Exact Expression for the Coherence Bandwidth of Rayleigh Fading Channels

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Abstract—Coherence bandwidth is an important characteristic of multipath fading channels, serving as a useful tool for wireless systems design. Regardless of its importance, the determination of the coherence bandwidth, so far, has relied on an empirical formula. In this correspondence, we derive an exact coherence-bandwidth formula for Rayleigh fading channels. The use of the new formula is illustrated by numerical results.

Index Terms—Coherence bandwidth, multipath channels, power delay profile (PDP), Rayleigh fading channels.

I. INTRODUCTION

MULTIPATH fading has significant impact on the system performance of wireless communications. Multipath fading channels are usually classified into flat fading and frequency-selective fading according to their coherence bandwidth relative to the bandwidth of transmitted signal. Coherence bandwidth is defined as the range of frequencies over which two frequency components remain a strong amplitude correlation. Physically, it defines the range of frequencies over which the channel can be considered “flat.” The analytic issue of coherence bandwidth was first studied by Jakes [1], where by assuming homogeneous scattering, he revealed that the coherence bandwidth of a wireless channel is inversely proportional to its rms delay spread. The same issue was subsequently studied by various authors [2]–[5]. Since many practical channel environments can significantly deviate from the homogeneous assumption, various measurements were conducted to determine multipath delay profiles and coherence bandwidths [6]–[8], aiming to obtain a more general formula for coherence bandwidth. Based on experimental data, Rappaport et al. [2] and Lee [9] modified Jakes’ formula to obtain their empirical formula for coherence bandwidth

\[ B_c = \begin{cases} 
\frac{1}{5\sigma \tau}, & \eta_0 = 0.9 \\
\frac{1}{5\sigma \tau}, & \eta_0 = 0.5 
\end{cases} \]  

(1)

where \( \sigma \tau \) denotes the rms delay spread. This formula resembles the original one by Jakes except a different scaling factor, which depends on the preset level \( \eta_0 \) for frequency correlation. Two typical values for the threshold level are \( \eta_0 = 0.9 \) and \( \eta_0 = 0.5 \), respectively. The aforementioned formula is widely used to determine the coherence bandwidth of a particular channel with given multipath profile, and many illustrating numerical examples can be found in the seminal textbook [10].

Though very useful, the expression (1) is an approximate formula in nature, representing a “ball park estimate,” as called by Rappaport [10]. It is, therefore, of practical and theoretical importance to derive an exact expression for the coherence bandwidth of a wireless channel with given power delay profile (PDP), which is the objective of this correspondence. As a convention, we will use \( \text{var}(r) \) to denote the variance of random variable \( r \), and use \( \text{cov}(r_1, r_2) \) to denote the cross correlation between \( r_1 \) and \( r_2 \).

II. FORMULATION

Let us consider a discrete multipath Rayleigh fading channel with tap weight coefficients \( \{ h_k \} \) and time delays \( \{ \tau_k \} \), such that its impulse response \( h(\tau) \) and frequency transfer function \( H(\omega) \) are given by

\[ h(\tau) = \sum_{k=0}^{D-1} h_k \delta(\tau - \tau_k) \]

\[ H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega \tau} d\tau = \sum_{k=0}^{D-1} h_k e^{-j\omega \tau_k} \]  

(2)

where we assume that \( \{ h_k \} \) are independent with PDP \( \{ P_0, P_1, \ldots, P_{D-1} \} \). Throughout the paper, we will use the superscripts *, T, and \( \dagger \) to denote conjugation, transposition, and conjugate transposition, respectively. Define vectors

\[ h = [h_0, h_1, \ldots, h_{D-1}]^T \]

\[ w_i = [\exp(j2\pi f_i \tau_0), \ldots, \exp(j2\pi f_i \tau_{D-1})]^T. \]  

(3)

Let \( x_i \) denote the frequency response at frequency \( f_i \), and let \( r_i \) denote its magnitude. Further let \( x \) denote the \( 2 \times 1 \) vector, such that

\[ x_i = H(\omega_i) = w_i^h, \quad i = 1, 2 \]

\[ x = [x_1, x_2]^T \]

\[ r_i = |x_i|. \]  

(4)

It follows from the independent Rayleigh fading assumption on \( \{ h_k \} \) that \( h \) is jointly complex Gaussian distributed with mean zero and covariance matrix \( R_h = E[hh^H] \). Specifically, we write

\[ h \sim \text{CN}(0, R_h) \]  

(5)

with

\[ R_h = \text{diag}\{P_0, P_1, \ldots, P_{D-1}\}. \]  

(6)

Paper approved by V. A. Aalo, the Editor for Diversity and Fading Channel Theory of the IEEE Communications Society. Manuscript received March 28, 2006; revised July 26, 2006. This work was supported by a strategic research grant from the City University of Hong Kong, Hong Kong, China, under Project 7001772.

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Digital Object Identifier 10.1109/TCOMM.2007.900596
Since \( x \) is linearly related to \( h \), it is also Gaussian, as shown by
\[
x \sim CN(0, R_x).
\]
We can determine the \((m, n)\)th entry of \( R_x \) from (4) yielding
\[
R_{x, mn} = w_m^\dagger R_h w_n, \quad m, n = 1, 2.
\] (8)

The coherence bandwidth of a fading channel is probed by sending two sinusoids, separated in frequency by \( \Delta f = f_1 - f_2 \) Hz, through the channel. The coherence bandwidth is defined as \( \Delta f \), over which the cross-correlation coefficient between \( r_1 \) and \( r_2 \) is greater than a preset threshold, say, \( \eta_0 = 0.9 \). Namely
\[
C_{r_1, r_2} = \frac{\text{cov}(r_1, r_2)}{\sqrt{\var(r_1) \var(r_2)}} = \eta_0
\] (9)
where \( \var(r_1) = \var(r_2) \). We need to determine \( \Delta f \) that meets this equation.

III. ANALYTIC EXPRESSION

By definition, we can write the cross correlation and variance explicitly to obtain
\[
C_{r_1, r_2} = \frac{E[r_1 r_2] - E[r_1] E[r_2]}{E[r_1^2] - (E[r_1])^2}.
\] (10)

Various moments of \( r_i \) can be expressed in terms of the covariance matrix \( R_x \). To this end, we define
\[
S = R_x^{-1}
\] (11)
and invoke the results of Miller [11] to write
\[
E[r_1 r_2] = \frac{\Gamma(1 + \alpha/2) \Gamma(1 + \beta/2) \text{det}(S)}{S_{11}^{1/2} S_{22}^{1/2}} (1 + \frac{\alpha}{2} + \frac{\beta}{2} + \lambda_{12}^2)
\] (12)
where \( 2F_1(a; b; c; x) \) is the Gaussian hypergeometric function.

Hence, we can write
\[
E[r_1 r_2] = 2F_1(1.5, 1.5; 1; \lambda_{12}^2) \pi \frac{\text{det}(S)}{4 S_{11}^{1/2} S_{22}^{1/2}}
\]
\[
E[r_1^2] = E[|x_1|^2] = P
\]
\[
E[r_1] = \sqrt{\frac{\pi P}{2}}.
\] (13)

Here, \( P \) denotes the total power, i.e., \( P = \sum_{k=0}^{D-1} P_k \).

We next show how to represent \( S \) explicitly in terms of the power delay profile of the fading channel. It is straightforward to use (8) to obtain
\[
R_x = \left[ \sum_{i=0}^{D-1} P_i e^{j2\pi \Delta f \tau_i} \right] \left[ \sum_{j=0}^{D-1} P_j e^{-j2\pi \Delta f \tau_j} \right] / P^2
\] (14)
and hence,
\[
S = \left[ \sum_{i=0}^{D-1} P_i e^{j2\pi \Delta f \tau_i} \right] \left[ \sum_{j=0}^{D-1} P_j e^{-j2\pi \Delta f \tau_j} \right] / P^2 - \sum_{i=0}^{D-1} \sum_{j=0}^{D-1} P_i P_j e^{j2\pi \Delta f(\tau_i - \tau_j)}
\] (15)
which, when applied to (11) and (13), produces
\[
\lambda_{12}^2 = \frac{\sum_{i=0}^{D-1} \sum_{j=0}^{D-1} P_i P_j e^{j2\pi \Delta f(\tau_i - \tau_j)}}{P^2}
\]
\[
E[r_1 r_2] = 2F_1(1.5, 1.5; 1; \lambda_{12}^2) \pi P (1 - \lambda_{12}^2)^2 / 4.
\] (16)

Note that both expressions depend on the frequencies only through their difference \( \Delta f \). We substitute (16) into (10), which leads to an explicit expression for the cross correlation
\[
C_{r_1, r_2} = 2F_1(1.5, 1.5; 1; \lambda_{12}^2) (1 - \lambda_{12}^2)^2 - 1 /
\]
\[
(4\pi - 1)
\]
\[
= 2F_1(-0.5, -0.5; 1; \lambda_{12}^2) - 1 /
\]
\[
(4\pi - 1)
\] (17)

On the second line, we have used [12, eq. (15.3.3)]. Note the cross-correlation function depends on \( \Delta f \) only through \( \lambda_{12}^2 \).

Since \( \lambda_{12}^2 \) is real as observed from (11), we can further write
\[
\lambda_{12}^2 = \sum_{i=0}^{D-1} \sum_{j=0}^{D-1} p_i p_j \cos(2\pi(\tau_i - \tau_j) \Delta f)
\] (18)
where \( p_i = P_i / P \). Accordingly, the coherence bandwidth for a given threshold \( \eta_0 \) is the value of \( \Delta f \), for which
\[
2F_1 \left( -\frac{1}{2}, -\frac{1}{2}; 1; \lambda_{12}^2 \right) - \frac{4}{\pi} \eta_0 - 1 = 0.
\] (19)

The expressions (18) and (19) reveal the dependence of the exact coherence bandwidth of a Rayleigh fading channel on its PDP. At this point, it would be helpful to obtain some intuitive understanding of (19). For a given \( \eta_0 \), \( \lambda_{12}^2 \) is fixed according to (19). Note from (16) that \( \lambda_{12}^2 \) is essentially the Fourier transform of the sample correlation function of PDP, thereby representing the PDP “spectrum” as a function of \( \Delta f \). Accordingly, a rough conclusion we can draw is that the smaller the rms delay spread, the wider the coherence bandwidth. Such explanation is intuitively appealing, but not rigorous. In fact, \( \lambda_{12}^2 \) depends on the structure of the PDP pattern rather than simply on its average characteristic—the rms delay spread. Therefore, in some application scenarios, the variation of coherence bandwidth with \( \eta_0 \) can exhibit fluctuations, as shown in [7, Figs. 6 and 9].

IV. RECURSIVE ALGORITHM

We need to solve the nonlinear equation (19) for the coherence bandwidth. It can be done by using the Newton–Raphson method. To this end, we define the function
\[
g(\Delta f) = 2F_1(-0.5, -0.5; 1; \lambda_{12}^2) - \xi_0 (4\pi - 1) - 1.
\] (20)

Then, the solution to coherence bandwidth can be obtained through the following iteration:
\[
\Delta f_{n+1} = \Delta f_n - \frac{g(\Delta f_n)}{g'(\Delta f_n)}
\] (21)

Here, \( g'(\Delta f) \) is the derivative of \( g \) with respect to \( \Delta f \) and can be calculated according to [12, eq. (15.2.1)]. The result is
TABLE I
MULTIPATH MODELS

<table>
<thead>
<tr>
<th>Tap</th>
<th>Indoor Office</th>
<th>Indoor Commercial</th>
<th>Outdoor Residence</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Relative Delay (ns)</td>
<td>Average Power (dB)</td>
<td>Relative Delay (ns)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>-2.6</td>
<td>50</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>-7.2</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>150</td>
<td>-8.7</td>
<td>200</td>
</tr>
<tr>
<td>5</td>
<td>250</td>
<td>-14.5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>300</td>
<td>-17.5</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>350</td>
<td>-20.3</td>
<td></td>
</tr>
</tbody>
</table>

given by

\[ g' (\Delta f) = \frac{-2F_1 ((1/2), (1/2); 2, \lambda \frac{2}{12})}{4} \times \sum_{i=0}^{D-1} \sum_{j=0}^{D-1} p_i p_j \sin (2\pi (\tau_i - \tau_j) \Delta f). \]  \hspace{1cm} (22)

To calculate the hypergeometric functions, one can directly call the Matlab function, or use the following series representation:

\[ 2F_1 (a, b; c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \]  \hspace{1cm} (23)

where \((x)_n = x(x+1) \cdots (x+n-1)\) denotes the ascending factorial with \((x)_0 = 1\).

V. NUMERICAL RESULTS

To illustrate the use of the new analytic formula, let us consider three examples.

Example 1: Let us reexamine [10, Example 5.5]. The PDP is defined by four components \{-20, -10, -10, 0\} dB located at 0, 1, 2, and 5 μs, respectively. The empirical estimation for the 90% coherence bandwidth is 14.55 kHz, and its 50% counterpart is 145.54 kHz [10]. We recalculate the bandwidth of the same channel using the exact formula. The result is given by 35.54 and 100.88 kHz, respectively. Clearly, the empirical formula is not accurate. That is what we expect since the coherence bandwidth, as implicitly defined through the nonlinear functions \(2F_1\) and \(\lambda^2\) in (19), is not expressible in terms of a simple reciprocal of the rms delay.

We further apply the new formula to more practical fading channel models, as proposed for personal communication systems (PCS) air interface standard [13].

Example 2: Three channel models proposed in [13] have a tap-delay-line structure with powers and delays tabulated in Table I. They represent typical operational environments for indoor office, indoor commercial, and outdoor residential, respectively. We calculate the coherence bandwidth using the exact and empirical formulas. The results for the three channels are depicted, respectively, in Figs. 1–3. Again, the inaccuracy of the empirical formula is observed.

We now compare the coherence bandwidth computed using (17) with its measured counterpart on the basis of real channel parameters.

Example 3: The real channel measurements were conducted in New York City and the results are reported in [7]. These results show the rationale of using independent Rayleigh distributions for multipath channels, and provide the measured coherence bandwidth of the relevant channels. From the average PDP measured at the Broome Street between Lafayette and Crosby [7, p. 1275, Fig. 3(a)], we can obtain the following discrete PDP.

Relative delay

\[ \text{Relative delay} \]  \hspace{1cm} (24)

Average power

\[ \text{Average power} = [1 \ 0.41 \ 0.15 \ 0.04 \ 0.02 \ 0.007 \ 0.001 \ 0.002], \]  \hspace{1cm} (25)

This data set is then applied to (19) to generate the theoretical coherence bandwidth. The results are graphed in Fig. 4 to compare with their counterparts obtained from real measurement. The 0.5- and 0.9-coherence bandwidth calculated using the empirical formula are also included for comparison. Good match between the theoretical result and the real measurement is observed demonstrating the validity of our theoretical formula. The results based on the empirical formula, on the contrary, considerably deviate from the theoretical and measured coherence bandwidth.
VI. CONCLUSION

In this correspondence, we obtained an exact formula for determining the coherence bandwidth of Rayleigh multipath fading channels. Numerical results show a good match between the theoretical results and the real measured correlation bandwidth. It is also shown that the conventional empirical formula produces results considerably deviating from their real measured counterparts.

REFERENCES