

Limber equation for luminosity dependent correlations

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Abstract

The passage from angular to spatial correlations, in the case of spatial clustering length depending on the average distance between nearby objects is studied. We show that, in a number of cases, the scaling law of angular correlation amplitudes, which holds for constant spatial clustering length, is still true also for a luminosity dependent spatial correlation. If the Limber equation is then naively used to obtain ‘the’ spatial clustering length from the angular function amplitude, a quantity close to the average object separation is obtained. The case of cluster clustering is explicitly considered.

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1 Introduction

Initial studies on clustering properties over large scales were performed using 2–dimensional samples (Totsuji & Kihara 1969, Peebles 1980; see also Sharp, Bonometto & Lucchin 1984). Spatial properties were then deduced from angular ones using the Limber equation (Limber 1953). However, even in the

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present epoch of large redshift samples, the technique used to pass from angular to spatial statistical properties is far from becoming obsolete. For instance, cluster samples are available and in preparation, selected with a fixed apparent magnitude limit. Even though the typical redshift of each cluster is given, the different intrinsic luminosity limit at different distances forces to cut off some information, to extract volume and intrinsic magnitude limited subsamples, where the spatial 2–point function can be directly estimated. Working out the angular function and using the Limber equation to extract spatial properties, might then seem a more convenient alternative approach. This approach was followed, e.g., by Dalton et al (1997), when discussing the clustering of clusters in catalogues extracted from the APM Galaxy Survey. In the final section we shall comment on the results of their analysis. The point is that this approach meets a different obstacle, as the cluster correlation length r_c may depend on cluster luminosities L (see Bahcall & West 1988 for a review of Abell cluster clustering data; see Nichol et al 1992, Dalton et al 1994, Croft et al 1997, for APM cluster clustering data; the whole set of data has been recently reanalysed by Lee & Park (1999), with the inclusion of X–ray catalog data).

Owing also to the different definition of clusters in different observational samples or simulations, it is usual to work out from the cumulative cluster luminosity functions $n(> L)$ the average distance between nearby objects $D_L \equiv n^{-1/3}(> L)$ and to seek the dependence of r_c on D_L . For $30 h^{-1}\text{Mpc} \lesssim D_L \lesssim 60 h^{-1}\text{Mpc}$ (h is the Hubble constant in units of 100 km/s/Mpc), the Bahcall & West (1988) conjecture that

$$r_c \simeq 0.4D_L \quad (1.1)$$

approaches most data, although, clearly, it cannot be considered an observational output. However, over greater scales, there is no agreement among results obtained from different samples and it is not clear how much this may depend on the different cluster definitions. For Abell clusters, however, eq. (1.1) approaches data up to $\sim 90 h^{-1}\text{Mpc}$. In this work, we shall assume the validity of eq. (1.1), as an example of gross discrepancy from assuming that r_c is L –independent.

In the presence of such kind of L –dependence, two questions arise: (i) As a byproduct of the Limber equation, a *scaling law* holds, among the coefficients of the angular functions for samples with different limiting magnitudes. Is such scaling law still valid? We shall show that the same scaling law holds in the case (1.1), as in the constant r_c case. (ii) If the Limber equation is then formally used to work out the spatial clustering length, which value of r_c is obtained? We shall show that a naive use of the standard Limber equation leads to working out an *apparent* correlation length, close to the average distance between nearby clusters for the whole sample.

As is obvious, our treatment applies to any kind of objects, not to clusters only.

Furthermore, the law (1.1) can be relaxed in various ways, obtaining related results, provided that the luminosity dependence of the clustering length (r_c) can be expressed through a function of the mutual distance of the objects belonging to an intrinsic luminosity limited set (D_L). Such generalizations will not be debated here. Our treatment also neglects departures from euclidean geometry and κ -correction. Their inclusion does not change the conclusions of the paper, although slightly complicating final expressions; of course, for several detailed applications, they are to be included, but, at this stage, they are unessential to make our points.

In order to show our points, we shall first need to reobtain the Limber equation in a way slightly different from textbooks (see, e.g., Peebles 1980a); this is the content of section 2 and 3. Using algorithms and expressions deduced in these sections, in section 4 we show the points made in the Introduction. Section 5 contains a few final remarks.

2 Sample definition

The Limber equation was introduced to work out the spatial 2-point correlation function from angular data assuming that: (i) the 2-point function is luminosity independent; (ii) there exists a universal luminosity function for the class of objects considered.

If the objects are galaxies, the luminosity function can be given the Schechter form (Schechter 1976):

$$\phi(L) = \frac{\phi^*}{L_*} \left(\frac{L}{L_*}\right)^{-\alpha} \exp\left(-\frac{L}{L_*}\right) \quad (2.1)$$

Here α (~ 1.1 – 1.2) is a phenomenological parameter; L_* is a typical galaxy luminosity, close to the top luminosity of the sample. The number density of galaxies with luminosity exceeding L reads

$$n(> L) = \int_L^\infty dL \phi(L) = \phi_* U(L/L_*) \quad (2.2)$$

with

$$U(\lambda) = \int_\lambda^\infty du u^{-\alpha} \exp(u) \quad (2.3)$$

and, henceforth, $\phi(L) = -dn(> L)/dL$.

If we deal with clusters or other kinds of objects, the expression (2.1) may no longer be a fair approximation. In the sequel, however, we shall only need

that $n(> L)$ can be expressed as in eq. (2.2), even though $U(\lambda)$ has not the expression (2.3). In eq. (2.2), ϕ_* should be expressed in $\text{Mpc}^{-3}h^3$ and approach the total number density for the class of objects considered and, of course, there must also exist a typical luminosity L_* , even though the distribution around it is not given by eq. (2.1). As previously stated, let us then define

$$D_L \equiv n^{-1/3}(> L) ; \quad n(> L) \equiv D_L^{-3} , \quad (2.4)$$

clearly $dn/dD = -3/D^4$.

The apparent luminosity of a source of intrinsical luminosity L , at a distance r from the observer, is $l = L/4\pi r^2$ and the *depth* d_* of a *sample* of objects, with apparent luminosity $l > l_m$, is defined so that

$$l_m = L_*/4\pi d_*^2 . \quad (2.5)$$

Hence, requiring that an object at distance r has $l > l_m$, means that its intrinsical luminosity

$$L > L_*(r/d_*)^2 . \quad (2.6)$$

For instance, the angular density of objects, in a sample of depth d_* , reads

$$n_\Omega(d_*) = \int_0^\infty dr r^2 \int_{L_*(r/d_*)^2}^\infty dL \phi(L) = \int_0^\infty dr r^2 D_{L_*(r/d_*)^2}^{-3} = d_*^3 G , \quad (2.7)$$

where

$$G = \int_0^\infty dq q^2 D_{L_*q^2}^{-3} = \frac{1}{2L_*^{3/2}} \int_0^\infty dL \sqrt{L} D_L^{-3} \quad (2.8)$$

is a universal constant, provided that $\phi(L)$ is a universal distribution, quite independently from the explicit form of $U(\lambda)$. According to eq. (2.7), $n_\Omega \propto d_*^3$, a well known volume effect.

3 The case of luminosity independent correlations

All above relations hold provided that the assumption (ii) is true. For a sample of objects of depth d_* , the angular 2–point function

$$w(\theta) = n_\Omega^{-2} \int_0^\infty dr_1 r_1^2 \int_{L_*(r_1/d_*)^2}^\infty dL_1 \phi(L_1) \int_0^\infty dr_2 r_2^2 \int_{L_*(r_2/d_*)^2}^\infty dL_2 \phi(L_2) \zeta(r_{12}, L_1, L_2) \quad (3.1)$$

is obtainable from the spatial 2–point functions $\zeta(r_{12}, L_1, L_2)$ and from the luminosity function $\phi(L)$.

Let us first review the standard case, when also the assumption (i) holds, *i.e.* $\zeta = (r_0/r)^\gamma$ with constant (luminosity independent) γ and r_0 (correlation length). In this case, changing integration variables, eq. (3.1) immediately yields:

$$w(\theta) = n_\Omega^{-2} r_0^\gamma \int_0^\infty dr_1 r_1^2 \int_{D_{L^*}(r_1/d_*)^2}^\infty dD_1 \frac{3}{D_1^4} \int_0^\infty dr_2 r_2^2 \int_{D_{L^*}(r_2/d_*)^2}^\infty dD_2 \frac{3}{D_2^4} r_{12}^{-\gamma} \quad (3.2)$$

and, therefore,

$$w(\theta) = n_\Omega^{-2} r_0^\gamma \int_0^\infty dr_1 r_1^2 D_{L^*}^{-3}(r_1/d_*)^2 \int_0^\infty dr_2 r_2^2 D_{L^*}^{-3}(r_2/d_*)^2 r_{12}^{-\gamma} \quad (3.3)$$

Herefrom, using eq. (2.7) and performing the changes of variables:

$$2r = r_1 + r_2, \quad q = r/d_*, \quad ud_*\theta = r_1 - r_2 \quad (3.4)$$

in order that $r_{12} \simeq ud_*\theta(q^2 + u^2)^{1/2}$, we work out

$$w(\theta) = A_\gamma \left(\frac{r_0}{d_*} \right)^\gamma \theta^{1-\gamma} \quad (3.5)$$

with

$$A_\gamma = c_\gamma \frac{\int_0^\infty dq q^{5-\gamma} D_{L^*}^{-6} q^2}{[\int_0^\infty dq q^2 D_{L^*}^{-3} q^2]^2} = 2 c_\gamma L_*^{\gamma/2} \frac{\int_0^\infty dL L^{2-\gamma/2} D_L^{-6}}{[\int_0^\infty dL \sqrt{L} D_L^{-3}]^2} \quad (3.6)$$

where

$$c_\gamma = \int_{-\infty}^{+\infty} du (1 + u^2)^{-\gamma/2} \quad (3.7)$$

is a purely numerical constant.

In above relations, the distances D_L were used, instead of the number densities $n(> L) = D_L^{-3}$, which might even seem more convenient, in order to facilitate the forthcoming passage to the case of luminosity dependent correlations.

Eq. (3.5) puts in evidence the *scaling properties* of the angular 2–point function. The only dependence on the depth of the sample (d_*) is factorized, as A_γ is a universal expression (still, provided that $\phi(L)$ is a universal distribution). It is often stated that such scaling is arises from the fact that r_0 does not depend on luminosities. Here we shall verify that such scaling is a more general property.

4 The case of luminosity–dependent correlations

In the case of galaxy clusters, we shall now assume that the Bahcall and West conjecture holds:

$$\xi(r_{12}, > L) = a (D_L/r_{12})^\gamma . \quad (4.1)$$

Here the numerical constant $a \simeq 0.4$ and $D_L^{-3} = n(> L)$ is the number density of clusters with luminosity $> L$ (see eq. 2.5). Eq. (4.1) is an *integral* correlation law, as it concerns all objects with luminosities above L . In principle, it is possible to define also a *differential* correlation law, holding if objects of intrinsic luminosities L_1 and L_2 only are considered. Such differential law is not to be measured, but we want to show that, the integral expression (4.1) follows from assuming that the differential law

$$\zeta(r_{12}, L_1, L_2) = \zeta_* (\sqrt{D_1 D_2}/r_{12})^\gamma \quad (4.2)$$

holds, provided that $a = \zeta_*(1 - \gamma/6)^{-2}$. In fact

$$\xi(r_{12}, > L) = n^{-2}(> L) \zeta_* \int_L^\infty dL_1 \phi(L_1) D_{L_1}^{\gamma/2} \int_L^\infty dL_2 \phi(L_2) D_{L_2}^{\gamma/2} r_{12}^{-\gamma} \quad (4.3)$$

and, changing variables as we did to pass from eqs. (3.1) to (3.2),

$$\xi(r_{12}, > L) = \frac{\zeta_* D_L^6}{r_{12}^\gamma} \left(\int_{D_L}^\infty dD \frac{3}{D^{4-\gamma/2}} \right)^2 = \zeta_* (1 - \gamma/6)^{-2} \left(\frac{D_L}{r_{12}} \right)^\gamma \quad (4.4)$$

according to eq. (4.1).

Let us now use the expression (4.2) to work out the angular function. In this case the correlation length has a precise dependence on luminosities. In spite of that, we shall find that eq. (3.5) is still formally true and, in particular $w(\theta)$ has the same *scaling properties* as in the case of luminosity independent r_0 .

In fact, if we replace the expression (4.2) in eq. (3.1) and perform the operations leading there to eqs. (3.2) and (3.3), we find

$$\begin{aligned} w(\theta) &= n_\Omega^{-2} \zeta_* \int_0^\infty dr_1 r_1^2 \int_{D_{L^*(r_1/d_*)}^2}^\infty dD_1 \frac{3}{D_1^{4-\gamma/2}} \int_0^\infty dr_2 r_2^2 \int_{D_{L^*(r_2/d_*)}^2}^\infty dD_2 \frac{3}{D_2^{4-\gamma/2}} r_{12}^{-\gamma} \\ &= n_\Omega^{-2} \zeta_* (1 - \gamma/6)^{-2} \int_0^\infty dr_1 r_1^2 D_{L(r_1/d_*)}^{-3+\gamma/2} \int_0^\infty dr_2 r_2^2 D_{L(r_2/d_*)}^{-3+\gamma/2} r_{12}^{-\gamma} \end{aligned} \quad (4.5)$$

Performing here again the changes of variables (3.4), we obtain

$$w(\theta) = n_\Omega^{-2} \zeta_*(1 - \gamma/6)^{-2} d_*^{6-\gamma} \theta^{1-\gamma} \int_0^\infty dq q^4 D_{L_* q^2}^{-6+\gamma} \int_{-\infty}^{+\infty} du (q^2 + u^2)^{-\gamma/2} \quad (4.6)$$

and, using eq. (2.7), we return to

$$w(\theta) = \tilde{A}_\gamma \left(\frac{\tilde{r}_0}{d_*} \right)^\gamma \theta^{1-\gamma} \quad (4.7)$$

provided that

$$\tilde{r}_0^\gamma \tilde{A}_\gamma = c_\gamma \frac{\int_0^\infty dq q^{5-\gamma} D_{L_* q^2}^{-6+\gamma}}{[\int_0^\infty dq q^2 D_{L_* q^2}^{-3}]^2} = 2 c_\gamma L_*^{\gamma/2} \frac{\int_0^\infty dL L^{2-\gamma/2} D_L^{-6+\gamma}}{[\int_0^\infty dL \sqrt{L} D_L^{-3}]^2} \quad (4.8)$$

(c_γ is defined in eq. 3.7).

Besides of finding the same scaling properties as in the case of L -independent r_0 , we can now try to answer a practical question: If we interpret angular results, possibly because of the scaling $w \propto d_*^\gamma$, as originating from a L -independent r_0 and work out the r_0 value, what value is obtained?

This question is answered by requiring that

$$r_0^\gamma A_\gamma = \tilde{r}_0^\gamma \tilde{A}_\gamma \quad (4.9)$$

and, using eqs. (3.6) and (4.8), this yields

$$r_0^\gamma = \frac{\int_0^\infty dL L^{2-\gamma/2} D_L^{-6+\gamma}}{\int_0^\infty dL L^{2-\gamma/2} D_L^{-6}}, \quad (4.10)$$

which is the *apparent* value of r_0 .

Eq. (4.10) can be used either with simulations, working out D_L from them and performing numerically the two integrations, or with analytical expressions. In the case of the Schechter law (2.1), using the function $U(\lambda)$ defined in eq. (2.3), it is easy to see that

$$r_0 = \phi_*^{1/3} \left[\frac{\int_0^\infty d\lambda \lambda^{2-\gamma/2} U_\alpha^{2-\gamma/3}(\lambda)}{\int_0^\infty d\lambda \lambda^{2-\gamma/2} U_\alpha^2(\lambda)} \right]^{1/\gamma}. \quad (4.11)$$

The expression in square brackets can be turned into:

$$[\Gamma(\alpha)]^{\gamma/3} v(\alpha, \gamma) \frac{\int_0^\infty dx \exp(-x) x^{\epsilon_1} \left(\int_0^\infty dt \exp(-t) \frac{t^{1-\alpha}}{x/(2-\gamma/3)+t} \right)^{2-\gamma/3}}{\int_0^\infty dx \exp(-x) x^{\epsilon_2} \left(\int_0^\infty dt \exp(-t) \frac{t^{1-\alpha}}{x/2+t} \right)^2}$$

γ	c_1	c_2
1.6	-1.95	3.29
1.8	-1.93	3.25
2.0	-1.91	3.20
2.2	-1.88	3.15

with $e_1 = 2 - \gamma/2 + (2 - \gamma/3)(1 - \alpha)$ and $e_2 = 2 - \gamma/2 + 2(1 - \alpha)$, while $v(\alpha, \gamma) = 2^{e_2+1}/(2 - \gamma/3)^{e_1+1}$. Using a Gauss-Laguerre integration algorithm, double integrals can be then turned into double summations and analytically evaluated. For the narrow α interval, for which such expression is needed, however, we can verify that the linear expression

$$r_0/\phi_*^{1/3} = c_1 * \alpha + c_2 \quad (4.12)$$

is better approximated than 0.2%.

In Table 1 we give c_1 and c_2 for the relevant γ interval. Using such values it is easy to see that the discrepancy between the correlation length r_0 and the average distance between nearby objects is expected to be $\lesssim 10\%$. Such result, clearly, depends on the use of the Schechter law (2.1), which also allows a comfortable numerical integration. We have considered a few other expressions, e.g. the expression deduced from the Press & Schechter (PS) cluster mass function, assuming a constant spectral index n_t for the transferred fluctuation spectrum and $L \propto M^2$. In most such cases, numerical integration only is possible and integration details are cumbersome. For the PS case, however, the expression (4.12) give results approximated up to some percents, provided that we assume $\phi_* \equiv 2\pi^{-1/2}n_t\nu^*$, where $\nu^* = \rho_m/M_*$ (ρ_m is the average matter density and M_* is the mass attributed to the cluster with typical luminosity L_*).

5 Final remarks

According to Peebles (1980a), the scaling relation (3.5) played an important role in testing that the angular correlations of galaxies in the catalogs do reflect the presence of a uniform spatial galaxy clustering, rather than something else, e.g. systematic errors due to patchy obscuration in the Milky Way. This work does not contradict such statement, corresponding to the point (ii) at the beginning of section 2. However, Hauser and Peebles (1973), in their seminal work on cluster clustering, made use of the scaling relation (3.5), as a test for the cluster correlation length $r_c \sim 30 h^{-1}\text{Mpc}$ they obtained for Abell clusters. This test, as we showed, has only a partial validity.

For the sake of curiosity, let us outline that, if the galaxy luminosity function values, $\phi^* \simeq 10^{-2}$ and $\alpha \simeq 1.2$ are used in eq. (4.11), we obtain a galaxy correlation length $r_g \simeq 6 h^{-1}\text{Mpc}$, for γ values in the interval 1.6–1.9. This output should not be overstressed, as it has been known for a long time that the average distance between nearby galaxies D_g and the galaxy correlation length r_g have close values. In the case of galaxies, there is little evidence of a linear relation between r_g values and D_g values for intrinsical magnitude limited samples. However, as galaxy luminosities essentially arise from stellar populations, they may be subject to significant variations, as their stellar and dust contents evolve on time scales certainly longer than observational times (see, e.g., Silva et al 1999). It is possible that the proximity of r_g and D_g indicates that, averaging luminosities over suitable time scales, a relation similar to clusters holds also for galaxies.

In the case of galaxy clusters, Dalton et al (1997) investigated the scaling properties of the angular functions for two different samples with apparent magnitudes $m > 18.94$ and $19.3 < m < 19.5$, for which they had obtained suitable $U(\lambda)$ expressions. Assuming a constant spatial clustering length r_c , they obtained that the predicted scaling properties hold, while $\gamma = 2.14$ and $r_c = 14.3 h^{-1}\text{Mpc}$. According to this work, finding a fair scaling does not require a constant r_c and, henceforth, any conclusion based on its existence is somehow premature.

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