Capital Pricing in Margin Periods of Risk and Repo KVA

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Abstract

The presence of hedging errors is practically a norm of derivatives businesses. Using the unhedgeable gap risk during a margin period of risk as a starting point, this article introduces a reserve capital approach to the hedging error and its inclusion in derivatives pricing and valuation. Specifically, we define economic capital associated with the gap risk hedging error and build the cost of capital into the Black-Scholes-Merton option pricing framework. An extended partial differential equation is derived, showing terms for expected gap loss and economic capital charge, corresponding to capital valuation adjustment -- KVA. For a repurchase agreement, economic capital is computed under a double-exponential jump-diffusion model, either estimated from historical data or calibrated to options smile. We find that the expected loss of a repo is very small and that cost of economic capital is the dominant component of the repo pricing spread. A repo therefore constitutes an ideal case to study economic capital and its valuation impact. The approach taken can be extended into margined OTC derivatives and more generally derivatives in incomplete markets.

Key words: repo, economic capital, hedging error, gap risk, margin period of risk, cost of capital, KVA, securities financing transaction,

1. Introduction

Options pricing lays its foundation on constructing a trading portfolio in underlying stock and money market (Black and Scholes 1973; Merton 1973) that dynamically hedges an option. A short option on a stock modeled as a geometric Brownian motion, for example, can be self-financed and hedged perfectly in that there is no hedging error. The option and its hedging portfolio therefore contribute zero market risk capital to its trading book. When the stock price is modeled as a jump diffusion process, the jumps can't be hedged. The hedging error is assumed diversifiable and would average out (Merton 1976) so there is no need to address risk capital, presumably non-zero. Generally, a hedging strategy can be optimized in theory to reduce the variance of the hedged portfolio in the incomplete market (Duffie and Richardson 1991, Karoui and Quenez 1995). The residual variance, however, has never been a formal part of pricing consideration, until the multifaceted post crisis reform of the regulatory risk capital framework arrives.

1 The views and opinions expressed herein are strictly the views and opinions of the author, and do not reflect those of his employer and any of its subsidiaries.
While a quick and practical response is to incorporate the strengthened regulatory capital requirements into existing derivatives pricing and valuation models, a fundamental revisit of derivatives pricing theories is in order. Post-crisis derivative pricing and risk management research has in fact advanced into a new regime where efforts from both academia and the industry have focused on studying secondary risk factors or frictions such as counterparty risk, funding, liquidity, and capital cost. These under-appreciated factors become sources of arbitrage barriers and challenge the foundation of the no-arbitrage belief. While Hull and White (2014) argue that the risk neutral price remains the best estimate of derivative’s fair value, Crepey (2011) questions whether the “one-price” rule is still valid and discusses many subtleties and recognize the need of redeveloping models from scratch.

Naturally, a revisionist approach is to expand the classic option hedging setting to accommodate these costs and re-derive the celebrated Black-Scholes-Merton partial differential equations (PDE). Burgard and Kjaer (2011, 2013) incorporate counterparty credit risk into the PDE framework and propose to treat counterparty default risk hedging errors as part of a derivatives desk's funding strategies and capture funding costs under FVA. The counterparty hedging errors are, however, results of an assumed exogenous recovery rate, for example 40%, and an outdated risk-free close-out amount calculation. Lou (2013) shows the hedging error can be eliminated by recognizing the endogenous recovery rate of par when the derivative economy is properly segregated such that, ignoring the gap risk, the residual cash flow of the economy can pay off the debt issued to finance the derivative and its hedges. Assuming the market value recovery, which is more closely aligned to current close-out amount protocol, an economic neutral hedge by the liability-side counterparty is presented that serves as an extension of the Black-Scholes framework to uncollateralized derivatives (Lou 2015).

While counterparty credit risk and funding costs are met with CVA (credit valuation adjustment) and FVA (funding valuation adjustment), whether cost of capital should be accounted as a cost of business or a new valuation adjustment becomes a heated topic. Green and Kenyon (2014) add a regulatory capital term to Burgard and Kjaer's PDE to define KVA, which Prampolini and Morini (2015) find it unnecessary as is achieved by replacing the junior bond with equity.

In this article, we set out to model and price hedging error associated with gap risk incurred during a margin period of risk (MPR). Margin period gap risk is common and pronounced in securities financing transactions and non-centrally cleared OTC derivatives. The gap risk is not hedgeable and is of non-reducible variance. In the standard market risk capital framework, capital is then required and cost of capital will incur and has to be part of pricing and valuation of derivatives. The main contribution of this article is to define a measure of economic capital corresponding to the hedging error and build capital financing cost into the Black-Scholes-Merton pricing framework. By choosing securities financing transactions where sufficient levels of haircuts essentially eliminate counterparty risk and derivatives funding risk, economic capital impact on pre-trade pricing and post-trade valuation can be well isolated and analyzed. Section 2 expands the Black-Scholes option economy for securities financing economy, its funding structure and hedging instruments. Replication portfolio and cash flow analysis are set up

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2 The funding cost on issued notes could be made higher to compensate for the gap risk, including inefficiencies in default settlements.
with the self-financing condition derived. Section 3 derives economic capital and capital valuation adjustment and links up with the repo economic model (Lou 2016). Section 4 shows repo pricing with examples. Section 5 concludes.

2. Modeling a Segregated Securities Financing Economy

We consider a capital market consisting of three classes of investors: the equity investor, the capital financer, and the liquidity provider. When underwriting a loan, these investors take on the equity, mezzanine (mezz), and the senior tranches respectively, which are cut to fit the expected loss (EL) of the loan, the unexpected loss (UL) corresponding to a given q-tile, and the remaining size of the loan. A loan for instance can be truncated into a 3% equity, 15% mezz, and 82% senior pieces according to 3% EL, and 15% UL on 18% VaR given 99.9 percentile confidence interval.

The liquidity provider looks for opportunities to invest its cash in near credit risk-free products for a finite term earning interest at the rate \( r_f \). As the credit risk premium is minimal, the spread over the risk-free rate \( r \) reflects the cost of fund (CoF) or liquidity premium of the investment. The equity investor takes any residual loss or gain of the loan in the form of interest margin or excess spread\(^3\). The capital financer only demands for the use of their capital a continuous dividend payout at a yield of \( r_k \).

A 1 year term loan is now tranched in size at C for capital (mezz), SR for liquidity provider, and 1-C-SR for equity, the loan originator's pricing sheet can be simply, \( NetCpn = C*r_k + SR*r_f + IntMargin \), where C is set to unexpected loss, 1-C-SR equals to expected loss (EL), expressed as percentages of the lending amount. Intuitively the earned interest on the loan needs to cover expected loss, capital financing cost, and the liquidity cost. NetCpn is nominal coupon netted off origination fee and other considerations such as operating costs and profit margins. The IntMargin is supposed to be sufficiently large to cover EL and is held by the equity investor.

Such a market structure, while rather foreign to derivatives pricing literature, is quite standard in insurance industry. In life insurance policy underwriting, for instance, insurers collect sufficient insurance premium to cover the policy pool's projected expected loss in the form of an economic reserve. Initial equity and retained earnings are captured in a special purpose financing vehicle that's used to pay any deficiency of the economic reserve. And a redundant reserve account is partially or fully funded that is last to be tapped into to pay insurance claims, much like the senior piece of the loan syndication.

In the classic option pricing framework, there is no need for such a capital market setup, as the dynamically replicated option portfolio as a whole has zero expected loss and zero unexpected loss (Shreve, 2004). In fact, we have \( EL=0, UL=0, SR=0 \) as it is self-financed. One could anticipate that when the secondary risk factors are considered rather than ignored, EL and UL would exist and need to be included in the option pricing framework. As an exploratory note, we adopt the BSM framework for securities financing transactions where funding and capital factors outweigh primary risks such as stock prices.

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\(^3\) Interest margin is a loan terminology while excess spread is a collateralized loan obligation (CLO) terminology. A loan syndicated with three classes of investors can be considered as a single asset CLO.
Suppose a fictitious bank B and its counterparty or customer C enter into a repurchase agreement (repo) where the bank lends an amount of \( N_p \) to C which sells stock to B at haircut \( h \) and will purchase it back on \( T \). \( N_p \) is a constant for a repo facility and the repo interest rate is \( r_p \). Let \( V_t \) be the pre-termination repo fair value. Write \( V = N_p + v_t, v_t \) the fair value of an embedded derivative in the repo. The bank has in place a hedging strategy \( \Delta_t \) in the underlying stock traded at price \( S_t \).

Economically, the reverse repo is a bilateral lending agreement that can be treated as a bilateral derivative on the stock with counterparty risk. Our intention is to segregate this lending activity into a standalone economy to identify hedging errors. We start with introducing a number of accounts associated with the bilateral defaultable option economy (Lou 2013) and add other necessary components to the economy.

**Bank account:** The segregated economy’s only investment option is a cash deposit account, with balance \( M_t \geq 0 \), earning the risk-free rate \( r(t) \).

**Stock financing account:** Stock financing is accessible in either repo or sec lending form. Assuming the reverse repo is long delta, so shares are borrowed from a sec lender to hedge the repo.

Let \( h \) be the haircut such that the economy needs to post cash collateral of \( L_s = (1+h)\Delta S \) to the sec lender, \( h > 0 \). The sec lender gives the borrower an interest rebate at a rate of \( r_s \) while the borrower pays a manufactured dividend if any to the lender. The difference between the rebate and the risk-free rate is stock’s borrowing cost. The margin account is revaluated and margined daily.

**Repo funding account:** This account represents the funding source of the lending amount of \( N_p \). This is from a liquidity provider who charges a cost of fund \( r_f \) for the use of liquidity.

**Economic capital reserve account:** This is a segregated reserve account, with its sole purpose to absorb any potential losses should the borrower default and collateral upon liquidation is not sufficient to pay off the loan. The segregated economy requires a reserve balance \( N_c \geq 0 \) deposited from the capital financier and is responsible for making dividend payment at the rate of \( r_k \). The reserve is set aside in a bank account, earning interest at \( r \).

**Debt account:** The segregated economy’s unsecured cash raising option for the remaining cash flow needs is a short term debt issued and recorded in a debt account with balance \( N_t \geq 0 \). The debt account has a unit price or par and is rolled at a short rate \( r_N(t), r_N(t) > r(t) \).

In Lou (2013) the pair of \((M_t, N_t)\) is used to capture the funding asymmetry between deposit and borrowing, satisfying \( M_t \cdot N_t = 0 \), so that cash flow generated in the economy would be used to pay down the debt first since \( r_N > r \). Here \( M_t \cdot N_t = 0 \) is not enforced. The residual cash flow of the transaction is an eligible claim paying resource for this class of notes.

**Repo margin account:** Repo terms are governed by the Master Repurchase Agreement (MRA) published by the Bond Market Association or the Global Master
Repurchase Agreement (GMRA) by the International Capital Market Association (ICMA). From counterparty credit risk mitigation point of view, repo principal $N_p$ is collateralized by the shares of stock.

Derivatives based securities financing transactions such as Total Return Swap (TRS) or Credit Default Swap (CDS) are governed by ISDA Master Agreement where the derivatives mark-to-market is collateralized under the Credit Support Annex (CSA). In securities financing transactions effected by TRS or CDS, the asset bought separately by the lender serves to collateralize its lending.

**Derivative collateral account:** Under both MRA and GMRA, a repo’s accrued interest is part of the margin while the present value of the funding interest is not. When it is placed in a trading book, the pv of the repo interest is therefore uncollateralized. Following the liability-side pricing theory (Lou 2015), a fictitious collateral account $L_t$ can be introduced to collateralize the pv at a cost of the borrower’s senior unsecured rate $r_c$.

With derivatives based SFTs, the same collateral account $L_t$ is used here but it pays only the risk-free rate $r$ when a full CSA is in place. Obviously if there is no CSA, the counterparty’s rate applies. Let $L_t$ denote the market value of collateral for $u_t$, posted by party C to B if positive, or by B to C if negative. For simplicity, we assume collateral is full, i.e., $u_t = L_t$.

Collateral for $V$ is therefore bifurcated into a repo margin account that handles the repo principal and the derivative collateral account. The economy only accepts or posts cash collaterals at zero haircut, and pays or receives $r_e$ or $r_c$ on the balance $L_t$. Furthermore the collateral process is efficient, prompt and never over-collateralized.

The repurchase agreement could terminate prior to its scheduled maturity on either party’s default. To facilitate discussion of economic capital, we assume collateral is segregated and not rehypotheticated. The borrower can reclaim its securities should the lender default. It follows that only the borrower’s default needs to be considered. This is consistent with standard loan analysis.

Write the wealth of the SFT economy $\pi_t$, $t\leq T$, as follows,

$$
\pi_t = M_t + (1 - \Gamma_t)(V - \Delta S + L - N - N_p),
$$

(1)

where $\Gamma_t = I(\tau \leq t)$ is C’s default indicator, $\tau$ the default time of C. Accounts excluding the cash are conditional to no termination, i.e., $\Gamma_t = 0$, so that all relevant quantities are to be understood as pre-default values. To shorten the formula, all $t$-subscripts have been dropped. Following the reduced form modeling approach, we further assume the default time is a stopping time having an adapted intensity process $\lambda$ defined in the usual probability triple $(\Omega, P, \mathcal{F})$. $M_t$ and $N_t$ are non-negative adapted stochastic processes.

At $t=0$, the wealth reduces to $\pi_0 = M_0 + h\Delta S_0 - N_0$. Let $M_0=0$ and $N_0=h\Delta S_0$ so that $\pi_0=0$, i.e., the initial capital of the economy starts out at zero. Note that $h\Delta S \geq 0$ as when $\Delta>0$, $h \geq 0$ for sec lending and if $\Delta<0$, $h \leq 0$ for repo. If the debt account is pegged to
the residual stock financing amount, \( N_t=h \Delta S_t \), the wealth of the pre-default economy reduces to the balance of the bank account, \( \pi_t=M_t \).

For \( \tau>\tau=0 \), during the normal course of business, the bank pursues a trading strategy to hedge the derivative and performs all necessary funding and credit support functions stipulated by the margin agreement. Excess cash is deposited into the bank account, debt, if any, is serviced and rolled as needed. Interests are collected and paid.

Specifically, over a small interval of time \( dt \), on the hedge front, shorting or rebalancing \( \Delta \) more shares of stock at the price of \( S_t+dS_t \) will see cash inflow of \( d\Delta(S_t+dS_t) \) amount. Since stocks are borrowed, additional money has to be posted to the sec lender. The debt account pays interest amount \( r_NdN dt \) and rolls into new issuance of \( N_t+dN_t \). The bank account accrues interest amount \( r_M dt \). On the collateral side, party C posts additional collateral amount \( dL_t \) in cash while being paid of interest amount \( r_L dL dt \). The repo funding account pays out \( r_N dt \) and the EC reserve account dividends out an amount of \( r_N dt \).

The wealth equation is written with all default effects implicitly built into the bank account. If a default happens before \( T \), i.e., \( \tau<T \), trades will have been settled without delay and resulting cash flow will be swiped into the bank account which becomes the only account active till \( T \). \( M_t \) may exhibit a jump at \( \tau \) as a result of default settlement. Put everything together, the economy’s pre-default financing equation, \( t<\min(T, \tau) \), follows,

\[
dM = r_M dt + (1-\Gamma)(r_p N_p dt + d\Delta(S + dS) + dN - r_N N dt + dNp - r_f N pdt - r_s N_c dt + dL - r_L dL dt - dL_s + r_s L_s dt)
\]

Noting that \( L_s=(1+h) \Delta S \) and \( L=v \), collect terms to arrive at

\[
dM = r_M dt + (1-\Gamma)((r_p - r_f) N_p dt + d\nu - \Delta dS - r_k N_c dt - r_e \nu dt + \bar{\gamma} \Delta S dt)
\]

where \( \bar{\gamma} = r_s (1+h) - r_k h \) is the economy’s effective stock financing rate.

Upon the borrower’s default at \( \tau \), the hedges can be unwound without loss as stock trades in established exchanges and counterparty risk is negligible. The stock financing account for the delta hedge also unwinds promptly with the overcollateralized amount of \( h \Delta \sigma S_t \) returned to pay down the issued note \( N_t \). So no net cash flow arises from stock hedge account \( \Delta \sigma S_t \), stock financing account \( L_s \), and the issued debt account \( N \). Furthermore the embedded derivative \( \nu_t \) offsets with the derivative collateral account \( L \).

On the repo, the lending amount at time \( \tau \) is \( N_p(\tau) \), with a collateral market value of \( N_p(\tau)/(1-h) \). Assuming that stock is liquidated at the end of the margin period of risk \( u \) at a liquidation discount \( g \) representing liquidity premium, repo account would settle at an amount of \( \min(N_p(\tau), \frac{1-g}{1-h} \frac{B_{\tau+u}}{B_{\tau}} N_p(\tau)) \). The settlement amount could result in a shortfall to the repo funding amount. The lender’s exposure is given by
where \( R \) is the applicable recovery rate of party C.

Between default time \( \tau \) and loss settlement at \( \tau+u \), the interest to the liquidity provider could keep on accruing. Accrual is ignorable for now as the accrual period is capped by MPR which is short, typically 5 business days for repos. Dividend to the capital financer can be suspended at the point of default, awaiting for default settlement. So the only cash flow during the MPR is the bank account accruing at \( r \).

As all accounts match up except the bank account, the wealth of the economy prior to completion of the default settlement is the balance of the bank account. If a loss has occurred, it will be deducted from the balance. If the balance should show a negative value, the segregated reserve account \( N_c \) will be drawn to cover the shortfall. So in the end, \( N_p \) is fully returned to the term liquidity provider, with the same confidence interval \( N_c \) is set up against.

Including the lagged default settlement, the differential wealth equation at time \( t \), \( t \geq u \), can be written as follows,

\[
d\pi = r \pi dt - d\Gamma(t-u)l(t) + (1-\Gamma(t))((r_p - r_f)N_p dt + d\nu - \Delta dS - r_k N_c dt - r_l \nu dt + \bar{r}_\nu \Delta dS dt) \tag{5}
\]

The \( d\Gamma \) term records default cash flow settlement at time \( t \) if \( t=\tau+u \). Assume that the stock price follows a geometric Brownian motion with its real world return \( \mu \) and volatility \( \sigma \), \( dS = \mu S dt + \sigma S dW \). Applying Ito’s lemma to \( \nu \), setting up delta hedge, and assuming deterministic short rates and default intensity, it follows,

\[
d\pi_t - r \pi dt = -d\Gamma(t-u)L(t) + (1-\Gamma)\lambda El(t) dt + (1-\Gamma)(\frac{\partial \nu}{\partial t} + \bar{r}_\nu \frac{\partial \nu}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \nu}{\partial S^2} - r_e \nu + (r_p - r_f)N_p - (r_k - r)N_c - \lambda El(t)) dt \tag{6}
\]

where \( El(t) = E[\left|l(t+u)\right|] \). The second term on the right hand side is a compensator to the first term, assuming that the loss at \( t+u \) is independent from the default, i.e., no specific wrong way risk. Setting the \( dt \) term on the right to zero results in the following PDE,

\[
\frac{\partial \nu}{\partial t} + \bar{r}_\nu \frac{\partial \nu}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \nu}{\partial S^2} - r_e \nu - \lambda El(t) - (r_k - r)N_c + (r_p - r_f)N_p = 0 \tag{7}
\]

The first four terms on the left hand side are same as in the liability-side pricing of derivatives (Lou 2015). The last three terms are repo specific adjustment terms. The fifth term is charge off or time decay of the expected loss during the MPR, sixth term cost of financing economic capital, and the last term the repo financing income.
The net wealth of the segregated economy, \( \pi_t \), represents a hedging error resulted from the gap loss \( l(t) \) upon a jump-to-default event. In the next section, we pinpoint the economic capital corresponding to the hedging error for an SFT economy, which enters the above PDE but has yet to be specified.

3. Measuring and Managing Hedging Error with Economic Capital

Now suppose that the embedded derivative fair value process \( \nu_t \) satisfies the PDE (7), the hedging error term (6) becomes

\[
-dA_t = d\pi_t - r\pi dt = -d\Gamma_{\tau-u} l(\tau + u) + (1 - \Gamma_{\tau}) El(t) \lambda dt.
\] (8)

For time \( s>0 \), \( dA_t \) can be integrated\(^4\) to get,

\[
A_s = \Gamma_{s} l(\tau) - \int_{0}^{s} (1 - \Gamma_{\tau}) El(t) \lambda dt.
\] (9)

Assuming default settlement loss is independent of the jump-to-default event, it can be easily verified that \( E[A_s]=0 \) with \( A_0=0 \) and that \( A_s \) is a martingale as for \( s>t>u \),

\[
A_s - A_t = I(t < \tau \leq s)(l(\tau) - \int_{t}^{s} El(y) \lambda dy).
\] (10)

\( A_s \) can be interpreted as the net hedging loss accumulation up to time \( s \), excluding its risk-free growth in the bank account. A natural way to look at the economic loss and capital requirement is to consider the terminal loss \( A_T \).

Define a VaR measure for \( A_T \) given a q-tile,

\[
VaR_A = \inf \{ x \in [0, +\infty) : \Pr(A_T > x) \leq 1 - q \}.
\] (11)

where the tail distribution of \( A_T \) is given by

\[
\Pr(A_T > x) = \int_{0}^{T} \Pr[l(\tau) > x + \int_{0}^{\tau} El(t) \lambda dt]dP(\tau)
\] (12)

Because \( E[A_T] \) is zero, economic capital is the same as \( VaR_A \), i.e., \( N_e=VaR_A \). Obviously, \( VaR_A \) can be defined at a forward time \( t \), assuming party \( C \) has not defaulted yet by then.

The EC definition above is for a fixed time \( T \), while the reserve account \( N_e(t) \) is set up starting from time zero. A simple adjustment is to discount it back, i.e., \( N_e=VaR_A \beta_T \), where \( \beta_T = \exp(-\int_{0}^{T} r ds) \).

\(^4\) Because of the delayed settlement, the jump term needs to be integrated from \( u \) to \( s+u \), while the survival term from \( 0 \) to \( s \).
Alternatively, multiply the risk-free deflator $\beta_t$ to equation (8) and integrate from $t$ to $T$ to arrive at, $\beta_T \pi_T - \beta_t \pi_t = \int_t^T \beta_s dA_s$. Because the deflator is continuous and $A_s$ is a martingale, the discounted hedging error is also a martingale. On a no-default path where $I_s=0$ and $dI_s=0$ for $t \leq s \leq T$, the path integral gives

$$\beta_T \pi_T - \beta_t \pi_t = \int_t^T \beta_s El(s + u) \lambda ds. \quad (13)$$

This shows that the discounted wealth of the economy accumulates continuously over time on a no-default path, at the rate of default intensity times the expected loss during the MPR.

On a path with default time $\tau < T$, it jumps due to default settlement that could lead to a loss, and the path integral becomes

$$\beta_T \pi_T - \beta_t \pi_t = \int_t^\tau \beta_s El(s + u) \lambda ds - \beta_t l(\tau + u) \quad (14)$$

Note that we always have $\beta_T \pi_T = \beta_t \pi_t$ for $\tau < T$.

Now we can define the loss of economic value at time $t$ for the full remaining duration of the trade and its VaR as follows,

$$\hat{\pi}_t = \pi_t - \frac{\beta_T}{\beta_t} \pi_T$$

$$VaR_t = \inf\{ x \in R : Pr(\hat{\pi}_t > x) \leq 1 - q \}. \quad (15)$$

Because $E[\hat{\pi}_t]$ is zero, $N_c(t) = VaR_t$. If the asset price dynamics does not depend on the default of the borrower, the tail probability of $\hat{\pi}_t$ becomes

$$Pr(\hat{\pi}_t > x) = \int_\tau^T E[I(\hat{\pi}_t > x) \mid \tau]dP_\tau(\tau), \quad (16)$$

where $dP_\tau(\tau)$ is probability of default at time $\tau$ as seen on time $t$, i.e., forward default curve,

$$E[I(\hat{\pi}_t > x) \mid \tau] = Pr(l \geq b_\tau),$$

$$b_\tau = \frac{\beta_T}{\beta_t} x + El \int_\tau^T \lambda \frac{\beta_T}{\beta_t} ds \quad (17)$$

When a loss is realized and the wealth is not sufficient to pay the loss, money in the reserve account will be drawn to cover loss. On an expectation basis, the wealth growth is sufficient to meet the loss, so economic capital is indeed for unexpected losses.
With economic capital defined, the PDE (eqt.7) is fully specified. Applying Feynman-Kac theorem and noting \( \nu(T)=0 \) arrive at the pricing formulae,

\[
\nu(t) = E_t \left[ e^{r_f T - \lambda r_p du} \left( s_p N_p(s) - s_k N_c(s) - \lambda El(s) \right) ds \right] 
\]

(18)

where \( s_p=r_p-r_f \) is the liquidity spread and \( s_k=r_k-r \) is the capital charge spread. Let \( U(t)=\nu^*(t)-\nu(t) \) be the valuation adjustment to the risk-free price \( \nu^* \)

\[
U(t) = CRA + EVA + KVA,
\]

\[
CRA = E_t^0 \left[ \int_t^T (r_e - r) \nu^*(s) e^{-\lambda s r_p du} ds \right],
\]

\[
EVA = E_t^0 \left[ \int_t^T \lambda El(s) e^{-\lambda s r_p du} ds \right],
\]

\[
KVA = E_t^0 \left[ \int_t^T s_k N_c(s) e^{-\lambda s r_p du} ds \right]
\]

(19)

In addition to the counterparty risk adjustment (CRA), fair value has two new adjustments, namely the economic value adjustment (EVA) and the capital value adjustment (KVA). There is no overlap between CRA and KVA as the latter is dedicated to the gap risk in the margin period of risk while the former does not consider MPR.

If expected loss \( El \) and economic capital \( N_c(t) \) are computed exogenously, for instance, by means of referencing a standardized schedule, the PDE can be solved by finite difference methods. Decomposition of CRA into bilateral, coherent CVA and FVA can be done by solving the PDE with shifts in each parties’ synthetic or cash curves (Lou 2015).

Because KVA itself is part of the fair value, the formulae for KVA is recursive in general. An implementation directly calculating KVA however risks to have open IR01 that could attract capital in the same way as FVA when defined imprecisely (Lou 2015b).

If the funding amount, expected loss, and economic capital are constant in time, then with a constant default intensity \( \lambda \), the fair value solution is given by \( \nu = (s_p N_p - s_k N_c - \lambda El) \text{apv} \). \text{apv} is the borrower's risky discounted annuity if the repo spread is uncollateralized, or risk-free discounted annuity if fully collateralized. For pre-trade pricing, setting \( \nu(t) \) to zero leads to the repo rate to be charged to client, we obtain the repo pricing formula,

\[
r_p = r_f + s_k \frac{N_c}{N_p} + \lambda \frac{El}{N_p}.
\]

(20)

For repos, \( El \) is usually very small due to the presence of haircuts, so EVA is small and repo rate \( r_p \) is dominated by cost of fund \( r_f \) and the charge on economic capital.

To obtain the formula above, we started by treating repo as a form of equity derivatives with a replication and financing scheme that accommodates capital for gap
risk. Because of its constant exposure, the derivatives based pricing degenerates into a computation of expected loss given default and the economic capital. In fact, the PDE is reduced to an ordinary differential equation \( \frac{d\nu}{dt} - r\nu + s_p N_p - s_k N_c - \lambda E\ell(t) = 0 \). Of course for constant position securities financing transactions, such as a TRS on a fixed number of shares of stock or fixed notional of bond, the full PDE applies\(^5\).

Alternatively, a repo can be seen as a form of counterparty credit derivatives and be priced following the conventional credit derivatives risk-neutral pricing approach. From a credit derivatives perspective, the lender sells a funded protection to the borrower\(^6\). Briefly, let \( \beta(t) \) be the applicable discount factor, the net present value (npv) of the loss, or default present value (dpv), is given by,

\[
dpv = E\left[ \int_0^T \beta(t)dl(t) \right] = \beta(T)E[l(T)] - \int_0^T E[l(t)]d\beta(t), \tag{21}
\]

The reverse repo is effectively a floating rate note on a rate index, commonly LIBOR for tenors of 1 month or longer. The index part of the repo interest rate gets back to par when discounted at the same index curve. The present value of a unit spread, or annuity denoted by \( apv \), is given by \( apv = E\left[ \int_0^T \beta(t)Q(t)dt \right] \), where \( Q(t) \) is party C's survival probability and the annuity is computed on a unit notional, suitable for a repo funding facility. The net present value of the repo (npv) is then \( npv = 1 - dpv + s_p*apv \) where \( s_p \) is the repo spread.

The pair of dpv and apv are essentially what the conventional credit derivatives pricing technique brings about. For a repo, however, these are not materially relevant, for dpv is very small due to the presence of significant haircuts and apv is very close to \( T \) when \( T \leq 1 \) year, unless with extreme short term discount rate and low quality credit counterparty. What matters to repos are cost of fund and how much capital is used at what cost, both of which are missing from the above. This shows again the inability of the risk-neutral pricing theory to capture gap risk and economic capital.

Lou (2016) presents a repo haircuts model and a companion economic capital model linking for the first time haircuts and economic capital to collateral security's price dynamics and borrower's credit spread dynamics. The exposure of an SFT is made equivalent to a secured term loan to a wholesale counterparty. The economic capital model takes into account asset risk, credit risk, wrong way risk, and market liquidity risk. General wrong way risk (WWR) is modeled by correlation between the asset return and the credit spread. The specific WWR is captured by a jump-on-default liquidation

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\(^5\) For collateralized OTC derivatives, margin period of risk exists and an initial margin account is called upon as a reserve. The PDE applies, with the funding term term \( N_p \) dropped and once expected loss and economic capital for the hedging error is linked to the derivatives. This will be left for future research.

\(^6\) If the borrower defaults and the lender suffers a loss after a repo settlement, the borrower's estate can be viewed as if having taken the opposite, i.e., benefitted from not having to pay back the full principal borrowed.
discount. Economic capital is defined as the unexpected loss taken either as the VaR or expected shortfall minus the expected loss. Historically estimated double exponential jump diffusion model shows that the regulatory capital undercuts economic capital in a risk-on range of haircuts and overstates economic capital in a low economic risk range.

As expected, the general WWR is weak and negligible, because of the frequent margining and the MPR is short. Consequently there is not a strong practical need to introduce stochastic default intensity. If the borrower credit spread is deterministic, then EL and EC can be computed as a deterministic function of time $t$. Repo fair value and valuation adjustment can be computed straightforwardly. If a dynamic spread such as the log-Ornstein-Uhlenbeck default intensity model is used and is independent of the asset price process, EL and EC can be computed separately and plugged into the fair value formula to arrive at repo fair value.

Note that the PDE is derived assuming a geometric Brownian motion. When stock price follows a jump-diffusion process, the PDE would involve fair value jumps corresponding to jumps in stock prices, and become a partial integral differential equation. The fair value expectation formula and valuation adjustments however remain valid as the jumps are reflected in the probability measure under which the expectation is taken.

4. Repo Pricing and Valuation Examples

Repos are mostly treated as secured loans for accounting purposes, valued at par and go on accrual. For broker-dealers acting as financing intermediaries between hedge funds and money market funds, repos are often treated as trading book positions and fair value accounting follows. Because repo tenors are short and risks embedded are gap risk in nature, repo valuation commonly reduces to taking the present value of the repo spread. Pre-trade repo pricing is no more interesting than post-trade valuation. A bank's repo desk for instance simply adds a spread margin on top of cost of fund obtained from its treasury or sourced elsewhere.

SFT counterparties, especially commercial banks and insurance companies whose books sit on the lending side and don't enjoy any offsetting effect, also consider regulatory capital implication. Under BASEL III, a credit risk exposure is calculated to the extent the trade haircut is less than the required regulatory haircut. A capital charge can be calculated by applying the firms' ROE target to the regulatory capital and added to the repo spread to form final pricing.

Exclusive use of regulatory capital in trade pricing is probably not its original intention. Lou (2016) shows that regulatory capital undercuts economic capital in a risk-on range of haircuts and overstates economic capital in a low economic risk range. Figure 1 shows a comparison of mark-to-market of a sample one year repo on SPX500 with a capital charge collected on economic capital and regulatory capital (RC), when traded at haircuts ranging from zero to 20%. The most noticeable difference lies in the low haircut region, where RC is significantly lower than EC, resulting in large overvaluation. In particular, at zero haircut, pv on EC is about 3.47 bp while pv on RC is
36.42 bp. This observation is important for derivatives (e.g. TRS) based securities financing which is often conducted at zero haircut.

Figure 1. Repo MTM with cost of economic capital (labelled 'EC-pv') compared with regulatory capital (labelled 'RC-pv') vs haircuts. 1y repo, 60 bp repo spread, 10% ROE, and $r_c = 3.1\%$.

Figure 2. Expected shortfall EC profile for a one year repo on SPX500 main equity under haircuts of 0%, 5%, and 10%. MPR=10 days.
Economic capital tends to decay through time as the remaining maturity of the trade shortens\(^7\). Figure 2 shows projected economic capital for the sample repo trade. The asset price model is a double-exponential jump-diffusion model estimated from daily history of SPX500 covering January 2008 to January 2013. The borrower is a ‘BBB’ party with its 5 year CDS traded at 250 bps level. With zero haircut, EC maintains elevated across the full duration, only tapering off approaching maturity. At 5% haircut, EC reduces by about half. At 10% haircut, EC starts at 0.72% and declines almost linearly to zero at maturity.

To highlight each risk factor’s contribution to repo fair value, valuation adjustments are shown in Table 1. KVA strikes out and dominates other adjustments. As expected, EVA is not a factor and can be dropped from the repo pricing formula (eqt. 20). CRA (CVA plus FVA) is very small as the counterparty credit risk is already mitigated by repo margining.

<table>
<thead>
<tr>
<th>HC</th>
<th>npv*</th>
<th>CRA</th>
<th>EVA</th>
<th>KVA</th>
<th>npv</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>59.78</td>
<td>0.73</td>
<td>0.07</td>
<td>55.51</td>
<td>3.47</td>
</tr>
<tr>
<td>5%</td>
<td>59.78</td>
<td>0.73</td>
<td>0.01</td>
<td>26.98</td>
<td>32.05</td>
</tr>
<tr>
<td>10%</td>
<td>59.78</td>
<td>0.73</td>
<td>0.00</td>
<td>4.09</td>
<td>54.96</td>
</tr>
</tbody>
</table>

In the above example, the repo has a fixed earned spread (repo rate minus cost of fund) of 60 bp, resulting in positive npv. In pre-trade pricing, the break-even earned spread can be solved by setting npv to zero, or equivalently by adding CRA, EVA and KVA. At 5% haircut, it will be 27.72 bp.

For post trade valuation, whether to include cost of capital in accounting fair value is debatable, as it is usually considered a cost outside of the revenue and income stream to which the fair value belongs. Others argue that, since it is charged by a party stepping in, it should be reflected in the fair value per IFRS 13. The more pressing question is what capital it is and what it accounts for. In our view, KVA should be based on economic capital and account for hedging errors or unhedgeable risks. A measure of forward economic capital cost can always be calculated and kept as a valuation reserve to be released over time, if not formally accepted into fair value.

The computation above shows an ideal case where a second order risk effect outweighs first order. Traditional risk neutral pricing is correct only to the extent that complete hedges can be constructed or hedging errors can be perfectly diversified to nil. Otherwise, economic capital will result and cost of capital has to be evaluated and treated properly. Charging clients for capital cost is not only necessary but also an extension of existing derivatives pricing theories where prices are computed on expectation under a risk neutral measure without consideration of risk capital.

\(^7\) Regulatory credit risk capital rule adopts a 1 year floor.
5. Conclusion

Recognizing the hedging error during margin periods of risk is unhedgeable, we set out to define economic capital and incorporate the capital financing cost into derivatives pricing and valuation. The Black-Scholes-Merton pricing framework is extended to cope with economic capital inherent of securities financing transactions. Two valuation adjustments - economic value added (EVA) and - capital valuation adjustment (KVA) are introduced. We find that EVA is very small and KVA dominates other valuation adjustments. At zero haircut, a one-year repo with 10 day MPR on main equity could command capital charge as large as 50 bp for a 'BBB' rated borrower. This is significant for derivatives based funding transactions such as TRS transacted at zero haircuts. Increased haircut reduces capital charge, e.g., to 4 bp at 10% haircut.

Repo's pricing is thus driven by cost of fund and economic capital charge, providing a perfect example for capital cost and KVA discussion. We define KVA as cost of economic capital (instead of regulatory capital), specifically associated with hedging errors or outright unhedgeable risks. Our approach in linking economic capital to hedging errors and pricing in its financing cost can be extended to margined OTC derivatives where margin periods of risk apply with a variable future mark-to-market exposure. It could also find applications in derivatives hedging and valuation in incomplete markets. These are left for future research.

References

Burgard, C. and M. Kjaer, 2013, Funding strategies, funding costs, Risk, December, pp 82-87.
Crepey, S., 2011, A BSDE approach to counterparty risk under funding constraints, working paper.


