Additive Envelopes of Continuous Functions

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Abstract

We present an iterative method for constructing additive envelopes of continuous functions on a compact set, with contact at a prespecified point. For elements of a class of submodular functions we provide closed-form expressions for such additive envelopes.

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1 Introduction

In this paper, we are concerned with finding an additive envelopes of a given continuous function on a compact set $X$, a problem that can be reduced to finding a best additive upper bound on $X$. We further require that the envelope has contact with $f$ at a specified point. This problem has interesting applications, one of which is briefly discussed in Section 5.

In response to Hilbert’s 13th problem A.N. Kolmogorov (1957) showed that it is possible to represent any continuous function $f$ of $n$ variables as a linear superposition of continuous functions of one variable and addition, i.e., $f(x_1, \ldots, x_n) = \sum_{k=1}^{2n+1} g \left( \sum_{l=1}^{n} \kappa_l \phi_k(x_l) \right)$ for some appropriate continuous functions $\phi_1, \ldots, \phi_{2n+1}, g$, and constants $\kappa_1, \ldots, \kappa_n$. If nonlinearities are excluded (i.e., $g(x) \equiv x$), as in our case, then one obtains a problem of best approximation via linear superposition, which is difficult for $n \geq 3$ (Khavinson, 1997). Our problem differs from the standard version in that our approximation is constrained to be without intersection, and with contact at a given point.

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2 Preliminaries

For some positive integer $n$, let $f \in C(\mathbb{R}^n, \mathbb{R})$ be a continuous real-valued function and let $X \subset \mathbb{R}^n$ be a nonempty compact set. We make the following additional assumptions.

**A1.** $0 \in X$.

**A2.** $f(0) = 0$.

These assumptions are without loss of generality. Indeed, given any $f \in C(\mathbb{R}^n, \mathbb{R})$ and any $\hat{x} \in X$ (so that $0 \in Y = X - \hat{x}$) the function $g \in C(\mathbb{R}^n, \mathbb{R})$ with $g(y) = f(y) - f(\hat{x})$ for all $y = x - \hat{x} \in Y$ is such that $g(0) = 0$. The function $g$ and the set $Y$ are simple translations of $f$ and $X$, and they satisfy A1–2.

A continuous real-valued function $\varphi \in C(X, \mathbb{R})$ is additive (on $X$) if there exist $n$ functions $\varphi_1, \ldots, \varphi_n \in C(X, \mathbb{R})$ such that

\[ \varphi(x) = \varphi_1(x_1) + \cdots + \varphi_n(x_n) \]

for any $x = (x_1, \ldots, x_n) \in X$. To emphasize its coordinate functions $\varphi_i$ we sometimes write an additive function $\varphi$ in the form $\varphi = [\varphi_i]$.

We consider the problem of finding a continuous real-valued function $\varphi \in C(X, \mathbb{R})$ which has the following properties.

**P1.** $\varphi$ is additive on $X$ (Additivity).

**P2.** $f \leq \varphi$ on $X$ (One-Sidedness).

**P3.** If $\hat{\varphi} \in C(X, \mathbb{R})$ is additive on $X$ and $f \leq \hat{\varphi}$ on $X$, then $\varphi \leq \hat{\varphi}$ on $X$ (Minimality).

**P4.** $\varphi(0) = 0$ (Contact at the Origin).

A function $\varphi \in C(X, \mathbb{R})$ that satisfies properties P1–3 is called an additive upper envelope of $f$ (on $X$). If $\varphi$ satisfies P1–3 with $f$ replaced by $-f$, then it is called an additive lower envelope of $f$ (on $X$). A tuple of functions $H = (\varphi, \hat{\varphi})$, where $\varphi$ is an additive upper envelope of $f$ and $\hat{\varphi}$ is an additive lower envelope of $f$, is called an additive envelope of $f$.

It is clear that in order to determine an additive envelope of $f$ we can restrict attention

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\[ ^{1}\text{We use the following definition of (uniform) continuity: } f \text{ is continuous on } X \text{ if, given any } x \in X, \text{ for any } \varepsilon > 0 \text{ there exists a } \delta = \delta(\varepsilon) > 0 \text{ such that } \hat{x} \in \{ \xi \in X : \|x - \xi\| < \delta \} \Rightarrow |f(\hat{x}) - f(x)| < \varepsilon, \text{ where } \|\cdot\| \text{ is a given norm on the Euclidean space } \mathbb{R}^n. \text{ Note that according to this definition any function is continuous whenever } X \text{ does not contain an open subset (e.g., when it is finite or countable).} \]
to the problem of finding an additive upper envelope of $f$. We say that an additive upper envelope of $f$ has \textit{contact at the origin} if P1–4 are satisfied. An additive (upper/lower) envelope $\varphi$ of $f$ is called \textit{weak} if instead of P3 the following weaker property holds.

\textbf{P3’}. If $\hat{\varphi} \in C(X, \mathbb{R})$ is additive on $X$ and $f \leq \hat{\varphi}$ on $X$, then $\hat{\varphi} \leq \varphi$ on $X$ implies that $\hat{\varphi} = \varphi$ on $X$ (Weak Minimality).

A function $\varphi$ that satisfies P1–2 and P4 is called an \textit{additive upper bound of $f$ (with contact at the origin)}.

\section{Main Result}

We start the construction of an upper envelope of $f$ on $X$ with an additive upper bound $\varphi^0 \in C(X, \mathbb{R})$ of $f$, that satisfies properties P1–2 and P4. Such a function always exists, since it is sufficient to take\footnote{For any $x \in \mathbb{R}^n$ and any $i \in \{1, \ldots, n\}$ we sometimes use the convenient notation $x = (x_i, x_{-i})$ and $B_i(x_i) = \{x_{-i} : (x_i, x_{-i}) \in X \text{ and } \|x_{-i}\|_{\infty} \leq |x_i|\}$, where $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.}

$$\varphi^0(x) = \sum_{i=1}^n \left( \max_{x_{-i} \in B_i(x_i)} \{ f(x_i, x_{-i}) \} \right),$$

(1)

for all $x \in X$.

\textbf{Remark 1} A better initial additive upper bound than the one suggested in (1) can obtained as follows. For any nonempty set $N \subseteq \{1, \ldots, n\}$ define

$$\Phi_N(x) = \begin{cases} \frac{1}{|N|} \sum_{i \in N} \left( \max_{x_{-i} \in B_i(x_i)} \left\{ f(x_i e_i + \sum_{j \in N \setminus \{i\}} x_j e_j) - f(\sum_{j \in N \setminus \{i\}} x_j e_j) \right\} \right) + \Phi_{N \setminus \{i\}}(x), & \text{if } |N| \geq 2, \\ f(0, \ldots, 0, x_j, 0, \ldots, 0), & \text{if } N = \{j\}, \end{cases}$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $i$-th vector of a standard Euclidean base in $\mathbb{R}^n$. Then

$$\tilde{\varphi}^0(x) = \Phi_{\{1, \ldots, n\}}(x)$$

is an additive upper bound of $f$ such that $f \leq \tilde{\varphi}^0 \leq \varphi^0$ on $X$. In particular, if $f$ is additive, it is straightforward to verify that $\tilde{\varphi}^0 = f$ on $X$.

In order to satisfy P3, we would like a procedure for tightening a given additive upper bound $\varphi^k \in C(X, \mathbb{R})$ that satisfies P1–2 and P4, where $k \in \{0, 1, \ldots\}$. Specifically, given such a function $\varphi^k = [\varphi^k]$ we now construct an \textit{improved} additive upper bound $\varphi^{k+1} = \sigma(\varphi^k) \in C(X, \mathbb{R}^n)$ as follows. For a given $i \in \{1, \ldots, n\}$ and $x \in X$, set

$$\tilde{\varphi}_i(x_i) = \max_{x_{-i} : (x_i, x_{-i}) \in X} \left\{ f(x_i, x_{-i}) - \sum_{j \neq i} \varphi^k_j(x_j) \right\}.$$  

(2)
Lemma 1 For any additive upper bound \( \varphi^k \) of \( f \), the function \( \varphi^{k+1} = \sigma(\varphi^k) \) is an improved additive upper bound of \( f \) in the sense that \( \varphi^{k+1} \leq \varphi^k \). More specifically, for any \( i \in \{1, \ldots, n\} \): (i) \( \varphi^{k+1}_i \leq \varphi^k_i \) on \( X \), and (ii) \( f \leq \varphi^{k+1}_i + \sum_{j \neq i} \varphi^k_j \) on \( X \).

Proof. We obtain from (2), by virtue of P1–2 and P4 with respect to \( \varphi^k \), that

\[
\varphi^k_i \geq \tilde{\varphi}_i \geq f - \sum_{j \neq i} \tilde{\varphi}_j^k \geq f - \sum_{j \neq i} \varphi^k_j
\]
on \( X \), whence

\[
\varphi^k_i \geq \varphi^{k+1}_i \geq f - \sum_{j \neq i} \tilde{\varphi}_j^{k+1} \geq f - \sum_{j \neq i} \varphi^k_j
\]
on \( X \), i.e., (i) and (ii) obtain. In addition, \( \varphi^{k+1} \in C(X, \mathbb{R}) \) is an additive upper bound of \( f \), since it satisfies P1–2 and P4.

\( \square \)

By setting \( \varphi^0 \) as in (1) we can construct a sequence \( \{\varphi^k\}_{k=0}^{\infty} \) of successively improved additive upper bounds of \( f \), where \( \varphi^{k+1} = \sigma(\varphi^k) \) for all \( k \geq 0 \). Lemma 1 implies that this sequence is monotonically decreasing and by P2 it is bounded from below by \( f \). Hence, by the monotone convergence theorem there exists a pointwise limit function \( \varphi^{\infty} = [\varphi_i^{\infty}] \) with \( \varphi_i^{\infty} = \lim_{k \to \infty} \varphi_i^k \), which may depend on the function \( \varphi^0 \), since \( \varphi^{\infty} = \sigma^{\infty}(\varphi^0) \).

Theorem 1 Given any additive upper bound \( \varphi^0 \) of \( f \), the limit \( \varphi^{\infty} = \sigma^{\infty}(\varphi^0) \) is in \( C(X, \mathbb{R}) \), and is a weak additive upper envelope of \( f \).

Proof. Let \( \varphi^0 \) be any additive upper bound of \( f \) with contact at the origin, and let \( \varphi^{\infty} = \sigma^{\infty}(\varphi^0) \) be its pointwise limit. By (2) the limit function \( \varphi^{\infty} = [\varphi_i^{\infty}] \) satisfies

\[
\varphi^{\infty}_i(x_i) = \sup_{x_{-i} : (x_i, x_{-i}) \in X} \left\{ f(x_i, x_{-i}) - \sum_{j \neq i} \varphi^{\infty}_j(x_j) \right\} \tag{3}
\]

for all \( i \). Since \( f \) is continuous on the compact set \( X \) by hypothesis, it is also uniformly continuous on \( X \): given any \( \varepsilon > 0 \) there exists a real number \( \delta(\varepsilon) > 0 \) such that

\[
\|x - \hat{x}\| \leq \delta(\varepsilon) \quad \Rightarrow \quad |f(x) - f(\hat{x})| \leq \varepsilon/2
\]

holds for all \( x, \hat{x} \in X \). Thus, as long as \( \|x - \hat{x}\| = \|(x_i, x_{-i}) - (\hat{x}_i, \hat{x}_{-i})\| \leq \delta(\varepsilon) \),

\[
\varphi^{\infty}_i(x_i) - \varphi^{\infty}_i(\hat{x}_i) \leq f(x_i, z_i(x_i)) - f(\hat{x}_i, z_i(x_i)) \leq \varepsilon, \tag{4}
\]
and

$$\varphi_i^\infty(\hat{x}_i) - \varphi_i^\infty(x_i) \leq f(\hat{x}_i, z_i(\hat{x}_i)) - f(x_i, z_i(x_i)) \leq \varepsilon, \quad (5)$$

where $z_i(x_i) = (z_{i,1}(x_i), \ldots, z_{i,i-1}(x_i), z_{i,i+1}(x_i), \ldots, z_{i,n}(x_i))$ is such that

$$\varphi_i^\infty(x_i) - \left( f(x_i, z_i(x_i)) - \sum_{j \neq i} \varphi_j^\infty(z_{i,j}(x_i)) \right) < \varepsilon/2.$$ 

Hence, using relations (4) and (5) we obtain that for any $x, \hat{x} \in X$,

$$\|x - \hat{x}\| \leq \delta(\varepsilon) \Rightarrow |\varphi_i^\infty(x_i) - \varphi_i^\infty(\hat{x}_i)| \leq \varepsilon,$$

so that the function $\varphi^\infty = [\varphi_i^\infty]$ is uniformly continuous on $X$. Clearly, $\varphi^\infty$ is an additive upper envelope of $f$ satisfying P1–2 and P4. In addition, $\varphi^\infty$ satisfies the weak-minimality property P3’ by construction. 

Remark 2 The weak additive envelopes identified by the limit of the sequence $\{\varphi^0, \varphi^1, \ldots\}$ are solutions of the system equations (3) for $i \in \{1, \ldots, n\}$. In contrast to the set of all additive upper bounds of $f$ (with or without contact at the origin), the set of additive envelopes given by the limits of our algorithm is generally not convex. Indeed, given any two solutions $\varphi^\infty = [\varphi_i^\infty]$ and $\hat{\varphi}^\infty = [\hat{\varphi}_i^\infty]$ their convex combination in general does not satisfy (3), since the supremum is subadditive.

Remark 3 The construction of the weak upper envelope provided in Section 3 is somewhat related to the levelling algorithm by Diliberto and Straus (1951), which in the limit provides the best approximation of an arbitrary function of two variables by a sum of two functions of one variable.

4 Some Exact Upper Envelopes

We now examine a class of submodular functions for which, instead of relying on the outcome of the limit process described in the last section, one can obtain an exact expression for a unique additive upper envelope.

Theorem 2 Let $X$ be a lattice, and suppose that $f = h \circ g$, where $h \in C(\mathbb{R}^n, \mathbb{R})$ is a submodular function, $g = (g_1, \ldots, g_n)$ with $g_i \in C(\mathbb{R}, \mathbb{R}_+)$ and $g_i(0) = 0$ for all $i \in \{1, \ldots, n\}$. Then assumptions A1–2 imply that the unique additive upper envelope of $f$ is submodular if for any $x, \hat{x} \in \mathbb{R}^n$: $h(x \land \hat{x}) + h(x \lor \hat{x}) \leq h(x) + h(\hat{x})$, where $x \land \hat{x} = (\min\{x_1, \hat{x}_1\}, \ldots, \min\{x_n, \hat{x}_n\})$ and $x \lor \hat{x} = (\max\{x_1, \hat{x}_1\}, \ldots, \max\{x_n, \hat{x}_n\})$. The set $X$ is a lattice if $x, \hat{x} \in X$ implies that $x \land \hat{x} \in X$ and $x \lor \hat{x} \in X$. 

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is given by
\[ \varphi(x) = [\phi_i(x_i)] = [h(0, \ldots, 0, g_i(x_i), 0, \ldots, 0)] \] (6)
for all \((x_1, \ldots, x_n) \in X\).

Proof. Since \(f(0) = 0\) by A2, \(h\) is submodular, all \(g_i\)'s vanish at the origin and take only nonnegative values, we have using (6) that
\[ f(x) = h(g_1(x_1), \ldots, g_n(x_n)) + 0 \leq \sum_{i=1}^{n} h(0, \ldots, 0, g_i(x_i), 0, \ldots, 0) = \varphi(x) \]
for all \(x \in X\). Hence, \(\varphi\) satisfies P1–2 and P4 constituting an additive upper bound of \(f\) with contact at the origin. We now show that \(\varphi\) also satisfies P3. This holds trivially true if \(X = \{0\}\). To obtain a contradiction, we thus assume that there is an \(x = (x_1, \ldots, x_n) \in X\), different from the origin, such that
\[ f(x) \leq \hat{\varphi}(x) < \varphi(x) \] (7)
for some additive upper envelope \(\hat{\varphi} = [\hat{\phi}_i]\) of \(f\) which satisfies P1–2 and P4. Since \(X\) is a lattice, \((0, \ldots, 0, x_i, 0, \ldots, 0) \in X\) for all \(i\). By P4 it is \(\hat{\phi}_i(0) = 0\), so that by P1–2 we have
\[ f(0, \ldots, 0, x_i, 0, \ldots, 0) = h(0, \ldots, 0, g_i(x_i), 0, \ldots, 0) \leq \hat{\phi}_i(x_i), \]
for all \(i \in \{1, \ldots, n\}\). As a result, by the definition of \(\varphi\) in (6) we obtain
\[ \varphi(x) = \sum_{i=1}^{n} f(0, \ldots, 0, x_i, 0, \ldots, 0) \leq \sum_{i=1}^{n} \hat{\phi}_i(x_i) = \hat{\varphi}(x), \]
a contradiction to (7). \(\blacksquare\)

Remark 4 The validity of Theorem 2 critically depends on the domain \(X\) being a lattice, which can be seen by considering a simple example. Let \(X = \{(x_1, x_2) : x_1 = x_2 \in [0, 1]\}\) be the diagonal of a two-dimensional unit square and let \(f(x) = \max\{x_1, x_2\}\) be a submodular function (of the form in Theorem 2) defined on \(\mathbb{R}^2\), then assumptions A1–2 are satisfied and (6) yields \(\varphi(x) = x_1 + x_2 = 2x_1 > x_1 = f(x)\) for all nonzero \(x = (x_1, x_2)\) in \(X\). However, this additive upper bound can be strictly improved upon by the additive upper envelope \(\varphi = [\varphi_i]\) of \(f\) with \(\varphi_1(x_1) = \lambda x_1\) and \(\varphi_2(x_2) = (1 - \lambda)x_2\), which for any real constant \(\lambda\) coincides with \(f\) on \(X\).

We now provide an application of Theorem 2 which can be used to determine additive envelopes of functions that take weighted averages as arguments, as is frequently the case in practical situations.
Corollary 1 Suppose that \( f = h \circ g \), where \( h : \mathbb{R} \to \mathbb{R} \) is a concave function, \( g = [g_i] \in C(\mathbb{R}^n, \mathbb{R}_+) \) is additive on \( X = [0, \bar{x}] \) for some \( \bar{x} \in \mathbb{R}_+^n \), and \( g_i(0) = 0 \) for all \( i \in \{1, \ldots, n\} \). Then, the unique additive upper envelope \( \varphi = [\varphi_i] \) of \( f \) is given by

\[
\varphi(x) = [\varphi_1(x_1), \ldots, \varphi_n(x_n)] = [(h \circ g_1)(x_1), \ldots, (h \circ g_n)(x_n)]
\]

for all \( (x_1, \ldots, x_n) \in X \).

**Proof.** Note first that \( f \) and \( X \) satisfy A1–2, and that \( X \) is a lattice. It can easily be verified that concavity of \( h \) implies that \( \hat{h}(y_1, \ldots, y_n) = \hat{h}(y_1 + \cdots + y_n) \) is submodular. Thus, Theorem 2 can be directly applied, since \( f(x) = \hat{h}(g_1(x), \ldots, g_n(x)) \) on \( X \).

**Examples.** Let \( c_1, c_2, \bar{x}_1, \bar{x}_2 > 0 \), and \( X = [0, \bar{x}_1] \times [0, \bar{x}_2] \). (i) For \( f(x_1, x_2) = \ln(1 + c_1 x_1 + c_2 x_2) \) the additive upper envelope \( \varphi = [\varphi_i] \) with contact at the origin is given by \( \varphi_i(x) = \ln(1 + c_i x_i) \). (ii) For \( f(x_1, x_2) = -(x_1^2 + x_2^2)^2 \) the additive upper envelope \( \varphi = [\varphi_i] \) is given by \( \varphi_i(x) = -x^{2e_i} \). (iii) For \( f(x_1, x_2) = [c_1 x_1^\rho + c_2 x_2^\rho]^{\frac{1}{\rho}} \), with \( \rho > 1 \), the additive upper envelope \( \varphi = [\varphi_i] \) is given by \( \varphi_i(x) = (c_i)^{\frac{2}{\rho}} x_i \).

5 Discussion

We have shown that an additive upper bound \( \varphi = [\varphi_i] \) of a continuous function \( f \) can satisfy the strong minimality property P3 only in certain special cases. In general, the weak-minimality property P3' is the most that can naturally be expected of any additive envelope of \( f \). This property can be readily interpreted in terms of weak Pareto dominance: if \( \varphi \) is an additive upper envelope of \( f \), then it is not possible to decrease one coordinate function of \( [\varphi_i] \) without increasing another one. Another way to interpret weak minimality is in terms of a solution to a variational problem: for any weak additive upper envelope \( \varphi \) of \( f \) there exists a probability distribution \( F \) defined on \( X \), such that \( \varphi \) solves

\[
\min_{\varphi \in C(X,\mathbb{R})} \int_X (\varphi(x) - f(x)) dF(x),
\]

subject to P1–2 and P4. This feature links the method in Section 3 to problems of equilibrium construction in economics. For instance, the construction of a minimizing sequence of additive upper bounds of \( f \) can be directly applied to an equilibrium construction by Strulovici and Weber (2004). They analyze the construction of a complete

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4The generic multiplicity of weak additive upper envelopes can be illustrated with the following simple example. Consider \( f(x_1, x_2) = x_1 x_2 \) on \( X = [-1, 1]^2 \). Then, any \( \varphi \in \{ |x_1|, |x_2|, x_1^2/2 + x_2^2/2 \} \) is a weak additive upper envelope of \( f \) satisfying P1–2, P3', and P4.
set of equilibria in a general multi-contracting problem between multiple principals and multiple agents. This setting was discussed first by Prat and Rustichini (2003) who provide nonconstructive existence proofs, based on concepts and findings by Bernheim and Whinston (1986). Lastly, we note that the equations (3) in Section 3 constitute necessary optimality conditions for (8). Further work is needed to determine a general solution for a given distribution $F$.

References


