UNIQUELY CIRCULAR COLOURABLE AND UNIQUELY FRACTIONAL COLOURABLE GRAPHS OF LARGE GIRTH

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ABSTRACT. Given any rational numbers \( r \geq r' > 2 \) and an integer \( g \), we prove that there is a graph \( G \) of girth at least \( g \), which is uniquely circular \( r \)-colourable and uniquely fractional \( r' \)-colourable. Moreover, the graph \( G \) has maximum degree bounded by a number which depends on \( r \) and \( r' \) but does not depend on \( g \).

1. INTRODUCTION

Suppose \( G \) is a graph with at least one edge and \( r \geq 2 \) is a rational number. A circular \( r \)-colouring of \( G \) is a mapping \( f : V(G) \to [0, r) \) such that for any edge \( xy \) of \( G \), \( 1 \leq |f(x) - f(y)| \leq r - 1 \). We say \( G \) is circular \( r \)-colourable if there is a circular \( r \)-colouring of \( G \). The circular chromatic number of \( G \) is defined as

\[
\chi_c(G) = \inf \{ r : G \text{ is circular } r \text{-colourable} \}.
\]

It is known that for any graph \( G \), \( \chi(G) = \lceil \chi_c(G) \rceil \). Hence the circular chromatic number of a graph is a refinement of its chromatic number.

Suppose \( f \) is a circular \( r \)-colouring of \( G \). Then for any \( c \in [0, r) \) and for \( \tau \in \{1, -1\} \), \( g : V(G) \to [0, r) \) defined as \( g(x) = [c + \tau f(x)]_r \) is also a circular \( r \)-colouring of \( G \). (For a real number \( x \) and a positive real number \( r \), we denote by \( [x]_r \) the remainder of \( x \) dividing \( r \), i.e., \( [x]_r \in [0, r) \) is the unique number for which \( x - [x]_r \) is a multiple of \( r \).) If \( f \) and \( g \) are \( r \)-colourings of \( G \) such that \( g(x) = [c + \tau f(x)]_r \) for some \( c \in [0, r) \) and \( \tau \in \{1, -1\} \), then we say \( f \) and \( g \) are equivalent circular \( r \)-colourings of \( G \), written as \( f \cong g \). It is obvious that ‘\( \cong \)’ is an equivalence relation. A graph \( G \) is called uniquely circular \( r \)-colourable if up to equivalence, there is only one circular \( r \)-colouring of \( G \). It is proved in [10] that for any rational \( r \geq 2 \), for any integer \( g \), there is a graph \( G \) of girth at least \( g \) which is uniquely circular \( r \)-colourable.

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Let \( I(G) \) be the family of independent sets of \( G \). A fractional colouring \( f \) of \( G \) is an assignment of nonnegative weights to independent sets of \( G \), i.e., a mapping \( f : I(G) \to \mathbb{R}^{\geq 0} \), such that for each \( x \in V(G) \), \( \sum_{I \in I(G)} f(I) = 1 \).

A fractional colouring \( f \) is called a fractional \( r \)-colouring of \( G \) if the sum \( \sum_{I \in I} f(I) \) is equal to \( r \). The fractional chromatic number of \( G \), denoted by \( \chi_f(G) \), is the least \( r \) such that \( G \) has a fractional \( r \)-colouring. We say that a graph \( G \) is uniquely fractional \( r \)-colourable if there is exactly one fractional \( r \)-colouring of \( G \). I.e., there is a fractional \( r \)-colouring \( f \) of \( G \) and if \( f' \) is a fractional \( r \)-colouring of \( G \), then \( f(I) = f'(I) \) for all \( I \in I(G) \). It is proved in [5] that for any rational \( r \geq 2 \), there is a uniquely fractional \( r \)-colourable graph of girth at least \( g \).

In this paper, we consider unique circular colourability and unique fractional colourability simultaneously. It is known [12] that for any graph \( G \), \( \chi_f(G) \leq \chi_c(G) \). On the other hand, it is not difficult to show that for any rationals \( 2 < r' \leq r \), there is a graph \( G \) with \( \chi_f(G) = r' \) and \( \chi_c(G) = r \). In this paper, we prove that for any rationals \( 2 < r' \leq r \), for any integer \( g \), there is a graph \( G \) of girth at least \( g \) such that \( G \) is uniquely fractional \( r' \)-colourable, and at the same time, uniquely circular \( r \)-colourable. In particular, \( \chi_f(G) = r' \) and \( \chi_c(G) = r \).

Both circular chromatic number and fractional chromatic number of a graph can be defined through graph homomorphisms. Suppose \( G \) and \( H \) are graphs. A homomorphism of \( G \) to \( H \) is a mapping \( f : V(G) \to V(H) \) such that \( \{ f(x), f(y) \} \in E(H) \) whenever \( \{ x, y \} \in E(G) \). A homomorphism of \( G \) to \( H \) is also called an \( H \)-colouring of \( G \). A graph \( G \) is said to be \( H \)-colourable if there exists a homomorphism of \( G \) to \( H \). A graph \( G \) is said to be uniquely \( H \)-colourable, if there exists an \( H \)-colouring \( f \) of \( G \) such that \( f \) is an onto homomorphism and for any other \( H \)-colouring \( f' \) of \( G \), \( f' \) is the composition \( f \circ \sigma \) of \( f \) with an automorphism \( \sigma \) of \( H \).

Note that a \( K_n \)-colouring of \( G \) is equivalent to an \( n \)-colouring of \( G \), and unique \( n \)-colourability of \( G \) is equivalent to unique \( K_n \)-colourability of \( G \). So the study of the chromatic number of a graph and unique colourability of a graph can be carried out in terms of graph homomorphisms. The same is true for the circular colouring.

For a pair of positive integers \( p, q \) such that \( p \geq 2q \). Let \( K_p^q \) be the graph which has vertices \( \{0, \cdots, p-1\} \) and in which \( \{i,j\} \) is an edge if and only if \( q \leq |i-j| \leq p-q \). A \( K_p^q \)-colouring of a graph \( G \) is also called a \( (p,q) \)-colouring of \( G \). It is known [12] and easy to see that for any graph \( G \), \( \chi_c(G) = \inf \{ \frac{p}{q} : G \text{ is } K_p^q \text{-colourable} \} \). It is also easy to show that a graph \( G \) is uniquely \( \frac{p}{q} \)-colourable if and only if it is uniquely \( K_p^q \)-colourable.

The fractional chromatic number of a graph can be defined through graph homomorphisms to Kneser graphs. Suppose \( n \geq 2k \) are positive integers. Let \( [n] = \{0,1,2,\cdots,n-1\} \) and denote by \( \binom{[n]}{k} \) the set of all \( k \)-subsets of \([n]\). The Kneser graph \( K(n,k) \) has vertex set \( V = \binom{[n]}{k} \) in which two vertices
A and B are adjacent if, when regarded as subsets of $[n]$, they do not intersect, i.e., $A \cap B = \emptyset$. A homomorphism $f$ from a graph $G$ to $K(n,k)$ is also called a $k$-tuple $n$-colouring of $G$. Such a homomorphism $f$ assigns to each vertex $x$ of $G$ a set $f(x)$ of $k$ colours, and if $x$ and $y$ are adjacent, then $f(x) \cap f(y) = \emptyset$, i.e., no colour is assigned to two adjacent vertices. It is known [9] that the fractional chromatic number of $G$ is $\chi_f(G) = \min\{\frac{n}{r} : G$ is $K(n,k)$-colourable $\}$. However, unique fractional $p/q$-colourability is different from unique $H$-colourability for any graph $H$ [5]. In particular, a uniquely $K(n,k)$-colourable graph $G$ may not be uniquely fractional $n/k$-colourable. This is due to the fact that a fractional $n/k$-colourable graph may not be $K(n,k)$-colourable. On the other hand, it is proved in [5] that if a graph $G$ is uniquely $K(pt,qt)$-colourable for some integer $t$, and moreover, for any integer $t'$, if $G$ is $K(pt',qt')$-colourable, then $G$ is uniquely $K(pt',qt')$-colourable, then $G$ is uniquely fractional $p/q$-colourable.

The purpose of this paper is to construct, for any $2 < \frac{p'}{q'} \leq \frac{p}{q}$, for any integer $g$, a graph $G$ of girth at least $g$ such that (1): $G$ is uniquely circular $\frac{p}{q}$-colourable, and (2): $G$ is uniquely fractional $\frac{p'}{q'}$-colourable.

2. MAIN RESULT AND SOME PRELIMINARIES

The main result of this paper is the following theorem:

**Theorem 1.** Given any two rational numbers $2 < r' \leq r$, for any integer $g$, there is a graph $G$ of girth at least $g$ such that $G$ is uniquely circular $r$-colourable and uniquely fractional $r'$-colourable. Moreover, the graph $G$ has maximum degree bounded by a number which depends on $r$ and $r'$ but does not depend on $g$.

To prove Theorem 1, we shall first relax the condition on large girth and prove that for any $2 < r' \leq r$, there is a graph $G'$ which is uniquely circular $r$-colourable, and also uniquely fractional $r'$-colourable. Assume $r = \frac{p}{q}$ and $r' = \frac{p'}{q'}$. If $\frac{p}{q} = \frac{p'}{q'}$, then $G' = K_{\frac{p}{q}}$ is uniquely circular $r$-colourable and uniquely fractional $r'$-colourable. Assume $\frac{p}{q} > \frac{p'}{q'}$. The graph which is uniquely circular $r$-colourable, and also uniquely fractional $r'$-colourable is constructed through graph product. For graphs $G$ and $H$, the categorical product $G \times H$ has vertex set $\{(x, y) : x \in V(G), y \in V(H)\}$. Two vertices $(x, y)$ and $(x', y')$ are adjacent in $G \times H$ if and only if $x$ and $x'$ are adjacent in $G$, $y$ and $y'$ are adjacent in $G$. We shall prove that if $t$ is a large enough integer, then the categorical product graph $K(p't, q't) \times K_{\frac{p}{q}}$ is uniquely circular $r$-colourable and uniquely fractional $r'$-colourable. The following lemma is easy.

**Lemma 2.** For any $2 < \frac{p'}{q'} \leq \frac{p}{q}$, if $t$ is a large enough integer, then $K(p't, q't) \times K_{\frac{p}{q}}$ is uniquely circular $\frac{p}{q}$-colourable.
Lemma 4 to show that up to equivalence, \( f \) is a maximum independent set. By Lemma 4, \( g \) is a core graph and no vertex \( f \) of \( C(H) \) is an automorphism, the graph \( C(H) \) is loopless. It is proved in [10] that if \( \chi(G) > \chi(C(H)) \) then \( G \times H \) is uniquely \( H \)-colourable. As \( \chi(C(K(p',q')t)) = (p' - 2q't) + 2 \) [6], it follows that if \( t > (\chi(C(K_q)) - 2) / (p' - 2q') \), then \( K(p',q')t \times K_q \) is uniquely \( K_q \)-colourable, and hence uniquely circular \( \frac{p'}{q'} \)-colourable.

**Lemma 3.** For any \( 2 < \frac{p'}{q'} < \frac{p}{q} \) and for any integer \( t \), \( K(p',q')t \times K_q \) is uniquely fractional \( \frac{p'}{q'} \)-colourable.

**Proof.** For each \( i \in \{0, 1, \cdots , p't - 1\} \), let \( I_i = \{x \in V(K(p',q't)) : i \in x\} \) (recall that each vertex of \( K(p',q't) \) is a \( q't \)-subset of \( \{0, 1, \cdots , p't - 1\} \)). Then \( I_i \) is a maximum independent set of \( K(p',q't) \) and \( I_i \times V(K_q) \) is an independent set of \( K(p',q't) \times K_q \). Let \( f : I(K(p',q't) \times K_q) \to [0, 1] \) be defined as \( f(I_i \times V(K_q)) = 1/q't \) for each \( i \in \{0, 1, \cdots , p't - 1\} \) and \( f(I) = 0 \) for any other independent set \( I \) of \( K(p',q't) \times K_q \). Then \( f \) is a \( \frac{p'}{q'} \)-fractional colouring of \( K(p',q't) \times K_q \). We need to prove that, up to equivalence, \( f \) is the unique fractional \( \frac{p'}{q'} \)-colouring of \( K(p',q't) \times K_q \).

**Lemma 4.** The independent sets \( I_i \times V(K_q) \) for \( i = 0, 1, \cdots , p't - 1 \) are the only maximum independent sets of \( K(p',q't) \times K_q \).

We shall delay the proof of Lemma 4 for a little while. Now we use Lemma 4 to show that up to equivalence, \( f \) is the unique fractional \( \frac{p'}{q'} \)-colouring of \( K(p',q't) \times K_q \).

Assume \( g \) is a fractional \( \frac{p'}{q'} \)-coloring of \( K(p',q't) \times K_q \). We need to prove that for any independent set \( U \) of \( K(p',q't) \times K_q \),

\[
g(U) = \begin{cases} 
1/q't & \text{if } U = I_i \times V(K_q) \text{ for some } i \in \{0, 1, \cdots , p't - 1\} \\
0 & \text{otherwise}.
\end{cases}
\]

It is well-known [9] that for any vertex transitive graph \( G \), \( \chi_f(G) = \frac{|V(G)|}{\alpha(G)} \) and for any optimal fractional colouring \( f \) of \( G \), \( f(I) = 0 \) if \( I \) is not a maximum independent set. By Lemma 4, \( I_i \times V(K_q) \) for \( i = 0, 1, \cdots , p't - 1 \) are the only maximum independent sets. Therefore \( g(I) = 0 \) if \( I \neq I_i \times V(K_q) \) for some \( i \in \{0, 1, \cdots , p't - 1\} \).
Assume there exists $I_i$ such that $g(I_i \times V(K_{\frac{q}{p}})) \neq 1/q't$. Without loss of generality, assume $g(I_i \times V(K_{\frac{q}{p}})) > 1/q't$. Since $\sum_{i=0}^{p't-1} g(I_i \times V(K_{\frac{q}{p}})) = \frac{p'}{q}$, there exist $I_{i_1} \times V(K_{\frac{q}{p}}), I_{i_2} \times V(K_{\frac{q}{p}}), \ldots, I_{i_{q'}} \times V(K_{\frac{q}{p}})$ such that $\sum_{i=1}^{q'} g(I_i \times V(K_{\frac{q}{p}})) < 1$. Let $x = \{i_1, \ldots, i_{q'}\} \in V(K(p't,q't))$. Since $I_{i_1} \times V(K_{\frac{q}{p}}), I_{i_2} \times V(K_{\frac{q}{p}}), \ldots, I_{i_{q'}} \times V(K_{\frac{q}{p}})$ are the only maximum independent sets containing $(x,a)$ for any $a \in V(K_{\frac{q}{p}})$, it follows that $\sum_{(x,a) \in I} g(I) = \sum_{i=1}^{q'} g(I_i \times V(K_{\frac{q}{p}})) < 1$, in contrary to the assumption that $g$ is a fractional colouring of $K(p't,q't) \times K_{\frac{q}{p}}$. Therefore,

$$g(I) = \begin{cases} 1/q't & \text{if } I = I_i \times V(K_{\frac{q}{p}}) \text{ for some } i \in \{0,1,\ldots,p't-1\} \\ 0 & \text{otherwise.} \end{cases}$$

i.e., $K(p't,q't) \times K_{\frac{q}{p}}$ is uniquely fractional $\frac{p'}{q}$-colourable.

\[\square\]

### 3. The proof of Lemma 4

Problems concerning independent sets of the categorical product of graphs have been studied in many papers. For example, Frankl [3] determined the maximum size of independent set of the categorical product of Kneser graphs. Ahlswede, Aydinian and Khachatrian [1] determined the size of the maximum independent set of the categorical product of certain generalized Kneser graphs. The size of the maximum independent set of the categorical product of a Kneser graph with a circular complete graph also follows from a result in [13] concerning the fractional chromatic number of such graphs. In Lemma 4, besides the size of a maximum independent set, we need to determine the structure of all maximum independent sets of the product of a Kneser graph with a circular complete graph. The proof given below is a refinement of the corresponding argument in [13].

Assume that $U$ is a maximum independent set of $K(p't,q't) \times K_{\frac{q}{p}}$ and $U \neq I_i \times K_{\frac{q}{p}}$ for any $i \in \{0,1,\ldots,p't-1\}$.

For each vertex $x$ of $K(p't,q't)$, let $U_x = \{y \in K_{\frac{q}{p}} : (x,y) \in U\}$.

**Claim 1.** If $\{x,x'\} \in E(K(p't,q't))$ and $U_x \neq \emptyset$, $U_{x'} \neq \emptyset$, then $|U_x| + |U_{x'}| \leq 2q$.

**Proof.** Assume $\{x,x'\} \in E(K(p't,q't))$ and $|U_x| + |U_{x'}| > 2q$. Since $U_x \neq \emptyset$ and $U_{x'} \neq \emptyset$, it is known [13] and easily to verify directly that there exist $a \in U_x$ and $b \in U_{x'}$ such that $\{a,b\} \in E(K_{\frac{q}{p}})$. Then $\{(x,a), (x',b)\} \in E(K(p't,q't) \times K_{\frac{q}{p}})$, in contrary to the assumption that $U$ is an independent set of $K(p't,q't) \times K_{\frac{q}{p}}$. \[\square\]
Claim 2. For any vertex $x$ of $K(p't, q't)$, either $|U_x| < 2q$ or $|U_x| = p$.

Proof. Assume to the contrary that there exists $x \in V(K(p't, q't))$ such that $2q \leq |U_x| < p$. By Claim 1, for all $y \in N(x)$, $U_y = \emptyset$. Therefore $U' = U \cup \{(x, a) : a \in K_{\frac{q'}{q}} - U_x\}$ is an independent set of $K(p't, q't) \times K_{\frac{q'}{q}}$. Since $|U_x| < p$, $U'$ is strictly larger than $U$. This is in contrary to our assumption that $U$ is a maximum independent set.

Claim 3. For any vertex $x$ of $K(p't, q't)$, either $U_x = V(K_{\frac{q'}{q}})$ or $U_x = \emptyset$.

Proof. Let $Y = \{x \in V(K(p't, q't)) : U_x = V(K_{\frac{q'}{q}})\}$. By Claim 1, for all $x \in N(Y)$, $U_x = \emptyset$. Let

$$U^* = U \cap (V(K(p't, q't)) - N[Y]) \times V(K_{\frac{q'}{q}}).$$

Then $U^*$ is an independent set of $(K(p't, q't) - N[Y]) \times K_{\frac{q'}{q}}$. If $U^* = \emptyset$, then we are done. Assume $U^* \neq \emptyset$.

For each independent set $Z$ of $K(p't, q't) - N[Y]$, $Z \cup Y$ is an independent set of $K(p't, q't)$, and hence has cardinality $|Z| + |Y| \leq \left\lfloor \frac{q't - 1}{q' - 1} \right\rfloor$. Therefore $\alpha(K(p't, q't) - N[Y]) \leq \left\lfloor \frac{q't - 1}{q' - 1} \right\rfloor - |Y|$. Since $\chi_f(K(p't, q't) - N[Y]) \leq \chi_f(K(p't, q't)) = \frac{p'}{q'}$, it follows that

$$|V(K(p't, q't) - N[Y])| \leq \alpha(K(p't, q't) - N[Y])\chi_f(K(p't, q't) - N[Y]) \leq \left( \frac{p't - 1}{q't - 1} \right) - |Y| \frac{p'}{q'}.$$

Since $\frac{p'}{q'} < \frac{p}{q}$, this implies that

$$|V(K(p't, q't) - N[Y])| q + |Y| p < \left( \frac{p't - 1}{q't - 1} \right) p = |I_i \times V(K_{\frac{q'}{q}})|. \quad (1)$$

Let $\kappa = \max\{|U_x| : x \in K(p't, q't) - N[Y]\}$. By Claim 2 and the definition of $Y$, we know that $\kappa < 2q$. If $\kappa \leq q$, then by (1),

$$|U| \leq |V(K(p't, q't) - N[Y])| q + |Y| p < |I_i \times V(K_{\frac{q'}{q}})|.$$

This is in contrary to the assumption that $U$ is a maximum independent set of $K(p't, q't) \times K_{\frac{q'}{q}}$.

Thus we may assume that $q < \kappa < 2q$. For $s = q + 1, q + 2, \ldots, 2q - 1$, let $Y_s = \{x \in V(K(p't, q't)) - N[Y] : |U_x| = s\}$.

Let $q + 1 \leq s_0 < s_1 < \cdots < s_m < 2q$ be the integers such that either $Y_{s_i} \neq \emptyset$ or $Y_{2q - s_i} \neq \emptyset$.

And let $Z_{s_i} = \{x \in V(K(p't, q't)) - N[Y] : |U_x| = 2q - s_i\}$

and $B = \{x \in V(K(p't, q't)) - N[Y] : |U_x| = q\}$. 

Then
\[ |U| = |Y|p + |B|q + \sum_{i=0}^{m} (|Y_{s_i}| + |Z_{s_i}|)q - \sum_{i=0}^{m} (|Z_{s_i}| - |Y_{s_i}|)(s_i - q). \]

Now we need the following lemma which is slightly different from Lemma 4.5 of [13], but can be proved the same way.

**Lemma 5.** Suppose \(a_0, \ldots, a_m\) and \(\beta_0, \ldots, \beta_m\) are real numbers such that \(\frac{\beta_0}{a_0} \geq \frac{\beta_{i+1}}{a_{i+1}}\) for \(i = 0, \ldots, m - 1\). If \(a_0, \ldots, a_m\) are real numbers satisfying \(\sum_{j=0}^{i} a_j x_j > 0\) for all \(0 \leq i \leq m\), then \(\sum_{j=0}^{i} \beta_j x_j > 0\) for all \(0 \leq i \leq m\).

Let \(x_i = |Z_{s_i}| - |Y_{s_i}|\), \(\beta_i = s_i - q\), \(\alpha_i = 2q - s_i\). Then \(\beta_i > 0\) and \(\alpha_i > 0\) for all \(i = 0, \ldots, m\) and
\[ |U| = |Y|p + |B|q + \sum_{i=0}^{m} (|Y_{s_i}| + |Z_{s_i}|)q - \sum_{j=0}^{m} \beta_j x_j. \]

If \(\sum_{j=0}^{i} \alpha_j x_j > 0\) for all \(i\), then by Lemma 5, \(\sum_{j=0}^{i} \beta_j x_j > 0\). This implies that
\[ |U| = |Y|p + |B|q + \sum_{i=0}^{m} (|Y_{s_i}| + |Z_{s_i}|)q - \sum_{j=0}^{m} \beta_j x_j \]
\[ < |Y|p + |B|q + \sum_{i=0}^{m} (|Y_{s_i}| + |Z_{s_i}|)q \]
\[ \leq |Y|p + |B|q + N(Y) \sum_{j=0}^{m} \beta_j x_j < |I| \times V(K_{p', q't}). \]

This is in contrary to the assumption that \(U\) is a maximum independent set of \(K(p', q't) \times K_{\frac{q}{4}}\).

Thus we assume that \(\sum_{j=0}^{i} \alpha_j x_j \leq 0\) for some \(0 \leq i \leq m\). Let \(U'\) the independent set of \(K(p', q't) \times K_{\frac{q}{4}}\) defined as
\begin{itemize}
  \item \(U'_x = V(K_{\frac{q}{4}})\) if \(x \in Y_{s_i}\) for some \(j \leq i\);
  \item \(U'_x = \emptyset\) if \(x \in Z_{s_j}\) for some \(j \leq i\);
  \item \(U'_x = U_x\) otherwise.
\end{itemize}

Then \(U'\) is an independent set of \(K(p', q't) \times K_{\frac{q}{4}}\) and
\[ |U'| = |U| - \sum_{j=0}^{i} |Z_{s_j}|(2q - s_j) + \sum_{j=0}^{i} |Y_{s_i}|(p - s_j) \]
\[ > |U| - \sum_{j=0}^{i} |Z_{s_j}|(2q - s_j) + \sum_{j=0}^{i} |Y_{s_i}|(2q - s_j) \]
\[ \geq |U|. \]

This is again in contrary to the assumption that \(U\) is a maximum independent set of \(K(p', q't) \times K_{\frac{q}{4}}\).

It follows from Lemma 3 that \(U = I \times V(K_{\frac{q}{4}})\) for some independent set \(I\) of \(K(p', q't)\). Since \(U\) is a maximum independent set of \(K(p', q't) \times K_{\frac{q}{4}}\), we conclude that \(I\) is a maximum independent set of \(K(p', q't)\) and hence \(I = \)
For arbitrary core graphs $H$, uniquely $H$-colourable graphs of large girth have been studied in many papers. As observed before, unique circular $p/q$-colourability of a graph is equivalent to the unique $K_{p/q}$-colourability of the graph. However, unique fractional $p'/q'$-colourability is not equivalent to unique $H$-colourability for any graph $H$. As noted in [5], if $t$ is large enough, then $K(p't, q't) \times K(p', q')$ is uniquely $K(p', q')$-colourable but not uniquely fractional $p'/q'$-colourable. For this reason, the existing results concerning uniquely $H$-colourable graphs of large girth cannot be applied directly to obtain Theorem 1. Nevertheless, the proof of Theorem 1 below is parallel to the existing probabilistic proofs concerning uniquely $H$-colourable graphs of large girth.

Suppose $F$ is an $n$ vertex graph with vertices $0, 1, \ldots, n - 1$. Given a positive integer $m$, we denote by $F[m] = F[K_m]$ the lexicographic product of $F$ and $K_m$. In other words, for each vertex $v$ of $F$, let $v[m]$ be a set of cardinality $m$. Then $F[m]$ has vertex set $\cup_{v \in V(F)} v[m]$ such that $x \in v[m]$ is adjacent to $x' \in v'[m]$ if and only if $\{v, v'\}$ is an edge of $F$.

It is proved in [9, 5] that for any integer $g$, there exists an integer $m$, such that $F[m]$ has a spanning subgraph $G$ of girth at least $g$ for which $V(G) = W_0 \cup W_1 \cup \cdots \cup W_{n-1}$ where $W_i = i[m]$ for each $i \in V(F)$.

1. For any edge $\{v, v'\}$ of $F$, for any $X \subseteq v[m]$, $Y \subseteq v'[m]$, if $|X| \geq m/40n$ and $|Y| \geq m/40n$, then there is an edge $(G)$ between $X$ and $Y$.

2. For any edge $\{v, v'\}$ of $F$, for any $X \subseteq v[m]$, $Y \subseteq v'[m]$ with $n \leq |X| = n|Y| \leq m/40$, there are less than $|Y|n^{10}/2$ edges between $X$ and $Y$.

3. For any edge $\{v, v'\}$ of $F$, for any vertex $x \in v[m]$, $x$ has at least $n^{10}/2$ neighbours in $v'[m]$.

4. Each vertex of $G$ has degree at most $5|V(F)|^{13}$.

If $\frac{p'}{q'} = \frac{p}{q}$, then let $F = K_{q'}$. If $2 < \frac{p'}{q'} < \frac{p}{q}$, then let $F = K(p't, q't) \times K_{q'}$, where $t$ is large enough so that $F$ is uniquely circular $\frac{p}{q}$-colourable. To prove Theorem 1, we shall prove that the spanning subgraph $G$ of $F[m]$ with properties (1) and (4) listed above is uniquely circular $\frac{p}{q}$-colourable and also uniquely fractional $\frac{p'}{q'}$-colourable. Property (5) implies that the maximum degree of $G$ is bounded by a number which does not depends on $g$ (but depends on $p/q$ and $p'/q'$). As unique circular $\frac{p}{q}$-colourability
is equivalent to unique $K_{p/q}$-colourability, the following lemma is a special case of Theorem 4 in [4].

**Lemma 6.** Suppose $G$ is a spanning subgraph of $F[m]$ with properties (1)-(4) listed above. Then $G$ is uniquely circular $\frac{p}{q}$-colourable.

**Lemma 7.** Suppose $G$ is a spanning subgraph of $F[m]$ with properties (1)-(4) listed above. Then $G$ is uniquely fractional $\frac{p}{q}$-colourable.

**Proof.** Since $G \subseteq F[m]$ and $F$ is fractional $\frac{p}{q}$-colourable, it follows that $G$ is fractional $\frac{p}{q}$-colourable. To prove that $G$ is uniquely fractional $\frac{p}{q}$-colourable, it suffices to show that each maximum independent set of $G$ is of the form $I[m]$ for a maximum independent set of $I$ of $F$.

Let $\alpha_F$ and $\alpha_G$ be the size of the maximum independent set of $F$ and $G$, respectively.

Since $G$ is a spanning subgraph of $F[m]$, we have $\alpha_G \geq \alpha_F m$. Assume $J \in I(G)$, $|J| = \alpha_G \geq \alpha_F m$. Let $v$ be a vertex of $F$, we denote by $\phi(v)$ the size of $v[m] \cap J$, i.e., $\phi(v) = |J \cap v[m]|$. Then, there exists an order of $V(F)$, \{v_1, v_2, \ldots, v_n\}, such that $\phi(v_1) \geq \phi(v_2) \geq \cdots \phi(v_n) \geq 0$. Since $\sum_{i=1}^n \phi(v_i) \geq \alpha_F m$, we have $\phi(v_1) \geq \frac{\alpha_F m}{m}, \phi(v_2) \geq \frac{\alpha_F m - m}{m}, \cdots, \phi(v_n) \geq \frac{\alpha_F m - (\alpha_F - 1)m}{m}$.

Let $I = \{v_1, v_2, \ldots, v_{\alpha_F}\}$. First we show that $I$ is an independent set of $F$.

If not, then there exists $v_i, v_j \in I$ such that $v_i, v_j \in E(F)$. Since $v_i[m] \cap J$ has size $\phi(v_i) \geq \frac{m}{m}$ and $v_j[m] \cap J$ has size $\phi(v_j) \geq \frac{m}{m}$, there are subsets $U$ of $v_i[m] \cap J$ and $W$ of $v_j[m] \cap J$ such that $|U| = |W| = \lceil \frac{m}{m} \rceil$. However, by Property (2), there exists an edge between $U$ and $W$, contrary to the assumption that $J$ is an independent set of $G$.

Next we show that $\phi(x_{\beta+1}) = 0$. Assume to the contrary that if $\phi(x_{\beta+1}) \neq 0$, i.e., $v_{\alpha_F+1}[m] \cap J \neq \emptyset$. Since $I \cup v_{\alpha_F+1}$ is not independent set of $F$, there exists a $v_i \in I$ such that $\{v_i, v_{\alpha_F+1}\}$ is an edge of $F$. By Property (3), each vertex in $J \cap v_{\alpha_F+1}[m]$ has at least $n^{10}/2$ neighbours in $v_i[m]$. As $J \cap v_{\alpha_F+1}[m] \neq \emptyset$ and $J$ is independent in $G$, it follows that $|v_i[m] - J| \geq n^{10}/2$. Let $W = v_i[m] - J$ and let $\beta = |W|$. Let $\ell = \phi(v_{\alpha_F+1})$. Since $\phi(v_{\alpha_F+1}) \geq \phi(v_j)$ for $j = \beta + 1, \ldots, n$, it follows that $\beta \leq \ell \cdot (n - \alpha) \leq \ell \cdot n$. So $\ell \geq \beta/n$.

Let $U \subseteq v_{\alpha_F+1}[m] \cap J$ be a subset of size $\beta/n$. Since each vertex of $U$ has at least $n^{10}/m$ neighbours in $v_i[m] - J = W$, we conclude that there are at least $\frac{n^{10}}{m}|U|$ edges between $U$ and $W$. This is in contrary to Property (3). Therefore $\phi(v_{\alpha_F+1}) = 0$, i.e., if $J$ is a maximum independent set of $G$, then $J = I[m]$ for some maximum independent set $I$ of $F$. And we have $I_i \times K_{p/q}$ for $i = 0, 1, \cdots, p't - 1$ are the only maximum independent set of $F$ with size $\lceil \frac{p't-1}{q} \rceil p$. Therefore, $J = (I_i \times K_{p/q})[m]$ for some $i = 0, 1, \cdots, p't - 1$. 


Since $G$ is a spanning subgraph of $F[m]$, $G$ is fractional $\frac{p'}{q'}$-colourable.

As $|V(F)| = \left(\frac{p'}{q'}t\right)p$, we have $|V(G)| = m\left(\frac{p'}{q'}t\right)p$, $\alpha_G = \alpha_F m = \left(\frac{p'}{q'}t-1\right)pm$, so $\chi_f(G) \geq \frac{|V(G)|}{\alpha_G} = \frac{p'}{q'}$. Thus we know that $\chi_f(G) = \frac{p'}{q'}$. Let $J_i$ be the maximum independent set of $G$ such that $J_i = (I_i \times K_{p'})[m]$ for $i = 0, 1, \ldots, p't - 1$.

Let $f : I(G) \to [0, 1]$ such that

$$f(U) = \begin{cases} 1/q't & \text{if } U = J_i \text{ for some } i \in \{0, 1, \ldots, p't - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

then we know that $f$ is a proper fractional $\frac{p'}{q'}$-colouring of $G$.

Next we want to show that for any fractional $\frac{p'}{q'}$-colouring $g$ of $G$, $g(I) = f(I)$ for any independent set $I$ of $G$. As $\chi_f(G) = \frac{|V(G)|}{\alpha_G}$, for any optimal fractional colouring $g$ of $G$, $g(I) = 0$ if $I$ is not a maximum independent set. As $J_i$ for $i = 0, 1, \ldots, p't - 1$ are the only maximum independent sets of $G$, we have $g(I) = 0$ if $I \neq J_i$ for some $i \in \{0, 1, \ldots, p't - 1\}$. It remains to show that for any fractional $\frac{p'}{q'}$-colouring $g$ of $G$,

$$g(U) = \begin{cases} 1/q't & \text{if } U = J_i \text{ for some } i \in \{0, 1, \ldots, p't - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

This part is similar to the proof of Lemma 3 and omitted. \qed

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