Can one validly use classical statistical inference in Open Quantum Systems?

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Abstract. A major problem to perform statistical inference in open quantum systems is the perturbation induced by the measurement process. However, at least theoretically, a suitable choice of the measurement process could provide a consistent approach through classical stochastic processes. This work proposes a method to perform statistical inference on open quantum systems represented by quantum Markov semigroups having a suitable classical reduction. The method is based on measurements associated to observables generating invariant abelian algebras.

1. Introduction

At this time, the advances in experimental physics and computer science allow us to obtain large amount of data, in a short period of time, to measure diverse systems modeled by quantum mechanics. For that reason, there exists the necessity to perform statistical inference in this field. A major inconvenience in this context is perturbations induced on the system by observations which may dramatically change its nature. This is a very serious difficulty which attacks the proper foundation of statistical analysis. To face this foundational challenge, a number of authors have generalized some statistical inference concepts to a non commutative framework, namely, Barndorff-Nielsen, Gill, Jupp, [12]. They emphasize the importance of concepts like measurement and instrument. The measurement represents
the observed physical magnitude, while the instrument is used to obtain a measurement and its action changes the system under observation. As a result, a non commutative inference is proposed. Although this is a valid attempt to overcome formal mathematical inconsistencies in quantum statistical inference, no clues are given to deal with concrete physical dynamics. In particular, the question of space and time scales in observation of a given dynamics are not considered. It is well-known, for instance, that decoherence phenomena appear in open quantum systems after an extremely short time of observation ($10^{-13} - 10^{-15}$ sec. for microscopic systems and $10^{-37} - 10^{-39}$ sec. for macroscopic bodies). Decoherence is connected with the collapse of wave functions, or, equivalently, off-diagonal terms of a given state (density matrix) are lost in the evolution of the quantum system. Thus, pure quantum inference seems far to be practically feasible, given that diagonalized density matrices are interpreted as classical states (probability measures) emerged in the system evolution.

However, this “emergence of a classical world” enables to perform classical inference on open quantum systems. Indeed, as it has been pointed out by several authors, among them Accardi, Lu and Volovich (see chapters 4 and 5 of [1], and the references in [17]), the construction of a mathematical model for an open quantum system through the so-called historically “Markov approximation” naturally leads to a Quantum Markov Semigroup (QMS) with a very peculiar structure. The dissipative part of the QMS-generator commutes with the so-called Hamiltonian part of it. That means that there exists a commutative sub-algebra of the algebra of observables which remains invariant under the action of the QMS. As a result, the semigroup restricted to that Abelian sub-algebra is simply a classical Markov semigroup which is associated to a classical stochastic process. That classical stochastic process gives an account of the evolution of the dissipative part of the phenomenon through a remarkable particular observable, opening a way to perform classical statistical inference on the model. This is the key idea ruling the current report.

This article considers open quantum systems represented by quantum Markov semigroups (QMS). The main goal consists in finding commutative algebras, generated by measurements on the system, which remain invariant under the semigroup action. As a result, the semigroup restricted to those algebras generate classical semigroups, known as classical reductions or unravelings of the given QMS. Thus, classical statistical inference is conceivable as soon as there exists a measurement which reduces the semigroup. Measurements are determined by the spectral measure of an observable $K$ and the von Neumann algebra generated by $K$ is the commutative algebra to be found. In some cases is possible to find more than one observable or measure-
ment that reduces the semigroup. In each one of those reductions, different classical estimation procedures can be used to identify the parameters of the dissipative part of the generator. In addition to that, this work allows to establish a connection between two different classical processes (discontinuous and continuous) and their own statistical inference theories through a quantum probability model.

It worth mentioning that a quantum Markov semigroup, in a sense, corresponds to the contraction (projection) of a quantum flow onto the algebra of endomorphisms of the main system. The quantum flow instead, is defined on the bigger algebra of endomorphisms of the total Hilbert space (which includes both, the system and the reservoir variables). The stochastic limit approach developed by Accardi, Lu and Volovich in [1] deals with this quantum flow, obtained from the total quantum dynamics via a suitable rescaling. In other words, the quantum flow is a dilation of the quantum Markov semigroup. As a result, classical stochastic processes can be obtained from their quantum counterparts at two different levels. Firstly, via the classical reduction of the quantum Markov semigroup by a sub-algebra of the system. We should classify these as inner reductions. Secondly, classical stochastic flows appear as restrictions of quantum flows on invariant Abelian sub-algebras of the total algebra (system plus reservoir) too. Here we are interested in the inner classical reduction since we assume that the available observables are defined on the main system.

The paper is organized as follows. Section 2 introduces the basic concepts that will be needed throughout the paper, and it is aimed at providing a quick start to non specialists. Section 3 contains an account on sufficient conditions to obtain classical reductions. Applications are considered in section 4. Finally, in section 5 different classical inferential methods are applied to particular examples inspired from Quantum Optics.

2. Basic Definitions and Results.

A Quantum Markov Semigroup (QMS) arises as the natural non commutative extension of the well known concept of Markov semigroup defined on a classical probability space. The motivation for studying a non commutative theory of Markov semigroups came, firstly, from Physics. They represent a mathematical model to describe the loss-memory evolution of a microscopic system in accordance with the quantum uncertainty principle. Consequently, the roots of the theory go back to the first researches on the so called open quantum systems (see [2] for an account of the theory), and have found its main non commutative tools in much older abstract results like the characterization of completely positive maps due to Stinespring [19].
Indeed, complete positivity contains a deep probabilistic notion expressed in the language of operator algebras: it is the core of mathematical properties of (regular versions of) conditional expectations. Thus, complete positivity appears as a keystone in the definition of a QMS.

Also, in classical Markov Theory, topology plays a fundamental role which goes from the basic setting of the space of states up to continuity properties of the semigroup. In particular, Feller property allows to obtain stronger results on the qualitative behavior of a Markov semigroup. In the non commutative framework, Feller property is expressed as a topological and algebraic condition. Namely, a classical semigroup satisfying the Feller property on a locally compact state space leaves invariant the algebra of continuous functions with compact support, which is a particular example of a $C^*$-algebra.

A non commutative version of Markov semigroups requires two basic ingredients: firstly, a $^*$-algebra $\mathcal{A}$, that means an algebra endowed with an involution $^*$ which satisfies $(a^*)^* = a$, $(ab)^* = b^*a^*$, for all $a, b \in \mathcal{A}$, in addition we assume that the algebra contains a unit $1$; and secondly, we need a semigroup of completely positive maps from $\mathcal{A}$ to $\mathcal{A}$ which preserves the unit. We will give a precise meaning to this below. We remind that positive elements of the $^*$-algebra are of the form $a^*a$, $(a \in \mathcal{A})$. A state $\varphi$ is a linear map $\varphi : \mathcal{A} \to \mathbb{C}$ such that $\varphi(1) = 1$, and $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$. An algebraic probability space is a couple $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a $^*$-algebra and $\varphi$ a state defined on it. In particular, if $\mathcal{H}$ is a separable complex Hilbert space and $\mathcal{A}$ is taken as the algebra of all bounded linear endomorphisms of $\mathcal{H}$, $L(\mathcal{H})$ say, then it is well-known that any normal state $\varphi$ is represented as $\varphi(a) = \text{tr} (\rho a)$ where $\rho$ is a positive trace-class operator with unit trace referred as a density operator (see [14] for instance). Thus, in that case the state is fully characterized by $\rho$ and the algebraic probability space will be denoted as $(\mathcal{H}, \rho)$ and referred as a von Neumann probability space. $L(\mathcal{H})$ is a particular case of a von Neumann algebra.

**Definition 1** Let $\mathcal{A}$ be a $^*$-algebra and $\mathcal{P} : \mathcal{A} \to \mathcal{A}$ a linear map. $\mathcal{P}$ is completely positive if for any finite collection $a_1, \ldots, a_n, b_1, \ldots, b_n$ of elements of $\mathcal{A}$ the element

$$\sum_{i,j} a_i^* \mathcal{P}(b_i^* b_j) a_j$$

is positive.

If $\mathcal{M}$ is a von Neumann algebra on a complex separable Hilbert space $\mathcal{H}$, its predual is denoted $\mathcal{M}_*$. The predual contains in particular all the normal states. As a rule, we will only deal with normal states $\varphi$ for which there exists a density matrix $\rho$, that is, a positive trace-class operator of $\mathcal{H}$ with unit trace, such that $\varphi(a) = \text{tr} (\rho a)$ for all $a \in \mathcal{A}$. 

Definition 2 A quantum sub-Markov semigroup, or quantum dynamical semigroup (QDS) on a \( \ast \)-algebra \( \mathcal{A} \) which has a unit \( 1 \), is a one-parameter family \( \mathcal{T} = (\mathcal{T}_t)_{t \in \mathbb{R}_+} \) of linear maps of \( \mathcal{A} \) into itself satisfying

(M1) \( \mathcal{T}_0(x) = x \), for all \( x \in \mathcal{A} \);

(M2) Each \( \mathcal{T}_t(\cdot) \) is completely positive;

(M3) \( \mathcal{T}_t(\mathcal{T}_s(x)) = \mathcal{T}_{t+s}(x) \), for all \( t, s \geq 0 \), \( x \in \mathcal{A} \);

(M4) \( \mathcal{T}_t(1) \leq 1 \) for all \( t \geq 0 \).

A quantum dynamical semigroup is called quantum Markov (QMS) if \( \mathcal{T}_t(1) = 1 \) for all \( t \geq 0 \).

If \( \mathcal{A} \) is a \( C^\ast \)-algebra, then a quantum dynamical semigroup is uniformly (or norm) continuous if it additionally satisfies

(M5) \( \lim_{t \to 0} \sup_{\|x\| \leq 1} \|\mathcal{T}_t(x) - x\| = 0 \).

If \( \mathcal{A} \) is a von Neumann algebra, (M5) is usually replaced by the weaker condition

(M5\( \sigma \)) For each \( x \in \mathcal{A} \), the map \( t \mapsto \mathcal{T}_t(x) \) is \( \sigma \)-weak continuous on \( \mathcal{A} \), and \( \mathcal{T}_t(\cdot) \) is normal or \( \sigma \)-weak continuous.

The generator \( \mathcal{L} \) of the semigroup \( \mathcal{T} \) is then defined in the \( w^\ast \) or \( \sigma \)-weak sense. That is, its domain \( D(\mathcal{L}) \) consists of elements \( x \) of the algebra for which the \( w^\ast \)-limit of \( t^{-1}(\mathcal{T}_t(x) - x) \) exists as \( t \to 0 \). This limit is denoted then \( \mathcal{L}(x) \).

The predual semigroup \( \mathcal{T}_\ast \) is defined on \( \mathcal{M}_\ast \) as \( \mathcal{T}_t(\varphi)(x) = \varphi(\mathcal{T}_t(x)) \) for all \( t \geq 0 \), \( x \in \mathcal{M} \), \( \varphi \in \mathcal{M}_\ast \). Its generator is denoted \( \mathcal{L}_\ast \).

In which follows, quantum statistical concepts will be defined within the framework of a von Neumann probability space.

Definition 3 Let \((\mathfrak{h}, \rho)\) be a von Neumann probability space. We say that \( M \) is a measurement of the system, in the state \( \rho \), taking values in the measurable space \((\Omega, \mathcal{F})\), if \( M \) is a function from the \( \sigma \)-algebra \( \mathcal{F} \) to \( \mathbb{R}\mathfrak{h}^+ \), the set of all non negative self-adjoint operators in \( \mathfrak{h} \), with the following properties:

1. \( M(\Omega) \) is the identity operator \( 1 \).

2. \( \sum_{i=1}^\infty M(A_i) = M(A) \), for all sequence \( \{A_1, A_2, \ldots\} \) of disjoint elements in \( \mathcal{F} \) satisfying \( A = \bigcup_{i=1}^\infty A_i \).

The infinite sum is defined in the sense of weak convergence of operators.
The result of a measurement $M$ in the system with state $\rho$ is a random variable $X$, with values in $(\Omega, \mathcal{F})$, with the following probability law:

$$P(X \in A) = \text{tr} (\rho M(A)).$$

Suppose that a Quantum Markov Semigroup $\mathcal{T}$ is given on the algebra $\mathcal{L}(\mathcal{H})$. A measurement compatible with $\mathcal{T}$ is defined as follows.

**Definition 4** A measurement on the space $(\Omega, \mathcal{F})$ is **compatible** with the QMS $(\mathcal{T}_t)_{t \geq 0}$, if and only if

$$[\mathcal{T}_t(M(A)), M(B)] = 0, \quad (1)$$

for all $t \geq 0$, and all $A, B \in \mathcal{F}$.

**Remark 1** In particular, an interesting class of measurements is provided by the spectral family of a self-adjoint operator. In that case, each operator $M(A)$ is a projection. More precisely, $M(A) = M(A)^* = M(A)^2$, for all $A \in \mathcal{F}$.

In which follows, our goal will consists in finding observables whose spectral families remain invariant under the action of a given quantum Markov semigroup.

### 3. Compatible measurements and classical reductions

**Definition 5** Let be given a self-adjoint operator $K$ and denote $M$ its spectral measure. We say that the measurement $M$ (or the operator $K$), **reduces classically** a quantum Markov semigroup $(\mathcal{T}_t)_{t \geq 0}$, if:

$$\mathcal{T}_t(W^*(M)) \subset W^*(M), \quad \text{for all } t \geq 0,$$

where $W^*(M) = W^*(K)$ is the von Neumann algebra generated by the spectral measure (respectively by $K$) via the Spectral Theorem. In other words, the algebra $W^*(K)$ is invariant under the action of the semigroup.

**Proposition 1** Let $M$ be the spectral measure of a self-adjoint operator $K$, where $W^*(K)$ is maximal, i.e. $W^*(K) = W^*(K)'$. The measurement $M$ is compatible with the semigroup $\mathcal{T}$ if and only if it reduces classically $\mathcal{T}$.

**Proof.** If the semigroup is classically reduced by $M$ then, in particular $\mathcal{T}_t(M(A))$ belongs to $W^*(K)$, so that it commutes with $M(B)$ for all $A, B \in \mathcal{F}$.
On the other hand, if a measurement $M$ is compatible with the semigroup, then for every pair of simple functions $f = \sum_{i=1}^{n} a_i 1_{A_i}$ and $g = \sum_{j=1}^{m} b_j 1_{B_j}$ we have:

$$[\mathcal{T}_t f(K), g(K)] = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j [\mathcal{T}_t M(A_i), M(B_j)] = 0.$$ 

And therefore, $\mathcal{T}_t(f(K)) \in W^*(K)$, for any simple function $f$ and $t \geq 0$, since $W^*(K) = W^*(K)'$ by hypothesis. Now, if $f \geq 0$ is any bounded measurable function, take any sequence of simple functions $(f_n)_{n \in \mathbb{N}}$ such that $f_n \uparrow g$. Therefore $f_n(K)$ is an increasing sequence (in the order of positive operators) which converges weakly to $f(K)$. Since $\mathcal{T}_t(\cdot)$ is normal, the sequence $(\mathcal{T}_t(f_n(K)))_n$ increases to $\mathcal{T}_t(f(K))$ in $W^*(K)$ so that $\mathcal{T}_t(W^*(K)) \subset W^*(K)$, for any $t \geq 0$, and the semigroup is classically reduced by $K$. \qed

**Remark 2** Let $M$ be the spectral measure of a self-adjoint operator $K$. The algebra $W^*(M) = W^*(K)'$ is maximal if and only if it has a cyclic vector. In other words, if there exists $\xi \in \mathfrak{h}$ such that $\{ f(K)\xi : f \in \mathfrak{b}\mathfrak{F} \}$ is dense in $\mathfrak{h}$ (see e.g. [14], Thm 4.77, page 183).

The following result is another simple application of the von Neumann Spectral Theorem, a detailed proof is given in [16],(see also [17]).

**Theorem 1** Consider a Quantum Markov Semigroup $(\mathcal{T}^\theta_t)_{t \geq 0}$, depending on some parameter $\theta$, given together with an initial state $\rho$. Let $M$ be the spectral measure of a self-adjoint operator $K$, which reduces the semigroup. Then there exists a classical markovian system $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_\theta, (X_t)_{t \geq 0})$ such that for every bounded borelian function $f$ we have

$$\mathcal{T}^\theta_t f(x) = \mathbb{E}^\theta_{\rho}(f(X_t)/X_0 = x),$$

for all $t \geq 0, x \in \text{Sp}(K)$, where $\mathcal{T}^\theta_t$ is defined by the restriction of $\mathcal{T}^\theta_t$ to $W^*(M)$ . In particular we have that

$$\mathbb{P}_\theta(X_t \in A) = \text{tr} \left( \rho \mathcal{T}^\theta_t(M(A)) \right).$$

Now, the problem is finding sufficient conditions for a measurement to be compatible. For that we need to introduce some notations. Given a bounded operator $Q$, we denote $\text{ad}_Q(\cdot) = [\cdot, Q]$. Furthermore we denote $\text{ad}_Q^2 = \text{ad}_Q(\text{ad}_Q(\cdot))$. Recall that $\mathcal{L}$ is the infinitesimal generator of the QMS with domain $D(\mathcal{L})$.

**Theorem 2** Let $(\mathcal{T}_t)_{t \geq 0}$ be a QMS on $\mathfrak{B}(\mathfrak{h})$. Let $M$ be the spectral measure of a self adjoint operator $K$ (not necessarily bounded), and assume that $W^*(M)$ is maximal. Then the following statements are equivalents:
(i) $M$ is compatible with the semigroup.

(ii) The manifold $D(\mathcal{L}) \cap W^*(M)$ is not trivial and for each $x \in D(\mathcal{L}) \cap W^*(M)$, we have $\mathcal{L}(x) \in W^*(M)$.

(iii) There exists a classical Markovian semigroup $(T_t)_{t \geq 0}$ over $Sp(K)$ such that for all $f \in L(\mathcal{L}(K))$,

$$T_t(f(K)) = \int_{Sp(K)} M(dx) T_t f(x).$$

In particular, if we have a QMS whose formal generator is:

$$\mathcal{L}(x) = i[H,x] - \frac{1}{2} \sum_k (L_k^* L_k x - 2 L_k^* x L_k + x L_k^* L_k)$$

and there exists a simple measurement $M$ such that the following two conditions are satisfied,

(i) $ad_{M(A)}^0(H) = 0$ for all $A \in F$ and

(ii) $ad_{M(A)}(L_k) = c_k(A)L_k$, where $c_k(A) = c_k^*(A)$, and $ad_{M(A)}(c_k(A)) = 0$ for all $A \in F$ and all $k \in \mathbb{N}$.

Then the measurement $M$ is compatible, i.e., it reduces classically the QMS.

Proof. If $M$ is simple and $W^*(M)$ is maximal, then $\mathcal{L}(W^*(M)) \subset W^*(M)$ if and only if $\mathcal{L}(x) \in W^*(M)'$ for any element $x \in W^*(M)$. (For details see [15]).

In particular, if the generator satisfies conditions (i) and (ii), we have
that:

\[
[\mathcal{L}(M(A)), M(A)] = [i[H, M(A)] - \frac{1}{2} \sum_k (L_k^a L_k x - 2L_k^a xL_k + xL_k^a L_k), M(A)] \\
= i\text{ad}_{M(A)}^0(H) - \frac{1}{2} \sum_k \left( [L_k^a L_k M(A), M(A)] - 2[L_k^a M(A)L_k, M(A)] + [M(A)L_k^a L_k, M(A)] \right) \\
= -\frac{1}{2} \sum_k \left( [L_k^a L_k, M(A)]M(A) - 2[L_k^a M(A)L_k, M(A)] + M(A)[L_k^a L_k, M(A)] \right) \\
= -\frac{1}{2} \sum_k \left( (L_k^a c_k(A)L_k - L_k^a c_k(A)L_k)M(A) - 2(L_k^a M(A)c_k(A)L_k - L_k^a c_k(A)M(A)L_k) + M(A)(L_k^a c_k(A)L_k - L_k^a c_k(A), M(A)L_k) \right) \\
= \sum_k (L_k^a M(A)c_k(A)L_k - L_k^a c_k(A)M(A)L_k) = 0
\]

\[
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\]

4. Some examples

4.1. Laser Model

The laser model (Light amplification by stimulated emission of radiation) describes an atom crossing an ideal resonator (a high quality cavity), and its energy attains two levels only (a so called two-level atom). Excitations of a mode of the quantized radiations field in the resonator correspond to photons staying in the cavity: they have a finite life-time and they interact with the incident atom.

Physical parameters are \( A \), the energy decay rate in the cavity; \( \nu \) the number of thermal excitations; and \( \omega \), the natural frequency. Consider \( \hbar = l^2(\mathbb{N}) \) with its canonical orthonormal basis \((e_n)_{n \in \mathbb{N}}\) and let the creation operator \( a^\dagger \) and the annihilation \( a \) be defined by: \( a^\dagger e_n = \sqrt{n+1}e_{n+1}, ae_0 = 0, ae_n = \sqrt{n}e_{n-1} \), for \( n \geq 1 \) and let \( \mathcal{N} = a^\dagger a \) be the number operator, so that \( \mathcal{N}e_n = ne_n \), for \( n \geq 1 \). The laser QMS is determined by the formal generator

\[
\mathfrak{L}(x) = i[\omega \mathcal{N}, x] - \frac{1}{2} A(\nu + 1)(a^\dagger ax - 2a^\dagger xa + xa^\dagger a)
\]
− \frac{1}{2} A\nu (aa^\dagger x - 2axa^\dagger + xaa^\dagger), \quad (2)

In [7] the following property has been proved.

**Proposition 2** This QMS is recurrent and ergodic for all $A, \nu \geq 0$.

Now, consider the spectral measure of the number operator, that is, $M(\{n\}) = |e_n\rangle \langle e_n|$. We can see that the properties (i) and (ii) of the theorem 2 are satisfied, that is

(i) Obviously $\text{ad}_{M(A)}(N) = 0$ since $M$ is the spectral measure of $N$;

(ii) $\text{ad}_{M(\{n\})}(a^\dagger) = [a^\dagger, |e_n\rangle \langle e_n|] = (|e_{n+1}\rangle \langle e_{n+1}| - |e_n\rangle \langle e_n|) a^\dagger = c^\dagger(\{n\})a^\dagger$ and $\text{ad}_{M(\{n\})}(a) = [a, |e_n\rangle \langle e_n|] = (|e_{n-1}\rangle \langle e_{n-1}| - |e_n\rangle \langle e_n|) a = c(\{n\})a$.

On the other hand, it is known that the von Neumann algebra generated by the number operator is maximal, or equivalently $W^*(M) = W^*(N) = W^*(N)'$.

Then, by Theorem 2 the number operator reduces the QMS. In addition, for any element $f(N) \in W^*(N)$ it holds

$$
L(f(N)) = i[\omega N, f(N)] - \frac{1}{2} A(\nu + 1)(a^\dagger af(N) - 2a^\dagger f(N)a + f(N)a^\dagger a) - \frac{1}{2} A\nu (aa^\dagger f(N) - 2af(N)a^\dagger + f(N)aa^\dagger).
$$

Therefore,

$$
L(f(N))e_n = (A\nu(n + 1)(f(n + 1) - f(n)) - A(\nu + 1)n(f(n) - f(n - 1)))e_n.
$$

This shows that the infinitesimal generator of $T_t$, the classical semigroup associated to $T_t|_{W^*(N)}$, is a Markov semigroup of a birth and death process with $\lambda_n = A\nu(n + 1)$ and $\mu_n = A(\nu + 1)n$ their birth and death rates respectively.

4.2. A Micromaser Model

A micromaser model (microwave amplification by stimulated emission of radiation) is represented here by a QMS $T_t$, with the following formal generator.

$$
\mathfrak{L}(x) = -\frac{\mu^2}{2}(a^\dagger ax - 2a^\dagger xa + xaa^\dagger) - \frac{\lambda^2}{2}(aa^\dagger x - 2axa^\dagger + xaa^\dagger) + R^2 \cos(\phi \sqrt{aa^\dagger})x \cos(\phi \sqrt{aa^\dagger}) + R^2(\sin(\phi \sqrt{aa^\dagger})S^x S \sin(\phi \sqrt{aa^\dagger}) - x), \quad (4)
$$

with $\mu^2 = \mu_0 \eta_T$ and $\lambda^2 = \gamma_0(n_T + 1)$.

In [7] the following properties for this model have been proved.
**Proposition 3** This QMS is ergodic and if $\lambda^2 < \mu^2$, then the QMS is recurrent. Furthermore, in the following cases we have:

- if $\lambda^2 > \mu^2$, the QMS is transient,
- if $\lambda^2 = \mu^2$, we have two possibilities: in the case $R\phi = 0$, the QMS is recurrent; if $R\phi \neq 0$, we obtain that the QMS is transient.

Similarly to the Laser model, this one can be reduced to a birth and death process too, with birth and death rates given by:

$$
\lambda_k = \lambda^2(k + 1) + R^2 \sin^2(\phi\sqrt{k + 1}), \quad \mu_k = \mu^2 k.
$$

(5)

For details see [7].

On the other hand, if we consider $h = L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \pi^{-1/4} \exp(-x^2/2)dx)$ and the generator given in (4), in the case $R = 0$, then:

$$
\mathcal{L}(x) = -\frac{\mu^2}{2} (a^\dagger ax - 2a^\dagger xa + xa^\dagger a) - \frac{\lambda^2}{2} (aa^\dagger x - 2axa^\dagger + xaa^\dagger),
$$

where $a$ and $a^\dagger$ are the annihilation and creator operators respectively, which are defined by:

$$
a^\dagger = \frac{q - ip}{\sqrt{2}}, \quad a = \frac{q + ip}{\sqrt{2}}.
$$

Consider $q$ and $p$ the position and momentum operators given by: $qf(s) = sf(s)$, $pf(s) = -if'(s)$, and the projection $E_\lambda$ defined by

$$
E_\lambda f(s) = \begin{cases} 
    f(s), & s < \lambda \\
    0, & s \geq \lambda
\end{cases}.
$$

It is known that $q$ has the following spectral decomposition:

$$
q = \int_{\mathbb{R}} \lambda dE_\lambda.
$$

In other words, the measurement $M(A)$ associated to this operator can be written as:

$$
M(A)f(s) = \begin{cases} 
    f(s), & s \in A \\
    0, & s \in A^c
\end{cases}.
$$

It is not difficult to compute the following commutation relations: $[a, f(q)] = (\sqrt{2})^{-1} f'(q)$ and $[a^\dagger, f(q)] = -(\sqrt{2})^{-1} f'(q)$ and it is known that $W^*(q)$ is maximal.

Furthermore, the following computation shows both, that $q$ reduces the semigroup and also that the generator restricted to $W^*(q)$ corresponds to the infinitesimal generator of an Ornstein-Ulhenbeck process.
\[ L(f(q)) = -\frac{\mu^2}{2} \left( a^+a f(q) - 2a^+f(q)a + f(q)a^+a \right) \]
\[ -\frac{\lambda^2}{2} \left( aa^+f(q) - 2af(q)a + f(q)aa^+ \right), \]
\[ = -\frac{\mu^2}{2} \left( a^+[a,f(q)] - [a^+f(q)]a \right) - \frac{\lambda^2}{2} \left( a[a^+,f(q)] - [a,f(q)]a^+ \right), \]
\[ = -\frac{\mu^2}{2} \left( a^+(\sqrt{2})^{-1}f'(q) + (\sqrt{2})^{-1}f'(q)a \right) \]
\[ -\frac{\lambda^2}{2} \left( -a(\sqrt{2})^{-1}f'(q) - (\sqrt{2})^{-1}f'(q)a^+ \right), \]
\[ = -\frac{\mu^2}{2\sqrt{2}} \left( a^+f'(q) + f'(q)a \right) - \frac{\lambda^2}{2\sqrt{2}} \left( -af'(q) - f'(q)a^+ \right), \]
\[ = -\frac{\mu^2}{2\sqrt{2}} \left( f'(q)a^+ - (\sqrt{2})^{-1}f''(q) + f'(q)a \right) \]
\[ + \frac{\lambda^2}{2\sqrt{2}} \left( f'(q)a + (\sqrt{2})^{-1}f''(q) + f'(q)a^+ \right), \]
\[ = -\frac{\mu^2}{2\sqrt{2}} \left( f'(q)\frac{2q}{\sqrt{2}} - (\sqrt{2})^{-1}f''(q) \right) \]
\[ + \frac{\lambda^2}{2\sqrt{2}} \left( \frac{2q}{\sqrt{2}} f'(q) + (\sqrt{2})^{-1}f''(q) \right), \]
\[ = \frac{\lambda^2 - \mu^2}{2} qf'(q) + \frac{\lambda^2 + \mu^2}{4} f''(q). \]

Similarly, we can deduce that the momentum operator \( p \) reduces classically the QMS to another Ornstein-Uhlenbeck process.

5. Statistics on classical reductions and simulations

Now, statistical inference methods for classical stochastic processes can be applied to the reduced dynamics, in order to estimate the parameters involved in the dissipative part of the quantum generator. In which follows, we show how to apply the above methods to both, birth and death, and Ornstein-Uhlenbeck processes derived from classical reductions of QMS.

5.1. Birth and death processes

We consider a birth and death process \((X_t)_{t \geq 0}\) with generator given as:

\[ L_\theta f(n) = \lambda_n(f(n + 1) - f(n)) - \mu_n(f(n) - f(n - 1)), \]
where \( \lambda_n = A\nu(n + 1) \) and \( \mu_n = A(\nu + 1)n \), with constants \( A, \nu > 0 \). The process has the following form:

\[
X_t = \sum_{n=0}^{\infty} \xi_n 1_{[T_n, T_{n+1}]}(t),
\]

(6)

where \( \{\xi_n\}_{t \geq 0} \) is a Markov chain with the following transition probability:

\[
\Pi(n, n + 1) = \frac{\lambda_n}{\lambda_n + \mu_n} = 1 - \Pi(n, n - 1),
\]

and \( \{T_n\}_{n \leq 0} \) are the jump times of the process whose increments are independent and exponentially distributed with rate \( q(\xi_n) = \lambda\xi_n + \mu\xi_n \).

It is known that this process is recurrent with invariant measure given by:

\[
\mu(\{n\}) = \frac{1}{\nu + 1} \left( \frac{\nu}{\nu + 1} \right)^n.
\]

(7)

Note that \( X_t \) is decomposed as the difference of two increasing processes, which can be written as: \( X_t = X_0 + N_t^+ - N_t^- \) with \( X_0 = \xi_0 \),

\[
N_t^+ = \sum_{n=1}^{\infty} (\xi_n - \xi_{n-1}) 1_{[T_n, \infty]}(t) 1_{\{\xi_n - \xi_{n-1} > 0\}},
\]

and

\[
N_t^- = \sum_{n=1}^{\infty} (\xi_{n-1} - \xi_n) 1_{[T_n, \infty]}(t) 1_{\{\xi_n - \xi_{n-1} < 0\}}.
\]

(\( N_t^+ \))\(_{t \geq 0} \) and (\( N_t^- \))\(_{t \geq 0} \) are counting processes with the following predictable compensators:

\[
\tilde{N}_t^+ = \int_0^t A\nu(X_{s^-} + 1) \, ds \quad \text{and} \quad \tilde{N}_t^- = \int_0^t A(\nu + 1)X_{s^-}I_{\{X_{s^-} > 0\}} \, ds.
\]

In figure 1 we show the graphs for given values of the parameters, changing the initial value of the processes. We can see that the initial value does not disturb the estimation, since the processes fast decrease to 0. This fact allows us, to consider processes starting from 0 in which follows.

For this model, our objective is to estimate the parameter \( \theta = (A, \nu) \). And using the Nelson–Aalen estimator (see [3]), we obtain the following:

\[
\widehat{A}\nu = \frac{1}{t} \int_0^t \frac{1}{(X_{s^-} + 1)} \, dN_s^+ \quad \text{and} \quad \widehat{A}(\nu + 1) = \frac{1}{t} \int_0^t \frac{I_{\{X_{s^-} > 0\}}}{X_{s^-}} \, dN_s^-.
\]

(8)
Fig. 1: \( A = 1, \nu = 1 \) and \( T = 100 \)
The first one is a consistent estimator, but the second is not a good estimator. This fact suggests that the new estimator should be based in the invariant measure as
\[
\frac{1}{\nu + 1} = \frac{1}{t} \int_0^t I\{X_s = 0\}. \tag{9}
\]
The Ergodic Theorem provides the consistency of the previous estimator. Additionally, we can compute the asymptotic convergence of both estimators.

**Proposition 4** Under the previous hypotheses, the estimator proposed defined in (8) satisfies:
\[
\sqrt{n} \left( \hat{A}^{(n)}(t) - A^+ \right) \overset{D}{\rightarrow} \sqrt{A \log(\nu + 1)} W_t,
\]
where $W$ is the Wiener processes.

The proof consists in applying the Martingale Central Limit Theorem.

**Proposition 5** The estimator proposed in 9 has the following property:
\[
\frac{1}{\sqrt{n}} \int_0^t \left( I\{X_s = 0\} - \frac{1}{\nu + 1} \right) ds \overset{D}{\rightarrow} \sqrt{\frac{2}{A(\nu + 1)^2} \log(\nu + 1)} W_t,
\]
for all $t > 0$.

In Table 1, the birth and death processes is simulated for $A = 1$ and $\nu = 1$. We can observe that when the time observation is increasing, the standard deviation of the error is significantly less. In an observation time $T = 10000$ the standard deviation is less than 0.05.

Now, in Table 2, we show the estimations obtained for $A$, $\nu$ y $A\nu$ for different values of $A$ and $\nu$ with theirs standard deviations of the error. This was performed for $T = 10000$.
5.2. Diffusion processes

There have been numerous researches on parameter estimation in diffusion processes proposed so far. Particularly, we stress the works of Lipster and Shiryaev [11] and Kutoyanz [10], where the parameters involved in the diffusion coefficient are estimated. These results have been improved by Florens-Zmirou [8], Dacunha-Castelle [6] and Sorensen [18]. Sorensen proposes a method in two steps. First, the diffusion coefficient is estimated. Secondly, a maximum likelihood estimation of the drift is obtained, assuming that the diffusion coefficient is known.

Here below, the parameters are estimated by two different methods. The first one uses the invariant measure, while the second follows Sorensen’s method.

Consider an Ornstein–Uhlenbeck process like the one obtained by classical reduction of the maser model, that is with generator:

$$Lf(x) = \frac{\lambda^2 + \mu^2}{4} f''(x) + \frac{\lambda^2 - \mu^2}{2} x f'(x)$$  \hspace{1cm} (10)

where $\lambda^2 = \gamma_c (nT + 1)$ and $\mu^2 = \mu_c nT$. The objective is to estimate the parameters $\gamma_c, \mu_c$ and $nT$. We can see that this model is not identified by the parameters of interest, although, the parameters $\lambda$ and $\mu$ identify the model.

This generator corresponds to a process $\xi_t$, satisfying the following stochastic differential equation:

$$d\xi_t = \frac{\lambda^2 - \mu^2}{2} \xi_t dt + \sqrt{\frac{\lambda^2 + \mu^2}{2}} dW_t,$$

where $W_t$ denotes the Wiener process.

<table>
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<th>$A_\nu$</th>
<th>$T$</th>
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Table 2: The estimation proposed for different values of $A$ and $\nu$ when $T = 10000$ with their respective standard deviations.
The solution is given by
\[
\xi_t = \exp \left( \frac{(\lambda^2 - \mu^2)t}{2} \right) \left( \xi_0 + \sqrt{\frac{\lambda^2 + \mu^2}{2}} \int_0^t \exp \left( \frac{(\mu^2 - \lambda^2)s}{2} \right) dW_s \right).
\]

And its expectation and variance can be deduced using integration by parts for stochastic integrals, obtaining:
\[
E(\xi_t) = \exp \left( \frac{\lambda^2 - \mu^2}{2}t \right) E(\xi_0) \tag{11}
\]
\[
Var(\xi_t) = \exp \left( \frac{(\lambda^2 - \mu^2)t}{2} \right) Var(\xi_0) + \frac{\lambda^2 + \mu^2}{2(\mu^2 - \lambda^2)} (1 - \exp((\lambda^2 - \mu^2)t)) \tag{12}
\]

Similarly to the previous case, we are interested in estimating the parameter \( \theta = (\lambda^2, \mu^2) \), for that we take \( \theta_1 = \frac{\lambda^2 + \mu^2}{2(\mu^2 - \lambda^2)} \) and \( \theta_2 = \frac{\lambda^2 + \mu^2}{2} \).

It is known that the quadratic covariance process has the following property
\[
\sum \frac{(\xi_{k+1}/n - \xi_{k/n})^2}{\sum_k/n \leq T} \xrightarrow{P} \int_0^T \frac{\lambda^2 + \mu^2}{2} ds = T \frac{\lambda^2 + \mu^2}{2} \tag{13}
\]

Using this approximation, we propose the following as an estimator of \( \theta_2 \):
\[
\hat{\theta}_2 = \frac{1}{T} \sum \frac{(\xi_{k+1}/n - \xi_{k/n})^2}{\sum_k/n \leq T} \tag{14}
\]
By construction, this estimator is consistent for \( \theta_2 \).

Now if the diffusion coefficient is known, we can estimate the drift coefficient by two different methods. The first one uses the invariant probability measure, which in this case corresponds to the law of a normal random variable with mean 0 and variance \( \frac{\lambda^2 + \mu^2}{2(\mu^2 - \lambda^2)} \) suggesting that the following be taken as estimator of \( \theta_1 \):
\[
\hat{\theta}_1(t) = \frac{1}{t} \int_0^t \xi_s^2 ds \tag{15}
\]

**Proposition 6** The estimator proposed is consistent for \( \theta_1 \).

**Proof.** The result is a direct consequence of the Ergodic Theorem, which yields
\[
\frac{1}{t} \int_0^t \xi_s^2 ds \xrightarrow{a.s.} \frac{\lambda^2 + \mu^2}{2(\mu^2 - \lambda^2)}, \text{if } t \to \infty
\]

The Martingale Central Limit Theorem provides the following corollary:
Corollary 1

\[
\frac{1}{\sqrt{n}} \int_0^{nt} \left( \xi^2 - \frac{\lambda^2 + \mu^2}{2(\mu^2 - \lambda^2)} \right) ds \overset{d}{\to} \frac{\lambda^2 + \mu^2}{(\mu^2 - \lambda^2)\sqrt{\mu^2 - \lambda^2}} W_t.
\]

Proof. The proof is obtained by directly applying Theorem VIII.3.65 in [9], taking \( g(x) = \frac{x^2}{x^2 - \mu^2} \) and \( f(x) = L(g(x)) = x^2 - \frac{x^2 + \mu^2}{2(\mu^2 - \lambda^2)}. \)

The second way is assuming that \( \theta_2 \) is known, as in the previous case, and using the maximum likelihood method. By Girsanov’s Theorem, the likelihood is given by

\[
L_T(\theta) = \exp \left( \int_0^T \frac{(\lambda^2 - \mu^2)X_s}{2\theta_2} dX_s - \frac{1}{2} \int_0^T \frac{(\lambda^2 - \mu^2)^2X_s^2}{4\theta_2} ds \right).
\]

Therefore, the maximum likelihood estimator for \( \theta_1^\Delta = \frac{\lambda^2 - \mu^2}{2} \) is

\[
\hat{\theta}_1^\Delta = \frac{X_T^2 - \hat{\theta}_2 T}{2\int_0^T X_s^2 ds}.
\]

To evaluate the behavior of this estimator, table 3 shows the estimations obtained for \( \theta_1, \theta_2, \lambda^2 \) and \( \mu^2 \) and their standard deviations, using the first method. This is shown for different values of \( \lambda^2 \) and \( \mu^2 \) and an observation time \( T = 10000 \).

Remark 3 For estimators with no asymptotic error distribution, the bootstrap method is used.

Table 4 provides the estimations obtained for an observation time at \( T = 10000 \), using the maximum likelihood method.
Table 4: The estimation proposed for $\lambda^2$, $\mu^2$, $\theta_1^\Delta$ and $\theta_2$ when $T = 10000$ with their respective standard deviations.

<table>
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6. Conclusions and outlook

As shown in this paper, classical reductions of quantum Markov semigroups provide a method for applying classical inference to analyze the dissipative part of such dynamics. The main dynamics still beyond this approach. Indeed, if one observes a physical magnitude $x$ which is in the algebra generated by the Hamiltonian ruling the main dynamics, the part which corresponds to the main dynamics in the generator of the semigroup vanishes. Thus, other techniques have to be used to estimate parameters appearing in the Hamiltonian.

Moreover, when two classical reductions are available, for instance in the case of the maser model, a comparison between two estimators for the same parameter should be important to obtain. This is the subject of a current research, which will be published elsewhere.

The Markov approximation corresponds to the method used primarily by a number of physicists to obtain a reduced dynamics focused on the main system. Different procedures have been used to obtain this: namely, the Kac-Zwanzig projection method, the weak coupling limit and other phenomenological approaches. Currently, the stochastic limit (see [1]) synthesizes and extend the above methods. Indeed, the stochastic limit focuses on the behaviour of the quantum flow obtained by a suitable rescaling and limit of the total dynamics of the system and the reservoir. Clearly, the restriction of the quantum flow to an invariant Abelian algebra provides a classical stochastic flow. The question of whether this classical process could be used in statistics depends on the nature of the observable available to the statistician. The method used in this paper assume that available observables are those of the main system, while the use of the quantum flow and its invariant sub algebras involves the reservoir. For that reason, we assumed the semigroup given at the outset no matter which method (phenomenological or ab initio) could have been used to obtain its generator. Undoubtedly, reference [1]
could stimulate further research in classical reductions of quantum flows this time related to measurement theory and observables defined in wider algebras. This is a forthcoming programme which deserves the name of Quantum Dynamical Tomography.

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References


