Archimedean t-norms in interval-valued fuzzy set theory

Glad Deschrijver
Fuzziness and Uncertainty Modeling Research Unit
Department of Applied Mathematics and Computer Science, Ghent University
Krijgslaan 281 (S9), B–9000 Gent, Belgium
Glad.Deschrijver@UGent.be
http://www.fuzzy.UGent.be

Abstract

In this paper the Archimedean property of t-norms on the lattice $\mathcal{L}^I$ is investigated, where $\mathcal{L}^I$ is the underlying lattice of interval-valued fuzzy set theory (Sambuc, 1975) and intuitionistic fuzzy set theory (Atanassov, 1983). We discuss also two additional variations on this property: the weak and strong Archimedean property.

Keywords: Archimedean, interval-valued fuzzy set, intuitionistic fuzzy set, limit property, t-norm, strongly Archimedean, weakly Archimedean, weak limit property

1 Introduction

Triangular norms on $[0, 1]$ were introduced in [16] and play an important role in fuzzy set theory (see e.g. [7, 10, 11] for more details). One of the most important properties that can be satisfied by t-norms on the unit interval is the Archimedean property, for example continuous t-norms can be fully characterized by means of Archimedean t-norms, the Archimedean property is closely related to additive and multiplicative generators [11, 13, 14].

Interval-valued fuzzy set theory [9, 15] is an extension of fuzzy theory in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree. Another extension of fuzzy set theory is intuitionistic fuzzy set theory introduced by Atanassov [1]. Intuitionistic fuzzy sets assign to each element of the universe not only a membership degree, but also a non-membership degree which is less than or equal to 1 minus the membership degree (in fuzzy set theory the non-membership degree is always equal to 1 minus the membership degree). In [6] it is shown that intuitionistic fuzzy set theory is equivalent to interval-valued fuzzy set theory and that both are equivalent to $L$-fuzzy set theory in the sense of Goguen [8] w.r.t. a special lattice $\mathcal{L}^I$.

In this paper we extend the Archimedean property to $\mathcal{L}^I$ and we investigate some necessary and sufficient conditions for a t-norm on $\mathcal{L}^I$ to be Archimedean.

2 Preliminary definitions

The underlying lattice $\mathcal{L}^I$ of interval-valued fuzzy set theory is given as follows.

Definition 2.1 We define $\mathcal{L}^I = (L^I, \leq_{L^I})$, where

$$L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 \leq x_2\},$$

and $[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2)$, for all $[x_1, x_2], [y_1, y_2]$ in $L^I$.

It can be shown that $\mathcal{L}^I$ is a complete lattice (see Lemma 2.1 in [6]).

Definition 2.2 [9, 15] An interval-valued fuzzy set ($\mathcal{L}^I$-fuzzy set) on $U$ is a mapping $A : U \to L^I$. 
Definition 2.3 [1] An intuitionistic fuzzy set on $U$ is a set

$$A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\},$$

where $\mu_A(u) \in [0, 1]$ denotes the membership degree and $\nu_A(u) \in [0, 1]$ the non-membership degree of $u$ in $A$ and where for all $u \in U$, $\mu_A(u) + \nu_A(u) \leq 1$.

An intuitionistic fuzzy set $A$ on $U$ can be represented by the $L^I$-fuzzy set $A$ given by

$$A : U \rightarrow L^I : u \mapsto [\mu_A(u), 1 - \nu_A(u)], \ \forall u \in U$$

In Figure 1 the set $L^I$ is shown. Note that to each element $x = [x_1, x_2]$ of $L^I$ corresponds a point $(x_1, x_2) \in \mathbb{R}^2$.

![Figure 1: The grey area is $L^I$.](image)

In the sequel, if $x \in L^I$, then we denote its bounds by $x_1$ and $x_2$, i.e. $x = [x_1, x_2]$. The smallest and the largest element of $L^I$ are given by $0_{L^I} = [0, 0]$ and $1_{L^I} = [1, 1]$. We define for further usage the sets $D = \{[x, x] \mid x \in [0, 1]\}$. Note that, for $x, y \in L^I$, $x \leq_L y$ is equivalent to $x \leq_{L^I} y$ and $x \neq y$, i.e. either $x_1 < y_1$ and $x_2 \leq y_2$, or $x_1 \leq y_1$ and $x_2 < y_2$. We denote by $x \ll_{L^I} y$: $x_1 < y_1$ and $x_2 < y_2$.

**Definition 2.4** A t-norm on a complete lattice $L = (L, \leq_L)$ is a commutative, associative, increasing mapping $T : L^2 \rightarrow L$ which satisfies $T(1_L, x) = x$, for all $x \in L$.

Let $T$ be a t-norm on a complete lattice $L = (L, \leq_L)$, then we denote $x^{(n)} = T(x, x^{(n-1)})$, for $n \in \mathbb{N} \setminus \{0, 1\}$.

We say that a t-norm $T$ on $L$ satisfies the residuation principle if and only if, for all $x, y, z$ in $L$,

$$T(x, y) \leq_L z \iff y \leq_L \sup \{\gamma \mid \gamma \in L \text{ and } T(x, \gamma) \leq_L z\}.$$ 

For t-norms on $L^I$, we consider the following special classes.

**Example 2.1** Let $T$ and $T'$ be t-norms on $[0, 1]$ such that $T \leq T'$, and let $\alpha \in [0, 1]$, then the following mappings are t-norms on $L^I$: for $x, y$ in $L^I$,

- $T_{T,T'}(x, y) = [T(x_1, y_1), T'(x_2, y_2)]$ (t-representable t-norms);
- $T_T(x, y) = [T(x_1, y_1), \max(T(x_1, y_2), T(x_2, y_1))]$ (pseudo-t-representable t-norms);
- $T_{T,T}(x, y) = [T(x_1, y_1), \max(T(\alpha, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))]$;
- $T_{T'}^I(x, y) = [\min(T(x_1, y_2), T(x_2, y_1)), T(x_2, y_2)].$

3 The Archimedean property for t-norms on $[0, 1]$

Denote $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Then Archimedean t-norms are defined as follows.

**Definition 3.1** [11, 12] Let $T$ be a t-norm on $[0, 1]$. We say that

- $T$ is Archimedean if
  $$(\forall (x, y) \in [0, 1]^2)(\exists n \in \mathbb{N}^*)(x^{(n)}T < y);$$
- $T$ has the limit property if
  $$(\forall x \in [0, 1])(\lim_{n \rightarrow +\infty} x^{(n)}T = 0).$$

An element $x \in [0, 1]$ is called a zero-divisor of $T$ if there exists an $y \in [0, 1]$ such that $T(x, y) = 0$.

For t-norms on the unit interval we have the following important characterizations of the Archimedean property.

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1In this paper $\mathbb{N}$ denotes the set of non-negative integers, i.e. $\mathbb{N} = \{0, 1, 2, \ldots\}$.
Theorem 3.1 [11, 12] A t-norm $T$ on $[0,1]$ is Archimedean if and only if it satisfies the limit property.

Theorem 3.2 [11, 12] Let $T$ be a t-norm on $[0,1]$. Then $T$ is Archimedean if and only if $T(x,x) < x$, for all $x \in [0,1]$, and, whenever $\lim_{x \to x^*} T(x,x) = x^*$, for some $x^* \in [0,1]$, there exists a $y^* \in [x^*,1]$ such that $T(y^*,y^*) = x^*$.

Theorem 3.3 [11, 12] Let $T$ be a continuous t-norm on $[0,1]$. Then $T$ is Archimedean if and only if $T(x,x) < x$, for all $x \in [0,1]$.

4 The Archimedean property for t-norms on $\mathcal{L}'$: necessary and sufficient conditions

We extend the definitions from the previous section to $\mathcal{L}'$. There are several possible extensions of the Archimedean property, which we call Archimedean, weak Archimedean and strong Archimedean property.

Definition 4.1 Let $T$ be a t-norm on $\mathcal{L}'$, $A = \{x \mid x \in \mathcal{L}'$ and $x_1 \in [0,1]\}$ and $A' = \{x \mid x \in \mathcal{L}'$ and $x_1 > 0$ and $x_2 < 1\}$. We say that

- $T$ is Archimedean if
  \[(\forall (x,y) \in A^2)(\exists n \in \mathbb{N}^*)(x^{(n)} \mathcal{T} < \mathcal{L}' y);\]
- $T$ is strongly Archimedean if
  \[(\forall (x,y) \in (\mathcal{L}' \setminus \{\mathcal{L}' , 1_{\mathcal{L}'}\})^2)(\exists n \in \mathbb{N}^*)(x^{(n)} \mathcal{T} < \mathcal{L}' y);\]
- $T$ is weakly Archimedean if
  \[(\forall (x,y) \in A^2)(\exists n \in \mathbb{N}^*)(x^{(n)} \mathcal{T} < \mathcal{L}' y).\]

Obviously, if a t-norm $T$ on $\mathcal{L}'$ is Archimedean, then it is weakly Archimedean, and if $T$ is strongly Archimedean, then it is Archimedean. The converse implications do not hold (counterexamples will be given in Section 5).

In [2, 3] the Archimedean property is defined for t-norms on a general bounded poset. If we apply this definition to $\mathcal{L}'$, then we obtain the following condition for a t-norm $T$ on $\mathcal{L}'$:

\[(\forall (x,y) \in (\mathcal{L}'^2)(\forall n \in \mathbb{N}^*)(x^{(n)} \mathcal{T} \geq \mathcal{L}' y) \implies (x = 1_{\mathcal{L}'}$ or $y = 0_{\mathcal{L}'}).\]

The following theorem shows that the Archimedean property defined using (1) corresponds to the Archimedean property given in Definition 4.1.

Theorem 4.1 Let $T$ be a t-norm on $\mathcal{L}'$. Then $T$ is Archimedean (in the sense of Definition 4.1) if and only if $T$ satisfies (1).

On the unit interval the Archimedean property is equivalent to the counterpart of (1) on the unit interval. Although the definition of the strong Archimedean property on $\mathcal{L}'$ is similar to the definition of the Archimedean property on the unit interval, on $\mathcal{L}'$ property (1) is not equivalent to the strong Archimedean property: Theorem 4.1 shows that (1) is equivalent to our definition of the Archimedean property.

Definition 4.2 Let $T$ be a t-norm on $\mathcal{L}'$ and $A' = \{x \mid x \in \mathcal{L}'$ and $x_1 > 0$ and $x_2 < 1\}$. We say that

- $T$ has the limit property if
  \[(\forall x \in A)(\lim_{n \to +\infty} x^{(n)} = 0_{\mathcal{L}'}).\]
- $T$ has the strong limit property if
  \[(\forall x \in \mathcal{L}' \setminus \{0_{\mathcal{L}'} , 1_{\mathcal{L}'}\})(\lim_{n \to +\infty} x^{(n)} = 0_{\mathcal{L}'});\]
- $T$ has the weak limit property if
  \[(\forall x \in A')(\lim_{n \to +\infty} x^{(n)} = 0_{\mathcal{L}'}).\]

Now we extend Theorem 3.1 to $\mathcal{L}'$.

Theorem 4.2 Let $T$ be a t-norm on $\mathcal{L}'$ and $A = \{x \mid x \in \mathcal{L}'$ and $x_1 \in [0,1]\}$. Then the following are equivalent:

(A1) $T$ is Archimedean;
(A2) $(\forall (x,y) \in A^2)(\exists n \in \mathbb{N}^*)(x^{(n)} \mathcal{T} \mathcal{S} \leq \mathcal{L}' y)$;
(A3) $\mathcal{T}$ satisfies the strong limit property;
(A4) $\mathcal{T}$ satisfies the limit property.

In a similar way the following theorem is proved.

**Theorem 4.3** Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$ and $A = \{x \mid x \in \mathcal{L}^I \text{ and } x_1 > 0 \text{ and } x_2 < 1\}$. Then the following are equivalent:

1. $\mathcal{T}$ is weakly Archimedean;
2. $(\forall (x, y) \in A^2)(\exists n \in \mathbb{N}^*)(x^{(n)} \triangleright_L x)$; 
3. $\mathcal{T}$ satisfies the weak limit property.

We will generalize Theorem 3.2 to $\mathcal{L}^I$. First we give a lemma.

**Lemma 4.4** Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$. If $\mathcal{T}$ does not satisfy the limit property, but satisfies

$$\forall x \in \mathcal{L}^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\})(\mathcal{T}(x, x) < \mathcal{L}^I x)$$

then there exists an $x^* \in \mathcal{L}^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$ such that one of the following holds:

1. $\lim_{x \triangleright_L x^*} \mathcal{T}([x_1, x_2], [x_1^*, x_2]) = x^*$;
2. $\lim_{x \triangleright_L x^*} \mathcal{T}(x, x) = x^*$ and $(\forall x \in \mathcal{L}^I)(x \triangleright_L x^* \implies \mathcal{T}(x, x) \triangleright_L x^*)$;
3. $\lim_{x \triangleright_L x^*} \mathcal{T}([x_1, x_2^*], [x_1, x_2]) = x^*$.

**Theorem 4.5** Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$. Then $\mathcal{T}$ satisfies the limit property if and only if $\mathcal{T}$ satisfies the following four properties:

1. $\mathcal{L}^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\})(\mathcal{T}(x, x) < \mathcal{L}^I x)$;
2. $\lim_{x \triangleright_L x^*} \mathcal{T}([x_1^*, x_2], [x_1, x_2^*]) = x^*$
   $$\implies (\exists y^* \in \{y \mid y \in \mathcal{L}^I \text{ and } x^* \triangleright_L y \leq \mathcal{L}^I [x_1^*, 1]\})(\mathcal{T}(y^*, y^*) = x^*)$$

The necessary and sufficient conditions for the Archimedean property given in Theorem 4.5 are much more complicated than for t-norms on $[0, 1]$, (see Theorem 3.2). However, we obtain the following implications.

**Corollary 4.6** Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$. Consider the following properties:

1. $\lim_{x \triangleright_L x^*} \mathcal{T}(x, x) = x^*$
   $$\implies (\exists y^* \in \{y \mid y \in \mathcal{L}^I \text{ and } x^* \triangleleft_L y \leq \mathcal{L}^I [x_1^*, 1]\})(\mathcal{T}(y^*, y^*) = x^*)$$

The following example shows that, unlike for t-norms on $[0, 1]$, for a t-norm $\mathcal{T}$ on $\mathcal{L}^I$ the properties (LP1) and (LP5) are not sufficient in order to prove that $\mathcal{T}$ satisfies the Archimedean property.

**Example 4.1** Let $x^* = [0, \frac{1}{2}] \in \mathcal{L}^I$ and let $\mathcal{T}$ and $\mathcal{T}'$ be the mappings defined by, for all $x, y$
in $[0, 1]$,
\[ T(x, y) = \max(0, x + y - 1), \]
\[ T'(x, y) = \begin{cases} 
xy, & \text{if } (x, y) \in [0, \frac{1}{2}]^2, \\
\frac{1}{2}(2x - 1)(2y - 1) + \frac{1}{2}, & \text{if } (x, y) \in ]\frac{1}{2}, 1][,2, \\
\min(x, y), & \text{else}.
\end{cases} \]

Then $T$ is the well known Łukasiewicz t-norm on $[0, 1]$ and $T'$ is a t-norm on $[0, 1]$.

Let $T : (L^I)^2 \to L^I$ the mapping defined by, for all $x, y$ in $L^I$,
\[ T(x, y) = [T(x_1, y_1), T'(x_2, y_2)]. \]

Then $T$ is a t-norm on $L^I$ which satisfies (LP1) and (LP5), but not (LP2). From Theorem 4.5 it follows that $T$ does not satisfy the limit property and thus, by Theorem 4.2, $T$ is not Archimedean.

For continuous t-norms it is possible to extend Theorem 3.3 to $L^I$. A mapping $F : L^I \to L^I$ is continuous if, for all $a \in L^I$,
\[(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in L^I) \quad (d(x, a) < \delta \implies d(F(x), F(a)) < \epsilon), \]
where $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$, for all $x, y$ in $L^I$ [4].

**Theorem 4.7** Let $T$ be a continuous t-norm on $L^I$. Then $T$ satisfies the Archimedean property if and only if (LP1) holds.

## 5 The Archimedean property for special classes of t-norms on $L^I$

In this section we discuss the Archimedean property in the case that the t-norm $T$ on $L^I$ satisfies additional properties. We will also check the Archimedean property for some special classes of t-norms, such as t-representable t-norms and pseudo-t-representable t-norms.

When $T$ satisfies some special conditions, the conditions of Theorem 4.5 can be simplified.

**Lemma 5.1** Let $T$ be a t-norm on $L^I$. Assume $T$ satisfies (LP1) and one of the following conditions:

1. $T(x, \sup(y, z)) = \sup(T(x, y), T(x, z))$, for all $x, y, z$ in $L^I$;
2. $T$ satisfies the residuation principle.

Then (LP2) is equivalent to:

- (LP2') $(\forall x^* \in \{x \mid x \in L^I \text{ and } x_1 = 0 \text{ and } x_2 > 0\})$
  \[ \left( \lim_{x_2 \to x_2^*} T([x_1^*, x_2], [x_1^*, x_2^*]) = x^* \right) \implies (\exists y^* \in \{y \mid y \in L^I \text{ and } x^* <_L y \leq_L [x_1^*, 1]\}) \]
  \[ (T(y^*, y^*) = x^*) \]

**Lemma 5.2** Let $T$ be a t-norm on $L^I$. Assume $T(D, D) \subseteq D$ and $T$ satisfies (LP1). Then (LP4) is equivalent to:

- (LP4') $(\forall x^* \in \{x \mid x \in L^I \text{ and } x_2 = 1 \text{ and } x_1 < 1\})$
  \[ \left( \lim_{x_1 \to x_1^*} T([x_1, x_2^*], [x_1, x_2]) = x^* \right) \implies (\exists y^* \in \{y \mid y \in L^I \text{ and } x^* <_L y \leq_L [x_2^*, x_2]\}) \]
  \[ (T(y^*, y^*) = x^*) \]

The following theorem follows from the previous lemmas.

**Theorem 5.3** Let $T$ be a t-norm on $L^I$. Assume $T(D, D) \subseteq D$ and $T$ satisfies one of the following conditions:

1. $T(x, \sup(y, z)) = \sup(T(x, y), T(x, z))$, for all $x, y, z$ in $L^I$;
2. $T$ satisfies the residuation principle.

Then $T$ satisfies the limit property if and only if (LP1), (LP2') and (LP3) hold.

**Theorem 5.4** Let $T$ be a t-norm on $L^I$. Assume $T$ satisfies one of the following conditions:

1. $T(x, \sup(y, z)) = \sup(T(x, y), T(x, z))$, for all $x, y, z$ in $L^I$;
2. $T$ satisfies the residuation principle.

Then (LP4) is equivalent to:

- (LP4') $(\forall x^* \in \{x \mid x \in L^I \text{ and } x_2 = 1 \text{ and } x_1 < 1\})$
  \[ \left( \lim_{x_1 \to x_1^*} T([x_1, x_2^*], [x_1, x_2]) = x^* \right) \implies (\exists y^* \in \{y \mid y \in L^I \text{ and } x^* <_L y \leq_L [x_2^*, x_2]\}) \]
  \[ (T(y^*, y^*) = x^*) \]
(ii) \( T \) satisfies the residuation principle.

If \( T([0,1],[0,1]) <_L [0,1] \) and \( T \) is weakly Archimedean, then \( T \) is Archimedean.

\[ T \]

**Theorem 5.5** Let \( T \) be a t-norm on \( L^I \). If \( T(D,D) \subseteq D \) and for all \( x \in D \setminus \{\{0,1\}\} \), it holds that \( (T(x,x))_1 > 0 \), then \( T \) is not strongly Archimedean.

\[ T \]

**Theorem 5.6** Let \( T_{T,T'} \) be a t-representable t-norm on \( L^I \) with representatives \( T \) and \( T' \). Then \( T_{T,T'} \) is not Archimedean, but \( T_{T,T'} \) is weakly Archimedean if and only if \( T \) and \( T' \) are Archimedean.

Note that from Theorem 5.6 it follows that there exist t-norms \( T \) on \( L^I \) such that \( T \) satisfies the residuation principle, \( T([0,1],[0,1]) = [0,1] \), \( T \) is weakly Archimedean, but not Archimedean. Indeed, let \( T \) and \( T' \) be left-continuous Archimedean t-norms on \([0,1] \), then the t-representable t-norm \( T_{T,T'} \) with representatives \( T \) and \( T' \) satisfies the residuation principle (see [5], Theorem 5). From Theorem 5.6 it follows that \( T_{T,T'} \) is weakly Archimedean, but not Archimedean.

\[ T \]

**Lemma 5.7** Let \( T \) be a t-norm on \([0,1] \) and \( \alpha \in [0,1] \). If the t-norm \( T_{T,T} \) on \( L^I \) is weakly Archimedean, then \( T \) is Archimedean.

\[ T \]

**Theorem 5.8** Let \( T \) be a t-norm on \([0,1] \) and \( \alpha \in [0,1] \). Then the t-norm \( T_{T,T} \) on \( L^I \) is Archimedean if and only if \( T \) is Archimedean. Moreover \( T_{T,T} \) is strongly Archimedean if and only if \( T \) is Archimedean and has zero-divisors.

\[ T \]

**Corollary 5.9** Let \( T \) be a t-norm on \([0,1] \) and \( \alpha \in [0,1] \). Then the t-norm \( T_{T,T} \) on \( L^I \) is weakly Archimedean if and only if \( T \) is Archimedean.

\[ T \]

**Corollary 5.10** Let \( T \) be a t-norm on \([0,1] \) and \( T_{T} \) a pseudo-t-representable t-norm on \( L^I \) with representant \( T \). Then

- \( T_{T} \) is Archimedean if and only if \( T \) is Archimedean;
- \( T_{T} \) is strongly Archimedean if and only if \( T \) is Archimedean and has zero-divisors.

\[ T \]

**Theorem 5.11** Let \( T \) be a t-norm on \([0,1] \). Then the t-norm \( T_{T} \) on \( L^I \) is not Archimedean, but \( T_{T} \) is weakly Archimedean if and only if \( T \) is Archimedean.

The results from Theorem 5.6, Theorem 5.8, Theorem 5.11 and Corollary 5.9 are summarized in Table 1.

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<th>weakly Archimedean</th>
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<td>( T_{T,T'} )</td>
<td>iff ( T ) and ( T' ) are Archimedean</td>
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<tr>
<td>( T_{T,T} (\alpha &lt; 1) )</td>
<td>iff ( T ) is Archimedean</td>
<td>iff ( T ) is Archimedean</td>
<td>iff ( T ) is Archimedean and has zero-divisors</td>
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<td>( T' )</td>
<td>iff ( T ) is Archimedean</td>
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Table 1: The Archimedean property for the different classes of t-norms

\[ T \]

6 Conclusion

In this paper we have introduced the Archimedean property for t-norms on \( L^I \). The Archimedean property given on the unit interval can be extended to \( L^I \) in several ways, which we called weak Archimedean property, Archimedean property and strong Archimedean property. For the weak Archimedean property and the Archimedean property we found a similar relationship with the (weak) limit property as on the unit interval. We extended some other necessary and sufficient conditions for the Archimedean property to \( L^I \). For continuous t-norms, we showed that a t-norm on \( L^I \) satisfies the Archimedean property if and only if it has no non-trivial idempotent elements. We also simplified some necessary and sufficient conditions for the Ar-
chimedean properties in the case that the t-norm considered has some additional properties. Finally we discussed the Archimedean property for special classes of t-norms on $\mathcal{L}^I$ based on t-norms on the unit interval: we found that only the generalized pseudo-t-representable t-norms (the t-representable t-norms excluded) can satisfy the Archimedean or the strong Archimedean property, the other classes we investigated can only be weakly Archimedean. Hence, from the three classes that we discussed, only generalized pseudo-t-representable t-norms which are not t-representable can be Archimedean in the sense of the definition given in [2, 3].

In a future paper we wish to investigate the relationship between the Archimedean property and additive or multiplicative generators of t-norms on $\mathcal{L}^I$. We will also investigate some other properties such as nilpotency, having zero-divisors, cancellative property, and many others.

References


