INFINITE TRANSLATION SURFACES WITH INFINITELY GENERATED VEECH GROUPS

PASCAL HUBERT, GABRIELA SCHMITHÜSEN

1. Introduction

After the fundamental work of Veech in 1989 ([Ve]), a lot of efforts have been done to understand Veech groups of translation surfaces of finite area. Infinitely generated Veech groups were discovered among them (see [HS], [Mc]).

Recently, work on translation surfaces of *infinite* area was done (see [Ho], [Va1], [HW]), which motivates our study. We work with a class of special translation surfaces that are coverings of the square torus ramified over the origin. They are called square tiled surfaces or origamis. These surfaces correspond to integer points in the moduli spaces of abelian differentials. Their Veech groups are subgroups of $SL_2(\mathbb{Z})$. There is an abundant literature on the subject (see for instance [HL2], [Sc2], [Sc3], [Sc4]). In this paper, we study Veech groups of a family of infinite origamis that are obtained as $\mathbb{Z}$-covers of finite origamis.

Theorem 1. *There exists a countable family of origamis (Y_n, α_n) of infinite area which arise as $\mathbb{Z}$-covers of genus 2 origamis and whose Veech groups are infinitely generated subgroups of $SL_2(\mathbb{Z})$.*

Moreover, we prove that the limit set of these Fuchsian groups is $\mathbb{P}^1(\mathbb{R})$ which means that they are Fuchsian groups of the first kind. We will see in the sequel of the paper that these families are completely explicit and we will give a more precise statement later. A slightly more general result is stated in Theorem 2. Our result provides a new phenomenon. In fact, if a subgroup of $SL_2(\mathbb{Z})$ that contains a hyperbolic element can be realized as the Veech group of a *finite* origami, then it is a lattice in $SL_2(\mathbb{R})$ (see [GJ]).

Reader's guide The strategy of the proof has commonalities with [HS] and [Mc]. One way to prove that a Fuchsian group is not finitely generated, is to show that it is not a lattice and that the limit set is equal to $\mathbb{P}^1(\mathbb{R})$. We check that some periodic direction is not parabolic,

which immediately implies that the group is not a lattice. The difficult part is to prove that the limit set is everything. We prove that the parabolic directions are dense. In our situation, the key ingredient is to check that the periodic directions with *one* cylinder on the finite origami are still parabolic on its $\mathbb{Z}$-cover. Then, it is clear that the orbits of the cusps corresponding to periodic directions with one cylinder are dense in $\mathbb{P}^1(\mathbb{R})$ which is enough to finish the proof.

In the last section of the paper, we use the embedding of the affine group in some linear group $\mathrm{GL}_k(\mathbb{Z})$ to find explicit subgroups of $\mathrm{GL}_k(\mathbb{Z})$ that are infinitely generated and intersection of two finitely generated groups.

2. Background

In this section, we recall classical facts about origamis and translation surfaces.

Translation surfaces A surface is a *compact translation surface*, if it can be obtained by edge-to-edge gluing of finitely many polygons in the plane using translations only. A compact translation surface induces a flat metric with singularities. It is a Riemann surface X endowed with a holomorphic 1–form whose zeroes are the singularities of the flat metric. There is a one to one correspondence between compact translation surfaces and compact Riemann surfaces equipped with a holomorphic 1–form.

Infinite translation surfaces are obtained from glueing countably infinitely many polygons again by edge-to-edge gluing via translations. The result $\hat{X}$ is in general not a surface, since the vertices of infinitely many polygons might glue to the same point on $\hat{X}$. Such a point is called an *infinite angle singularity*. A punctured neighbourhood of an infinite angle singularity is a $\mathbb{Z}$-cover of the punctured disk. Let X be $\hat{X}$ with all infinite angle singularities removed. Then X is an *infinite translation surface*. As in the compact case X is a Riemann surface endowed with a holomorphic 1–form, which defines a flat metric on X. $\hat{X}$ is the completion of X with respect to this metric (see e.g. [Va2] for a more detailed introduction to infinite translation surfaces). In a slight abuse of notations, we will sometimes also refer to $\hat{X}$ as the

translation surface, in order to keep notations simple.

Veech groups Given any translation surface (X, ω), an *affine diffeomorphism* is an orientation preserving homeomorphism of X that permutes the singularities of the flat metric and acts affinely on the polygons defining X. The group of affine diffeomorphisms is denoted by $\mathrm{Aff}(X, \omega)$. The image of the derivation

$$d : \begin{cases} \mathrm{Aff}(X, \omega) \to \mathrm{GL}_2(\mathbb{R}) \\ \qquad f \mapsto df \end{cases}$$

is called the *Veech group*. In the sequel, it is denoted by $\Gamma(X)$ ($\mathrm{SL}(X, \omega)$ is an other frequently used denotation for it). If (X, ω) is a finite translation surface, then $\Gamma(X)$ is a Fuchsian group. If (X, ω) is an infinite translation surface as defined above and $\hat{X}$ is its completion, we define $\Gamma(\hat{X}) = \Gamma(X)$.

Parabolic elements A *cylinder* on a compact translation surface X is a maximal connected set of homotopic simple closed geodesics. If the genus of X is greater than one, then every cylinder is bounded by saddle connections. A cylinder has a width (or circumference) x and a height y. The *modulus* of a cylinder is $\mu = x/y$.

An affine diffeomorphism is *parabolic* if the absolute value of the trace of its derivative is equal to 2. We know from Veech's paper (see [Ve]) that there is a canonical way to construct parabolic elements in the affine group.

Let (X, ω) be a translation surface of finite area. Assume that it has a decomposition into metric cylinders for the horizontal direction with commensurable moduli, then the Veech group $\mathrm{SL}(X, \omega)$ contains

$$Df = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

where c is the least common multiple of the moduli.

A direction (slope) is *parabolic* if the surface is decomposed in this direction into cylinders with commensurable moduli. This means that there is a parabolic affine diffeomorphism fixing this direction. A parabolic direction is a *one-cylinder direction*, if the surface is decomposed into one cylinder in this direction.

Origamis define special cases of translation surfaces. There are several definitions of origamis (algebraic, geometric, combinatorial). We will switch between the following different possible descriptions (each with the appropriate equivalence relation):

- collection of Euclidean unit squares with gluing rules,

- covering $p : \hat{X} \to E$ of the torus E ramified over at most one point called ∞; here a (ramified) covering is a map which is continuous, open and discrete;
- monodromy map $F_2 \to S_d$ or $F_2 \to \mathrm{Sym}(\mathbb{Z})$, which leads to a transitive action,
- transitive pairs of permutations (σ_h, σ_v) in S_d or in $\mathrm{Sym}(\mathbb{Z})$, where σ_h describes the horizontal gluings and σ_v describes the vertical gluings and
- subgroups U of F_2. $F_2 = F_2(x, y)$ is the free group in the 2 generators x and y. It is identified with the fundamental group of the one-punctured torus $E_I^* = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$; x is identified with the horizontal and y with the vertical standard generator of $\pi_1(E_I^*)$.

If the origami is built from finitely many squares than it is called a *finite origami*; otherwise it is an *infinite origami*. The equivalence above is carried out for finite origamis in details in Section 1 of [Sc4]; the same proofs work for infinite origamis.

For finite origamis, $X = \hat{X}$ is a translation surface; for infinite origamis, we have to remove the infinite-angle singularities and obtain an infinite translation surface X. The decomposition into Euclidean unit squares makes it to a *square-tiled surface*. We call the points in $p^{-1}(\infty)$ *cusps*. They are precisely the points coming from vertices of the Euclidean squares and all singularities are among them.

Monodromy Recall that an unramified covering $p : X \to Y$ naturally defines an action from the right of the fundamental group $\pi_1(Y)$ on the fibre E of a base point y on Y. Namely for $x_1, x_2 \in X$ with $p(x_1) = p(x_2) = y$, $c \in \pi_1(Y)$ we have:

$$x_1 \cdot c = x_2 \iff \text{ the lift of } c \text{ in } x_1 \text{ ends in } x_2.$$

This action defines the *monodromy map* $m : \pi_1(Y) \to \mathrm{Sym}(E)$:

$$m(c) = (x_1 \mapsto x_1 \cdot c) \text{ for } c \in \pi_1(X)$$

Observe that the monodromy map defined in this way in general is not a group homomorphism, but suffices the condition $m(c_1 \cdot c_2) = m(c_2) \circ m(c_1)$. However if the image of m is abelian, as it will almost always be the case in this article, this is equivalent to being a group homomorphism.

Veech groups of finite origamis The Veech group of a finite origami is a lattice in $\mathrm{SL}_2(\mathbb{R})$(see [GJ]). More precisely, the Veech group of an origami is commensurable to $\mathrm{SL}_2(\mathbb{Z})$, i.e. they share a subgroup which is of finite index in both groups. Moreover, if (X, ω) is an origami and if we denote by X^* the complement in X of the preimages of the origin of the torus, then the Veech group of (X^*, ω) is a subgroup of finite index of $\mathrm{SL}_2(\mathbb{Z})$. As an origami is a covering of the torus, every direction of rational slope is a parabolic direction.

3. AN INFINITE FAMILY OF INFINITE TRANSLATION SURFACES

In this section we introduce the family $Z_{a,b}^\infty$ of infinite square-tiled surfaces. They occur as limit cases of sequences $(Z_{a,b}^k)_{k \in \mathbb{Z}}$ of finite ones.

Definition 1. *Let $Z_{a,b} = Z_{a,b}^1$ $(a, b \in \mathbb{N}_0, a \geq 2)$ be the origami drawn in Figure 1. More precisely, the corresponding covering map to the torus is described by the two permutations*

$$\sigma_h = \begin{pmatrix} 1 & 2 & \ldots & a \end{pmatrix}, \quad \sigma_v = \begin{pmatrix} 1 & a & a+1 & \ldots & a+b \end{pmatrix}.$$

Let $Z_{a,b}^*$ be the punctured surface obtained from $Z_{a,b}$ by removing all the vertices of the squares. Its fundamental group $U_{a,b} = \pi_1(Z_{a,b}^*)$ is:

$$U_{a,b} \quad = \quad < g_1, g_2, g_3, h_i, l_j | \ i \in \{1, \ldots, a-2\}, j \in \{1, \ldots, b\} >$$

with $\ g_1 = yx^{-(a-1)}, \quad g_2 = x^{a-1}y^{b+1}, \quad g_3 = x^a,$

$h_i = x^i y x^{-i}, \qquad l_j = x^{a-1} y^j x y^{-j} x^{-(a-1)}.$

We have chosen the base point of $\pi_1(Z_{a,b}^*)$ in the square labelled by 1.
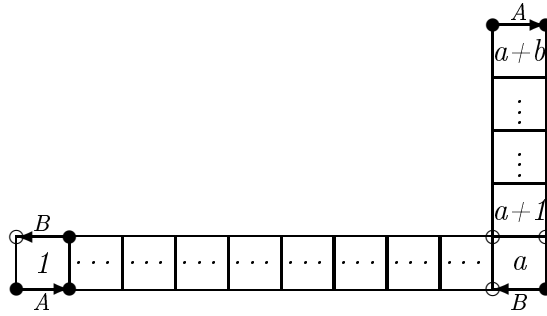

Figure 1: The origami $Z_{a,b}$. Edges labelled by the same letter and unlabelled opposite edges are identified.

Observe that the genus of the surface $Z_{a,b}$ is 2, if $(a, b) \neq (2, 0)$. More precisely, the genus g of $Z_{a,b}$ and the number n of zeroes can be read off from Figure 1 and Figure 2 differing four cases:

- Case I: $a = 2,\ b = 0 \quad \Rightarrow \quad g = 1$ and $n = 0$ (Figure 2).
- Case II, III: $a = 2,\ b \geq 1 \quad$ or $\quad a \geq 3,\ b = 0$
  $$\Rightarrow \quad g = 2 \text{ and } n = 1 \quad \text{(Figure 2)}.$$
- General case: $a \geq 3,\ b \geq 1$
  $$\Rightarrow \quad g = 2 \text{ and } n = 2 \quad \text{(Figure 1)}.$$

In Case II and Case III we have one zero of angle 6π and in the general case two zeroes each of angle 4π.
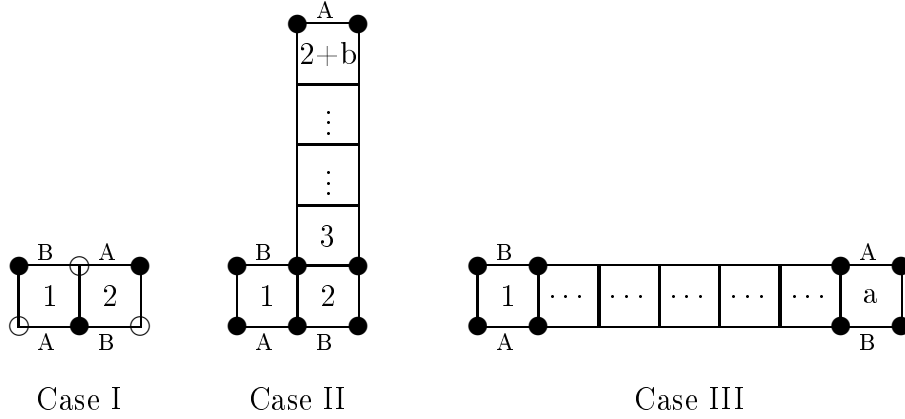

Figure 2: *The three special cases for* $Z_{a,b}$:
Case I: $a = 2$, $b = 0$,   *Case II:* $a = 2$, $b \geq 1$,   *Case III:* $a \geq 3$, $b = 0$.

Now, we build the surface $Z_{a,b}^k$ ($k \in \mathbb{N}$) from $Z_{a,b}$ as follows: We slit $Z_{a,b}$ along the edges A and B (compare Figure 1 and 2). We then take k copies of the two-slit surface and label them by the elements in $\mathbb{Z}/k\mathbb{Z}$. We glue A-edges to A-edges and B-edges to B-edges according to the following rules.

Gluing Rules:

- *Crossing the A-edge of the copy labelled by l in the direction bottom to top leads to the copy labelled by $l + 1$.*
- *Crossing the B-edge in the direction bottom to top leads to the copy labelled by $l - 1$.*

The result is a connected translation surface without boundary (see Figure 3). By construction $Z_{a,b}^k$ comes with a covering map $p^k = p_{a,b}^k$: $Z_{a,b}^k \to Z_{a,b}$ of degree k which is ramified at most over the vertices of the squares. Its group of Deck transformations is $\mathbb{Z}/k\mathbb{Z}$.

Definition 2. *Let $p^k = p_{a,b}^k : Z_{a,b}^k \to Z_{a,b}$ be the k-fold cover given by the monodromy map*

$$m^k = m_{a,b}^k : \quad U_{a,b} \;\to\; \mathrm{Sym}(\mathbb{Z}/k\mathbb{Z}), \quad \textit{defined on the generators as follows:}$$
$$g_1 \;\mapsto\; (z \mapsto z - 1), \quad g_2 \;\mapsto\; (z \mapsto z + 1),$$
$$g_3, \; h_i, \; l_j \;\mapsto\; \mathrm{id} \qquad (i \in \{1, \dots, a - 2\}, \; j \in \{1, \dots, b\})$$

Here $\mathrm{Sym}(\mathbb{Z}/k\mathbb{Z})$ denotes the symmetric group of $\mathbb{Z}/k\mathbb{Z}$ and g_1, g_2, g_3, the h_i's and the l_j's are the generators from Definition 1.
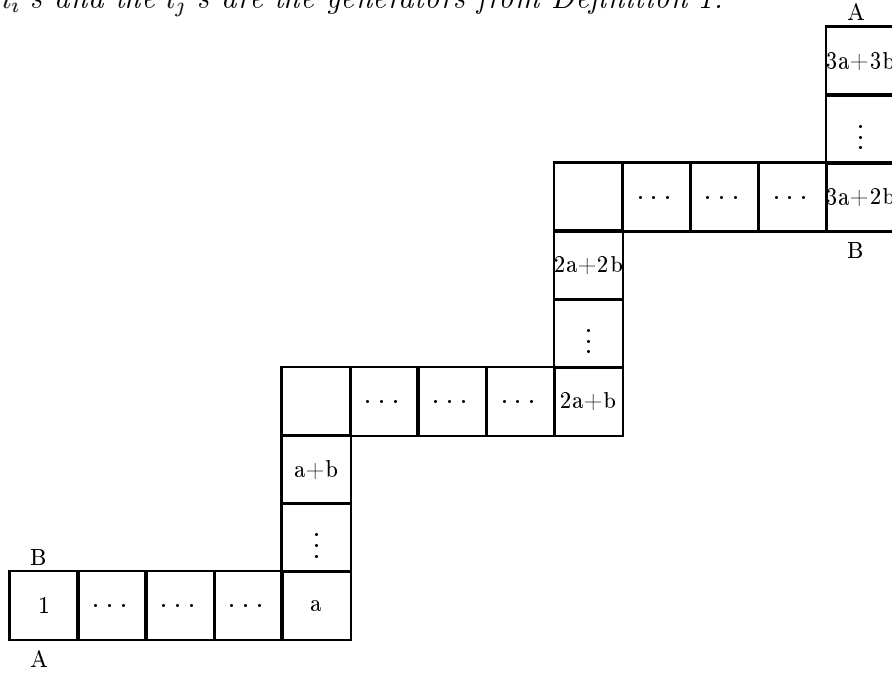

Figure 3: The origami $Z_{a,b}^k$ for $k = 3$. Edges labelled by the same letter and unlabelled opposite edges are identified.

Finally, we obtain a connected infinite translation surface $Z_{a,b}^\infty$ by gluing infinitely many copies of the two-slit surface which are labelled by the integer numbers according to the same gluing rules from above.

Definition 3. *Let $p^\infty = p_{a,b}^\infty : Z_{a,b}^\infty \to Z_{a,b}$ be the $\mathbb{Z}$-cover given by the monodromy map*

$$m^\infty = m_{a,b}^\infty : \quad U_{a,b} \to \mathrm{Sym}(\mathbb{Z}), \quad \textit{defined on the generators as follows:}$$
$$g_1 \;\mapsto\; (z \mapsto z - 1), \quad g_2 \;\mapsto\; (z \mapsto z + 1),$$
$$g_3, \; h_i, \; l_j \;\mapsto\; \mathrm{id} \quad (i \in \{1, \dots, a - 2\}, \; j \in \{1, \dots, b\})$$

$\mathrm{Sym}(\mathbb{Z})$ denotes the symmetric group of $\mathbb{Z}$, $Z_{a,b}^{\infty}{}^*$ is the punctured surface $Z_{a,b}^\infty \backslash \{\textit{vertices of the squares}\}$.

4. Monodromy of the infinite origami

There is a simple way, to determine the monodromy of the map p^∞ from Definition 3: Let c be a closed path on $Z_{a,b}^*$ and let $\tilde{c}$ be a lift of c on $Z_{a,b}^{\infty *}$. It follows immediately from the gluing rules on Page 6 that $\tilde{c}$ ascends one copy, whenever c intersects the A-slit or the B-slit in a positive crossing and it descends one copy, whenever it intersects in a negative crossing. Here A and B carry the orientation indicated in Figure 1 and *positive* and *negative crossing* is defined as shown in Figure 4.

positive crossing   negative crossing

Figure 4: Oriented crossings.

From this we obtain the following Lemma.

Lemma 4. *Let c be a closed curve on $Z_{a,b}^*$ and let w be the corresponding element in $\pi_1(Z_{a,b}^*)$ written as word in the generators from Definition 1.*

  i) $m^\infty(c) = \sharp$ *of positive crossings* $- \sharp$ *of negative crossings, where we consider crossings of c with the slits A and B.*
  ii) $m^\infty(c) = -\sharp_{g_1}(w) + \sharp_{g_2}(w)$*;
     here $\sharp_{g_i}(w)$ is the number of g_i's in w with g_i^{-1} counting as -1.*

Recall that the cusps of $Z_{a,b}^k$, resp. $Z_{a,b}^\infty$, are the points in $Z_{a,b}^k \backslash Z_{a,b}^{k\;*}$, resp. $Z_{a,b}^\infty \backslash Z_{a,b}^{\infty *}$. Using Lemma 4 we easily obtain their type. For this we consider small positively oriented simple loops around the cusps.

Corollary 5. *For a cusp P let l_P be a small positively oriented loop around P. We distinguish the four different cases described after Definition 1 and shown in Figure 1 and Figure 2:*

General case: If $P = \bullet$, the monodromy of l_P is 1; if $P = \circ$, it is -1. Thus on $Z_{a,b}^k$ both cusps have precisely one preimage which is a zero of angle $2k\pi$. Hence the genus of $Z_{a,b}^k$ is $2k$.
On $Z_{a,b}^\infty$ the singularities have each precisely one preimage which is an infinite angle singularity. The genus of $Z_{a,b}^\infty$ is infinite.

Case (I): If $P = \circ$, the monodromy of l_P is 2; if $P = \bullet$, it is -2. Thus each cusp has one preimage on $Z_{a,b}^k$, if k is odd and two preimages, if k is even. The genus of $Z_{a,b}^k$ is k, if k is odd and $k-1$, if k is even.

On $Z_{a,b}^\infty$ each cusp of $Z_{a,b}$ has 2 preimages which are infinite angle singularities. $Z_{a,b}^\infty$ has infinite genus.

Case (II) and Case (III): For $P = \bullet$, the monodromy of l_p is 0. Hence the maps $p_{a,b}^k$ and $p_{a,b}^\infty$ are unramified even above the cusps. In particular the cusp $\bullet$ on $Z_{a,b}$ has k preimages on $Z_{a,b}^k$ and infinitely many preimages on $Z_{a,b}^\infty$, each of angle 6π respectively. The genus of $Z_{a,b}^k$ is $k + 1$ and $Z_{a,b}^\infty$ is again an infinite genus surface.

5. The Veech groups

In the following, $Z_{a,b}$, $Z_{a,b}^k$ and $Z_{a,b}^\infty$ are always endowed with the translation structure coming from the square tiling; therefore, throughout all notations we will omit to explicitly denote the translation structure.

Observe that the $\mathbb{Z}$-module spanned by the development vectors of the saddle connection on $Z_{a,b}^\infty$ is equal to $\mathbb{Z}^2$. Hence we do not have to distinguish between $Z_{a,b}^\infty$ and the punctured surface $Z_{a,b}^{\infty\,*}$, since the affine groups $\text{Aff}(Z_{a,b}^\infty)$ and $\text{Aff}(Z_{a,b}^{\infty\,*})$ coincide and we have $\Gamma(Z_{a,b}^{\infty\,*}) = \Gamma(Z_{a,b}^\infty) = \text{GL}^+(Z_{a,b}^\infty) = \text{SL}(Z_{a,b}^\infty)$. If $(a,b) \neq (2,0)$ or $k \geq 3$, the same is true for the translation surface $Z_{a,b}^k$. In particular we have $\Gamma(Z_{a,b}^k) = \Gamma(Z_{a,b}^{k\,*})$ for $(a,b) \neq (2,0)$ or $k \geq 3$.

The surfaces $Z_{2,0}^k$ ($k \in \mathbb{N}$, k odd) appear in [He, 4.3] as origamis NSt_k. They occur as the smallest normal cover of the "stair-origamis" St_k, which consists of stairs of k squares, whose opposite edges are identified. Hence the Veech groups $\Gamma(Z_{2,0}^k)$ contain the subgroup of the respective $\Gamma(St_k)$ (see e.g. [Sc3, 3.1(5)]). The Veech groups $\Gamma(St_k)$ were calculated in [Sc3]. They are $\Gamma(2)$, if k is even, and

$$(1) \qquad \Gamma \;=\; \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,|\, a + c \text{ and } b + d \text{ odd}\},$$

if k is odd. We will see in Remark 6 that $\Gamma(Z_{2,0}^k) = \Gamma$ for all k. Equations for curves in these infinite families have been examined in [LS].
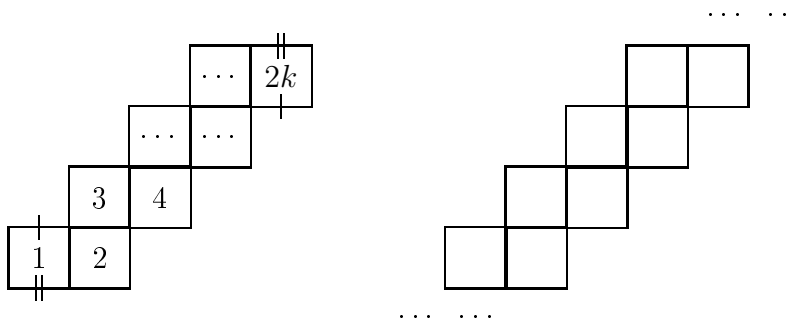
Figure 5: *The origami $Z_{2,0}^k$ for k finite and k infinite. Edges with the same label and unlabelled opposite edges are identified.*

Remark 6. *Let $k \in \mathbb{N}_{\geq 3} \cup \{\infty\}$. The Veech group $\Gamma(Z_{2,0}^k) = \Gamma$, where Γ is the group defined in* (1).

Proof. Observe that the surface $Z_{2,0}^k$ decomposes in the horizontal and the vertical direction in cylinders of length 2. Thus the Veech group contains the two matrices

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

These two matrices generate $\Gamma(2)$, hence we have $\Gamma(Z_{2,0}^k) \supseteq \Gamma(2)$. Observe furthermore that the fundamental group $N_k := \pi_1(Z_{2,0}^{k*})$ is a normal subgroup of F_2 and we have:

$$N_k = <<x^2, y^2, (xy)^k>>, \text{ if } k \text{ is finite, } \text{ and } N_\infty = <<x^2, y^2>>,$$

where $<< \cdot >>$ denotes the normal subgroup generated by the elements between the brackets. The automorphism defined by $x \mapsto y$ and $y \mapsto x^{-1}$ stabilises N_k for all $k \in \mathbb{N}_{\geq 3} \cup \{\infty\}$. Thus

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is in $\Gamma(Z_{2,0}^k)$ (see e.g. [Sc2, Lemma 2.8]). Since Γ is generated by $\Gamma(2)$ and S, we have that $\Gamma(Z_{2,0}^k)$ contains Γ. Finally, Γ has index 3 in $\mathrm{SL}_2(\mathbb{Z})$. Thus it suffices to show that $\Gamma(Z_{2,0}^k) \neq \mathrm{SL}_2(\mathbb{Z})$. Consider the automorphism $\gamma : x \mapsto y, y \mapsto xy$. Then $\gamma(y^2) \notin N_n$ for all n. Thus γ does not stabilise N_k. Since N_k is normal, composing by a conjugation leads again to an element which is not in the stabiliser. Thus the image of γ under the projection $\mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$ is not in the Veech group $\Gamma(Z_{2,0}^k)$. This finishes the proof. $\square$

It follows from Remark 6 that in the case $(a, b) = (2, 0)$ the Veech group of $Z_{a,b}^\infty$ is a lattice. We will see in Theorem 1 that this is not the case in general. We assume from now on always that $(a, b) \neq (2, 0)$.

Theorem 1. *Let* $\Gamma_{a,b}^\infty = \Gamma(Z_{a,b}^\infty)$ *be the Veech group of the infinite translation surface* $Z_{a,b}^\infty$ *with* $(a, b) \neq (2, 0)$. *If* a *or* b *is even, then* $\Gamma_{a,b}^\infty$ *is an infinitely generated subgroup of* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. We show in Lemma 7 that $\Gamma_{a,b}^\infty$ has infinite index in $\mathrm{SL}_2(\mathbb{Z})$. It remains to show that the limit set of the Fuchsian group $\Gamma_{a,b}^\infty$ is dense on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. In order to prove this, we proceed as follows:

- We show in Corollary 9 that, whenever $v \in \mathbb{R}^2$ is a one-cylinder direction of $Z_{a,b}$, then $Z_{a,b}^\infty$ decomposes in direction v into cylinders isometric to the one cylinder on $Z_{a,b}$. It follows that the Veech group $\Gamma_{a,b}^\infty$ contains parabolic matrices with eigenvector v. Thus $z_v = \frac{p_v}{q_v}$ is a cusp of $\Gamma_{a,b}^\infty$, where p_v, q_v are the x- and y-coordinates of v, i.e. $v = \binom{p_v}{q_v}$.

- We show in Lemma 10 that $v = \binom{2}{1}$ is a one-cylinder direction of $Z_{a,b}$, if a is even and $v = \binom{1}{2}$ is a one-cylinder direction, if b is even. It follows that $\gamma(v)$ is a one-cylinder direction of $Z_{a,b}$ for all γ in the Veech group $\Gamma(Z_{a,b})$. By the preceding item it follows that all the points $z_{\gamma(v)}$ are cusps of $\Gamma_{a,b}^\infty$. Since $\Gamma(Z_{a,b})$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$, they lie dense in $\overline{\mathbb{R}}$.

$\square$

We now fill in the details of the proof of Theorem 1.

Lemma 7. *For any* $a, b \in \mathbb{N}_0$ *with* $a \geq 2$, $(a, b) \neq (2, 0)$ *the Veech group* $\Gamma_{a,b}^\infty$ *of the translation surface* $Z_{a,b}^\infty$ *has infinite index in* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. We proceed as follows: we consider the translation surface

$$(2) \qquad Y_{a,b}^\infty = Z_{a,b}^\infty \cdot B, \text{ where } B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

This means that $Y_{a,b}^\infty$ arises from $Z_{a,b}^\infty$ by decomposing each chart in the atlas with the affine map $z \mapsto B \cdot z$. Then $Y_{a,b}^\infty$ and $Z_{a,b}^\infty$ are affine equivalent and $\Gamma(Y_{a,b}^\infty) = B\Gamma(Z_{a,b}^\infty)B^{-1}$. We show that the Veech group of $Y_{a,b}^\infty$ does not contain any power of

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus $\Gamma(Y_{a,b}^\infty)$ – and therefore also $\Gamma(Z_{a,b}^\infty)$ – has infinite index in $\mathrm{SL}_2(\mathbb{Z})$.

By construction $Y_{a,b}^\infty$ comes with a $\mathbb{Z}$-covering $p_\infty' : Y_{a,b}^\infty \to Y_{a,b}$, where $Y_{a,b} = Z_{a,b} \cdot B$. The surface $Y_{a,b}$ is equivalently obtained by composing the origami map $Z_{a,b} \to E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ with the affine (but not holomorphic!) map $E \to E$ which is the descend of $\mathbb{C} \to \mathbb{C}, z \mapsto B \cdot z$. Hence, $Y_{a,b}$ is again an origami with monodromy $\sigma' = \sigma \circ \gamma_B^{-1}$, where σ is the monodromy of the origami map $Z_{a,b} \to E$ and γ_B is a lift of B to $\mathrm{Aut}(F_2)$. We choose $\gamma_B : F_2 \to F_2, x \mapsto y^{-1}, y \mapsto yx$. Using the permutations $\sigma_h = \sigma(x)$ and $\sigma_v = \sigma(y)$ given in Definition 1, we obtain the monodromy of $Y_{a,b}$.

$$
\begin{aligned}
\sigma'(x) &= \sigma(\gamma_B^{-1}(x)) = \sigma(xy) = \sigma(y) \circ \sigma(x) = \sigma_v \circ \sigma_h \\
&= \begin{pmatrix} 1 & 2 & \ldots & a-1 & a+1 & \ldots & a+b \end{pmatrix} \\
\sigma'(y) &= \sigma(\gamma_B^{-1}(y)) = \sigma(x^{-1}) = \sigma_h^{-1} = \begin{pmatrix} a & a-1 & \ldots & 1 \end{pmatrix}
\end{aligned}
$$

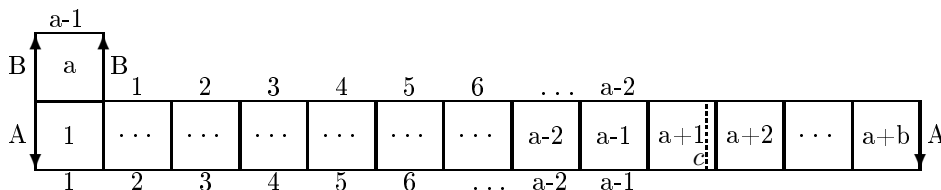The corresponding square-tiled surface is drawn in Figure 6.

Figure 6: The origami $Y_{a,b} = Z_{a,b} \cdot B$. Edges labelled with the same number or letter and unlabelled opposite edges are identified.

$Y_{a,b}$ consists of two horizontal cylinders, one of length $a + b - 1$ and one of length 1. The two oriented segments A and B on $Z_{a,b}$ become segments in direction $B \cdot \binom{1}{0} = \binom{0}{-1}$ and $B \cdot \binom{-1}{0} = \binom{0}{1}$ on $Y_{a,b}$, respectively. A is the unique horizontal segment on $Z_{a,b}$, which is intersected by the path starting in Square 1 induced by y^{-1} (see Figure 1). Hence it becomes the unique vertical segment on $Y_{a,b}$, which is intersected by the path starting in Square 1 induced by $\gamma_B(y^{-1}) = x^{-1}y^{-1}$. This is the vertical edge between the square labelled with 1 and the square labelled with $a + b$. Similarly one obtains that B becomes the vertical edge which bounds the square labelled with a from both sides.

Similarly as before (see Lemma 4), the monodromy map m'^∞ of the infinite cover $p_\infty' : Y_{a,b}^\infty \to Y_{a,b}$ is obtained as follows: Let c be a closed curve on $Y_{a,b}$. Its monodromy $m'^\infty(c)$ is the number of oriented intersections with the segments A and B. The orientations of A and B are indicated by the arrows in Figure 6.

Let us suppose for the moment that $b \geq 1$. We fix in the following the vector $v_1 = \binom{0}{1}$.

Let c now be a closed curve on $Y_{a,b}^*$ with developing vector v_1 which runs through the square labelled by $a + 1$ (see Figure 6). It does not cross any cutting slit. Hence each lift $\hat{c}$ of c on $Y_{a,b}^{\infty *}$ is also a closed curve with developing vector v_1. Suppose now that $T^n \in \Gamma(Y_{a,b}^\infty)$ for some $n \in \mathbb{N}_{\geq 1}$ and let $\hat{f}$ be an affine diffeomorphisms of $Y_{a,b}^\infty$ with derivative T^n. By passing to a power if necessary, we may assume that $n > a + b$. Then $\hat{c}_2 = \hat{f}(\hat{c})$ is a closed curve with developing vector

$$v_2 \; = \; T^n \cdot v_1 \; = \; \binom{n}{1}.$$

Its image $c_2 = p'_\infty(\hat{c}_2)$ on $Y_{a,b}^*$ is then as well a closed curve with developing vector v_2. Furthermore its monodromy $m'^\infty(c_2)$ is 0. Observe from Figure 6 that first of all any closed curve with developing vector v_2 intersects a horizontal saddle connection. Thus we may assume that the starting point of c_2 lies on a boundary of the two horizontal cylinders. Then again since its developing vector is v_2, c_2 hits at most one of the two slits and it intersects it always in the same direction. Furthermore it intersects at least once, since $n > a + b - 1$. Therefore $m'^\infty(c_2)$ is either positive, if c_2 lies in the bottom cylinder, or it is negative, if c_2 lies in the top cylinder. In any case it is not 0. Contradiction!

Let us finally consider the case $b = 0$, see Figure 7. We replace the vector $v_1 = \binom{0}{1}$ by the vector $v_1' = \binom{1}{1}$ and the closed curve c by the closed curve with developing vector v_1' which starts on the lower edge of Square 1. The claim follows by similar arguments as before.
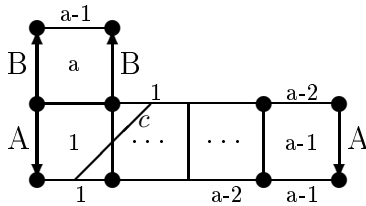

Figure 7: The origami $Y_{a,b} = Z_{a,b} \cdot B$, if $b = 0$. Edges labelled with the same number or letter and unlabelled opposite edges are identified.

$\square$

Lemma 8. *In the following we fix a direction $v = \binom{p}{q}$. Let k be the number of cylinders of $Z_{a,b}$ in direction v and let $c_1, \ldots, c_k$ be geodesics*

in direction v which are core curves of the cylinders. We then have:

$$\sum_{i=1}^{k} m(c_i) = 0,$$

where $m = m_{a,b}^{\infty}$ is the monodromy of the cover $p_{a,b}^{\infty} : Z_{a,b}^{\infty *} \to Z_{a,b}^{*}$.

Proof. We consider the first return map for the union $A \cup B$ in direction v. In order to be more explicit we fix some $r \in \mathbb{R} \backslash \mathbb{Q}$ with $0 < r < \frac{1}{q}$, q points $P_1, \ldots, P_q$ on the segment A and q points $P_{q+1}, \ldots, P_{2q}$ on the segment B (see Figure 8) such that

- $\vec{O_1 P_i} = (r + \frac{i-1}{q}) \cdot \binom{1}{0}$ for $i \in \{1, \ldots, q\}$, where O_1 is the lower left vertex of the square labelled by 1.
- $\vec{O_2 P_i} = (r + \frac{i-1-q}{q}) \cdot \binom{1}{0}$ for $i \in \{q+1, \ldots, 2q\}$, where O_2 is the lower left vertex of the square labelled by a.
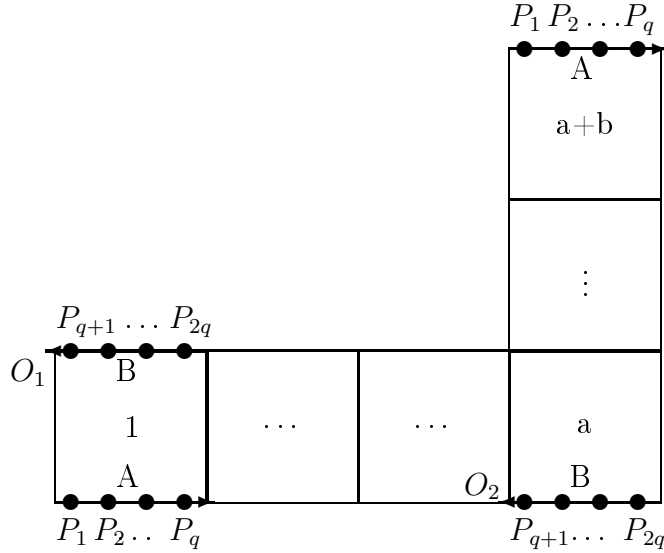

Figure 8: The flow in direction v acts on the points P_i.

The geodesic flow in direction v defines a permutation p_v of the points $P_1, \ldots, P_{2q}$. Mapping z to the permutation $(p_v)^z$ gives an action of $\mathbb{Z}$ on them. The orbits of this action are in one-to-one correspondence with the geodesic cylinders on $Z_{a,b}$ in direction v and thus as well in one-to-one correspondence with the given geodesics $c_1, \ldots, c_k$. In particular we may choose the geodesics c_j, such that they start in one of the points P_i.

Let now c be a closed geodesic in direction v starting from the point P_i. Then c intersects the slits A or B precisely in the points P_j. Let

us assume for the moment, that the geodesic c runs "from the bottom to the top", i.e. that the y-coordinate q of v is positive. Then the monodromy $m(c)$ increases, whenever c runs through A and it decreases whenever c runs through B. We define $\chi(P_j) = 1$ for $j \in \{1, \ldots, q\}$ and $\chi(P_j) = -1$ for $j \in \{q+1, \ldots, 2q\}$. It follows that in the case $q > 0$ we obtain:

$$(3) \qquad\qquad m(c) = \sum_{P_j \in \text{ orbit}(P_i)} \chi(P_j)$$

Similarly, we obtain for $q < 0$:

$$(4) \qquad\qquad m(c) = \sum_{P_j \in \text{ orbit}(P_i)} -\chi(P_j)$$

Since $\sum_{j=1}^{2q} \chi(P_j) = \sum_{j=1}^{q} 1 + \sum_{j=q+1}^{2q} -1 = 0$, the claim follows from (3) and (4).

$\square$

From Lemma 8 we immediately obtain the following corollary.

Corollary 9. *If $v = \binom{p}{q}$ is a one-cylinder direction on $Z_{a,b}$ and c is a closed geodesic in direction v, then we have:*

    i) *The monodromy $m(c)$ is 0.*
    ii) *$Z_{a,b}^{\infty}$ decomposes in direction v into cylinders isometric to the one cylinder on $Z_{a,b}$.*
    iii) *The Veech group $\Gamma(Z_{a,b}^{\infty})$ contains a parabolic element in $SL_2(\mathbb{Z})$ with eigenvector $v = \binom{p}{q}$.*
    iv) *p/q is a cusp of $\Gamma(Z_{a,b}^{\infty})$.*

Observe that so far, we have not used the prerequisite that a or b is even. We will need this now in the last step, where we find a one-cylinder direction on $Z_{a,b}$.

Lemma 10. *Let*

$$v = \binom{1}{2} \quad and \quad v' = \binom{2}{1}.$$

If b is even, then v is a one-cylinder direction on $Z_{a,b}$. If a is even, then v' is a one-cylinder direction.

Proof. Let us first assume that b is even. Consider the geodesic path c_v on the one-square translation surface E with developing vector $v = \binom{1}{2}$ starting in the midpoint M. This actually is a closed curve on the punctured surface E^*.
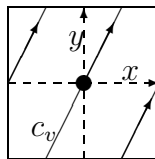
Figure 9: The closed curve c_v on the once punctured torus E^* with
$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

As element of $\pi_1(E^*, M)$ it is equal to yxy (compare Figure 9). Hence its monodromy with respect to the covering $Z^*_{a,b} \to E^*$ is

$$
\begin{aligned}
m(yxy) \quad &= \quad p_v \circ p_h \circ p_v \\
&= \quad \begin{pmatrix} 1 & a & a+1 & \dots & a+b \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & \dots & a \end{pmatrix} \\
&\qquad \circ \begin{pmatrix} 1 & a & a+1 & \dots & a+b \end{pmatrix} \\
&\overset{b \text{ even}}{=} \quad (\; 1 \; a \; a+2 \; a+4 \; \dots \; a+b \\
&\qquad\quad 2 \; 3 \; \dots \; a-1 \; a+1 \; a+3 \; \dots \; a+b-1)
\end{aligned}
$$

Since $m(yxy)$ acts transitively, it follows that we have only one cylinder in direction v.

In the case that a is even the proof works similarly: The geodesic path $c_{v'}$ on E with developing vector $v' = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ starting in M is equal to xyx in $\pi_1(E^*, M)$. Its monodromy is:

$$
\begin{aligned}
m(xyx) &= p_h \circ p_v \circ p_h \\
&= \quad \begin{pmatrix} 1 & 2 & \dots & a \end{pmatrix} \circ \begin{pmatrix} 1 & a & a+1 & \dots & a+b \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & \dots & a \end{pmatrix} \\
&\overset{a \text{ even}}{=} (1 \; 3 \; 5 \; \dots \; a-1 \; a+1 \; a+2 \; \dots \; a+b \; 2 \; 4 \; \dots \; a).
\end{aligned}
$$

$\square$

6. Generalization

In this section, we describe a general construction producing infinite origamis with large Veech groups.

Let $\mathcal{O}$ be a finite origami. Let A and B be two horizontal segments of length 1, each of them joining preimages of the origin of the torus (in other words, the end points of the horizontal segments A, B have integral coordinates). We assume that A and B are not homologous. We cut the surface along these segments. We obtain a translation surface with boundary. We call the part of the boundary where the vertical flow can be defined for some positive time the *positive boundary*. Its complement is the *negative boundary*. This yields a partition of the boundary: $(A, +)$, $(A, -)$, $(B, +)$, $(B, -)$.

We construct an infinite origami $\mathcal{O}^\infty$ which is a $\mathbb{Z}$-cover of $\mathcal{O}$. Consider a countable number of copies of $\mathcal{O}$ cut along A and B. On each copy

$\mathcal{O}_i$, there are slits $(A_i, +)$, $(A_i, -)$, and $(B_i, +)$, $(B_i, -)$. For all $i \in \mathbb{Z}$, we glue $(A_i, -)$ with $(A_{i+1}, +)$ and $(B_i, +)$ with $(B_{i+1}, -)$. We denote by $\mathcal{O}^\infty$ the infinite origami obtained by this construction.

We note that, as A and B are not homologous, the surface $\mathcal{O}^\infty$ is connected. In fact, $\mathcal{O} \backslash A \cup B$ is connected and, by construction, every level is connected to the previous one by two segments. Assuming that A and B are homologous is necessary to get an interesting object. Otherwise the construction produces a non connected surface homeomorphic to $\mathcal{O} \times \mathbb{Z}$.

Theorem 2. *If there is a one-cylinder direction on $\mathcal{O}$, then the Veech group of $\mathcal{O}^\infty$ is either a lattice or infinitely generated. Its limit set is equal to $\mathbb{P}^1(\mathbb{R})$.*

The proof is omitted because it is mutatis mutandis the same as for the family $Z_{a,b}^\infty$.

This theorem provides many examples. We know that finite origamis with one-cylinder directions are dense in every connected components of every stratum of moduli spaces of holomorphic differentials (see [KZ]). Moreover, by results of Hubert-Lelièvre ([HL1]) and McMullen ([Mc]), every origami of genus 2 with one singularity has a one-cylinder direction.

Open questions

- There are origamis without one-cylinder decompositions. For instance, the Veech group of $Z_{3,1}^\infty$ has two cusps. Each of them corresponds to directions in which the surface is decomposed into two cylinders. We don't know whether the limit set of the Veech group of $Z_{3,1}^\infty$ is $\mathbb{P}^1(\mathbb{R})$ or a Cantor set.
- When Theorem 2 holds, it seems difficult to give a general criterion to decide whether the group is a lattice or infinitely generated.

7. THE VEECH GROUPS AS SUBGROUPS OF $\mathrm{GL}_k(\mathbb{Z})$

Let X be a closed surface, μ a translation structure on X, S a finite set which contains the singularities of μ and $X^* = X \backslash S$. Recall that the natural action of the affine group $\mathrm{Aff}(X^*, \mu)$ on its first homology $H_1(X^*, \mathbb{Z})$ defines an embedding of $\mathrm{Aff}(X^*, \mu)$ into $\mathrm{Aut}(H_1(X^*, \mathbb{Z}))$. Let k be the rank of $\pi_1(X^*)$, i.e. $k = 2g + n - 1$, where g is the genus of X and n is the number of elements in S. Any choice of an isomorphism $\varphi : H_1(X^*, \mathbb{Z}) \rightarrow \mathbb{Z}^k$ defines an isomorphism $\varphi_* : \mathrm{Aut}(H_1(X^*, \mathbb{Z})) \xrightarrow{\cong} \mathrm{GL}_k(\mathbb{Z})$. Thus we can describe the affine group as a

subgroup of $\mathrm{GL}_k(\mathbb{Z})$. If X^* does not have nontrivial translations, then the derivative map $D : \mathrm{Aff}^+(X^*, \mu) \to \Gamma(X^*, \mu)$ is an isomorphism. Altogether, in this case we obtain an embedding:

$$(5) \quad h_\varphi : \Gamma(X^*, \mu) \xrightarrow{D^{-1}} \mathrm{Aff}^+(X^*, \mu) \hookrightarrow \mathrm{Aut}(H_1(X^*, \mathbb{Z})) \overset{\varphi_*}{\cong} \mathrm{GL}_k(\mathbb{Z}).$$

This embedding depends on φ only up to conjugation.

We now return to the translation surfaces $Z_{a,b}$ and $Z_{a,b}^\infty$ defined in Section 3. As before we omit the notation of the translation structures. Let S be the set of the integral points on $Z_{a,b}$. The rank k of $\pi_1(Z_{a,b}^*)$ is the number of squares plus 1, i.e. $k = a + b + 1$. We assume that $(a, b) \neq (2, 0)$. In Remark 13 we will see that the Veech group $\Gamma(Z_{a,b}^\infty)$ is a subgroup of $\Gamma(Z_{a,b})$. The aim of this section is to describe its isomorphic image $h_\varphi(\Gamma(Z_{a,b}^\infty))$ in $\mathrm{GL}_k(\mathbb{Z})$ for a suitable choice of $\varphi : H_1(Z_{a,b}^*, \mathbb{Z}) \to \mathbb{Z}^k$. We follow the construction in [Sc1, Section 7.1]. In particular, we show that $h_\varphi(\Gamma(Z_{a,b}^\infty))$ is the intersection of two finitely generated subgroups of $\mathrm{GL}_k(\mathbb{Z})$. Since $\Gamma(Z_{a,b}^\infty)$ itself is infinitely generated by Theorem 1, this is an example for $\mathrm{GL}_k(\mathbb{Z})$ not having the Howson property. Recall that a group has the Howson property, if the intersection of any two finitely generated subgroups is finitely generated (see e.g. [BB]).

Proposition 11. *Let $a \geq 2$, $b \geq 0$, $(a, b) \neq (2, 0)$ and $k = a + b + 1$. The Veech group $\Gamma(Z_{a,b}^\infty)$ embeds to a subgroup of $\mathrm{GL}_k(\mathbb{Z})$, which is the intersection of two finitely generated subgroups of $\mathrm{GL}_k(\mathbb{Z})$. More precisely we have for a suitable choice of $\varphi : H_1(Z_{a,b}^*, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^k$:*

$$h_\varphi(\Gamma(Z_{a,b}^\infty)) \;=\; h_\varphi(\Gamma(Z_{a,b})) \,\cap\, H,$$

$$\text{with} \quad H = \left\{ \begin{pmatrix} a_{1,1} & 0 & \dots & 0 \\ a_{2,1} & a_{2,2} & \dots & a_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k,1} & a_{k,2} & \dots & a_{k,k} \end{pmatrix} \in \mathrm{SL}_k(\mathbb{Z})\mid a_{i,j} \in \mathbb{Z} \right\}$$

and h_φ defined as in (5).

Part of Proposition 11 can be concluded from Corollary 7.3 in [Sc1]. Nevertheless, we include the whole proof adapted to our present situation.

Proof. First of all, we show in Remark 12 that $Z_{a,b}^*$ has no nontrivial translations. Therefore we actually obtain an embedding $h_\varphi : \Gamma(Z_{a,b}) \hookrightarrow \mathrm{GL}_k(\mathbb{Z})$ as described in the section before Proposition 11.

We then show in Remark 13 that any affine diffeomorphism of $Z_{a,b}^\infty$ descends to an affine diffeomorphism of $Z_{a,b}$. Thus $\Gamma(Z_{a,b}^\infty)$ is a subgroup of $\Gamma(Z_{a,b})$. Finally we choose an isomorphism $\varphi : H_1(X^*, \mathbb{Z}) \to \mathbb{Z}^k$ and show in Corollary 15 that an element $A \in \Gamma(Z_{a,b})$ lifts to $Z_{a,b}^\infty$ if and only if $h_\varphi(A)$ stabilises the subspace U of $\mathbb{Z}^k$ spanned by all standard basis vectors other than the first one. This is equivalent to A being in the group H. $\qquad \square$

Remark 12. *If $(a,b) \neq (2,0)$, then $Z_{a,b}^*$ has no nontrivial translations.*

Proof. Recall that $Z_{a,b}^*$ has a nontrivial translation if and only if the fundamental group $\pi_1(Z_{a,b}^*, M_i)$ and $\pi_1(Z_{a,b}^*, M_j)$ define the same subgroup of F_2 for the midpoints M_i and M_j of two squares. Let M_1 be the midpoint of the square labelled by 1 (see Figure 1). Then we have yx is in $\pi_1(Z_{a,b}^*, M_1)$. For any other midpoint M_i we have $yx \notin \pi_1(Z_{a,b}^*, M_i)$. Hence there are no nontrivial translations. Observe that this argument works as well for the special cases II and III (see Figure 2). In Case I, i.e. $(a,b) = (2,0)$, we in fact have a translation on $Z_{a,b}^*$. $\qquad \square$

Remark 13. *Any affine diffeomorphism of $Z_{a,b}^{\infty*}$ descends to $Z_{a,b}^*$ via the covering map $p^\infty : Z_{a,b}^{\infty*} \to Z_{a,b}^*$. In particular we have $\Gamma(Z_{a,b}^\infty) \subseteq \Gamma(Z_{a,b})$.*

Proof. Here we again use the description of the origamis by their corresponding subgroups of F_2:

$$
\begin{aligned}
U \quad &= \quad U_{a,b} \;=\; \pi_1(Z_{a,b}^*, M_1) \;\subseteq\; F_2, \\
&\quad \text{where } M_1 \text{ is the midpoint of Square 1}, \\
U^\infty \;&=\; \pi_1(Z_{a,b}^{\infty*}, \hat{M}_1) \;\subseteq\; U, \\
&\quad \text{where } \hat{M}_1 \text{ is a preimage of } M_1 \text{ on } Z_{a,b}^\infty.
\end{aligned}
$$

We show that $\mathrm{Norm}_{F_2}(U^\infty) = U$. Since we have that

$$
\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U^\infty) \quad \subseteq \quad \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(\mathrm{Norm}_{F_2}(U^\infty))
$$

(see e.g. [Sc3, Remark 3.1]), the claim follows e.g. from [Sc2, Lemma 2.8(2)].

We use the same notations as in Definition 3. Since the cover $p^\infty : Z_{a,b}^{\infty*} \to Z_{a,b}^*$ is normal, U^∞ is the kernel of m^∞. In particular U^∞ is a normal subgroup of U. Therefore $\mathrm{Norm}_{F_2}(U^\infty)$ contains U. Suppose now that there is some $w \in \mathrm{Norm}_{F_2}(U^\infty)$ which is not in U. Then there exists a square on $Z_{a,b}$ labelled by $i \neq 1$, such that a closed path starting in the midpoint M_i lifts to $Z_{a,b}^\infty$ if and only if its image in F_2 is in U^∞. Observe from Figure 1 and Figure 2 that for each square other

than Square 1 at least one of the paths x,y, yxy^{-1} or $x^{-1}yx$ is closed and lifts to $Z_{a,b}^\infty$. But they are all three not in $\pi_1(Z_{a,b}^*, M_1)$ and thus in particular not in U^∞. It follows that $\mathrm{Norm}_{F_2}(U^\infty) = U$. $\square$

We now want to choose an isomorphism $\varphi : H_1(Z_{a,b}^*, \mathbb{Z}) \to \mathbb{Z}^k$. Instead of this, we may choose as well an isomorphism $\hat{\varphi} : \pi_1(Z_{a,b}^*) \to F_k = F(x_1, \ldots, x_k)$, the free group in the $k = a + b + 1$ generators x_1, ..., x_k. For matter of convenience we start with the somehow natural isomorphism

$$\hat{\varphi}' :\ \pi_1(Z_{a,b}^*)\ \to\ F_k, \qquad \begin{aligned} g_i &\ \mapsto\ x_i & \text{for } i \in \{1, \ldots, 3\}, \\ h_i &\ \mapsto\ x_{3+i} & \text{for } i \in \{1, \ldots, a-2\}, \\ l_i &\ \mapsto\ x_{1+a+i} & \text{for } i \in \{1, \ldots, b\}, \end{aligned}$$

respectively with its descend $\varphi' : H_1(Z_{a,b}^*, \mathbb{Z}) \to \mathbb{Z}^k$. Here the g_i's, h_i's and l_i's are as in Definition 1. We will later modify φ' by a base change in order to obtain a nicer subgroup in $\mathrm{GL}_k(\mathbb{Z})$.

Lemma 14. *Let A be in the Veech group $\Gamma(Z_{a,b})$ and consider the embedding $h_{\varphi'} : \Gamma(Z_{a,b}) \hookrightarrow GL_k(\mathbb{Z})$ defined in (5).*

$$A \text{ is in } \Gamma(Z_{a,b}^\infty)\ \Leftrightarrow\ h_{\varphi'}(A) \text{ stabilises } V' = \{\begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} \in \mathbb{Z}^k | a_1 - a_2 = 0\}.$$

Proof. Recall that $Z_{a,b}$ has no nontrivial translations. Let $f = f_A$ be the unique affine homeomorphism of $Z_{a,b}$ with derivative A. Recall further that f defines an outer automorphism $[f_*]$ of the fundamental group U of $Z_{a,b}^*$ and that f lifts to $Z_{a,b}^{\infty *}$ if and only if $[f_*]$ preserves the conjugacy class of U^∞, i.e. for each automorphism $f_\star$ in the class of $[f_*]$ we have that $f_\star(U^\infty)$ is conjugate to U^∞. Since U^∞ is normal in U this is in this case equivalent to $f_\star(U^\infty) = U^\infty$. Furthermore we have the following commutative diagram, where $\mathrm{proj} : F_k \to \mathbb{Z}^k$ is the natural projection:

$$\begin{array}{ccccc} U\ =\ \pi_1(Z_{a,b}^*) & \xrightarrow[\cong]{\hat{\varphi}'} & F_k & \xrightarrow{\mathrm{proj}} & \mathbb{Z}^k \\ {\scriptstyle f_\star}\big\downarrow & & & & \big\downarrow {\scriptstyle h_{\varphi'}(A)} \\ U\ =\ \pi_1(Z_{a,b}^*) & \xrightarrow[\cong]{\hat{\varphi}'} & F_k & \xrightarrow{\mathrm{proj}} & \mathbb{Z}^k \end{array}$$

By Lemma 4 we have that

$$U^\infty = \{w \in U | \ m^\infty(w) = \sharp_{g_2}(w) - \sharp_{g_1}(w) = 0\},$$

where $\sharp_{g_i}(w)$ is the number of occurrences of g_i in w; g_i^{-1} is counted negative. It follows that U^∞ is the preimage of V', i.e. $U^\infty = (\mathrm{proj} \circ$

$\hat{\varphi}')^{-1}(V')$. Thus $f_\star(U^\infty) = U^\infty$ if and only if $h_{\varphi'}(A)(V') = V'$. This proves the claim. $\qquad\square$

Observe that V' defined in in Lemma 14 is the following $(k-1)$ - dimensional submodule of $\mathbb{Z}^k$:

$$
V' \;=\; < \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} >
$$

Let B be the matrix

$$
B \;=\; \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix} .
$$

In the following l_B and $l_{B^{-1}}$ are the linear maps $z \mapsto B \cdot z$ and $z \mapsto B^{-1} \cdot z$, respectively. Then we have $l_B(V) = V'$, where

$$
V \;=\; < \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} > .
$$

Corollary 15. *Let $\varphi = l_{B^{-1}} \circ \varphi'$. Let furthermore H be the group defined in Proposition 11. Then we have for all $A \in \Gamma(Z_{a,b})$:*

$$
A \text{ is in } \Gamma(Z_{a,b}^\infty) \quad \Leftrightarrow \quad h_\varphi(A) \in H.
$$

Proof. Let us consider the isomorphism $\varphi_* : \mathrm{Aut}(H_1(Z_{a,b}^*, \mathbb{Z})) \to \mathrm{GL}_k(\mathbb{Z})$ induced by φ. Then we have for any automorphism $\bar{f}$ of $H_1(Z_{a,b}^*, \mathbb{Z})$

the following commutative diagram:

$$
\begin{array}{ccccc}
H_1(Z_{a,b}^*, \mathbb{Z}) & \xrightarrow{\varphi'} & \mathbb{Z}^k & \xrightarrow{l_{B^{-1}}} & \mathbb{Z}^k \\
\bar{f} \downarrow & & \varphi'_*(\bar{f}) \downarrow & & \downarrow \varphi_*(\bar{f}) \\
H_1(Z_{a,b}^*, \mathbb{Z}) & \xrightarrow{\varphi'} & \mathbb{Z}^k & \xrightarrow{l_{B^{-1}}} & \mathbb{Z}^k
\end{array}
$$

Hence $\varphi_* = l_{B^{-1}} \circ \varphi'_* \circ l_B$. It follows from the Definition of h_φ and $h_{\varphi'}$ in (5) that $h_\varphi = l_{B^{-1}} \circ h_{\varphi'} \circ l_B$. Since $l_B(V) = V'$, we have for all $A \in \Gamma(Z_{a,b})$ that $h_\varphi(A)(V) = V \iff h_{\varphi'}(A)(V') = V'$. The claim now follows from Lemma 14, since H is the stabiliser of V in $\mathrm{GL}_k(\mathbb{Z})$. $\square$

Observe finally that H is finitely generated, since it is generated by the elementary matrices contained in H together with an arbitrary matrix in H having determinant -1. This finishes the proof of Proposition 11.

Example 16. *Let us consider the surface $Z_{3,0}^\infty$ over the basis surface $Z_{3,0}$ (see Figure 10). The Veech group $\Gamma(Z_{3,0}^\infty)$ is isomorphic to*

$$
< C_1, C_2, C_3 > \quad \cap \quad H
$$

$$
with \quad C_1 \;\; = \;\; \begin{pmatrix} -1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & -2 & 1 \end{pmatrix},
$$

$$
C_3 \;\; = \;\; \begin{pmatrix} 1 & 1 & 0 & -1 \\ -2 & 0 & 2 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \quad in \quad \mathrm{SL}_4(\mathbb{Z})
$$

$$
and \quad H \;\; = \;\; \left\{ \begin{pmatrix} a_{1,1} & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \in \mathrm{SL}_4(\mathbb{Z}) \mid a_{i,j} \in \mathbb{Z} \right\}.
$$

This is obtained as follows: The fundamental group U of the basis surface $Z_{3,0}$ is isomorphic to F_4. More precisely, we have:

$$
U \;\; = \;\; < g_1 = yx^{-2}, \quad g_2 = x^2 y, \quad g_3 = x^3, \quad h_1 = xyx^{-1} >
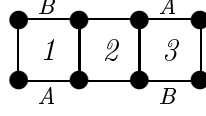$$

Figure 10: The translation surface $Z_{3,0}$.

One can calculate the Veech group $\Gamma(Z_{3,0})$ e.g. with the algorithm described in [Sc2] and obtains:

$$\Gamma(Z_{3,0}) \quad = \quad < A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \ A_3 = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \ >$$

This is an index 3 subgroup of $SL_2(\mathbb{Z})$, which consists of all elements which are congruent to I or A_1 modulo 2.

The following lifts γ_1, γ_2 and γ_3 of the matrices A_1, A_2 and A_3 to $\mathrm{Aut}^+(F_2)$ are all in $\mathrm{Stab}(U)$:

$$\begin{aligned}
\gamma_1 : \quad & x \mapsto x, & y &\mapsto x^2 y x^{-1} \\
\gamma_2 : \quad & x \mapsto xy^{-2}, & y &\mapsto y, \\
\gamma_3 : \quad & x \mapsto y^2 x^{-1}, & y &\mapsto yx^{-1}
\end{aligned}$$

Their restriction to U give the following automorphisms:

$$\begin{aligned}
\gamma_1 : \quad & g_1 \mapsto g_2 g_3^{-1}, & g_2 &\mapsto g_3 h_1, \\
& g_3 \mapsto g_3, & h_1 &\mapsto g_3 g_1 \\
\gamma_2 : \quad & g_1 \mapsto g_1 g_2 g_1 h_1^2, & g_2 &\mapsto h_1^{-2} g_1^{-1}, \\
& g_3 \mapsto h_1^{-2} g_1^{-1} g_2^{-1} g_3 g_2^{-1} g_1^{-1}, & h_1 &\mapsto h_1 \\
\gamma_3 : \quad & g_1 \mapsto g_2^{-1} g_3 g_2^{-1} g_1^{-1}, & g_2 &\mapsto g_1 g_2 g_3^{-1} g_2 g_1 h_1, \\
& g_3 \mapsto g_1 g_2 g_3^{-1} g_2 g_1 h_1^2, & h_1 &\mapsto g_1 g_2 g_3^{-1} g_1^{-1}
\end{aligned}$$

These automorphisms induce the following matrices in $GL_k(\mathbb{Z})$:

$$C_1' \ = \ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ C_2' = \begin{pmatrix} 2 & -1 & -2 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -2 & -2 & 1 \end{pmatrix},$$

$$C_3' \ = \ \begin{pmatrix} -1 & 2 & 2 & 0 \\ -2 & 2 & 2 & 1 \\ 1 & -1 & -1 & -1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

By the construction we have that A_i acts on the homology by C_i', where we have chosen the basis of $H_1(Z_{3,0}^*, \mathbb{Z})$ to be the images of the four closed curves g_1, g_2, g_3 and h_1. Using the notations in Proposition 11 and Lemma 14, we have that $h_{\varphi'}(A_i) = C_i'$. Conjugating with the matrix B^{-1} from the proof of Corollary 15 gives the desired matrices $C_i = B^{-1}C_i'B = h_\varphi(A_i)$. Then the statement follows from Proposition 11.

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LATP, CASE COUR A, FACULTÉ DES SCIENCES SAINT JÉRÔME, AVENUE ESCADRILLE NORMANDIE NIEMEN, 13397 MARSEILLE CEDEX 20, FRANCE
E-mail address: `hubert@cmi.univ-mrs.fr`

INSTITUTE FOR ALGEBRA AND GEOMETRY, UNIVERSITY OF KARLSRUHE, 76128 KARLSRUHE, GERMANY
E-mail address: `schmithuesen@mathematik.uni-karlsruhe.de`