Linear stochastic differential-algebraic equations with constant coefficients

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Abstract

We consider linear stochastic differential-algebraic equations with constant coefficients and additive white noise. Due to the nature of this class of equations, the solution must be defined as a generalised process (in the sense of Dawson and Fernique). We provide sufficient conditions for the law of the variables of the solution process to be absolutely continuous with respect to Lebesgue measure.

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1 Introduction

A Differential-Algebraic Equation (DAE) is, essentially, an Ordinary Differential Equation (ODE) $F(x, \dot{x}) = 0$ that cannot be solved for the derivative $\dot{x}$. The name comes from the fact that in some cases they can be reduced to a two-part system: A usual differential system plus a "nondifferential" one (hence "algebraic", with some abuse of language), that is

$$\begin{cases} \dot{x} = f(x, y) \\ 0 = g(y, z) \end{cases}$$

for some partitioning of x into variables x, y, z. In general, however, such a splitting need not exist.

In comparison with ODE's, these equations present at least two major difficulties: the first lies in the fact that it is not possible to establish general existence and uniqueness results, due to their more complicate structure; the second one is that DAE's do not regularise the input (quite the contrary), since solving them typically involves differentiation in place of integration. At the same time, DAE's are very important objects, arising in many application fields; among them we mention the simulation of electrical circuits, the modelling of multibody mechanisms, the approximation of singular perturbation problems arising e.g. in fluid dynamics, the discretisation of partial differential equations, the analysis of chemical processes, and the problem of protein folding. We refer to Rabier and Rheinboldt [9] for a survey of applications.

The class of DAE's most treated in the literature is, not surprisingly, that of linear equations, which have the form

$$A(t)\dot{x}(t) + B(t)x(t) = f(t) \ ,$$

with $x, f \colon \mathbb{R}^+ \to \mathbb{R}^n$ and $A, B \colon \mathbb{R}^+ \to \mathbb{R}^{n\times n}$. When A and B are constant matrices the equation is said to have constant coefficients.

Recently, there has been some incipient work (Schein and Denk [11] and Winkler [12]) on Stochastic Differential-Algebraic Equations (SDAE). In view to incorporate to the model a random external perturbation, an additional term is attached to the differential-algebraic equation, in the form of an additive noise (white or coloured). The solution will then be a stochastic process instead of a single function.

Since the focus in [11] and [12] is on numerical solving and the particular applications, some interesting theoretical questions have been left aside in these papers. Our long-term purpose is to put SDAE into the mainstream of stochastic calculus, developing as far as possible a theory similar to that of stochastic differential equations. In this first paper our aim is to investigate the solution of linear SDAE with constant coefficients and an additive white noise, that means

$$A\dot{x}(t) + Bx(t) = f(t) + \Lambda\xi(t) \ ,$$

where ξ is a white noise and A, B, Λ are constant matrices of appropriate dimensions. We shall first reduce the equation to the so-called Kronecker Canonical Form (KCF), which is easy to analyse, and from whose solution one can recover easily the solution to the original problem. Unfortunately, it is not possible to extend this approach to the case of linear SDAE with varying coefficients, just as happens in the deterministic case, where a bunch of different approaches have been proposed. Among these, the most promising in our opinion is that of Rabier and Rheinboldt [8].

Due to the simple structure of the equations considered here, it is not a hard task to establish the existence of a unique solution in the appropriate sense. However, as mentioned before, a

DAE does not regularise the input $f(t)$ in general. If white noise, or a similarly irregular noise is used as input, then the solution process to a SDAE will not be a usual stochastic process, defined as a random vector at every time t, but instead a "generalised process", the stochastic analogous of a Schwartz generalised function.

The paper is organised as follows: in the next section we shall provide a short introduction to linear DAE's and to generalised processes. In the third section we shall define what we mean by a solution to a linear SDAE and in Section 4 we shall provide a sufficient condition for the existence of a density of the law of the solution. In the final Section 5 we shall discuss a simple example arising in the modelling of electrical circuits.

Superscripts in parentheses mean order of derivation. All function and vector norms throughout the paper will be L^2 norms.

2 Preliminaries on DAE and generalised processes

In this section we briefly introduce two topics: the (deterministic) differential-algebraic equations and the generalised processes. An exhaustive introduction on the first topic can be found in Rabier and Rheinboldt [9], while the basic theory of generalised processes can be found in Dawson [1], Fernique [2], or Chapter 3 in Gel'fand and Vilenkin [3].

2.1 Differential-Algebraic Equations

Consider an implicit autonomous ODE,

$$F(x, \dot{x}) = 0 \ , \tag{2.1}$$

where $F := F(x,p) : \mathbb{R}^{n \times n} \to \mathbb{R}^n$ is a sufficiently smooth function. If the partial differential $D_pF(x,p)$ is invertible at every point (x_0, p_0), one can easily prove that the implicit ODE is locally reducible to an explicit ODE. If $D_pF(x_0,p_0)$ is not invertible, two cases are possible: either the total derivative $DF(x_0,p_0)$ is onto $\mathbb{R}^n$ or it is not. In the first case, and assuming that the rank of $D_pF(x,p)$ is constant in a neighbourhood of (x_0,p_0), (2.1) is called a *differential-algebraic equation*, while in the remaining cases one speaks of an ODE with a singularity at (x_0,p_0).

A *linear DAE* is a system of the form

$$A(t)\dot{x} + B(t)x = f(t) \ , \quad t \geq 0 \ , \tag{2.2}$$

where $A(t), B(t) \in \mathbb{R}^{n \times n}$ and $f(t) \in \mathbb{R}^n$. The matrix function $A(t)$ is assumed to have a constant (non-full) rank for any t in the interval of interest. (Clearly, if $A(t)$ has full rank for all t in an interval, then the DAE reduces locally to an ODE.) In the simplest case, when A and B do not depend on t, we have a *linear DAE with constant coefficients*, and an extensive study of these problems has been developed. The theory starts with the definition of a regular matrix pencil:

Definition 2.1 *Given two matrices $A, B \in \mathbb{R}^{n \times n}$, the* matrix pencil (A, B) *is the function $\lambda \mapsto \lambda A + B$, for $\lambda \in \mathbb{R}$. It is called a* regular matrix pencil *if* $\det(\lambda A + B) \neq 0$ *for some λ.*

A classical result, due to Weierstrass and Kronecker, states that the matrices of a regular matrix pencil can be simultaneously transformed into a convenient *canonical form*, as stated in the following proposition (see e.g. Gripentrog and März [4] for the proof):

Proposition 2.2 *Given a regular matrix pencil (A, B), there exist nonsingular matrices P and Q and integers $0 \le d, q \le n$, with $d + q = n$, such that*

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix} \quad \text{and} \quad PBQ = \begin{pmatrix} J & 0 \\ 0 & I_q \end{pmatrix}$$

where I_d, I_q are identities of dimensions d and q, $N = blockdiag(N_1, \dots, N_r)$, with N_i the $q_i \times q_i$ matrix

$$N_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

and J is in Jordan canonical form.

In what follows, we shall always assume that

(H.1) $\quad (A, B)$ is a regular matrix pencil

so that the proposition above applies.

Notice that the matrix N is nilpotent, with nilpotency index given by the dimension of its largest block. This nilpotency index of the matrix N in this canonical form is a characteristic of the matrix pencil and we shall call it the *index* of the equation (2.2).

From a given DAE with constant coefficients A and B that satisfy (H.1), multiplying from the left by P and defining the new variables $y = Q^{-1}x$, we get a new linear DAE with matrices PAQ and PBQ in place of A and B, which can be easily solved (see Section 3). The regularity of the solution depends directly on the index of the equation.

Remark 2.3 *Without hypothesis (H.1), a linear DAE may possess an infinity of solutions or no solution at all, depending on the right-hand side. Consider for instance,*

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \dot{x}(t) + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

with any fixed initial condition.

2.2 Generalised processes

Let $\mathcal{D}'$ be the space of distributions (generalised functions) on some open set $U \subset \mathbb{R}$, that is, the dual of the space $\mathcal{D} = \mathcal{C}_c^\infty(U)$ of smooth functions with compact support defined on U. A *random distribution* on U, defined on the probability space $(\Omega, \mathcal{F}, P)$, is a measurable mapping $X : (\Omega, \mathcal{F}) \to (\mathcal{D}', \mathcal{B}(\mathcal{D}'))$, where $\mathcal{B}(\mathcal{D}')$ denotes the Borel σ-field, relative to the strong dual topology (equivalently, the weak-$\star$ topology). Denoting by $\langle X(\omega), \phi\rangle$ the action of the distribution $X(\omega) \in \mathcal{D}'$ on the test function $\phi \in \mathcal{D}$, it holds that the mapping $\omega \mapsto \langle X(\omega), \phi\rangle$ is measurable from $(\Omega, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, hence a real random variable $\langle X, \phi\rangle$ on $(\Omega, \mathcal{F}, \mathcal{P})$. The law of X is determined by the law of the finite-dimensional vectors $(\langle X, \phi_1\rangle, \dots, \langle X, \phi_d\rangle)$, $\phi_i \in \mathcal{D}$, $d \in \mathbb{N}$.

The sum of random distributions X and Y on $(\Omega, \mathcal{F}, \mathcal{P})$, defined in the obvious manner, is again a random distribution. The product of a real random variable α and a random distribution, defined by $\langle \alpha X, \phi\rangle := \alpha\langle X, \phi\rangle$, is also a random distribution. The derivative of a random distribution, defined by $\langle X', \phi\rangle := -\langle X, \phi'\rangle$, is again a random distribution.

Given a random distribution X, the mapping $X\colon \mathcal{D} \to L^0(\Omega)$ defined by $\phi \mapsto \langle X, \phi \rangle$ is called a *generalised stochastic process*. This mapping is linear and continuous with the usual topologies in $\mathcal{D}$ and in the space of all random variables $L^0(\Omega)$. Notice that we can safely overload the meaning of the symbol X.

The *mean functional* and the *correlation functional* of a random distribution are the deterministic distribution $\phi \mapsto \mathrm{E}[\langle X, \phi\rangle]$ and the bilinear form $(\phi, \psi) \mapsto \mathrm{E}[\langle X, \phi\rangle\langle X, \psi\rangle]$, respectively, provided they exist.

A simple example of random distribution is *white noise* ξ, characterised by the fact that $\langle \xi, \phi\rangle$ is centred Gaussian, with correlation functional $\mathrm{E}[\langle \xi, \phi\rangle\langle \xi, \psi\rangle] = \int_{\mathbb{R}} \phi(s)\psi(s)\, ds$. In particular, $\langle \xi, \phi\rangle$ and $\langle \xi, \psi\rangle$ are independent if the supports of ϕ and ψ are disjoint. Whenever this property holds true for a process, we will say that it takes *independent values on disjoint sets*. In this paper we will use as the base set the half-line $U =]0, +\infty[$. White noise on U coincides with the Wiener integral with respect to a Brownian motion W: Indeed, if ϕ is a test function, then

$$\langle \xi, \phi\rangle = \int_0^\infty \phi(s)\, dW_s \tag{2.3}$$

in the sense of equality in law. More precisely, the Wiener integral is defined as the extension to $L^2(\mathbb{R}^+)$ of white noise (see Kuo [6] for a construction of the Wiener integral as extension of white noise). Now, integrating by parts in (2.3), we can write

$$\langle \xi, \phi\rangle = -\int_0^\infty W(s)\dot{\phi}(s)\, ds = -\langle W, \dot{\phi}\rangle \ ,$$

so that ξ is the derivative of the Brownian motion W as random distributions. A random distribution is *Gaussian* if every finite-dimensional projection is a Gaussian random vector. This is the case of white noise and Brownian motion.

Further results on random distributions and generalised stochastic processes can be found for instance in the classical papers by Dawson [1] and Fernique [2].

3 The generalised process solution

Consider the equation

$$A\dot{x} + Bx = f + \Lambda\xi \ , \tag{3.1}$$

where A and B are $n \times n$ real matrices, $f\colon \mathbb{R}^+ \to \mathbb{R}$ is continuous, Λ is a $n \times m$ constant matrix, and ξ is a m-dimensional white noise: $\xi = (\xi_1, \dots, \xi_m)$, with ξ_i independent one-dimensional white noises.

We first reduce the equation to Kronecker Canonical Form (KCF), see Proposition 2.2: There exist regular matrices P and Q and integers $0 \le d, q \le n$, with $d + q = n$, such that

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix} \quad \text{and} \quad PBQ = \begin{pmatrix} J & 0 \\ 0 & I_q \end{pmatrix}$$

where I_d, I_q are identities of dimensions d and q, $N = \operatorname{blockdiag}(N_1, \dots, N_r)$, with N_i the $q_i \times q_i$ matrix

$$N_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

and J is in Jordan canonical form. We can assume that the blocks of J corresponding to the eigenvalue 0, if any, are located in the lower rows of J.

Multiplying equation (3.1) by P from the left, and defining the new variables $y = Q^{-1}x$, we get

$$\begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix} \dot{y} + \begin{pmatrix} J & 0 \\ 0 & I_q \end{pmatrix} y = f + \Lambda\xi \ , \tag{3.2}$$

where for simplicity we use again f and Λ to denote the function Pf and the new "diffusion" matrix $P\Lambda$.

System (3.2) can be split into two parts. The first one is a classical stochastic differential system of dimension d, and the second one is an "algebraic stochastic system" of dimension q. Denoting by u and v the variables in the first and the second part respectively, by b and c the related partitioning of the vector function f, and by (Σ, R) the corresponding splitting of Λ into matrices of dimensions $d \times m$ and $q \times m$, we can write the two systems as follows:

$$\begin{pmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_d \end{pmatrix} + J \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} + \Sigma \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \ , \tag{3.3}$$

$$N \begin{pmatrix} \dot{v}_1 \\ \vdots \\ \dot{v}_q \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_q \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_q \end{pmatrix} + R \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \ . \tag{3.4}$$

We refer to u as the *differential variables* and to v as the *algebraic variables.*

For any initial vector $u(t_0) = u^0 \in \mathbb{R}^d$, and $t_0 > 0$, the solution to (3.3) can be written, in the sense of equality in law, as

$$u(t) = e^{-J(t-t_0)} \left[u^0 + \int_{t_0}^t e^{J(s-t_0)} b(s)\, ds + \int_{t_0}^t e^{J(s-t_0)} \Sigma\, dW(s) \right] \ , \quad t \in]0, \infty[\ ,$$

where $W(t)$ is a m-dimensional standard Wiener process. It can also be expressed, if desired, as a generalised process:

For $\phi \in \mathcal{C}_c^\infty(]0, \infty[)$,

$$\begin{aligned} \langle u, \phi \rangle = & \int_0^\infty e^{-J(t-t_0)} \Big(u^0 + \int_{t_0}^t e^{J(s-t_0)} b(s)\, ds \Big) \phi(t) dt \\ & + \int_0^\infty \left[\int_{t_0}^t e^{-J(t-s)} \Sigma\, dW(s) \right] \phi(t) dt \ . \end{aligned} \tag{3.5}$$

On the other hand, system (3.4) consists of a number of decoupled blocks, which are easily solved by backward substitution. For instance, for the first block,

$$N_1 \begin{pmatrix} \dot{v}_1 \\ \vdots \\ \dot{v}_{q_1} \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_{q_1} \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{q_1} \end{pmatrix} + R_1 \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \ , \tag{3.6}$$

R_1 representing the sub-matrix of R consisting of the first q_1 rows, and $c = (c_1, \ldots, c_{q_1})$ the corresponding part of vector c, a recursive calculation gives the following generalised process solution, for a given $\phi \in \mathcal{C}_c^\infty(]0, +\infty[)$:

$$\langle v_j, \phi \rangle = \sum_{k=j}^{q_1} \Big\langle c_k + \sum_{\ell=1}^m (R_1)_{k,\ell} \xi_\ell, \phi^{(k-j)} \Big\rangle \ , \quad j = 1, \ldots q_1 \ , \tag{3.7}$$

which can be expressed with the help of a standard Wiener process W as

$$(3.8)\qquad \langle v_j,\phi\rangle = \sum_{k=j}^{q_1}\Big[\int_0^\infty c_k(t)\phi^{(k-j)}(t)\,dt + \sum_{\ell=1}^{m}(R_1)_{k,\ell}\int_0^\infty \phi^{(k-j)}\,dW_\ell(t)\Big]\,,\quad j=1,\dots q_1\,.$$

The remaining blocks are treated in the same way to get the whole vector y solving the KCF system (3.2).
We can thus state the following result:

Proposition 3.1 *Under assumption (H.1) of Section 2, equation (3.1) admits a unique generalised process solution $x = Qy$, where Q is the matrix determined by the Kronecker canonical form and $y = (u, v)$, with u the solution to system (3.3), given by expression (3.5), and v the solution to system (3.4).*

4 The law of the solution

In the previous section we have seen that the solution to a linear SDAE with regular pencil and additive white noise can be explicitly given as a functional of the input noise. From the modelling viewpoint, the law of the solution is the important output of the model. Using the explicit form above, one can try to investigate the features of the law in which one might be interested.

To illustrate this point, we shall write down the joint law of the solution vector evaluated at a fixed arbitrary test function ϕ and we shall investigate some absolute continuity properties we do not aim at a very general statement, but instead we want to show the sort of arguments that can be used in each particular instance). For notational simplicity, let us assume that the differential system has dimension $d = 2$ and that the algebraic part consists of a unique block of nilpotency index 2. We will also assume $b = 0$ and $c = 0$ in (3.3) and (3.4); the general case can be studied similarly. Thus, we are dealing with the following system (4.1)-(4.2):

$$(4.1)\qquad \begin{pmatrix}\dot u_1\\ \dot u_2\end{pmatrix} + J\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}\sigma_{11} & \dots & \sigma_{1m}\\ \sigma_{21} & \dots & \sigma_{2m}\end{pmatrix}\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix}\,,$$

$$(4.2)\qquad \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\dot v_1\\ \dot v_2\end{pmatrix} + \begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}\rho_{11} & \dots & \rho_{1m}\\ \rho_{21} & \dots & \rho_{2m}\end{pmatrix}\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix}\,.$$

Given a test function $\phi\in\mathcal{C}_c^\infty(]0,+\infty[)$, the solution to the differential part (4.1), with initial condition $u^0 = (u_1^0, u_2^0)$ at time $t_0 > 0$, is

$$\langle u_i,\phi\rangle = \int_0^\infty (e^{-J(t-t_0)})_{ik}u_k^0\phi(t)dt + \int_0^\infty\Big[\int_{t_0}^t (e^{-J(t-s)})_{ik}\sigma_{k\ell}dW^\ell(s)\Big]\phi(t)dt$$

for $i = 1, 2$, (see (3.5)), whereas the solution to the algebraic part (4.2) is given by

$$(4.3)\qquad \langle v_1,\phi\rangle = \int_0^\infty (\rho_{1\ell}\phi(t) + \rho_{2\ell}\dot\phi(t))\,dW^\ell(t)$$

$$(4.4)\qquad \langle v_2,\phi\rangle = \int_0^\infty \rho_{2\ell}\phi(t)\,dW^\ell(t)\,,$$

(see (3.7)), with the convention of summation over repeated indices, that we will keep applying in the sequel without explicit mention.

By standard computations, for any given test function ϕ, the random vector $\langle (u_1,u_2,v_1,v_2),\phi\rangle$ has a Gaussian distribution, with expectations

$$E[\langle u_i,\phi\rangle] = \int_0^\infty (e^{-J(t-t_0)})_{ik} u_k^0 \phi(t)dt \ , \quad i=1,2$$
$$E[\langle v_1,\phi\rangle] = E[\langle v_2,\phi\rangle] = 0$$

and covariances

$$\text{(4.5)} \qquad \mathrm{Cov}[\langle u_i,\phi\rangle,\langle u_j,\phi\rangle] = \sum_{\ell=1}^m \int_0^\infty \Big[\int_s^\infty \phi(t)\sum_{k=1}^2 (e^{-J(t-s)})_{ik}\sigma_{k\ell}\,dt\Big] \times \Big[\int_s^\infty \phi(t)\sum_{k=1}^2 (e^{-J(t-s)})_{jk}\sigma_{k\ell}\,dt\Big]\,ds \ , \quad i,j=1,2$$

$$\text{(4.6)} \qquad \mathrm{Cov}[\langle v_1,\phi\rangle,\langle v_1,\phi\rangle] = \sum_{\ell=1}^m \int_0^\infty \Big((\rho_{1\ell})^2\phi(t)^2 + (\rho_{2\ell})^2\dot\phi(t)^2\Big)\,dt$$

$$\text{(4.7)} \qquad \mathrm{Cov}[\langle v_2,\phi\rangle,\langle v_2,\phi\rangle] = \sum_{\ell=1}^m \int_0^\infty (\rho_{2\ell})^2\phi(t)^2\,dt$$

$$\text{(4.8)} \qquad \mathrm{Cov}[\langle v_1,\phi\rangle,\langle v_2,\phi\rangle] = \sum_{\ell=1}^m \int_0^\infty \rho_{1\ell}\rho_{2\ell}\phi(t)^2\,dt$$

$$\text{(4.9)} \qquad \mathrm{Cov}[\langle u_i,\phi\rangle,\langle v_1,\phi\rangle] = \sum_{\ell=1}^m \int_0^\infty \Big[\int_s^\infty \phi(t)\sum_{k=1}^2 (e^{-J(t-s)})_{ik}\sigma_{k\ell}\,dt\Big]\Big[\rho_{1\ell}\phi(s)+\rho_{2\ell}\dot\phi(s)\Big]\,ds$$

$$\text{(4.10)} \qquad \mathrm{Cov}[\langle u_i,\phi\rangle,\langle v_2,\phi\rangle] = \sum_{\ell=1}^m \int_0^\infty \int_s^\infty \Big[\phi(t)\sum_{k=1}^2 (e^{-J(t-s)})_{ik}\sigma_{k\ell}\,dt\Big]\rho_{2\ell}\phi(s)\,ds \ .$$

Consider first the algebraic variables alone. Let us write (4.3) in terms of the white noises ξ:

$$\langle v_1,\phi\rangle = \langle \rho_{1\ell}\xi_\ell,\phi\rangle + \langle \rho_{2\ell}\xi_\ell,\dot\phi\rangle$$
$$\langle v_2,\phi\rangle = \langle \rho_{2\ell}\xi_\ell,\phi\rangle \ .$$

Denoting $\rho_s := (\rho_{s1},\dots,\rho_{sm})$, and $\langle \xi,\phi\rangle := \langle (\xi_1,\dots\xi_m),\phi\rangle$ (as row vectors), we have in matrix form

$$\begin{pmatrix}\langle v_1,\phi\rangle\\ \langle v_2,\phi\rangle\end{pmatrix} = \begin{pmatrix}\rho_1 & \rho_2\\ \rho_2 & 0\end{pmatrix}\begin{pmatrix}\langle \xi,\phi\rangle^\perp\\ \langle \xi,\dot\phi\rangle^\perp\end{pmatrix} ,$$

and therefore

$$\mathrm{Cov}\Big[\langle v_1,\phi\rangle,\langle v_2,\phi\rangle\Big] = \begin{pmatrix}\rho_1 & \rho_2\\ \rho_2 & 0\end{pmatrix}\mathrm{Cov}\Big[\langle \xi,\phi\rangle,\langle \xi,\dot\phi\rangle\Big]\begin{pmatrix}\rho_1 & \rho_2\\ \rho_2 & 0\end{pmatrix}^\perp .$$

It is easily checked that for all $i=1,\dots,m$,

$$\text{(4.11)} \qquad \mathrm{Cov}\Big[\langle \xi_i,\phi\rangle,\langle \xi_i,\dot\phi\rangle\Big] = \begin{pmatrix}\|\phi\|^2 & 0\\ 0 & \|\dot\phi\|^2\end{pmatrix} ,$$

which is a nonsingular matrix for every $\phi \not\equiv 0$. Taking into account that ξ_i are centred and independent of each other, we find that $\mathrm{Cov}\Big[\langle \xi,\phi\rangle,\langle \xi,\dot\phi\rangle\Big]$ is nonsingular for every $\phi\not\equiv 0$.

Hence, we see that if ρ_2 is not the zero vector, the joint law of $\langle v_1,\phi\rangle$ and $\langle v_2,\phi\rangle$ is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^2$; if $\|\rho_2\|=0$ and $\|\rho_1\|\neq 0$, then $\langle v_2,\phi\rangle$ is degenerate and $\langle v_1, \phi\rangle$ is absolutely continuous; and if $\|\rho_2\| = \|\rho_1\| = 0$, then the joint law degenerates to a point.

This sort of elementary analysis, with validity for any test function ϕ, can be carried out for algebraic blocks of nilpotency index up to 4. We can thus summarise these arguments in the following proposition:

Proposition 4.1 *Let y be the generalised process solution to the linear SDAE in Kronecker Canonical Form (3.2). Let $(v_1, \dots, v_{q_1})$ be the generalised process solution to the algebraic subsystem*

$$N_1 \begin{pmatrix} \dot{v}_1 \\ \vdots \\ \dot{v}_{q_1} \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_{q_1} \end{pmatrix} = \begin{pmatrix} \rho_{11} & \cdots & \rho_{1m} \\ \vdots & & \vdots \\ \rho_{q_1 1} & \cdots & \rho_{q_1 m} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} , \tag{4.12}$$

with $q_1 \le 4$, and let r be the greatest row index such that $\|\rho_r\| \neq 0$.
Then, for every test function $\phi \in \mathcal{C}_c^\infty(]0,\infty[)$, $\langle (v_1, \dots, v_r), \phi\rangle$ is a Gaussian absolutely continuous random vector, and $\langle (v_{r+1}, \dots, v_{q_1}), \phi\rangle$ degenerates to a point.

For blocks of higher index, the support of the joint law can of course be determined without special difficulty for any specific test function ϕ. In general, the entries of the covariance matrix of the white noise and its derivatives up to order k, evaluated at a test function ϕ, can be written in a compact form as

$$\mathrm{Cov}(\langle \xi, \phi\rangle, \dots, \langle \xi^{(k)}, \phi\rangle)_{ij} = \mathrm{R\scriptsize{E}}\Big[(-1)^{\frac{|i-j|}{2}}\Big] \|\phi^{((i+j)/2)}\|^2 , \tag{4.13}$$

where RE means the real part. In case this covariance matrix is nonsingular, the absolute continuity result of Proposition 4.1 is valid for that fixed ϕ, as stated in the next proposition.

Proposition 4.2 *Let $(v_1, \dots, v_{q_1})$ be the generalised process solution to the algebraic subsystem (4.12), without any restriction on its dimension q_1, and let $\phi \in \mathcal{C}_c^\infty(]0,\infty[)$ be a test function such that the covariance matrix $\mathrm{Cov}(\langle \xi, \phi\rangle, \dots, \langle \xi^{(q_1-1)}, \phi\rangle)$, given in (4.13), is nonsingular. Let r be the greatest row index such that $\|\rho_r\| \neq 0$.*
Then $\langle (v_1, \dots, v_r), \phi\rangle$ is a Gaussian absolutely continuous random vector, and $\langle (v_{r+1}, \dots, v_{q_1}), \phi\rangle$ degenerates to a point.

For the differential variables alone, there are well known conditions for the their joint absolute continuity (e.g. Hörmander conditions, see for instance Nualart [7], Theorem 2.3.2). These conditions put into play the matrix J together with the matrix Σ and allows absolute continuity of the law of a subset $u_{i_1}, \dots, u_{i_k}$ of differential variables even in the case when the matrix Σ does not have full rank.

Let us now consider the joint law of an algebraic and a differential variable. We will not attempt here to arrive at a general criterion similar to Hörmander conditions; we just find a sufficient condition for absolute continuity involving only the entries of Λ.

We simplify notation by assuming that $m = 4$ (the case $m > 4$ can be derived with simple changes), and that

$$\mathrm{rank} \begin{pmatrix} \rho_{11} & \cdots & \rho_{14} \\ \rho_{21} & \cdots & \rho_{24} \end{pmatrix} = 2 .$$

Let us start by considering the joint law of the five variables $\langle (u_1, u_2, v_1, v_2, \dot{v}_2), \phi\rangle$. It is immediate to prove that the joint law of $\langle (v_1, v_2, \dot{v}_2), \phi\rangle$ is absolutely continuous with respect to

Lebesgue measure, using (4.11). From this, in order to obtain the joint absolute continuity of the five variables (and therefore of the solution to (4.1)-(4.2)), it will be sufficient to prove that the conditional law of $\langle (u_1, u_2), \phi \rangle$, given $\langle (v_1, v_2, \dot{v}_2), \phi \rangle$, is absolutely continuous for any given test function ϕ.

Let us assume, without loss of generality, that the 2×2 minor

$$\Theta = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} \\ \rho_{2,1} & \rho_{2,2} \end{pmatrix}$$

is non-degenerate. Multiplying equation (4.2) by Θ^{-1}, we get

$$\begin{aligned} \Theta^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} + \Theta^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \Theta^{-1} \begin{pmatrix} \rho_{1,1} & \cdots & \rho_{1,4} \\ \rho_{2,1} & \cdots & \rho_{2,4} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_4 \end{pmatrix} \\ &= \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \Theta^{-1} \begin{pmatrix} \rho_{1,3} & \rho_{1,4} \\ \rho_{2,3} & \rho_{2,4} \end{pmatrix} \begin{pmatrix} \xi_3 \\ \xi_4 \end{pmatrix} . \end{aligned}$$

Solving for ξ_1 and ξ_2, we obtain

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \Theta^{-1} \begin{pmatrix} \dot{v}_2 + v_1 \\ v_2 \end{pmatrix} - \Theta^{-1} \begin{pmatrix} \rho_{1,3} & \rho_{1.4} \\ \rho_{2,3} & \rho_{2,4} \end{pmatrix} \begin{pmatrix} \xi_3 \\ \xi_4 \end{pmatrix} . \tag{4.14}$$

For the remainder of the section, $(u_1, u_2) \in \mathrm{Eq}[\theta]$ will mean that (u_1, u_2) is the solution to (4.1), with the right hand side given by a generic two-dimensional generalised process $\theta = (\theta_1, \theta_2)$. There is no difficulty in defining the solution of such an equation since it can be regarded, for each random element $\omega \in \Omega$, as a deterministic linear differential equation with distributional input, which is a well known object. Substituting (4.14) into equation (4.1), we get

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathrm{Eq} \left[\begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix} \Theta^{-1} \begin{pmatrix} \dot{v}_2 + v_1 \\ v_2 \end{pmatrix} + \widetilde{\Theta} \begin{pmatrix} \xi_3 \\ \xi_4 \end{pmatrix} \right]$$

where

$$\widetilde{\Theta} = \begin{pmatrix} \sigma_{1,3} & \sigma_{1,4} \\ \sigma_{2,3} & \sigma_{2,4} \end{pmatrix} - \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix} \Theta^{-1} \begin{pmatrix} \rho_{1,3} & \rho_{1,4} \\ \rho_{2,3} & \rho_{2,4} \end{pmatrix} .$$

It follows that the law of $\langle (u_1, u_2), \phi \rangle$ conditioned to $\langle (v_1, v_2, \dot{v}_2), \phi \rangle$ coincides with the law of $\langle (\tilde{u}_1, \tilde{u}_2), \phi \rangle$, with

$$\begin{pmatrix} \widetilde{u}_1 \\ \widetilde{u}_2 \end{pmatrix} \in \mathrm{Eq} \left[\begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix} \Theta^{-1} w + \widetilde{\Theta} \begin{pmatrix} \eta_3 \\ \eta_4 \end{pmatrix} \right]$$

and where w is a constant vector and $\langle (\eta_3, \eta_4), \phi \rangle$ is a Gaussian absolutely continuous random vector, with some non-singular covariance matrix C. In a more compact form, we can write

$$\begin{pmatrix} \widetilde{u}_1 \\ \widetilde{u}_2 \end{pmatrix} \in \mathrm{Eq} \left[a + \widetilde{\Theta} C \begin{pmatrix} \xi_3 \\ \xi_4 \end{pmatrix} \right] \tag{4.15}$$

with a constant vector a and the two-dimensional white noise $(\xi_3, \xi_4))^T := C^{-1}(\eta_3, \eta_4))^T$.

It is immediate to see that $\tilde{\Theta}$ is the Schur complement of the matrix Θ in

$$\Lambda = \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} & \sigma_{1,4} \\ \sigma_{2,1} & \sigma_{2,2} & \sigma_{2,3} & \sigma_{2,4} \\ \rho_{1,1} & \rho_{1,2} & \rho_{1,3} & \rho_{1,4} \\ \rho_{2,1} & \rho_{2,2} & \rho_{2,3} & \rho_{2,4} \end{pmatrix} . \tag{4.16}$$

Therefore (see e.g. Horn and Johnson [5], page 21),

$$\det(\tilde{\Theta}) = \frac{\det(\Lambda)}{\det(\Theta)} .$$

Thus, assuming that Λ in (4.16) is a non-degenerate matrix, we obtain that the matrix $\tilde{\Theta}C$ in (4.15) is non-singular, and it is well known that the solution $(\tilde{u}_1, \tilde{u}_2)$ is a stochastic process with absolutely continuous law when applied to any test function $\phi \not\equiv 0$. We conclude the absolute continuity of $\langle (u_1, u_2, v_1, v_2), \phi \rangle$.

The case $m > 4$ can be obtained with similar computations and we can state the following final result:

Theorem 4.3 *Under the assumption (H.1) of Section 2, if the rank of the matrix*

$$\Lambda = \begin{pmatrix} \sigma_{1,1} & \dots & \sigma_{1,m} \\ \sigma_{2,1} & \dots & \sigma_{2,m} \\ \rho_{1,1} & \dots & \rho_{1,m} \\ \rho_{2,1} & \dots & \rho_{2,m} \end{pmatrix}$$

is equal to 4, then the law of the unique solution to the SDAE (4.1)-(4.2) at a test function $\phi \not\equiv 0$ *is absolutely continuous with respect to Lebesgue measure on* $\mathbb{R}^4$.

5 Example: An electrical circuit

In this last section we shall present an example of linear SDAE's arising from a problem of electrical circuit simulation.

An electrical circuit is a set of devices connected by wires. Each device has two or more connection *ports*. A wire connects two devices at specific ports. Between any two ports of a device there is a *flow* (current) and a *tension* (voltage drop). Flow and tension are supposed to be the same at both ends of a wire; thus wires are just physical media for putting together two ports and they play no other role.

The circuit topology can be conveniently represented by a network, i.e. a set of nodes and a set of directed arcs between nodes, in the following way: Each port is a node (taking into account that two ports connected by a wire collapse to the same node), and any two ports of a device are joined by an arc. Therefore, flow and tension will be two quantities circulating through the arcs of the network.

It is well known that a network can be univocally described by an incidence matrix $A = (a_{ij})$. If we have n nodes and m arcs, A is the $m \times n$ matrix defined by

$$a_{ij} = \begin{cases} +1, & \text{if arc } j \text{ has node } i \text{ as origin} \\ -1, & \text{if arc } j \text{ has node } i \text{ as destiny} \\ 0, & \text{in any other case.} \end{cases}$$

At every node i, a quantity d_i (positive, negative or null) of flow may be supplied from the outside. This quantity, added to the total flow through the arcs leaving the node, must equal the total flow arriving to the node. This conservation law leads to the system of equations $Ax = d$, where x_j, $j = 1, \dots, n$, is the flow through arc j.

A second conservation law relates to tensions and the cycles formed by the flows. A *cycle* is a set of arcs carrying nonzero flow when all external supplies are set to zero. The *cycle space* is thus $\ker A \subset \mathbb{R}^n$. Let B be a matrix whose columns form a basis of the cycle space, and let

$c \in \mathbb{R}^n$ be the vector of externally supplied tensions to the cycles of the chosen basis. Then we must impose the equalities $B^\top u = c$, where u_j, $j = 1, \ldots, n$, is the tension through arc j.

Once we have the topology described by a network, we can put into play the last element of the circuit modelling. Every device has a specific behaviour, which is described by an equation $\varphi(x, u, \dot{x}, \dot{u}) = 0$ involving in general flows, tensions, and their derivatives. The system $\Phi(x, u, \dot{x}, \dot{u}) = 0$ consisting of all of these equations is called the *network characteristic*. For instance, typical simple two-port (linear) devices are the *resistor*, the *inductor* and the *capacitor*, whose characteristic (noiseless) equations, which involve only their own arc j, are $u_j = Rx_j$, $u_j = L\dot{x}_j$, and $x_j = C\dot{u}_j$, respectively, for some constants R, L, C. Also, the *current source* (x_j constant) and the *voltage source* (u_j constant) are common devices.

Solving an electrical circuit therefore means finding the currents x and voltage drops u determined by the system

$$\begin{cases} Ax = d \\ B^\top u = c \\ \Phi(x, u, \dot{x}, \dot{u}) = 0 \end{cases}$$

Example 5.1 *Let us write down the equations corresponding to the circuit called* LL-cutset *(see [10], pag. 60), formed by two inductors and one resistor, which we assume submitted to random perturbations, independently for each device. This situation can be modelled, following the standard procedure described above, by the stochastic system*

$$\begin{cases} x_1 = -x_2 = x_3 \\ u_1 - u_2 + u_3 = 0 \\ u_1 = L_1\dot{x}_1 + \tau_1\xi_1 \\ u_2 = L_2\dot{x}_2 + \tau_2\xi_2 \\ u_3 = Rx_3 + \tau_3\xi_3 \end{cases} \tag{5.1}$$

where ξ_1, ξ_2, ξ_3 are independent white noises, and τ_1, τ_2, τ_3 are non-zero constants which measure the magnitude of the perturbations. With a slight obvious simplification, we obtain from (5.1) the following linear SDAE:

(5.2)

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & L_1 & 0 \\ 0 & 0 & 0 & L_2 \end{pmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} R^{-1} & -R^{-1} & 1 & 0 \\ -R^{-1} & R^{-1} & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\tau_3 R^{-1} \\ 0 & 0 & \tau_3 R^{-1} \\ -\tau_1 & 0 & 0 \\ 0 & -\tau_2 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix},$$

Let us now reduce the equation to KCF. To simplify the exposition, we shall fix to 1 the values of τ_i, R and L_i. (A physically meaningful magnitude for R and L_i would be of order 10^{-6} for the first and of order 10^4 for the latter. Nevertheless the structure of the problem does not change with different constants.) The matrices P and Q, providing the desired reduction (see Proposition 2.2), are

$$P = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & -1 \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Indeed, multiplying (5.2) by P from the left and defining $y = Q^{-1}x$, we arrive to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \dot{y}(t) + \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} y(t) = \begin{pmatrix} -\tau_1 & \tau_2 & -\tau_3 \\ -\tau_1 & -\tau_2 & -\tau_3 \\ 0 & 0 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \tag{5.3}$$

We see that the matrix N of Section 3 has here two blocks: A single zero in the last position ($\dot{y}_4$) and a 2-nilpotent block affecting $\dot{y}_2$ and $\dot{y}_3$. We have therefore an index 2 SDAE. From Propositions 4.1, 4.2 and Theorem 4.3, we can already say that, when applied to any test function $\phi \neq 0$, the variables y_4, y_2 and y_1, as well as the vectors (y_1, y_2) and (y_1, y_4), will be absolutely continuous, whereas y_3 degenerates to a point.

In fact, in this case, we can of course solve completely the system: The differential part is the one-dimensional classical SDE

$$\dot{y}_1 + \frac{1}{2} y_1 = -\tau_1 \xi_1 + \tau_2 \xi_2 - \tau_3 \xi_3 \ , \tag{5.4}$$

and the algebraic part reads simply

$$\begin{cases} \dot{y}_3 + y_2 = -\tau_1 \xi_1 - \tau_2 \xi_2 - \tau_3 \xi_3 \\ y_3 = 0 \\ y_4 = \tau_3 \xi_3 \ . \end{cases} \tag{5.5}$$

The solution to (5.3) can thus be written as

$$\begin{aligned} y_1(t) &= e^{-(t-t_0)/2} \Big[y(t_0) + \int_{t_0}^{t} e^{-(s-t_0)/2} (-\tau_1 dW_1 + \tau_2 dW_2 - \tau_3 dW_3)(s) \Big] \\ y_2 &= -\tau_1 \xi_1 - \tau_2 \xi_2 - \tau_3 \xi_3 \\ y_3 &= 0 \\ y_4 &= \tau_3 \xi_3 \ , \end{aligned}$$

where W_1, W_2, W_3 are independent Wiener processes whose generalised derivatives are ξ_1, ξ_2 and ξ_3. Multiplying by the matrix Q we finally obtain the value of the original variables:

$$\begin{aligned} x_1(t) = -x_2(t) &= -\tfrac{1}{2} y_1(t) \\ u_1 &= -\tfrac{1}{4} y_1 - \tfrac{1}{2} y_2 - \tfrac{3}{4} y_4 \\ u_2 &= \tfrac{1}{4} y_1 - \tfrac{1}{2} y_2 \ , \end{aligned}$$

with $x_1(t_0) = -\frac{1}{4} y_1(t_0)$ a given intensity at time t_0.

It is clear that the current intensities, which have almost surely continuous paths, are much more regular than the voltage drops, which are only random distributions.

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