Dynamic Anti-Windup Compensator Design Considering Behavior of Controller State

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Abstract
The purpose of this paper is to study the determination of $L_2$ gain performance index in a design method of anti-windup compensator. Since the purpose of the anti-windup compensation is to control the controller state, we introduce a new $L_2$ gain performance index that contains both the plant state and the controller state. Local control design technique which is based on the circle criterion is used for anti-windup compensator synthesis, while global control performance analysis is conducted using Popov criterion. Better response is obtained by tuning the two weighting matrices on the states of the performance index, which is demonstrated by simulation and experiment examples.

Key words: Actuator Saturation, Anti-Windup Compensator, Controller State, $L_2$ Gain Performance Index, Absolute Stability, Linear Matrix Inequality

1. Introduction
Actuator saturation is present almost everywhere in the practical control systems. Common examples of actuator saturation are the voltage limits in electrical actuators and flow rate limits in hydraulic actuators. The phenomenon resulting from an active input saturation is referred to windup. Originally, windup problems were encountered when using PI/PID controllers(1) where the integral states of the controller are wrongly updated. Later on, it is well known that windup not only occurs in integral controller, but whenever the controller contains badly damped or unstable modes. Further, this phenomenon is referred to as controller windup.

Many researches have been conducted to overcome the windup problem. In the last two decades, anti-windup scheme has been a major research interest in dealing with actuator saturation. The main idea of the anti-windup scheme is to design the linear controller by ignoring the saturation nonlinearities first, then add an anti-windup compensator to stabilize the closed-loop system and to minimize the adverse effects of the input saturation on the closed-loop performance. There have been many studies about the anti-windup compensator design. Various kinds of anti-windup compensators have been proposed and a general framework is given in Ref. (1). A recent overview of many anti-windup techniques is provided and their connections to one another are stated in Ref. (2).

It is desirable that the anti-windup compensator not only guarantees the stability of the closed-loop system subject to the actuator saturation, but also recovers as much as possible the performance lost during saturation. The important result in establishing this notion was...
stated in Refs. (3) and (4), where the $L_2$ gain was picked as an appropriate induced system norm to measure the anti-windup compensator’s performance\(^5\). In Ref. (6), the problem of synthesizing fixed-order anti-windup compensator, which meets the $L_2$ performance bound has been addressed. It is also shown that for stable plants, there always exists an anti-windup compensator of order greater than or equal to that of the plant.

The anti-windup compensator design often assumes the saturation as a sector-bounded nonlinearity and absolute stability conditions such as circle and Popov Criterion are applied for the stability and performance analysis. Although the anti-windup meets the stability and performance condition requirements, in some cases the target response of the anti-windup scheme tends to be sluggish and more study is necessary. Reducing sector bound from 1 can significantly improve the performance of anti-windup compensated system\(^7\), but the calculated performance level is achievable as long as the magnitude of the output from the controller is in the finite interval defined by sector bound and saturation bound. Recently, attempt to modify the anti-windup scheme is given in Ref. (8), where the basic idea is to apply anti-windup when the performance of the saturated system faces substantial degradation by allowing actuators to remain in the saturated region longer.

In the previous studies, the state variables of the plant and the controller have not been considered directly for the $L_2$ gain performance index. Since the controlled object seen from the anti-windup compensation is the feedback system composed of the plant and the controller, and the main purpose of the compensation is to control the states of the controller, we consider that tracking performance may be improved by using a new $L_2$ gain performance index that contains the controller and the plant state. In this paper, we give the new $L_2$ gain performance index and derive appropriate convex $L_2$ gain LMIs for the synthesis of dynamic anti-windup compensator. In order to show the usefulness of the proposed design method, we apply this method to an experiment setup of a belt drive control system and a numerical example taken from the literature.

2. Problem Statement

Consider a linear plant $\tilde{P}$ with state-space representation

\[
\begin{align*}
\dot{x}_p &= A_p \dot{x}_p + B_{pw} w + B_{pu} \hat{u} \\
\hat{y} &= C_y x_p
\end{align*}
\]  

(1)

where $\dot{x}_p \in \mathbb{R}^{n_p}$ is the plant state, $\hat{u} \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is the exogenous input (possibly containing disturbance, reference and measurement noise) and $\hat{y} \in \mathbb{R}^{n_y}$ is the measured output. Matrices $A_p$, $B_{pw}$, $B_{pu}$, and $C_y$ are real constant matrices of appropriate dimensions. Pairs $(A_p, B_{pu})$ and $(C_y, A_p)$ are assumed to be controllable and observable, respectively. Assume that an unconstrained controller $\tilde{K}$ has been designed in order to guarantee some performance requirements and the stability of the closed-loop system in the absence of control input saturation, with state-space representation

\[
\begin{align*}
\dot{x}_c &= A_c \dot{x}_c + B_{cw} w + B_{cy} \hat{y} \\
\hat{u} &= C_y x_c + D_{cw} w + D_{cy} \hat{y}
\end{align*}
\]  

(3)

where $\dot{x}_c \in \mathbb{R}^{n_c}$ is the controller state, and $\hat{u} \in \mathbb{R}^{n_u}$ is the controller output. The desirable linear feedback system is depicted in Fig. 1. The bar of the variables denotes the linear model.

Suppose, the magnitude of the control input is bounded by the saturation function $\Phi$, as a result, the control input is not always the same as the controller output. The plant and the

![Fig. 1 Linear feedback system](image-url)
controller of the saturated feedback system are described by
\[
\dot{x}_p = A_p x_p + B_p w + B_p u
\]
(5)
\[
y = C_g x_p
\]
(6)
\[
\dot{x}_c = A_c x_c + B_c w + B_c y
\]
(7)
\[
\tilde{u} = C_c x_c + D_c w + D_c y
\]
(8)
\[
u = \Phi(\tilde{u}) = [\phi_1(\tilde{u}_1), \ldots, \phi_n(\tilde{u}_n)]^T
\]
(9)
where
\[
\phi_i(\tilde{u}_i) = \text{sign}(\tilde{u}_i) \min(|\tilde{u}_i|, u_{sat}(i))
\]
(10)
The saturated feedback system is depicted in Fig. 2.

When the saturation occurs, the feedback system behaves as an open-loop system. Then, the control error does not become zero, and the state of the controller \(K\), especially the integrator state, increases excessively. As a result, the system has a large over-shoot or it might be unstable in the worst case. In addition, even in the absence of dynamic controller elements, saturation nonlinearities can trigger undesired oscillations which is due to inappropriate plant state and therefore called plant windup\(^{(9)}\). In order to prevent this, an anti-windup compensator is added to the controller as shown in Fig. 3. Namely, the difference of the input and the output of the saturation function \(\tilde{u} - u\) is used to modify the states of the controller \(K\), in order to maintain stability and to recover, as much as possible, the performance lost during saturation. The controller with the anti-windup compensator \(K\) is described by
\[
\dot{x}_c = A_c x_c + B_c w + B_c y + v
\]
(11)
\[
\tilde{u} = C_c x_c + D_c w + D_c y
\]
(12)
where \(v \in \mathbb{R}^{n_c}\) is the output of the anti-windup compensator AWC which is described by
\[
\dot{x}_{AW} = A_{AW} x_{AW} + B_{AW} (\tilde{u} - u)
\]
(13)
\[
v = C_{AW} x_{AW} + D_{AW} (\tilde{u} - u)
\]
(14)
where \(x_{AW} \in \mathbb{R}^{n_{AW}}\) is the anti-windup compensator state.

Figure 4 shows the system configuration for the design. The difference between the states of the plants \(\bar{P}(s)\) and \(P(s)\) is denoted by \(\Delta x_p = x_p - \bar{x}_p\), and that of the controllers \(\bar{K}(s)\) and \(K(s)\) is denoted by \(\Delta x_c = x_c - \bar{x}_c\). These are introduced to estimate the differences of the states of the actual feedback system and the linear model.

In the previous studies (see, e.g.,\(^{(6)}\),\(^{(10)}\)–\(^{(12)}\)), the \(L_2\) gain from \(w\) to \(z\) is used as the performance index, namely, the problem is to minimize \(\gamma\) subject to
\[
\|z\|_2 < \gamma \|w\|_2
\]
(15)
with guaranteed internal stability and the following output $z$ is used to estimate the control performance.

$$z = r - y$$

(16)

where $r$ denotes for reference input. The closed-loop system can be globally stabilized for stable plants by applying the systematic design procedures in many cases, but the tracking performance tends to be sluggish sometime. In this case, we observe by simulation that the integrator state seems to be compensated too much in the sense that $x_c$ is fairly far from $\bar{x}_c$. Therefore, it seems desirable to control $x_c$ so that it may not go far from $\bar{x}_c$. Attempt to modify the controller state of saturated system to mimic that of unsaturated system is conducted in Ref. (13) where design for dynamic compensator is given only for the case where the plant is open loop stable and the different approach with this paper is used. Since inappropriate plant state can trigger undesired oscillations, even in the absence of dynamic controller elements(9), it seems necessary to consider the plant state $x_p$ in the performance index.

As far as we know, there has been no study in the literature that considers the controller state $x_c$ and the plant state $x_p$, with $\Delta x_c$ and $\Delta x_p$ as the performance output $z$. Considering that the windup phenomena are caused by the controller and the plant having badly damped or unstable mode, it is reasonable to include the state of the controller and the plant in the performance index. Therefore, we will consider the next $z$.

$$z = \begin{bmatrix} W_p \Delta x_p \\ W_c \Delta x_c \end{bmatrix}$$

(17)

The constant weights $W_p$ and $W_c$ are used for the tuning of the balance between the plant-state difference $\Delta x_p$ and the controller-state difference $\Delta x_c$. Thus our problem is to minimize $\gamma$ subject to Eq. (15) where $z$ is defined by Eq. (17) with internal stability. In the next section, we will derive an LMI-based synthesis procedure.

### 3. Anti-windup Compensator Design

By using the deadzone function

$$d = \Psi(\tilde{u}) = \tilde{u} - \Phi(\tilde{u})$$

(18)

the system in Fig. 4 can be represented as that of Fig. 5. The system $\mathcal{H}(s)$ in this figure is described by

$$\dot{x}_h = A_h x_h + B_{hw} w + B_{hd} d + B_{hv} v$$

$$\tilde{u} = C_{hu} x_h + D_{huv} w$$

$$z = C_{hc} x_h + D_{hcz} w$$

(19)

(20)

(21)
where $x_h \in \mathbb{R}^{n_h}$, $n_h = 2(n_p + n_c)$ and

$$A_h = \begin{bmatrix} A_{cl} & 0 \\ 0 & A_{cl} \end{bmatrix}, \quad A_c = \begin{bmatrix} A_p + B_{pu}D_{cy}C_p & B_{pu}C_c \\ B_{cy}C_p & A_c \end{bmatrix}$$

$$B_{hw} = \begin{bmatrix} B_{pw} + B_{pu}D_{cw} \\ B_w \\ 0 \\ 0 \end{bmatrix}, \quad B_h = \begin{bmatrix} -B_{pu} \\ 0 \\ -B_{pu} \\ 0 \end{bmatrix}, \quad B_{hw} = \begin{bmatrix} I \\ 0 \\ I \\ 0 \end{bmatrix}$$

$$C_{hw} = \begin{bmatrix} D_{cy}C_p & C_c & 0 & 0 \\ 0 & 0 & W_p & 0 \\ 0 & 0 & W_z & 0 \end{bmatrix}^T, \quad D_{hw} = D_{cw}, \quad x_h = \begin{bmatrix} x_p^T & x_c^T & \Delta x_p^T & \Delta x_c^T \end{bmatrix}^T$$

For the performance index defined in Eq. (17), the matrices $C_h$ and $D_{haw}$ are defined as follow

$$C_{hc} = \begin{bmatrix} 0 & 0 & W_p & 0 \\ 0 & 0 & 0 & W_z \end{bmatrix}, \quad D_{hcw} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Further, the system in Fig. 5 can be represented as that of Fig. 6 with the system $\mathcal{N}(s)$ being described by

$$\dot{x} = Ax + B_aw + B_ad$$

$$\tilde{u} = C_x x + D_{aw}w$$

$$z = C_z x + D_{cw}w$$

where $x \in \mathbb{R}^n$, $n = n_h + n_{aw}$ and

$$A = A_u + H_1^T \Lambda G_1, \quad B_d = B_{ad} + H_1^T \Lambda G_2$$

$$A_u = \begin{bmatrix} A_h & 0 \\ 0 & 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} B_{hw} \\ 0 \end{bmatrix}, \quad B_{ud} = \begin{bmatrix} B_{hd} \\ 0 \end{bmatrix}$$

$$C_u = \begin{bmatrix} C_{hw} & 0 \end{bmatrix}, \quad D_{uw} = D_{hw}, \quad C_z = \begin{bmatrix} C_{hc} & 0 \end{bmatrix}, \quad D_{cw} = D_{hcw}$$

$$x = \begin{bmatrix} x_p^T & x_c^T & \Delta x_p^T & \Delta x_c^T \end{bmatrix}^T, \quad \Lambda = \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & I \\ 0 & I & 0 & I & 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Fig. 6 Nonlinear feedback system with sector condition

Considering that $\Psi(\cdot)$ satisfies the next sector condition in the finite interval $[-\Xi, \Xi]$ with $\Xi = (1/(1 - \kappa_i))u_{sat}(i)$, as illustrated in Fig. 7,

$$\Psi(\tilde{u})^T W[\Psi(\tilde{u}) - \kappa \tilde{u}] \leq 0$$

where
Fig. 7 Deadzone function and the sector condition

\[ W = \text{diag}[w_1, w_2, \ldots, w_n] > 0 \]

\[ \kappa = \text{diag}[\kappa_1, \kappa_2, \ldots, \kappa_n], \quad 0 \leq \kappa_i \leq 1 \]

Then, the next theorem guarantees the \( L_2 \) gain condition described in Eq. (15) for the nonlinearity that satisfies Eq. (26).

**Theorem 1** For a given \( \kappa \), if there exist a positive-definite symmetric matrix \( Q \in \mathbb{R}^{n \times n} \), a diagonal matrix \( T = \text{diag}[t_1, t_2, \ldots, t_n] > 0 \), and a scalar \( \gamma > 0 \) that satisfies the next matrix inequality,

\[
Y + H^T \Lambda G + G^T \Lambda^T H < 0
\]

Where

\[
Y = \begin{bmatrix}
QA_w + A_w Q & B_w & B_{od} T + QC_w & QC_z \\
* & -\gamma I & D_{uw} \kappa & D_z^T \\
* & * & -2T & 0 \\
* & * & * & -\gamma I \\
\end{bmatrix}
\]

\[
H = \begin{bmatrix}
H_1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
G_1 Q & G_2 T & 0 \\
\end{bmatrix}
\]

Then, the feedback system described in Eqs. (23)∼(25) with

\[
\Lambda = \begin{bmatrix}
A_{uw} & B_{uw} \\
C_{uw} & D_{uw} \\
\end{bmatrix}
\]

is asymptotically stable and the \( L_2 \) gain from \( w \) to \( z \) is less than \( \gamma \) when the condition \( |\tilde{u}_i| \leq (1/(1 - \kappa_i)) u_{\text{sat}(i)}, \quad i = 1, 2, \ldots, n_u \) holds.

**Proof 1** Consider a Lyapunov quadratic function

\[
V(x) = x^T P x, \quad P = P^T > 0, \quad P \in \mathbb{R}^{n \times n}
\]

In order to show that the closed-loop system of Fig. 6 is asymptotically stable and the \( L_2 \) gain from \( w \) to \( z \) is less than \( \gamma \), we may show that the Lyapunov function in Eq. (32) satisfies the next dissipation inequality.

\[
\frac{dV}{dt} < \gamma w^T w - \frac{1}{\gamma} z^T z
\]

By using the sector condition Eq.(26) and the S-procedure, we obtain

\[
\frac{dV}{dt} + \frac{1}{\gamma} z^T z - \gamma w^T w - 2d^T X (d - k \tilde{u}) < 0
\]

Define \( \xi = \begin{bmatrix} x^T & w^T & d^T \end{bmatrix}^T \), the left term of this last equation can be written in the form

\[
\xi^T \begin{bmatrix}
A^T P + PA + \frac{1}{\gamma} C_c^T C_z & PB_w + \frac{1}{\gamma} C_c^T D_{cw} & PB_d + C_t^T \kappa X \\
* & -\gamma I + \frac{1}{\gamma} D_{uw}^T D_{uw} X & D_{uw}^T \kappa X \\
* & * & -2X \\
\end{bmatrix} \xi
\]
Since Eq. (35) must hold for all $x$, $w$, and $d$, the matrix $P > 0$ must satisfy the next constraint
\[
\begin{bmatrix}
A^T P + PA + \frac{1}{\gamma} C_1^T C_z & PB_w + \frac{1}{\gamma} C_2^T D_{zw} & PB_d + C_0^T kX \\
\star & -\gamma I + \frac{1}{\gamma} D_{zw}^T D_{zw} & D_{zw}^T kX \\
\star & \star & -2X
\end{bmatrix} < 0 \quad (36)
\]
Applying the Schur complement to Eq. (36), we obtain
\[
\begin{bmatrix}
A^T P + PA & PB_w & PB_d + C_0^T kX & C_2^T \\
\star & -\gamma I & D_{zw}^T kX & D_{zw}^T \\
\star & \star & -2X & 0 \\
\star & \star & \star & -\gamma I
\end{bmatrix} < 0 \quad (37)
\]
Applying a simple congruence transformation block-diag $\{P^{-1}, I, X^{-1}, I\}$ to Eq. (37) and define $Q = P^{-1}$, $T = X^{-1}$, we have
\[
\begin{bmatrix}
Q A^T + AQ & B_w & B_d T + QC_0^T k & QC_2^T \\
\star & -\gamma I & D_{zw}^T kX & D_{zw}^T \\
\star & \star & -2T & 0 \\
\star & \star & \star & -\gamma I
\end{bmatrix} < 0 \quad (38)
\]
Equation (38) can be written as
\[
\begin{bmatrix}
Q A^T + AQ & B_w & B_d T + QC_0^T k & QC_2^T \\
\star & -\gamma I & D_{zw}^T kX & D_{zw}^T \\
\star & \star & -2T & 0 \\
\star & \star & \star & -\gamma I
\end{bmatrix} +
\begin{bmatrix}
H_1^T \Lambda G_1 Q + QG_1^T \Lambda^T H_1 & 0 & 0 \\
\star & 0 & 0 & 0 \\
\star & \star & 0 & 0 \\
\star & \star & \star & 0
\end{bmatrix} < 0 \quad (39)
\]
Define $H = \begin{bmatrix} H_1 & 0 & 0 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} G_1 Q & 0 & G_2 T & 0 \end{bmatrix}$, then Eq. (39) can be written as Eq. (27)

The relation in Eq. (27) is a BMI with respect to variable $Q$ and $\Lambda$. Based on the elimination of matrix variables\(^{(14)}\), this BMI condition can be solved with the following steps:

**Procedure 1 (Anti-windup compensator construction):**

**Step 1)** For the given $\kappa$ and $n_{aw}$, find a solution to $R, S, T$ and minimize $\gamma$ to the following set of LMIs
\[
\begin{bmatrix}
R A^T_h + A_h R & B_{hw} & B_{hw} T + R C_0^T k & R C_2^T \\
\star & -\gamma I & D_{hw}^T kX & D_{hw}^T \\
\star & \star & -2T & 0 \\
\star & \star & \star & -\gamma I
\end{bmatrix} L \leq 0 \quad (40)
\]
\[
\begin{bmatrix}
S A^T_h + A_h S & B_{hw} & S C_2^T \\
\star & -\gamma I & D_{hw}^T \\
\star & \star & -\gamma I
\end{bmatrix} < 0 \quad (41)
\]
\[
R = \begin{bmatrix} R_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} = R^T > 0 \quad (42)
\]
\[
S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} = S^T > 0 \quad (43)
\]
\[
R_{11} - S_{11} > 0 \quad (44)
\]
with

\[ L = \text{block-diag}(\bar{W}_H, I, I) \]  
\[ \bar{W}_H = \begin{bmatrix} I_{n_p} & 0 & 0 \\ 0 & I_{n_c} & 0 \\ 0 & 0 & I_{n_p} \\ 0 & -I_{n_c} & 0 \end{bmatrix} \]  

(45)

(46)

**Step 2)** Using the solution \((R, S, \gamma)\) from Step 1, define \(N \in \mathbb{R}^{n_h \times n_w}\) as a solution of the following equation

\[ NN^T = RS^{-1}R - R \]  

and define the matrix \(M \in \mathbb{R}^{n_w \times n_w}\) as

\[ M = I + N^TR^{-1}N \]  

(47)

(48)

then construct the matrix \(Q \in \mathbb{R}^{n_h \times n_h}\)

\[ Q = \begin{bmatrix} R & N \\ N^T & M \end{bmatrix} \]  

(49)

**Step 3)** With \(T\) and \(Q\) from previous steps, solve the LMI in Eq. (27) to obtain \(\Lambda\).

**Theorem 2** For the given plant \(P\), the controller \(K\), an integer \(n_w\) and a solution \(R, S, T, \gamma\) of the Step 1 of the above procedure. The solution \(\Lambda\) from Step 3 defines the matrices of anti-windup compensator Eq. (31) that guarantees asymptotic stability and that the \(L_2\) gain of the feedback system Eqs. (23)~(25) from \(w\) to \(z\) being less than \(\gamma\) when the condition \(|\tilde{u}_i| \leq (1/(1-\kappa_i))u_{sat}(i), \ i = 1, 2, \ldots, n_u\) holds.

**Proof 2** Based on the elimination of matrix variables\(^{(14)}\), denote \(W_H\) and \(W_G\) any matrices whose columns form bases of the null space of \(G\) and \(H\), respectively, then Eq. (27) is solvable for \(\Lambda\) if and only if

\[ W_H^TYW_H < 0 \]  

\[ W_G^TYW_G < 0 \]  

(50)

(51)

**Condition 1** \((W_H^TYW_H < 0)\) The null space of \(H\) is in the form

\[ W_H = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \]  

(52)

Therefore the right side of (50) can be written as

\[ \begin{bmatrix} W_{H1}^T(\dot{Q}A_{u} + A_{u}Q)W_{H1} & W_{H1}^TB \star & W_{H1}^TB_{cd}T + W_{H1}^TC_{d}^T \star & W_{H1}^TC_{z}^T \\ \\ \star & -\gamma I & \star & \star & \\ \star & \star & -2T & \star & \\ \star & \star & \star & -\gamma I \end{bmatrix} \]  

(53)

Define

\[ Q = \begin{bmatrix} R & N \\ N^T & M \end{bmatrix} \]  

(54)
where $R \in \mathbb{R}^{n_1 \times n_2}, N \in \mathbb{R}^{n_1 \times n_3}, M \in \mathbb{R}^{n_2 \times n_4}$, we have

\[
W_{H1}^T A_o Q W_{H1} = \begin{bmatrix} \bar{W}_{H1}^T & 0 \\ A_h & 0 \\ 0 & N^T M \end{bmatrix} = \bar{W}_{H1} A_h R \bar{W}_{H1} 
\]

(55)

\[
W_{H1}^T B_w = \begin{bmatrix} \bar{W}_{H1}^T & 0 \\ B_{hw} & 0 \end{bmatrix} = \bar{W}_{H1} B_{hw} 
\]

(56)

\[
W_{H1}^T B_{ad} T = \begin{bmatrix} \bar{W}_{H1}^T & 0 \\ B_{hd} & 0 \end{bmatrix} T = W_{H1}^T B_{ad} T 
\]

(57)

\[
W_{H1}^T Q C_o & = \begin{bmatrix} \bar{W}_{H1}^T & 0 \\ R & N^T M \end{bmatrix} \begin{bmatrix} C^T_{hw} & 0 \end{bmatrix} = \bar{W}_{H1}^T R C^T_{hw} 
\]

(58)

\[
W_{H1}^T Q C_z & = \begin{bmatrix} \bar{W}_{H1}^T & 0 \\ R & N^T M \end{bmatrix} \begin{bmatrix} C^T_{h_c} & 0 \end{bmatrix} = \bar{W}_{H1}^T R C^T_{h_c} 
\]

(59)

So Eq. (50) can be written as Eq. (40)

Condition 2 ($\bar{W}_{G1}^T Y W_G < 0$) Define

\[
G = \begin{bmatrix} G_1 \bar{Q} & 0 & G_2 T & 0 \end{bmatrix} = G_o \bar{Q} 
\]

(60)

with $G_o = \begin{bmatrix} G_1 & 0 & G_2 & 0 \end{bmatrix}$ and $\bar{Q} = \text{diag}(Q, I, T, I)$. The right side of Eq. (51) can be written as

\[
\bar{W}_{G1}^T Y W_G = \frac{\bar{W}_{G1}^T \bar{Q} - \bar{Q}^{-1} Y \bar{Q}^{-1}}{\bar{Q} W_G} = \bar{W}_{G1}^T Y W_G 
\]

(61)

Define $P = Q^{-1}$

\[
\tilde{Y} = \begin{bmatrix} A_o^T P + P A_o & PB_{hw} & PB_{hd} T + C_o^T \kappa & C_z^T \\
\star & -\gamma I & D_{hw}^T \kappa & D_{h_c}^T \\
\star & \star & -2 T & 0 \\
\star & \star & \star & -\gamma I 
\end{bmatrix} 
\]

(62)

Further $W_{Go}$ can be written as

\[
W_{Go}^T = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & I 
\end{bmatrix} = \begin{bmatrix} W_{G1} & 0 & 0 \\
0 & I & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{bmatrix}, \quad W_{G1} = \begin{bmatrix} I 
\end{bmatrix} 
\]

(63)

where $W_{G1} \in \mathbb{R}^{n_1 \times n_2 \times n_3 \times n_4}$. The right side of Eq. (51) can be written as

\[
\begin{bmatrix} W_{G1}^T A_o^T P W_{G1} + W_{G1}^T P A_o W_{G1} & W_{G1}^T P B_{hw} \\
\star & -\gamma I \\
\star & \star & -\gamma I 
\end{bmatrix} 
\]

(64)

Define

\[
P = \begin{bmatrix} P_{11} & P_{12} \\
P_{12}^T & P_{22} 
\end{bmatrix} 
\]

(65)

where $P_{11} \in \mathbb{R}^{n_1 \times n_1}, P_{12} \in \mathbb{R}^{n_1 \times n_3}, P_{22} \in \mathbb{R}^{n_2 \times n_2}$, we have

\[
W_{G1}^T P A_o W_{G1} = \begin{bmatrix} I & 0 \\
0 & I \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\
P_{12}^T & P_{22} 
\end{bmatrix} \begin{bmatrix} A_h & 0 \\
0 & 0 \end{bmatrix} = P_{11} A_h 
\]

(66)

\[
W_{G1}^T P B_{hw} = \begin{bmatrix} I & 0 \\
0 & I \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\
P_{12}^T & P_{22} 
\end{bmatrix} \begin{bmatrix} B_{hw} \end{bmatrix} = P_{11} B_{hw} 
\]

(67)

\[
W_{G1}^T C_z^T = \begin{bmatrix} I & 0 \\
0 & C_z^T \end{bmatrix} = C_z^T 
\]

(68)
Therefore Eq. (51) can be written as
\[
\begin{bmatrix}
A^T_{h} P_{11} + P_{11} A_h & P_{11} B_{sw} & C_{h c}^T \\
* & -\gamma I & D_{c m}^T \\
* & * & -\gamma I
\end{bmatrix} < 0
\]
(69)

Applying a simple congruence transformation block-diag \((P_{11}^{-1}, I, I)\) and define \(P_{11}^{-1} = S\), Eq. (69) can be written as Eq. (41).

**Condition 3 (relation between matrices \(R\) and \(S\))** Matrices \(R\) and \(S\) have the following relation
\[
Q^{-1} = \begin{bmatrix} R & N \\ N^T & M \end{bmatrix}^{-1} = P = \begin{bmatrix} S^{-1} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}
\]
(70)

Next, by the formula for the inversion of block matrices\(^{(15)}\), the upper left block of \(P\) needs to satisfy
\[
P_{11} = S^{-1} = R^{-1} + R^{-1}N\left(M - N^TR^{-1}N\right)^{-1}N^TR^{-1}
\]
(71)
\[
S = R - NM^{-1}N^T
\]
(72)

Therefore, the relation (70) holds if the following conditions are met
\[
R - S = NM^{-1}N^T \geq 0
\]
(73)
\[
\text{rank}(R - S) = \text{rank}(NM^{-1}N^T) \leq n_{sw}
\]
(74)

Define
\[
R = \begin{bmatrix} R_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}
\]
with \(R_{11} \in \mathbb{R}^{n_{sw} \times n_{sw}}, S_{11} \in \mathbb{R}^{n_{sw} \times n_{sw}}, S_{12} \in \mathbb{R}^{n_{sw} \times (n_{h} - n_{sw})}, S_{22} \in \mathbb{R}^{(n_{h} - n_{sw}) \times (n_{h} - n_{sw})}\), and \(R_{11} = R_{11}^T > 0, S_{11} = S_{11}^T > 0, S_{22} = S_{22}^T > 0\), this gives
\[
R - S = \begin{bmatrix} R_{11} - S_{11} & 0 \\ 0 & 0 \end{bmatrix}
\]
(75)

Ensuring the following conditions holds
\[
R_{11} - S_{11} > 0
\]
(76)
the satisfaction of the relation Eq. (73) will follow. It is shown that \(\text{rank}(R_{11} - S_{11}) = n_{sw}\), and therefore the relation in Eq. (74) is satisfied. Further, by defining
\[
M = I + N^TR^{-1}N
\]
(77)
then pre and post-multiplying by \(R\) and substituting the selection for \(M\), Eq. (71) can be written as Eq. (47)

**Remark 1** Note that if the above design problem is satisfied for \(\kappa = I\), the sector condition is satisfied for the deadzone function globally. Therefore, the system with the deadzone function is guaranteed to be globally asymptotically stable.

4. Popov Stability and Performance Analysis

The stability and dissipativity of the above anti-windup compensator synthesis is established through the use of quadratic Lyapunov function, \(V(x) = x^TPx\). The resulting stability condition could be conservative. In order to reduce the conservativeness of the quadratic Lyapunov approach it would be preferable to use a different type of Lyapunov function, such as

\[\text{...}\]
Lur'e Lyapunov function. However, such choice of Lyapunov function does not lead to a convex problem for the anti-windup compensator synthesis. In this paper, we will use the Lur'e Lyapunov function for analysis purposes. We consider the Lur'e Lyapunov function

\[ V(x) = x^T P x + 2 \sum_{i=1}^{n_u} \int_0^x \Psi(\tilde{u}_i) Y_{(i,j)} \tilde{d} \tilde{u}_i \]  

(78)

with \( P = P^T > 0 \), \( P \in \mathbb{R}^{n \times n} \), and \( Y \) is a positive diagonal matrix.

**Theorem 3** For a given \( \kappa \), if there exist a symmetric positive definite matrix \( P \in \mathbb{R}^{n \times n} \), diagonal positive definite matrix \( W \in \mathbb{R}^{n \times n} \), diagonal matrix \( Z \in \mathbb{R}^{n \times n} \), and a scalar \( \gamma > 0 \) satisfying

\[
\begin{bmatrix}
PA + A^T P & PB_w & -\gamma I & D_w^T & C^T_w \\
PB_w & PB_d + C^T_w k w + A^T C^T Z & 0 & 0 & 0 \\
-\gamma I & 0 & -2W + ZC_u B_d + B_d^T C^T_u Z & 0 & 0 \\
D_w^T & 0 & 0 & -\gamma I & 0 \\
C^T_w & 0 & 0 & 0 & 0 \\
\end{bmatrix} < 0
\]

(79)

then the system described by Eqs. (23)–(25) is asymptotically stable and the \( L_2 \) gain from \( w \) to \( z \) is less than \( \gamma \) when the condition \( |\tilde{u}_i| \leq (1 / (1 - \kappa_i)) u_{sat}(i), \quad i = 1, 2, \ldots, n_u \) holds.

**Proof 3** The proof is similar, mutatis mutandis, to that of Theorem 1.

5. Examples

5.1. Example 1

Let us consider the model of the belt-drive experiment setup of Fig. 8(16). The coefficient matrices of the plant \( \mathcal{P}(s) \) and the controller \( \mathcal{K}(s) \) are given by

\[
\begin{bmatrix}
A_p & B_{pu} & B_{pw} \\
C_p & D_{pu} & D_{pw} \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-3.042e4 & -44.05 & 2.535e3 & 3.661 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
2.375e4 & 34.30 & -1.980e3 & -6.259 & 0 & 282.1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_c & B_{cw} & B_{cy} \\
C_c & D_{cw} & D_{cy} \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & -1 \\
0 & 1 & 1 & 1 \\
2.772 & -6.768e3 & 70.57 & -70.57 \\
\end{bmatrix}
\]

where \( x_p = [\theta_2^T, \omega_2^T, \theta_1^T, \omega_1^T]^T \). \( \theta_1 \) is the angular of the drive disk, \( \theta_2 \) is the angular of the load disk, and \( \omega_1 \) and \( \omega_2 \) are their angular velocities, respectively. We will consider the angular control problem of the load disk. The control input \( u \) is the input voltage to the motor \([V]\), and it satisfies

\[ |u(t)| \leq 5 \]  

(80)

Fig. 8 Configuration of the belt-drive system
The proposed anti-windup compensator synthesis is applied to the model of the belt-drive experiment. We choose the compensator order \( n_w \) the same as the plant order \( n_p \). Synthesizing the anti-windup compensator using sector bound \( \kappa = 1 \) means assuring global asymptotic stability by circle criterion. However, in this example we obtain that there is no major differences in the output response for all performance indices. Therefore, we reduce the sector bound \( \kappa \) and Table 1 shows the achievable performance for different values of the sector bound \( \kappa \). It can be seen that reducing the sector bound from \( \kappa = 1 \), can significantly improve the performance of anti-windup compensator. Choosing the value of the sector bound \( \kappa = 0.9 \) the achievable performance is around 0.99%, 0.67%, and 0.95% smaller compared with the achievable performance using \( \kappa = 1 \) for \( z = \Delta x_c \), \( z = \Delta x_p \), and \( z = r - y \), respectively. Further, the calculated performance level is achievable as long as the magnitude of the output from nominal controller is less than \((1/ (1 - \kappa)) \tilde{u}_{\text{sat}}(t)\).

Synthesizing the anti-windup compensator using sector bound \( \kappa = 0.9 \), we obtain the responses shown in Fig. 9 by simulation, for the step reference input \( r = 2\pi (= 6.2832) \) [rad]. From Fig. 9, the step response without the compensator has the largest overshoot and the convergence is slow. The overshoot is suppressed in the cases of \( z = \Delta x_p \) and \( z = r - y \). For \( z = \Delta x_c \), the response is fast but the overshoot still remain. Further, as expected that the controller state response for the case of \( z = \Delta x_c \) are more similar to that of the linear feedback system, as shown in Fig. 10.

![Fig. 9 Output y(t) of belt drive system](image-url)

![Fig. 10 State of the integrator](image-url)

Next, we apply the performance index defined in Eq. (17) and by tuning the weighting matrices \( W_p \) and \( W_c \) we can obtain better response, as shown in Fig. 11. In this example, we
obtain a good response by setting $W_p : W_c = 1 : 70$ with the anti-windup compensator given as

\[
\begin{bmatrix}
A_{aw} & B_{aw} \\
C_{aw} & D_{aw}
\end{bmatrix} =
\begin{bmatrix}
-54.971 & -119.990 & 361.970 & -3080.082 & -1.845 \\
27.109 & -95.726 & 98.320 & -2559.247 & -0.314 \\
-140.609 & 102.744 & -190.259 & 1307.702 & 3.160 \\
59.191 & -73.979 & 2605.465 & -22988.489 & -29.512 \\
-1.101 & -2.402 & 5.801 & -74.986 & -0.105 \\
0.028 & 0.0623 & -0.152 & 1.953 & -0.0119
\end{bmatrix}
\]

The experiment results for the belt-drive system are shown in Fig. 12. A good agreement between simulation and experiment results can be seen in Fig. 11 and Fig. 12. Further, although circle criterion only guarantee local asymptotic stability, it will be shown in the following section that global asymptotic stability is guaranteed by Popov criterion for this feedback system augmented with the obtained anti-windup compensator.

5.2. Example 2

In order to study the robustness of our method, we apply our method to the academic problem used in Ref. (12). The coefficients of the plant and the controller are given by

\[
\begin{bmatrix}
A_p & B_{pu} & B_{pu} \\
C_p & D_{pu} & D_{pu}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-10 & a & 0 & 10 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_c & B_{cw} & B_{cw} \\
C_c & D_{cw} & D_{cw}
\end{bmatrix} =
\begin{bmatrix}
-80 & 0 & 2.5 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & -2.5 & 1 & 0 \\
-9450 & 3375 & 337.5 & 0 & -135
\end{bmatrix}
\]
where \( a = -10 \) for the nominal plant \( P \) and \(-0.01\) for the perturbed plant \( \tilde{P} \).

Analyzing the performance of the anti-windup closed-loop system for different sector bound as in Example 1, we choose sector bound \( \kappa = 0.8 \). Synthesizing the anti-windup compensator based on the nominal plant and \( n_{aw} = n_p \), we obtain the responses shown in Fig. 13(a) for the perturbed plant case. Figure 13(a) shows oscillatory behavior for the feedback system without anti-windup compensator. The system can be stabilized by adding the anti-windup compensator. Interestingly, the tracking performance for the case \( z = \Delta x_c \) is worse compared with the other performance index. It is probable due to plant dynamics that is not considered in this performance index, where the plant has badly damped mode for the perturbed plant case.

![Fig. 13 Output \( y(t) \) for example 2](image)

Next, apply our proposed anti-windup compensator synthesis by tuning the weighting matrices \( W_p \) and \( W_c \). After several trial and error, we obtain a good response by setting \( W_p : W_c = 1 : 20 \), with anti-windup compensator is given as

\[
\begin{bmatrix}
A_{aw} & B_{aw} \\
C_{aw} & D_{aw}
\end{bmatrix} = \begin{bmatrix}
-41.249 & 796.265 & 0.562 \\
-159.385 & -39447.7 & 42.909 \\
0.057 & -2.447 & -0.0057 \\
-0.011 & 0.502 & -0.0005 \\
-0.068 & 3.472 & -0.0027
\end{bmatrix}
\]

The response of the saturated system with this anti-windup compensator is shown in Fig. 13(b). Further, it is shown that the global asymptotic stability is guaranteed by Popov criterion, as shown in the following section.

5.3. Popov Stability and Performance Analysis

By reducing the sector bound from \( \kappa = 1 \) means that the global asymptotic stability cannot be guaranteed, and the asymptotic stability is guaranteed for the magnitude of the controller output \( \dot{u} \) to be less that \( u_{sat}(1 - \kappa) \), see Fig. 7. However, it is well-known that in some cases the resulting stability condition by the circle criterion are extremely conservative. Therefore, the \( L_2 \) gain and stability of the closed-loop system augmented with the obtained anti-windup compensator for the case \( z = W_p \Delta x_p : W_c \Delta x_c \) is further analyzed using Theorem 3. Figure 14 shows the \( L_2 \) gain as a function of \( \Xi \) for Examples 1 and 2. It can be seen from this figure that global stability is guaranteed.

6. Conclusion

A design method of a dynamic anti-windup compensator for good tracking performance is presented. The approach is based on the circle criterion and \( L_2 \) performance index that leads to LMI. It is shown that by considering the \( L_2 \) gain performance index that contains both the plant state and the controller state, we can obtain good tracking performance. Reducing the sector bound value from 1, has the potential for performance improvement, while the global stability is guaranteed by Popov criterion. The belt-drive experiment setup and the numerical
example taking from the literature are used to demonstrate the effectiveness of the proposed approach.

References

(16) -, Model 220: Industrial plant emulator, Educational Control Products (ECP).