomorphism averaging map, that each of the subspaces $\mathcal{H}^{b(m),j,m}$ is an eigenspace corresponding to the eigenvalue (12). In this way, *the sequences j are responsible for the area assigned to the 2-surface S of the horizon by the bulk Quantum Geometry, whereas the sequences b represent the intrinsic quantum degrees of freedom of the horizon.* This is the key conclusion of the ABCK framework.

III. THE COMBINATORIC FORMULATION OF THE PROBLEM

From now our treatment starts to defer from that of [8]. Our aim is to define the number of the quantum states of the horizon which contribute to the entropy in a form of a clear and precise combinatoric formula.

Recall that the classical area a of the horizon is fixed as

$$a \ = \ 4\pi\gamma\ell_{\rm Pl}^2 k, \quad k \in \mathbb{N}. \tag{24}$$

Calculation of the horizon entropy consists in counting those quantum horizon states, elements of the Hilbert space $\mathcal{H}_{S,\rm phys}$ (21) labeled by the finite sequences defined in (15) (the number n of the entries can be arbitrary) which correspond to subspaces $\mathcal{H}^{b(m),j,m}$ in (23) such that the area operator eigenvalue a_S^j satisfies the following inequality[3]

$$a_S^j \ = \ 8\pi\gamma\ell_{\rm Pl}^2 \sum_{i=1}^n \sqrt{j_i(j_i+1)} \ \leq \ a. \tag{25}$$

Combinatorially, the task amounts to counting the finite sequences $(b_1, ..., b_n)$ of elements of $\mathbb{Z}_k$, such that the following two conditions (i) and (ii) are satisfied:

$$(i) \qquad \sum_{i=1}^n b_i = 0\,, \tag{26}$$

(ii) there exist two sequences: a sequence $(j_1, ..., j_n)$ consisting of non-zero elements of $\frac{1}{2}\mathbb{N}$ which satisfies the inequality (25), and a sequence $(m_1, ..., m_n)$ such that

[3] Assuming just inequality instead of considering an interval $[a-\delta a, a+\delta a]$ simplifies the calculation. Knowing the result for all a one can always consider the interval. The result, at least in our case, is the same.

$$b_i = -2m_i \mod k\,, \tag{27}$$
$$\text{and} \quad m_i \in \{-j_i, -j_i+1, ..., j_i\}\,, \tag{28}$$

for every $i = 1, , , n$.

This recipe can be simplified. It is easy to eliminate the j_i's by noting that, given a sequence $(m_1, ..., m_n)$, there is a sequence $(j_1, ..., j_n)$ such that (28) and (25) if and only if

$$8\pi\gamma\ell_{\rm Pl}^2 \sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} \;\leq\; a. \tag{29}$$

Next, it follows from (26, 27) that

$$\sum_{i=1}^{n} m_i \;=\; n'\frac{k}{2}\,, \tag{30}$$

with some $n' \in \mathbb{Z}$. But then,

$$\begin{aligned} a \;=\; 4\pi\gamma\ell_{\rm Pl}^2 k \;&\geq\; 8\pi\gamma\ell_{\rm Pl}^2 \sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} > \qquad (31)\\ &> 8\pi\gamma\ell_{\rm Pl}^2 \sum_{i=1}^{n} |m_i| \;\geq\; 8\pi\gamma\ell_{\rm Pl}^2 |\sum_{i=1}^{n} m_i| \;= \qquad (32)\\ &=\; 4\pi\gamma\ell_{\rm Pl}^2 k|n'|. \qquad (33)\end{aligned}$$

Hence

$$\sum_{i=1}^{n} m_i = 0, \quad \text{and also} \quad \sum_{i=1}^{n} |m_i| \leq \frac{k}{2}\,. \tag{34}$$

Finally, notice that given a sequence $(b_1, ..., b_n)$ of elements of $\mathbb{Z}_k$ there is exactly one sequence $(m_1, ..., m_n)$ of elements of $\frac{1}{2}\mathbb{Z}$ such that (27) and (34) holds. In this way, we have derived the following formula for the entropy counting:

The entropy S of a quantum horizon of the classical area a according to Quantum Geometry and the Ashtekar-Baez-Corichi-Krasnov [8] framework is

$$S \;=\; \ln N(a), \tag{35}$$

where $N(a)$ is 1 plus the number of all the finite sequences $(m_1, ..., m_n)$ of non-zero elements of $\frac{1}{2}\mathbb{Z}$, such that the following equality and inequality are satisfied:

$$\sum_{i=1}^{n} m_i \;=\; 0, \qquad \sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} \;\leq\; \frac{a}{8\pi\gamma\ell_{\rm Pl}^2}, \tag{36}$$

where γ is the Barbero-Immirzi parameter of Quantum Geometry. The extra term 1 above comes from the trivial sequence.

IV. THE ENTROPY CALCULATIONS

To find an upper bound for the number $N(a)$ introduced in the previous section (recall that $a = 4\pi\gamma\ell_{\rm Pl}^2 k$, and $k \in \mathbb{N}$), consider the set M_k^+ of all the finite sequences $(m_1, ..., m_n)$ of non-zero elements of $\frac{1}{2}\mathbb{Z}$ which satisfy the inequality in (34), but do not necessarily sum to zero,that is,

$$M_k^+ \ := \ \{(m_1, ..., m_n) \ \Big| \ n \in \mathbb{N}, \ 0 \neq m_1, ..., m_n \in \frac{1}{2}\mathbb{Z}, \ \sum_{i=1}^n |m_i| \ \leq \ \frac{k}{2}\}\,. \tag{37}$$

Let N_k^+ be the number of elements of M_k^+ plus 1 (the empty sequence). Certainly,

$$N(a) \ \leq \ N_k^+ \tag{38}$$

Next, since k is arbitrarily fixed integer, let it become a variable of the sequence $(N_0^+, N_1^+, ..., N_k^+, ...)$. To establish a recurrence relation satisfied by the sequence $(N_0^+, N_1^+, ..., N_k^+, ...)$, notice that if $(m_1, ..., m_n) \in M_{k-1}^+$, then both $(m_1, ..., m_n, \frac{1}{2})$, $(m_1, ..., m_n, -\frac{1}{2}) \in M_k^+$. In the same way, for arbitrary natural $0 < l \leq k$,

$$(m_1, ..., m_n) \in M_{k-l}^+ \ \Rightarrow \ (m_1, ..., m_n, \pm\frac{1}{2}l) \in M_k^+. \tag{39}$$

Obviously, if we consider all $0 < l \leq k$, and all the sequences $(m_1, ..., m_n) \in M_{k-l}^+$, then the resulting $(m_1, ..., m_n, \pm\frac{1}{2}l)$ set the entire set M_k^+. Also, for two different $l \neq l'$,

$$(m_1, ..., m_n, \pm\frac{1}{2}l) \ \neq \ (m_1, ..., m_n, \pm\frac{1}{2}l'). \tag{40}$$

This proves the following recurrence relation,

$$N_k^+ \ = \ 2N_{k-1}^+ + ... + 2N_0^+ + 1. \tag{41}$$

The (unique) solution is

$$N_k^+ \ = \ 3^k. \tag{42}$$

In conclusion,

$$N(a) \leq 3^k\,. \tag{43}$$

To find a lower bound for $N(a)$, we use the inequality

$$\sqrt{|m_i|(|m_i|+1)} \ \leq \ |m_i| + \frac{1}{2}, \tag{44}$$

and consider the number N_k^- equal to 1 plus the number of elements in the set

$$M_k^- \ := \ \{(m_1, ..., m_n) \ \Big| \ n \in \mathbb{N}, \ 0 \neq m_1, ..., m_n \in \frac{1}{2}\mathbb{Z}, \ \sum_{i=1}^n (|m_i| + \frac{1}{2}) \ \leq \ \frac{k}{2}\} \tag{45}$$

Notice that this time, ignoring the constraint that the elements m_i of each sequence sum to zero (34) makes an in-equivalence relation between $N(a)$ and N_k^- apriori not known. But let us postpone this problem for a moment. Using the same construction as above, we find the recurrence relation satisfied by N_k^-,

$$N_k^- \;=\; 2N_{k-2}^- + ... + 2N_0^- + 1. \tag{46}$$

The unique solution is

$$N_k^- \;=\; \frac{2}{3}2^k + \frac{(-1)^k}{3}\,. \tag{47}$$

A lower bound for $N(a)$ is the number $N_k'^-$ of the elements of M_k^- which additionally satisfy $m_1 + ... + m_n = 0$,

$$N_k'^- \;\leq N(a). \tag{48}$$

A statistical physics argument giving the value of the desired number $N_k'^-$ is as follows (this argument is due to Meissner, who also provided an exact proof). We can think of each sequence $(m_1, ..., m_n)$ as of a sequence of random steps on a line. The total length of each path is bounded by k owing to the inequality in the definition of the set M_k^-. The number of sequences in M_k^- of a given, fixed value of the sum

$$m_1 + ... + m_n = \delta \tag{49}$$

depends on δ. The average value of δ is $\bar{\delta} = 0$. For large values of k, the number of the paths corresponding to the random walk distance δ should be given by the Gaussian function $\frac{C}{\sqrt{k}}e^{-\frac{\delta^2}{k}}\;\; N_k^-$. In particular, the value

$$N_k'^- \;= \frac{C}{\sqrt{k}} N_k^- \tag{50}$$

corresponds to $\delta = 0$.

Summarizing,

$$\frac{C}{\sqrt{k}} N_k^- \;\leq\; N(a) \leq N_k^+, \tag{51}$$

where the numbers N_k^- and N_k^+ were calculated in (43, 47). Therefore the entropy is bounded in the following way

$$\frac{\ln 2}{4\pi\gamma\ell_{\rm Pl}^2} a + o(a) \leq S(a) \leq \frac{\ln 3}{4\pi\gamma\ell_{\rm Pl}^2} a. \tag{52}$$

We are in a position now, to compare our results with those of [8]. The mistake made in [8] in the combinatorial formulation (see (44) in that paper) is an assumption, that one can order the j_i labels in (23) such that $j_1 \leq ... \leq j_n$, and impose this as a constraint on 'admissible' states we count. As a consequence, the number of the states contributing to the entropy was underestimated: the analysis of [8] implied that for large a the leading contribution to the number of sequences comes only from the sequences $(m_1, ..., m_n) = (\pm\frac{1}{2}, ..., \pm\frac{1}{2})$. The resulting number $S_{\rm bh}(a)$ proposed for the value of the entropy was

$$S_{\rm bh}(a) \;=\; \frac{\ln 2}{4\pi\sqrt{3}\gamma\ell_{\rm Pl}^2} a + o(a). \tag{53}$$

The comparison with our (52) shows that *the contribution from the higher half integers* $|m_i| > \frac{1}{2}$ *can not be neglected.*

Next consequence of the estimate (52) is a necessary condition for the agreement of the entropy $S(a)$ with the Bekenstein-Hawking entropy $a \,/\, 4\ell_{\rm Pl}^2$; the condition is that the value of the Barbero-Immirzi parameter γ is bounded in the following way,

$$\frac{\ln 2}{\pi} \;\leq\; \gamma \;\leq \frac{\ln 3}{\pi}. \tag{54}$$

On the other hand, the value proposed for the Barbero-Immirzi parameter by counting the sequences $(\pm 1/2, ..., \pm 1/2)$ was

$$\gamma \;=\; \frac{\ln 2}{\pi\sqrt{3}}. \tag{55}$$

In the accompanying paper [10] Meissner solved completely the corrected combinatorial problem formulated in the previous section. His method is remarkably powerful and the calculation is rigorous. He proved that

$$S(a) \;=\; \frac{\gamma_M}{\gamma}\frac{a}{4\ell_{\rm Pl}^2} - \frac{1}{2}\ln a + O(1), \tag{56}$$

where γ_M is a constant calculated in his paper.

An interesting historical remark is (it has been unraveled to us recently, after our results were derived) that before coming to their final result (53, 55), Ashtekar, Baez and Krasnov had established certain estimate. The inequality had been

$$S_{\rm bh}(a) \;\leq\; \frac{\gamma_0}{\gamma}\frac{a}{4\ell_{\rm Pl}^2} + o(a) \tag{57}$$

where the constant γ_0 had been defined as a real solution to the equation

$$\frac{1}{2} - \sum_{0\neq j\in\frac{1}{2}\mathbb{N}} e^{-2\pi\gamma_0\sqrt{j(j+1)}} = 0. \tag{58}$$

(But this result emerged as an upper bound only, and finally it was replaced by the improper value in (53,55).) Surprisingly, this is just the way Meissner [10] defined his constant γ_M in his *exact* formula for the entropy with our correct counting method.

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