



Research article

On the upper semicontinuity of global attractors for damped wave equations

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Abstract: We provide a new proof of the upper-semicontinuity property for the global attractors admitted by the solution operators associated with some strongly damped wave equations. In particular, we demonstrate an explicit control over semidistances between trajectories in the weak energy phase space in terms of the perturbation parameter. This result strengthens the recent work by Y. Wang and C. Zhong [7].

Keywords: Upper-semicontinuity; global attractor; strongly damped wave equation

Mathematics Subject Classification: 35B41, 35L71, 35Q74, 35L20

1. Introduction

In this short article, we revisit the recent work of [7] who examine the upper-semicontinuity properties of the family of global attractors associated with the strong damping perturbation of weakly damped wave equations. Such equations are used in modeling non-Hookean viscoelastic materials. Here, the strong damping term $-\varepsilon\Delta u_t$ present in such equations indicates that we are accounting for the strain *rate* in the material, in addition to other forces. The upper-semicontinuity result in [7] shows that the global attractors do not “blow-up” as the perturbation parameter vanishes. Hence, the asymptotic behavior of the solutions is stable. What we offer here improves this result by communicating that the difference of trajectories corresponding to the perturbation problem and the limit problem, emanating from the same initial data, can be estimated in terms of the perturbation parameter ε in the topology associated with the weak energy phase space of the model problems.

Let Ω be a bounded domain in \mathbb{R}^3 with boundary $\partial\Omega$ of class C^2 . We consider the semilinear strongly damped wave equation,

$$u_{tt} - \varepsilon\Delta u_t + u_t - \Delta u + f(u) = 0 \quad \text{in } (0, \infty) \times \Omega, \tag{1.1}$$

where $0 \leq \varepsilon \leq 1$ represents the diffusivity of the momentum. The equation is endowed with Dirichlet

boundary condition,

$$u|_{\partial\Omega} = 0 \quad \text{on} \quad (0, \infty) \times \partial\Omega, \quad (1.2)$$

and with the initial conditions

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \quad \text{at} \quad \{0\} \times \Omega. \quad (1.3)$$

For the nonlinear term, we assume $f \in C^2(\mathbb{R})$ satisfies the sign condition

$$\liminf_{|s| \rightarrow \infty} f'(s) > -\lambda_1, \quad (1.4)$$

where $\lambda_1 > 0$ denotes the first eigenvalue of the Dirichlet–Laplacian, and we assume the growth assumption holds, for all $s \in \mathbb{R}$,

$$|f''(s)| \leq \ell(1 + |s|), \quad (1.5)$$

for some positive constant ℓ . We will refer to equations (1.1)–(1.3) under assumptions (1.4)–(1.5) as Problem \mathbf{P}_ε , for $\varepsilon \in [0, 1]$.

It is now well-known that the model problems admit globally defined weak-solutions in the (weak) energy phase space

$$\mathcal{H}_0 := H_0^1(\Omega) \times L^2(\Omega)$$

and, for each $\varepsilon \in [0, 1]$, a global attractor \mathcal{A}_ε is compact in \mathcal{H}_0 and bounded in

$$\mathcal{H}_1 := (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega).$$

Furthermore, when $\varepsilon > 0$, the operator associated with the linear part of the abstract Cauchy problem generates an *analytic* semigroup on \mathcal{H}_0 . On these results we mention the following references [1–5].

The main result in this paper is the following:

Theorem 1.1. *The family of global attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon \in [0,1]}$ is upper-semicontinuous in the topology of \mathcal{H}_0 in the following explicit sense: there is a constant $C > 0$ independent of ε in which*

$$\text{dist}_{\mathcal{H}_0}(\mathcal{A}_\varepsilon, \mathcal{A}_0) := \sup_{a \in \mathcal{A}_\varepsilon} \inf_{b \in \mathcal{A}_0} \|a - b\|_{\mathcal{H}_0} \leq C \sqrt{\varepsilon}.$$

A word about notation: we will often drop the dependence on x and even t from the unknown $u(x, t)$ writing only u instead. The norm in the space $L^p(\Omega)$ is denoted $\|\cdot\|_p$ except in the common occurrence when $p = 2$ where we simply write the $L^2(\Omega)$ norm as $\|\cdot\|$. The $L^2(\Omega)$ product is simply denoted (\cdot, \cdot) . Other Sobolev norms are denoted by occurrence; in particular, since we are working with the homogeneous Dirichlet boundary conditions (1.2), in $H_0^1(\Omega)$, we will use the equivalent norm

$$\|u\|_{H_0^1} = \|\nabla u\|.$$

Given a subset B of a Banach space X , denote by $\|B\|_X$ the quantity $\sup_{x \in B} \|x\|_X$. In many calculations C denotes a *generic* positive constant which may or may not depend on several of the parameters involved in the formulation of the problem. Finally, for each $\varepsilon \in [0, 1]$, and $t \geq 0$, we denote by $S_\varepsilon(t)$ the semigroup of solution operators acting on \mathcal{H}_0 defined through the weak solution,

$$S_\varepsilon(t)(u_0(x), u_1(x)) := (u_\varepsilon(t, x; u_0, u_1), \partial_t u_\varepsilon(t, x; u_0, u_1)),$$

where u_ε here denotes the weak solution to Problem \mathbf{P}_ε .

The next section contains a proof of Theorem 1.1.

2. Continuity properties of the global attractors

Following [6, Section 10.8], the type of perturbation examined in this article is called *regular* because both classes of Problem \mathbf{P}_ε ($\varepsilon > 0$ and $\varepsilon = 0$) lie in the same phase space; in particular, the family of global attractors, $\{\mathcal{A}_\varepsilon\}_{\varepsilon \in [0,1]}$, lies in \mathcal{H}_0 . Hence, we will utilize [6, Theorem 10.16].

Proposition 2.1. *Assume that for $\varepsilon \in [0, \varepsilon_0)$ the semigroups S_ε each admit a global attractor \mathcal{A}_ε and that there exists a bounded set X such that*

$$\bigcup_{\varepsilon \in [0, \varepsilon_0)} \mathcal{A}_\varepsilon \subset X.$$

If in addition the semigroup S_ε converges to S_0 in the sense that, for each $t > 0$, $S_\varepsilon(t)x \rightarrow S_0(t)x$ uniformly on bounded subsets Y of the phase space H , i.e.,

$$\sup_{x \in Y} \|S_\varepsilon(t)x - S_0(t)x\|_H \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

then

$$\text{dist}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

We now arrive at our first result.

Lemma 2.2. *Let $T > 0$. There exists a constant $C = C(\|\mathcal{A}_\varepsilon\|_{\mathcal{H}_1}, T) > 0$ such that for all $\zeta_0 \in \mathcal{A}_\varepsilon$ and for all $t \in [0, T]$, there holds, for all $\varepsilon \in (0, 1]$,*

$$\|S_\varepsilon(t)\zeta_0 - S_0(t)\zeta_0\|_{\mathcal{H}_0} \leq C\sqrt{\varepsilon}. \quad (2.1)$$

Proof. Let B be a bounded set on \mathcal{H}_0 and $T > 0$. Let $\zeta_0 = (u_0, u_1) \in \mathcal{A}_\varepsilon$. For $t > 0$, let

$$\zeta^+(t) = (u^+(t), u_t^+(t)) \quad \text{and} \quad \zeta^0(t) = (u^0(t), u_t^0(t)),$$

denote the corresponding global solutions of Problem \mathbf{P}_ε and Problem \mathbf{P}_0 , respectively, on $[0, T]$, both with the (same) initial data ζ_0 . For all $t \in (0, T]$, set

$$\begin{aligned} \bar{\zeta}(t) &:= \zeta^+(t) - \zeta^0(t) \\ &= (u^+(t), u_t^+(t)) - (u^0(t), u_t^0(t)) \\ &=: (\bar{u}(t), \bar{u}_t(t)). \end{aligned}$$

Then $\bar{\zeta}$ and \bar{u} satisfy the equations

$$\begin{cases} \bar{u}_{tt} - \varepsilon \Delta \bar{u}_t + \bar{u}_t - \Delta \bar{u} + f(u^+) - f(u^0) = -\varepsilon \Delta u_t^0 & \text{in } (0, \infty) \times \Omega \\ \bar{u}|_{\partial\Omega} = 0 & \text{on } (0, \infty) \times \partial\Omega \\ \bar{\zeta}(0) = \mathbf{0} & \text{at } \{0\} \times \Omega. \end{cases} \quad (2.2)$$

After multiplying the equation (2.2)₁ by $2\bar{u}_t$ in $L^2(\Omega)$, we estimate the new product to arrive at the differential inequality,

$$\frac{d}{dt} \left\{ \|\bar{u}_t\|^2 + \|\nabla \bar{u}\|^2 \right\} + 2\varepsilon \|\nabla \bar{u}_t\|^2 + 2\|\bar{u}_t\|^2$$

$$\begin{aligned}
&= -2(f(u^1) - f(u^0), \bar{u}_t) - 2\varepsilon(\nabla u_t^0, \nabla \bar{u}_t) \\
&\leq C\|\nabla \bar{u}\|^2 + \|\bar{u}_t\|^2 + \varepsilon\|\nabla u_t^0\|^2 + \varepsilon\|\nabla \bar{u}_t\|^2.
\end{aligned} \tag{2.3}$$

The constant $C = C(L, \Omega) > 0$ is due to the local Lipschitz condition of $f : H_0^1 \rightarrow L^2$ following assumptions (1.4) and (1.5), as well as the embedding $H_0^1 \hookrightarrow L^2$.

It suffices to find an appropriate bound for $\|\nabla u_t^0(t)\|^2$. Indeed, since the global attractor for Problem \mathbf{P}_0 consists of strong solutions (\mathcal{A}_0 is bounded in \mathcal{H}_1), we are allowed to test/multiply the *weakly* damped wave equation in $L^2(\Omega)$ by $-2\Delta u_t^0(t)$. To this end we obtain,

$$\begin{aligned}
\frac{d}{dt} \{ \|\nabla u_t^0\|^2 + \|\Delta u^0\|^2 \} + 2\|\nabla u_t^0\|^2 &\leq 2|(f'(u^0)\nabla u^0, \nabla u_t^0)| \\
&\leq \|f'(u^0)\nabla u^0\|^2 + \|\nabla u_t^0\|^2 \\
&\leq \|f'(u^0)\|_{L^3}^2 \|\nabla u^0\|_{L^6}^2 + \|\nabla u_t^0\|^2 \\
&\leq \|u^0\|_{H^1}^4 \|u^0\|_{H^2}^2 + \|\nabla u_t^0\|^2.
\end{aligned}$$

Integrating this inequality over $[0, T]$ yields the desired bound,

$$\int_0^T \|\nabla u_t^0(s)\|^2 ds \leq C, \tag{2.4}$$

where the constant $C = C(\|\mathcal{A}_0\|_{\mathcal{H}_1}, T) > 0$, depends on the bound on \mathcal{A}_0 in \mathcal{H}_1 (through the initial condition) and on $T > 0$.

Now returning to inequality (2.3), we integrate

$$\frac{d}{dt} \{ \|\bar{u}_t\|^2 + \|\nabla \bar{u}\|^2 \} \leq \|\bar{u}_t\|^2 + C\|\nabla \bar{u}\|^2 + \varepsilon\|\nabla u_t^0\|^2 \tag{2.5}$$

over $[0, T]$ and apply the bound (2.4) to the last term on the right-hand side to produce the claim (2.1). This completes the proof. \square

Remark 2.3. The above result (2.1) establishes that, on compact time intervals, the difference between trajectories of Problem \mathbf{P}_ε , $\varepsilon \in (0, 1]$, and Problem \mathbf{P}_0 , originating from the same initial data on $\mathcal{A}_\varepsilon \subset \mathcal{H}_1$, can be controlled, explicitly, in terms of the perturbation parameter ε in the topology of \mathcal{H}_0 .

The well-known upper-semicontinuity result in Proposition 2.1 now follows for our family of global attractors.

Conflict of Interest

The author declares no conflicts of interest in this paper.

References

1. A. V. Babin and M. I. Vishik, *Attractors of evolution equations*, North-Holland, Amsterdam, 1992.
2. Alexandre N. Carvalho and Jan W. Cholewa, *Attractors for strongly damped wave equations with critical nonlinearities*, Pacific J. Math. **207** (2002), 287-310.

3. Alexandre N. Carvalho and Jan W. Cholewa, *Local well posedness for strongly damped wave equations with critical nonlinearities*, Bull. Austral. Math. Soc. **66** (2002), 443-463.
4. V. Pata and M. Squassina, *On the strongly damped wave equation*, Comm. Math. Phys. **253** (2005), 511-533.
5. Vittorino Pata and Sergey Zelik, *A remark on the damped wave equation*, Commun. Pure Appl. Anal. **5** (2006), 609-614.
6. James C. Robinson, *Infinite-dimensional dynamical systems*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.
7. Yonghai Wang and Chengkui Zhong, *Upper semicontinuity of global attractors for damped wave equations*, Asymptot. Anal. **91** (2015), 1-10.



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