Symmetry detection through local skewed symmetries

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We explore how global symmetry can be detected prior to segmentation and under noise and occlusion. The definition of local symmetries is extended to affine geometries by considering the tangents and curvatures of local structures, and a quantitative measure of local symmetry known as symmetry is introduced, which is based on Mahalanobis distances from the tangent-curvature states of local structures to the local skewed symmetry state-subspace. These symmetry values, together with the associated local axes of symmetry, are spatially related in the local skewed symmetry field (LSSF). In the implementation, a fast, local symmetry detection algorithm allows initial hypotheses for the symmetry axis to be generated through the use of a modified Hough transform. This is then improved upon by maximizing a global symmetry measure based on accumulated local support in the LSSF—a straight active contour model is used for this purpose. This produced useful estimates for the axis of symmetry and the angle of skew in the presence of contour fragmentation, artifacts and occlusion.

Keywords: skewed symmetry, symmetry axis, local symmetries, Hough transform, perceptual organization

INTRODUCTION AND MOTIVATION

Symmetry, like the other properties described in the Gestalt principles of perceptual grouping\(^1\), represents information redundancy which may be used to overcome noise and occlusion. The reason why it is rarely used in computational perceptual organization\(^1,2\) is that symmetry is much harder to detect compared to other grouping properties such as continuity and token similarity. Nevertheless, our visual system seems to be able to recover symmetry from images even when the symmetry is badly distorted\(^3\).

Two-dimensional symmetry can be exploited in various ways in computer vision. For example, the symmetry axis transforms and smoothed local symmetries\(^4\) are intended for 2D shape description. Other applications to symmetry detection, or asymmetry detection, include medical image analysis\(^6\) and feature detection\(^7\).

Much of the research literature is concerned with bilateral symmetry detection. Abstract mathematical theory\(^8,9\) has been used, but in most cases is only intended to handle perfect set of figures. On the other hand, Marola\(^10\) catered for deviations from perfect symmetry. Others adopted different approaches, e.g. Labonte et al.\(^11\), who detected symmetry in textured patterns over a number of stages. Directional correlations were used by Masuda et al.\(^12\) to extract partial symmetry, while Reisfeld et al.\(^7\) employed a low-level symmetry operator to detect small salient features.

Under normal viewing conditions, however, we are more likely to encounter skewed symmetry\(^13\), which is planar bilateral symmetry transformed affinely when viewed under weak perspective\(^14\).

Brady and Yuille\(^15\) proposed maximizing a compactness measure to backproject a closed image contour displaying skewed symmetry, i.e. parallelograms are interpreted as squares, and ellipses as circles. Friedberg\(^16,17\) described a method based on moments to detect skewed symmetry axes of 2D shapes, which assumed prior segmentation of objects and hence difficult to apply to real images. Glachet et al.\(^18\) proposed a technique for estimating symmetric axes in line drawings under full perspective projection, involving recovery of accurate straight line segments from the convex-hull of the figure, pairing them and predicting the symmetry parameters through the use of cross-ratios. Ponce\(^19\) exploited straight-spined Brooks Ribbons and their curvature relationships to pick out symmetrical pairs of points. Yuen\(^20\) suggested carrying out Hough transforms on the sets of mid-points of point-pairs lying at a particular orientation, and doing this over a series of discretized orientations, but did not implement it. Mukherjee et al.\(^21\) exploited bitangents in objects to define affine bases, from which relative invariants are used to compare pairs of contours with...
bitangents for symmetry. The transformation matrix relating the affine bases is then tested for conformity with the 3-DOF structure required for affine-transformed reflections. Van Gool et al. constructed the arc length space (ALS)\(^{22}\) to analyse symmetric contours, and also studied their invariant signatures\(^{23}\).

Many of these methods apply only to objects which are easy to segment, have complete contours or are special cases, and do not consider circumstances when symmetry will be useful to perceptual grouping, such as in cases where objects are occluded, or have fragmented contours.

In this paper, a local approach to skewed symmetry detection is adopted to handle occlusion. Sensitivity to noise inherent in local measurements is coped with through non-committal classification, uncertainty analysis, and arriving at global decisions by accumulation of independent, local support.

THEORETICAL FRAMEWORK

Transformations of 2D reflectional symmetry

Image contours of planar objects may be considered under three classes of 2D transformations: scaled Euclidean, affine or projective. A 2D scaled Euclidean transformation requires four parameters for unique specification (scale, rotation and translation in 2D), an affine transformation requires six (including magnitude and axis of pure shear) and a projective transformation requires eight (additional location of horizon to specify the perspective distortion).

As symmetry is similar to having two views of half the object, symmetry bases under these transformations require less parameters to specify:

- **Scaled Euclidean**: Two parameters are required for the axis of symmetry. The lines joining symmetric points are perpendicular to this line.
- **Affine**: Three parameters are required to additionally define the angle of skew, which is not perpendicular to the axis of symmetry.
- **Projective**: Four parameters are needed, because instead of an angle of skew, we have a vanishing point for lines of skew.

A symmetric shape under these transformations is shown in Figure 1. Since skewed symmetry and stereo are similar, the two cases share similar difficulties such as the correspondence problem. A common task after identifying a shape as belonging to affine or projective symmetry is to find the transformation under its class which maps the shape to one exhibiting scale Euclidean symmetry. This is known as backprojection or deskewing.

PROPERTIES OF SKEWED SYMMETRY

The basic properties of skewed symmetry are presented in a number of articles\(^{13,24,25}\). Some of these properties which are incorporated into key ideas in this paper are mentioned below.

Consider a symmetric shape \(S\) and its affine-transformed contour \(S'\) (e.g. Figures 1a, b, respectively).

**Property 1** Tangents of a symmetric pair of points belonging to \(S'\) will intersect at the symmetry axis if they are not parallel, in which case they intersect at the symmetry axis at infinity (Figure 2a).

**Property 2** The orientation of the skewed axis of symmetry for \(S'\) must lie between the extrapolated directions of the two tangents at every pair of symmetric points on \(S'\) (Figure 2b).

Property 2 constrains the direction of the axis of symmetry at every intersection point of tangents belonging to symmetric contour point pairs, and is also invariant under projective transformation.

The following proposition states that the angle of skew may be calculated based on the tangent directions at every such intersection point:

**Proposition 1** Given only the directions of tangents belonging to a symmetric edgel pair in an image, and the direction of the skewed symmetry axis, we can compute a unique direction of skew resulting from a general affine view of the object, given by:

\[
\tan \theta = \frac{1}{2} (\cot \gamma - \cot \beta)
\]

where \(\theta\), \(\beta\) and \(\gamma\) are as shown in Figure 3.

**Proof** Referring to Figure 3:

\[
k = d \tan \theta
\]

but:

\[
\frac{d}{\tan \gamma} - \frac{d}{\tan \beta} = 2k
\]

and eliminating \(k\):

\[
\tan \theta = \frac{1}{2} \left( \frac{1}{\tan \gamma} - \frac{1}{\tan \beta} \right) = \frac{1}{2} (\cot \gamma - \cot \beta)
\]

Since there is only one direction of skew per symmetry axis under affine transformation, it follows that the direction of skew computed for all tangent-pair intersections lying on the symmetry axis should theoretically be the same.

Backprojection may be used to deskew \(S'\). Although

![Figure 1](image1.png)
it may appear that the planar orientation of the shape is uniquely determined, this is not true, and only a one-parameter family of shapes with orthogonal symmetry axes may be recovered.

**Local skewed symmetries**

There are two justifications for the use of local measures:

- local measures are less sensitive to occlusions compared to global ones; and
- the correspondence problem is reduced by comparing pairs of contour points and their derivatives rather than comparing larger sets of points.

These two factors are important when attempting to include symmetry as a viable grouping technique in computational perceptual organization.

For a point on a curve, its description may be expressed as a Taylor series expansion. The zeroth order term gives the location of the point, the first order term the orientation of the tangent, the second order term the curvature, etc. The locations of two skewed symmetric points provide two constraints on two parameters: the angle of skew and the mid-point between them (the axis of symmetry must pass through this mid-point). Orientation information gives another constraint to fully define the axis of symmetry by Property 1, which states that the intersection of tangents to a pair of skewed symmetric contour points lie on the axis of symmetry.

Having uniquely defined the symmetry basis consisting of an axis of symmetry and an angle of skew for the pair of points, we can verify the basis through a further constraint based on curvature:

**Proposition 2** Suppose \( \kappa_a \) and \( \kappa_b \) are the curvatures (inverses of the radii of curvature \( R_a \) and \( R_b \) shown in Figure 3) and \( \sigma_a \) and \( \sigma_b \) are the respective angles between the curve normals and the axis of skew, as shown in Figure 4. Then, if the two points are skewed symmetric, the following must hold:

\[
\frac{\kappa_a}{\kappa_b} = \left( \frac{\cos \sigma_a}{\cos \sigma_b} \right)^3
\]  

The result is similar to that given by Ponce, but the derivation is obtained independently via a different approach, and shown in the appendices. We can extend this constraint verification process to the highest order derivative which the curve is locally differentiable. This allows us to extend the definition of local symmetries to different transformation groups:

**Definition 1** A local symmetry for a particular transformation group refers to a pair of contour points defining a unique symmetry basis within that group. The required number of parameters is obtained through the derivatives of the contours at those points. Additionally, the symmetry basis must also be consistent with at least one other independent constraint provided by the next higher order derivative.

Notice that the smoothed local symmetries proposed by Brady and Asada satisfy this definition, since a symmetry basis under scaled Euclidean transformations may be uniquely defined by the positions of two points; the requirement that the orientations of the tangents to
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the contours must be equal with respect to the line joining the points, provides the additional independent constraint. We can also define local skewed symmetries, for which the positions and the tangents at the symmetric contour points define a unique symmetry basis under affine transformations, and the curvatures provide the extra constraint to be satisfied in the form of (5).

Another important point to note is that pairs of points on straight line segments are not distinguishable because there are no second derivatives—indeed, every pair of points on two straight line segments are local skewed symmetries. This is analogous to the local aperture problem because of the lack of distinguishable points*.

Symmetry

The reliability of derivatives extracted from image features degrades considerably with increasing order (of the derivatives), and attempting to classify pairs of points on contours as local skewed symmetries is unlikely to be accurate. It is therefore proposed that a continuous measure of local symmetry be adopted based on (5).

It is possible to rewrite (5) as $\kappa_a \cos^3 \sigma_a = \kappa_b \cos^3 \sigma_b$, and select the terms on both sides of the equation as state variables. In Figure 5, the state space with state variables $v = \kappa_a \cos^3 \sigma_a$ and $u = \kappa_b \cos^3 \sigma_b$ is shown for a pair of contour points (the formulation is modified from that originally proposed earlier). The state of the contour-point pair will map onto a single spot $(u_0)$ in this state space, and the straight line $v = u$ is the subspace for which local skewed symmetry holds. Noise perturbs the positions of contour-point pairs in this state space, and therefore the proposed symmetry measure is based on the shortest distance to the subspace. Since the error variances are not the same nor necessarily independent for both state variables, then on the assumption that the errors are Gaussian†, Mahalanobis distances‡ are used instead of Euclidean distances.

Assuming that the covariance matrix $K$ between $u$ and $v$ may be obtained (e.g. via automated B-spline fitting; see a later section), then the Mahalanobis distance squared $d^2$ is given by:

$$d^2 = (u - u_0)' K^{-1} (u - u_0)$$

(6)

where $u = [u \ v]'$ and $u_0$ is the observed state of the contour-point pair. The shortest Mahalanobis distance to the local skewed symmetry subspace may be found by substituting $v = u$ into (6), and minimizing the quadratic equation. These steps are further elaborated in the appendix.

*We will, however, be able to determine if line segments, are symmetrical if we know the exact end points, without the need for curvature.
†The errors in observation data (orientation, etc.) are assumed to be Gaussian. If these errors are small, then considering only first order terms in the Taylor series expansion of the expressions for the state variables, the errors in the state variables are also approximately Gaussian.

This leads us to the following redefinition for symmetricity:

**Definition 2** Given the shortest Mahalanobis distance $d$ from the observed state of the pair of contour points $a$ and $b$, to the local skewed symmetry subspace within the state-space spanned by $u$ and $v$ as described above, the symmetricity between $a$ and $b$ is given by:

$$\Psi_{a,b} = \frac{1}{1 + kd^2}$$

(7)

where $k$ is a empirical constant determining the scale of the Mahalanobis distance.

Local skewed symmetry field

To spatially represent the symmetry evaluation for each pair of contour points in an intuitive and useful way, we define a local skewed symmetry field for the set of contours in an image:

**Definition 3** The local skewed symmetry field (LSSF) for a set of contours is formed by:

- Assigning the symmetricity for a pair of points on the contours to the location of the mid-point of the line joining the pair of points. If more than one pair of points maps to this location, only the pair with highest symmetricity is retained. This is the magnitude of the field at the location.
- Assigning a direction vector representing the direction of the local axis of symmetry; this is termed the 'first direction' of the LSSF.
- Assigning another direction vector parallel to the line joining the pair of joints, representing the direction of skew; the 'second direction'.

Figure 5 In the above symmetricity state-space, the straight line $v = u$ represents the local skewed symmetry subspace. The symmetricity between a pair of contour points is inversely proportional to $(1 + \text{the shortest Mahalanobis distance squared} (d^2) \text{from their observed combined state position (} u_0 \text{) to the subspace})$
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To obtain curvature information to calculate the local skewed symmetry field, cubic B-splines are fitted to the output from the Canny edge-detector.

Figure 6 shows two planar symmetric objects to which cubic B-splines are manually fitted to the edge maps (obtained from the Canny edgefinder) in order to obtain the curvatures for computation of the local skewed symmetry fields.

The magnitude and first direction components of the local skewed symmetry fields are shown in Figure 7. Of importance is the fact that the peaks and ridges in the fields are the suggested local skewed symmetries. In the total absence of errors, these ridges will be analogous to the symmetry set for smoothed local symmetries. The symmetry values represent estimations of the likelihood for pairs of points to be local symmetries, and provide some robustness when noise is introduced.

These properties allow it to degrade more gracefully than if we were to use global measures or strict local measures. This makes it useful as a tool for recovering the global axis of symmetry from imperfect local skewed symmetries.

Recent research has also independently proposed a similar field for local bilateral symmetries, which is applied to locating useful points for fixation.

IMPLEMENTATION

Symmetry detection strategy

In images of objects with complicated contours, it is hard to automatically infer the axis of symmetry purely from its LSSF, much less do so in the presence of noise and occlusion. A vast amount of computation will be required to analyse and compare different portions of the LSSF. Our strategy is therefore, to:

1. Obtain initial hypotheses for the axis of symmetry from a fast but approximate symmetry detection process. The symmetry detection process has to use local measures as well to avoid problems of occlusion and incomplete contours.
2. Compute the local skewed symmetry field with the aid of B-splines (to obtain curvature information from the edge map of the image).
3. Improve the accuracy of the axis of symmetry and angle of skew by maximizing a global symmetry measure from local support in the LSSF.

Steps 1 and 2 can in principle be carried out in parallel.

Initial symmetry detection (Hough transform) technique

A novel, fast method is introduced for the initial symmetry detection. It is also consistent with our occlusion-insensitive requirements in that it allows local structures to vote independently for the global axis of symmetry and angle of skew, but in this case, the local structures are limited to low-curvature segments and straight edges.

Using the edge map of the image as input, the symmetry detection process comprises of four steps, as shown in Figure 9:

1. Extrapolate tangents of edgels in an array. This is shown in Figure 10. Each entry in the array contains data on the directions and numbers of tangents passing through it.
2. Recover locations of strong intersections. Entries with more than one registered direction are classified as intersection points. Each intersection point is given a strength corresponding to the number of votes for the second strongest direction, and only points above
a specified intersection threshold are retained. By recovering only strong intersections, we are effectively considering only the tangents belonging to low-curvature segments. The locations of these intersections are more accurate than those obtained from high-curvature segments.

3. Carry out a reduced Hough transform on the intersection points. The variation of the Hough transform used here parameterizes lines by the normal vector from the origin to the line, expressed in polar coordinates \((d, \theta)\). Each intersection point maps to a half-sinusoidal curve in Hough space, as may be seen in Figure 11. The additional knowledge of Property 2 allows us to reduce the Hough transform by coding only the relevant portions of the sinusoids representing the angular sweep between the directions of the extrapolated tangents, since the property states that the orientation of the axis of symmetry must lie between the extrapolation directions of tangents belonging to a skewed symmetric edge pair. This reduces the computation time as well as ignores false hypotheses such as those joining the tips of a serrated edge. The locations of multiple intersections in the Hough space represent hypotheses for symmetry axis, which is based on Property 1. Since there is no guarantee that the axis of symmetry has the largest number of intersections on it, we have to look at a number of strong hypotheses rather than just one.

4. Estimate angle of skew and rank the hypotheses. Ranking the hypotheses is important, since the more likely axes of symmetry should be passed onto the next stage first. This is achieved in a dual-purpose manner by predicting the angle of skew per hypothesis for the axis of symmetry. It is possible to calculate the angle of skew from just the directions of tangents at one intersection point and knowledge of the axis of symmetry, based on (1). Since we will have a number of intersection points contributing votes \(\theta_k\) for the angle of skew with different degrees of accuracy, we will sum \(\theta_k\) modified by an optimal set of weights \(a_k\) (obtained through sensitivity analysis and Lagrange multipliers) as expressed:

\[
\hat{\theta} = \sum_{k=1}^{N} a_k \theta_k
\]

where \(\hat{\theta}\) is the optimal estimate. A detailed derivation is shown in the appendices. The sample variance may be obtained as well:

\[
\hat{\sigma}^2 = \frac{1}{N-1} \sum_{k=1}^{N} a_k^2 (\theta_k - \hat{\theta})^2
\]

This sample variance will be high for collinear intersection points which do not have similar predictions for the angle of skew, and hence can be used to reject false hypotheses. Therefore, the hypotheses are ranked in ascending order of their sample variances.
Calculating the local skewed symmetry field

A number of sample points per polynomial piece are then extracted with tangent and curvature information. Each sample point is compared with the rest in \( \frac{1}{2} NA(N - 1) \) operations (where \( N \) is the number of sample points) to derive the LSSF in three arrays of the same size as the original images, containing separately the symmetricity values, and the two direction data. Although computationally expensive, this method has the advantage of not involving any occlusion-sensitive parameters.

Maximization of symmetry measure in the local skewed symmetry field

Having obtained a ranked list of hypotheses for the axis of symmetry, we are able to calculate an overall symmetry measure for each of the hypothesis by averaging the effective symmetricity values of the points in the LSSF crossed by the axis of symmetry. Taking \( S \) as the global symmetry measure and \( P \) as the number of sample points, we have:

\[
S = \frac{1}{P} \sum_{i=1}^{P} \Psi_{i0}
\]

where \( \Psi_{i0} \) is the effective value at simple point \( i \) with respect to the global axis direction \( \theta \), \( \Psi_i \) is the absolute symmetricity value measured at sample point \( i \), and \( \theta_i \) is the direction of the local axis of symmetry in the LSSF at sample point \( i \).

We then proceed to locate the position of maximum symmetry measure for the axis of symmetry by initializing a straight active contour model\(^{25} \) (a ‘pole’). This ‘pole’, unlike normal ‘snakes’, is driven by effective symmetricity values which change according to the orientation of the pole (see Figure 12).

Once the location of maximum symmetry measure has been found, the angle of skew may be estimated based on the weighted average of the second directions in the LSSF (local angles of skew) scaled by their covariances.

B-SPLINE CURVE FITTING

B-spline curve fitting is used to recover smoothed local curvatures and their variance estimations, although any robust curvature estimation algorithms may be substituted. The information is used to compute the symmetricity values as mentioned previously. This is further elaborated on elsewhere\(^{25} \).

Present techniques used in automatic fitting of B-splines by Medioni and Yasumoto\(^{31} \) suffer from unstable estimated derivatives, since the method assumes that the control points of the spline are evenly spread out. In our experiments, a partially automated technique is used. Chains of edgels are first automatically selected, followed by the manual specification of the control point locations along the spline parameters, and finally, least squares solutions are obtained for the spline fits. The implementation details are described elsewhere\(^{25} \). This technique has the advantage that the spline fits are smoother and more accurate than the automated method, and it is also possible to obtain the covariances of estimated derivatives.

Figure 11 The Hough space in (a) shows the result of carrying out a standard Hough Transform on the intersection points shown in Figure 10. Knowledge of the directions of tangents at the intersections allows reduction of the complexity of the Hough space (b), from Property 2.

Figure 12 Straight active contour model (a ‘pole’) being used to locate the position in the LSSF where the global symmetry measure is maximized. The ‘forces’ on the pole are based on effective symmetricity values (defined in (11)) which depend on both the absolute symmetricity as well as the angles between the first directions of the field and the orientation of the pole. The pole is initialized at the location of the each hypothesis obtained from the symmetry detection algorithm forming the first stage.
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effective symmetricity values. The new hypotheses for
the axes of symmetry may then be ranked again by their
global symmetry measure $S$. However, this does not
mean we reject all but one of the hypotheses, since
different symmetrical objects, or separate symmetrical
components, may be present.

ANALYSIS OF RESULTS

Preliminary results

The initial symmetry detection algorithm was tested on
a number of synthetic and real images with straight and
curved contours. The results show reasonable perfor-
mance by the initial technique, in most cases the
location of the axis is found to a satisfactory degree,
and often with an angle of skew of adequate accuracy.
The images in Figure 13 show the results obtained just
after the first stage of the algorithm.

Figure 14 shows back-projection of the relatively
intact edge maps belonging to the images in the top
row of Figure 13, to a canonical frame with orthogonal
symmetry axes.

However, there is a possibility of having extraneous
intersection points from other structures located on the
predicted axis of symmetry, and their votes are likely to
badly distort the estimation for the angle of skew. This
occurs in some of the lower ranked hypotheses, where
slight displacements of the axis of symmetry resulted in
large errors for the angle of skew. Ranking the
hypotheses is also not always accurate, and therefore
there is a need to use the LSSF to improve the
robustness of such estimations. Some of these poorer
hypotheses are used in the final stage of the algorithm to
demonstrate the ability of the algorithm to handle such
errors.

Besides the easily segmented objects shown in
Figure 6, we have also calculated the LSSF for two
other scenes of objects where shadows and mild
occlusion are present, which result in fragmented
contours and unwanted artifacts. These are shown in
Figure 15.

Since accurate results are obtained in general from
the initial algorithm, lower ranked hypotheses have
been used to demonstrate the use of the 'pole' in the
maximization of the global symmetry measure in the
LSSF, as mentioned earlier. The errors in both the
positions of the axes of symmetry and the angles of
skew are minimized accurately. These examples are
shown in Figure 16.

Overall, the results obtained for the thinner and
better segmented objects are particularly good. For
objects which have significant thicknesses and those in
scenes which include shadows, occlusion or noise, the
technique still return useful estimates for the symmetry
axes.

DISCUSSION

Both the initial symmetry detection algorithm and the
later stages share similar advantages. The algorithm
uses low-level data directly to extract symmetry. It is
local in nature and does not depend upon accurate
grouping of edgels into complete contours, nor does it

Figure 13 Results obtained from the
first symmetry detection stage indicate
that the first stage is performing
reasonably. However, the accuracy is
not robust enough to noise and
occlusion, such as is the case of (d)
and (f). Hence the need to use the
LSSF

Figure 14 Backprojection of the rela-
tively intact edge maps of (a), (b) and
(c) in Figure 13 to a canonical frame
where the symmetry axes are orthogo-

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require segmentation of the subject (as is necessary for a moment-based approach). Instead, it allows individual pairs of points to vote for the symmetry axis and angle of skew, and hence it is resistant to occlusion and fragmentation. Noise and quantization errors are handled by uncertainty analysis in the first stage, and by adopting a continuous measure of local symmetry (symmetricity). The sample variance obtained for the angle of skew from the first stage and the global symmetry measure obtained in the final stage both provide means of measuring the overall symmetry. This means that it is possible to gauge how symmetric the object is under the estimated set of axes, and hence to be able to reject false hypotheses. The initial symmetry detection algorithm also has a speed advantage without having to adopt global measures, unlike other algorithms which are normally slow if they use local measures, or suffer from having to test a discretized set of orientations.

Some of the problems faced with our implementation are described below. The initial symmetry detection algorithm is unable to handle symmetric objects comprising solely of parallel symmetric edges, e.g. a rectangle (in fact, it interprets a rectangle as a skewed rhombus). The performance of this stage is also proportional to the number of unique intersection points.
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points of symmetric edgel pairs. Hence, the algorithm favours gently curved objects or those with many non-parallel symmetric edges.

Two difficulties are encountered in the second stage of the algorithm. The first problem is speed, since every pair of sample points must be compared, which makes it slow. The other problem is that the curvature measurements are sensitive to errors, and the B-spline fits must be smooth in orientation to have stable curvature values. In some cases, it may be better to ignore the symmetricity value and maximize the global symmetry measure using only the first directions in the LSSF. Finally, if a scene is over-cluttered with details, the large number of overlapping entries in the LSSF will reduce the coherency of the entries which contribute to the overall global symmetry.

CONCLUSIONS AND FUTURE WORK

Although many methods have been presented for symmetry detection, they do not handle occlusion, fragmentation or noise well, which makes them unsuitable candidates to detect the symmetries needed in conditions where perceptual organization is most useful.

We have presented a new method which does not depend on segmentation of objects or any accurate determination of occlusion-sensitive global measures (e.g. moments). Instead, independent support from local structures may be combined to achieve global symmetry detection in the presence of noise and even occlusion via the local skewed symmetry field. Preliminary results obtained have been promising.

The next stage of development will be to allow coherent neighbouring elements in the LSFF to reinforce each other in order to suppress non-generic local skewed symmetries, i.e. those which occur only instantaneously at two particular points on two contours, thereby relieving the problem faced when there are too many overlapping entries.

Given a method for detecting global skewed symmetry axes from an image, we will proceed to investigate the recovery of the image contours from which the symmetry axes were formed, and to improve the accuracy of the estimated contours based on symmetrical redundancy. The eventual goal will be to accurately detect and segment symmetrical objects in an image, in reverse to the conventional method of segmenting the objects first followed by recovery of the symmetry axes. It is hoped that this will be of particular use to computational perceptual organization, since it has been shown that symmetry is both a local and a global property, and is therefore useful in a grouping process.

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APPENDIX 1: LOCAL SYMMETRY

RELATION OF CURVATURES TO TANGENT ANGLES

Referring to Figure 17, we consider small angles $\gamma_a$ and $\gamma_b$. Then:

\[
\begin{align*}
 k_a &= R_\alpha \gamma_a \sin (\tau + \gamma_a) \\
 k_b &= R_\beta \gamma_b \sin (\sigma + \gamma_b) \\
 l &= R_\gamma \gamma_a \cos (\tau + \gamma_b) = R_\gamma \gamma_b \cos (\sigma + \gamma_b)
\end{align*}
\]

(12)

Now let:

\[
\begin{align*}
 m &= k_a - l \tan \tau \\
 n &= k_b - l \tan \sigma
\end{align*}
\]

For skewed symmetry:

\[
\begin{align*}
 m &= n \\
 &= k_b - k_a = l (\tan \sigma - \tan \tau) \\
 &= \tan \sigma - \tan \tau = \tan (\sigma + \gamma_b) - \tan (\tau + \gamma_a) \\
 &= \tan \sigma - \tan (\sigma + \gamma_b) = \tan \tau - \tan (\tau + \gamma_a)
\end{align*}
\]

Using the trigonometric formula $\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$:

\[
\begin{align*}
 [1 + \tan \sigma \tan (\sigma + \gamma_b)] \tan \gamma_b \\
 &= [1 + \tan \tau \tan (\tau + \gamma_a)] \tan \gamma_a
\end{align*}
\]

Figure 17 Notation for proof. (a) Symmetry base system; (b) close-up of symmetric point pair
This approximates to:
\[(1 + \tan^2 \sigma) \gamma_b = (1 + \tan^2 \tau) \gamma_a\]
\[\sec^2 \sigma \gamma_b = \sec^2 \tau \gamma_a\]
Substituting into (12), we get:
\[R_h \sec^2 \sigma \cos \tau = \cos^3 \tau\]
\[R_a \sec^2 \tau \cos \sigma = \cos^3 \sigma\]
\[(13)\]

APPENDIX 2: MINIMIZATION OF MAHALANOBIS DISTANCE

The multivariate Gaussian probability density function for the state-space shown in Figure 5 is given by:
\[p(\mathbf{s}) = \frac{1}{(2\pi)|\mathbf{K}|^{1/2}}e^{-1/2(\mathbf{s} - \mathbf{m})'^{-1}(\mathbf{s} - \mathbf{m})}\]
\[(14)\]
where \(\mathbf{m}\) is the true state vector of a pair of contour-points. The Mahalanobis distance is given by the power term of the exponential:
\[d^2 = (\mathbf{s} - \mathbf{m})'^{-1}(\mathbf{s} - \mathbf{m})\]
\[(15)\]
Since this is a proper distance function, it is possible to use the function to measure the distance between any two points in the state-space, and in this application we consider the minimum Mahalanobis distance from the observed state to the local skewed symmetry subspace \(\nu = v\).

For:
\[\mathbf{K}^{-1} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad u_o = \kappa_o(\cos \sigma_o)^3, \quad v_o = \kappa_o(\cos \sigma_o)^3\]
\[d^2 = a(u_o - u)^2 + 2b(u_o - u)(v_o - v) + c(v_o - v)^2\]
substituting \(v = u\) and equating \(d^2\) for \(d^2 = 0:\)
\[\min \quad d^2 = \left[ a - \frac{(a + b)^2}{a + 2b + c} \right] u_o^2 + \left[ c - \frac{(b + c)^2}{a + 2b + c} \right] v_o^2 + \left[ b - \frac{(a + b)(b + c)}{a + 2b + c} \right] u_o v_o\]
\[(16)\]
\[(17)\]

APPENDIX 3: SENSITIVITY ANALYSIS IN ANGLE OF SKEW CALCULATIONS

Referring to Figure 3, we have from (1):
\[\tan \theta = \frac{1}{2} (\cot \gamma - \cot \beta)\]
or as a function:
\[\theta = g(\beta, \gamma)\]
\[(18)\]
Expanding and differentiating:
\[\frac{\partial g}{\partial \theta} = \frac{1}{1 + \frac{1}{4}(\cot \gamma - \cot \beta)^2} \frac{1}{2} \cosec^2 \beta\]
This applies to \(\gamma\) as well, and simplifying gives:
\[\frac{\partial \theta}{\partial \beta} = \frac{2}{\sin^2 \gamma [4 + (\cot \gamma - \cot \beta)^2]}\]
\[\frac{\partial \theta}{\partial \gamma} = -\frac{2}{\sin^2 \gamma [4 + (\cot \gamma - \cot \beta)^2]}\]
\[(19)\]
\[(20)\]
From \(N\) intersection points lying on the axis of symmetry, we derive observations of the angle of skew \(\theta_k, k = 1, \ldots (N)\) using (1). These estimates have an error:
\[\theta_k = \theta + \delta \theta_k\]
\[(21)\]
where \(\theta\) is the exact value of the angle of skew. \(\delta \theta_k\) is the error introduced by \(\delta \theta_k\) and \(\delta \gamma_k\). Each estimate and error are assumed independent.

We seek a set of parameters \(a_i\) such that the estimate for the angle of skew is optimal:
\[\hat{\theta} = \sum_{k=1}^{N} a_i \theta_k\]
\[\nabla_{\mathbf{a}} \text{Var} \{\hat{\theta}\} - \lambda \nabla_{\mathbf{a}} \left( \sum_{k=1}^{N} a_k \right) = 0\]
\[\sum_{k=1}^{N} a_k = 1\]
\[(23)\]
which are \(N + 1\) linear equations in the same number of unknowns.

Solving this will give:
\[a_i = \frac{1}{\sigma_i^2} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \]
\[(24)\]
where \(\sigma_i^2 = \text{E}\{\delta \theta_i^2\}\) is the variance of the errors, with \(E\{\delta \theta_j \delta \theta_k\} = 0\) for \(j \neq k\).

We can combine this with perturbation analysis:
\[\delta \theta = \frac{\partial g}{\partial \theta} \delta \beta + \frac{\partial g}{\partial \gamma} \delta \gamma\]
\[= m_\beta \delta \beta + m_\gamma \delta \gamma\]
\[(25)\]
with substitutions to \(m_i\)'s for convenience.
Hence:
\[E\{\delta \theta_i^2\} = E\{(m_\beta \delta \beta_i + m_\gamma \delta \gamma_i)^2\}\]
\[= (m_\beta^2 + m_\gamma^2) \sigma_i^2\]
\[(26)\]
where \(\sigma_i^2 = E\{\delta \beta_i^2\} = E\{\delta \gamma_i^2\}\), assuming that \(\delta \beta_i, \delta \gamma_i\) are independent with zero means and have statistically the same distribution.
Putting this into (24) gives:

\[ a_i = \frac{1}{\sum_{k=1}^{n} \frac{1}{m_i^2 + m_j^2}} \]

which are the weights needed to obtain the optimal estimate for the angle of skew.

REFERENCES