The uses of homogeneous barycentric coordinates in plane euclidean geometry

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Abstract. The notion of homogeneous barycentric coordinates provides a powerful tool of analysing problems in plane geometry. In this paper, we explain the advantages over the traditional use of trilinear coordinates, and illustrate its powerfulness in leading to discoveries of new and interesting collinearity relations of points associated with a triangle.

1. Introduction

In studying geometric properties of the triangle by algebraic methods, it has been customary to make use of trilinear coordinates. See, for examples, [1], [2], [3], [4], [5]. With respect to a fixed triangle ABC (of side lengths a, b, c, and opposite angles α, β, γ), the trilinear coordinates of a point is a triple of numbers proportional to the signed distances of the point to the sides of the triangle. The late Jesuit mathematician Maurice Wong has given [5] a synthetic construction of the point with trilinear coordinates $\cot\alpha : \cot\beta : \cot\gamma$, and more generally, in [4] points with trilinear coordinates $a^{2n}x : b^{2n}y : c^{2n}z$ from one with trilinear coordinates $x : y : z$ with respect to a triangle with sides a, b, c. On a much grandiose scale, Kimberling [2], [3] has given extensive lists of centres associated with a triangle, in terms of trilinear coordinates, along with some collinearity relations.

The present paper advocates the use of homogeneous barycentric coordinates instead. The notion of barycentric coordinates goes back to Möbius. See, for example, [6]. With respect to a fixed triangle ABC, we write, for every point P on the plane containing the triangle,

$$P = \frac{1}{\triangle ABC}((\triangle PBC)A + (\triangle PCA)B + (\triangle PAB)C),$$

and refer to this as the barycentric coordinate of P. Here, we stipulate that the area of a triangle XYZ be zero if X, Y, Z are collinear, positive if the orientation of the vertices is counter - clockwise, and negative otherwise. In terms of barycentric coordinates, there is the basic area formula.

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Proposition 1 (Bottema [7]). *If the vertices of a triangle $P_1P_2P_3$ have homogeneous coordinates $P_i = x_iA + y_iB + z_iC$, then the area of the triangle is*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \triangle.$$

Often, it is convenient to consider the homogeneous coordinates of P, namely,

$$\triangle PBC : \triangle PCA : \triangle PAB.$$

Here is a useful mechanical interpretation: the homogeneous barycentric coordinates of P are "weights" at A, B, C such that the "center of mass" is precisely at P. For example, every point P on the line BC has homogeneous coordinates of the form

$$0 : CP : PB,$$

in terms of signed lengths of directed segments.

It is clear that a point with homogeneous barycentric coordinates $x : y : z$ has trilinear coordinates $(x/a) : (y/b) : (z/c)$. Conversely, a point with trilinear coordinates $u : v : w$ has homogeneous barycentric coordiantes $au : bv : cw$.

2. Traces of a point on the sides of a triangle

2.1. *Coordinates of traces.* The first advantage of homogeneous barycentric coordinates is that the coordinates of the traces of a point on the sides of the reference triangle can be read off easily. Let $P = x : y : z$ in homogeneous barycentric coordinates. The lines AP, BP, CP intersect the lines BC, CA, AB respectively at the points X, Y, Z with homogeneous coordinates

$$\begin{aligned} X &= 0 : y : z, \\ Y &= x : 0 : z, \\ Z &= x : y : 0. \end{aligned} \tag{1}$$

See figure 1. Conversely, if X, Y, Z have homogeneous coordinates given by (1) above, then the cevians AX, BY, and CZ are concurrent at a point with homogeneous barycentric coordinates $x : y : z$.

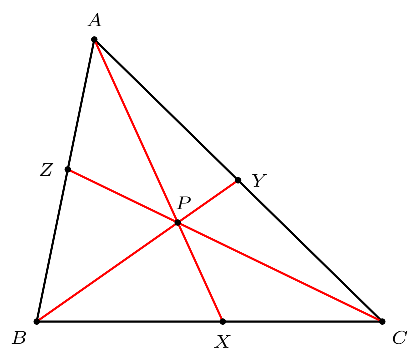

Figure 1. The traces of a point

2.2. *An example: the Nagel point.* The Nagel point N_a is the point of concurrency of the cevians joining each vertex to the point of contact of the excircle on its opposite side. The existence of this point is clear from noticing that the points of contact X, Y, Z have homogeneous coordinates

$$\begin{array}{rcccccc} X & = & 0 & : & s-b & : & s-c, \\ Y & = & s-a & : & 0 & : & s-c, \\ Z & = & s-a & : & s-b & : & 0. \end{array} \tag{2}$$

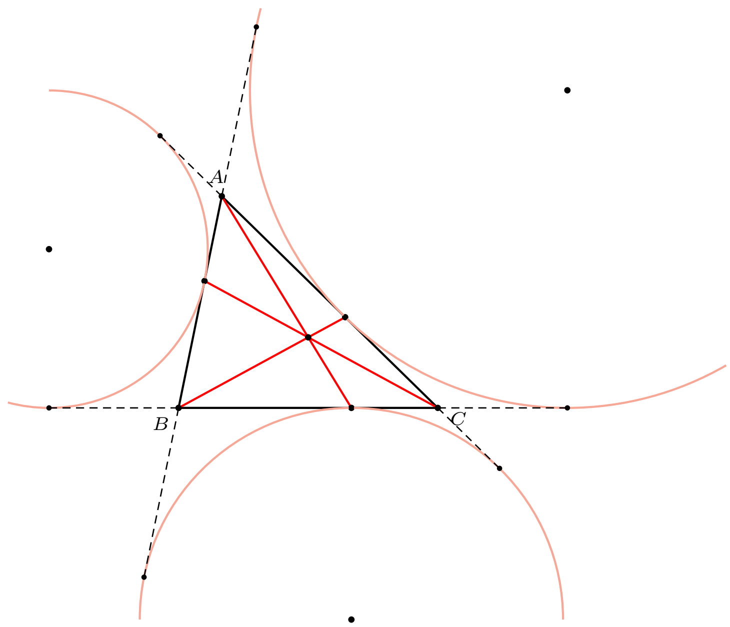

Figure 2. The Nagel point

In barycentric coordinates, this is

$$\begin{aligned} N_a &= (s-a)A + (s-b)B + (s-c)C \quad \text{normalized} \\ &= s(A+B+C) - (aA+bB+cC) \quad \text{normalized} \\ &= s(3G) - (2s)I \quad \text{normalized} \\ &= 3G - 2I. \end{aligned}$$

Here, 'normalized' means dividing the expression by the sum of the coefficients. From this we conclude that the Nagel point N_a divides the segment IG externally in the ratio $IN_a : N_aG = -2 : 3$. This is, of course, a well known result. See, for example [8].

2.3. *Some notable centres.* In Table 1, we list the homogeneous barycentric coordinates of some notable centres associated with a triangle ABC, with sides a, b, c, and semiperimeter $s = \frac{1}{2}(a+b+c)$.

centre	Symbol	Homogeneous barycentric coordinates
Centroid	G	1:1:1
Incentre	I	$a : b : c$
Excentres	I_A	$-a : b : c$
	I_B	$a : -b : c$
	I_C	$a : b : -c$
Gergonne point	G_e	$(s-b)(s-c) : (s-c)(s-a) : (s-a)(s-b)$
Nagel point	N	$s-a : s-b : s-c$
Symmedian point	K	$a^2 : b^2 : c^2$
Exsymmedian points	K_A	$-a^2 : b^2 : c^2$
	K_B	$a^2 : -b^2 : c^2$
	K_C	$a^2 : b^2 : -c^2$
Circumcentre	O	$a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)$
Orthocentre	H	$(a^2+b^2-c^2)(c^2+a^2-b^2)$ $: (b^2+c^2-a^2)(a^2+b^2-c^2)$ $: (c^2+a^2-b^2)(b^2+c^2-a^2)$

Table 1. Homogeneous barycentric coordinates of some notable points.

3. Multiplication

3.1. *Multiplication of points on a line.* The following proposition provides a simple construction for the product of two points on a segment, and leads to the fruitful notion of multiplication of points of the plane not on any of the lines defining the reference triangle.

Proposition 2. *Let X_1, X_2 be two points on the line BC, distinct from the vertices B, C, with homogeneous coordinates $0 : y_1 : z_1$ and $0 : y_2 : z_2$. For $i = 1, 2$, complete parallelograms $AK_iX_iH_i$ with K_i on AB and H_i on AC. The line joining the intersections of BH_1 and CK_2, and of BH_2 and CK_1, passes through the vertex A, and intersects BC at a point X with homogeneous coordinates $0 : y_1y_2 : z_1z_2$.*

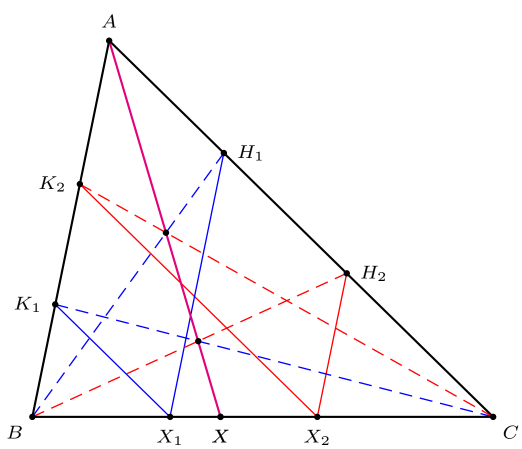

Figure 3. Multiplication of points

Proof. Consider the cevians BH_1 and CK_2. Clearly,

$$\frac{CH_1}{H_1A} = \frac{CX_1}{X_1B} = \frac{y_1}{z_1},$$

and

$$\frac{AK_2}{K_2B} = \frac{CX_2}{X_2B} = \frac{y_2}{z_2}.$$

By Ceva's theorem, the unique point X on BC for which the cevians AX, BH_1, and CK_2 are concurrent is given by

$$\frac{BX}{XC} \cdot \frac{CH_1}{H_1A} \cdot \frac{AK_2}{K_2B} = 1.$$

From this, $BX : XC = z_1z_2 : y_1y_2$, and X is the point $0 : y_1y_2 : z_1z_2$.

A similar calculation shows that this is the same point for which the cevians AX, BH_2, and CK_1 are concurrent. □

3.2. *Multiplication of points in a plane.* Consider two points P_i, $i = 1, 2$, with nonzero homogeneous barycentric coordinates $x_i : y_i : z_i$. By applying Proposition 2 to the traces on each of the three sides of the reference triangle, we obtain three points

$$\begin{array}{rcccccc} X &=& 0 &:& y_1y_2 &:& z_1z_2, \\ Y &=& x_1x_2 &:& 0 &:& z_1z_2, \\ Z &=& x_1x_2 &:& y_1y_2 &:& 0. \end{array} \tag{3}$$

The cevians AX, BY, CZ intersect at a point with homogeneous barycentric coordinates

$$x_1x_2 : y_1y_2 : z_1z_2.$$

We shall denote this point by $P_1 \cdot P_2$, and call it the product of P_1 and P_2 (with respect to triangle ABC).

3.3. *An abelian group structure.* The multiplication of points considered above clearly defines an abelian group structure on the set $\mathcal{G}$ of all points with nonzero homogeneous barycentric coordinates, i.e., points not on any of the lines defining the reference triangle. The centroid G is the multiplicative identity, since it has homogeneous barycentric coordinates $1 : 1 : 1$. The inverse of a point $P = x : y : z$ is precisely its isotomic conjugate ([4]), with homogeneous barycentric coordinates $1/x : 1/y : 1/z$. For this reason, we shall denote by P^{-1} the isotomic conjugate of P.

3.4. *Isogonal conjugates.* The isogonal conjugate ([4]) of a point $P = x : y : z$ (with nonzero homogeneous barycentric coordinates) is the point

$$P^* = \frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}.$$

Note that $P \cdot P^* = a^2 : b^2 : c^2$. This is the symmedian point of the triangle, and can be constructed from the incentre I as $I^2 = I \cdot I$. It is also the isogonal conjugate of the centroid $G = 1 : 1 : 1$.

3.5. *Examples.* A second advantage of the use of homogeneous barycentric coordinates is that a factorization of the coordinates entails a construction procedures of the point in question in terms of simpler ones. Consider, for example, the problem of locating in the interior of triangle ABC a point whose distances from the sides are in the proportion of the respective exradii. This is the point with trilinear coordinates

$$r_A : r_B : r_C = \frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c},$$

and appears as X_{57} in [2]. In homogeneous barycentric coordinates, this is the point

$$a \cdot \frac{1}{s-a} : b \cdot \frac{1}{s-b} : c \cdot \frac{1}{s-c}.$$

As such, it can be constructed as the product $I \cdot G_e$ of the incentre I and the Gergonne point G_e (see table 1).

4. The square root construction

4.1. *The square root of a point.* Let P be a point in the interior of triangle ABC, with traces X, Y, Z on the sides. The square root of P is an interior point Q such that $Q \cdot Q = P$. To locate such a point Q, construct circles with respective diameters BC, CA, and AB, intersecting the respective perpendiculars to the sides through X, Y, Z (respectively) at X'', Y'', Z''. Construct the bisectors of the right angles $BX''C$, $CY''A$, $AZ''B$ intersecting the sides BC, CA, AB at X', Y', Z'. The cevians AX', BY', and CZ' are concurrent at the requisite square root Q. This follows easily from the lemma below.

Lemma 3. *Let X be a point on a segment BC. Suppose the perpendicular through X intersects the circle with diameter BC at X''. Construct the bisector of the right angle $BX''C$ to intersect BC at X'. Then*

$$\left(\frac{BX'}{X'C}\right)^2 = \frac{BX}{XC}.$$

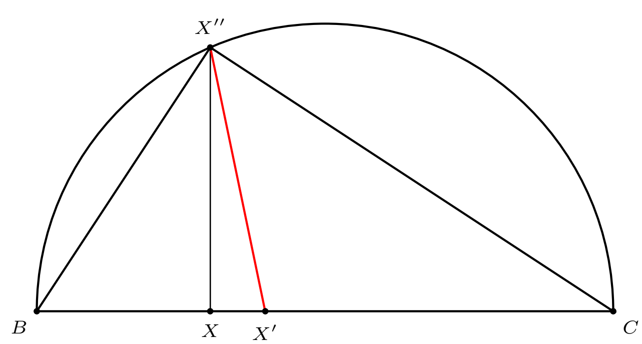

Figure 4. Square root of a point

Proof. Since $X''X'$ bisects the right angle $BX''C$, we have

$$\frac{BX'}{X'C} = \frac{X'B}{X'C}.$$

It follows that

$$\left(\frac{BX'}{X'C}\right)^2 = \frac{X'B^2}{X'C^2} = \frac{BC \cdot BX}{BC \cdot XC} = \frac{BX}{XC}.$$

This completes the proof of the lemma. □

4.2. *Examples.* For example, given a triangle, to construct the point Q whose distances from the sides are proportional to the square roots of the lengths of these sides. The trilinear coordinates of Q being $\sqrt{a} : \sqrt{b} : \sqrt{c}$, the homogeneous barycentric coordinates are given by $a^{3/2} : b^{3/2} : c^{3/2}$. This point is the square root of the point $a^3 : b^3 : c^3$, which is $I \cdot K$.

As another example, consider the point with homogeneous coordinates

$$\sin\frac{\alpha}{2} : \sin\frac{\beta}{2} : \sin\frac{\gamma}{2}.$$

Since

$$\sin^2\frac{\alpha}{2} = \frac{(s-b)(s-c)}{bc} = \frac{(s-a)(s-b)(s-c)}{abc} \cdot \frac{a}{s-a},$$

we have

$$\sin^2\frac{\alpha}{2} : \sin^2\frac{\beta}{2} : \sin^2\frac{\gamma}{2} = \frac{a}{s-a} : \frac{b}{s-b} : \frac{c}{s-c},$$

and the point in question can be constructed as the geometric mean of the incentre $I = a : b : c$ and the Gergonne point $G_e = 1/(s-a) : 1/(s-b) : 1/(s-c)$.

5. More examples of collinearity relations

A third advantage of homogeneous barycentric coordinates is the ease of obtaining interesting collinearity relations of points associated with a triangle. We have already seen one example in §2.2. Here is another interesting example of the use of homogeneous coordinates.

5.1. *Equal-intercept point.* Given a triangle, to construct a point P the three lines through which parallel to the sides of a given point cut out equal intercepts. In general, the line through $P = xA + yB + zC$ parallel to BC cuts out an intercept of length $(1-x)a$. It follows that the three intercepts parallel to the sides are equal if and only if

$$1-x : 1-y : 1-z = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}.$$

The right hand side clearly gives the homogeneous barycentric coordinates of the isotomic conjugate of the incentre I. It follows that

$$I^{-1} = \frac{1}{2}[(1-x)A + (1-y)B + (1-z)C] = \frac{1}{2}(3G - P).$$

From this, $P = 3G - 2I^{-1}$, and can be easily constructed as the point dividing the segment $I^{-1}G$ externally in the ratio $I^{-1}P : PG = 3 : -2$. See figure 5. Note that an easy application of Proposition 1 shows that the line joining P, G and I^{-1} does not contain I unless the triangle is isosceles.

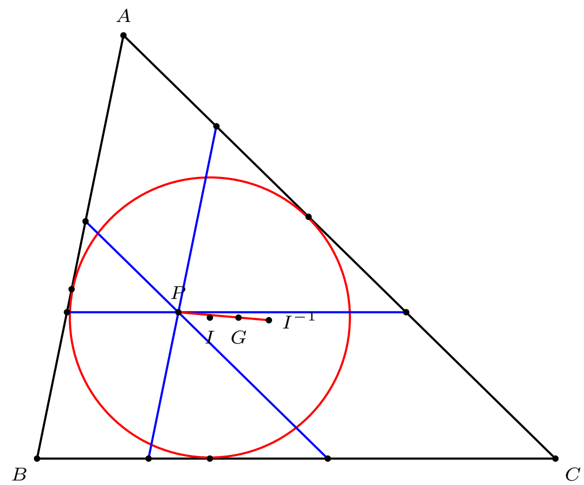

Figure 5. Equal-intercept point

In fact, many of the collinearity relations in [2], [3] can be explained by manipulating homogeneous barycentric coordinates. We present a few more examples.

5.2. *The point with trilinear coordinates* $b+c : c+a : a+b$. Consider, for example, the 'simplest unnamed centre' X_{37} in [2], with trilinear coordinates $b+c : c+a : a+b$. In homogeneous barycentric coordinates, this is $a(b+c) : b(c+a) : c(a+b)$. Thus, $X_{37} = I \cdot P$, where P is the point with homogeneous barycentric coordinates $b+c : c+a : a+b$. This happens to be the point X_{10} of [2], the Spieker centre of the triangle, the incentre of the triangle formed by the midpoint of the sides of the given triangle. Without relying on this piece of knowledge, a direct, simple calculation with the barycentric coordinates leads to an easy construction of this point. It is indeed the point which divides the segment IG externally in the ratio $3 : -1$.

$$\begin{aligned} P &= (b+c)A + (c+a)B + (a+b)C \quad \text{normalized} \\ &= (a+b+c)(A+B+C) - (aA+bB+cC) \quad \text{normalized} \\ &= (2s)(3G) - (2s)I \quad \text{normalized} \\ &= \frac{1}{2}(3G - I). \end{aligned}$$

5.3. *The Mittenpunkt.* Consider the Mittenpunkt X_9 with trilinear coordinates $s-a : s-b : s-c$, or homogeneous barycentric coordinates $a(s-a) : b(s-b) : c(s-c)$. While this can certainly be interpreted as the product $I \cdot N_a$ of the incentre I and the Nagel point N_a, we consider the barycentric coordinates:

$$\begin{aligned} & a(s-a)A + b(s-b)B + c(s-c)C \quad \text{normalized} \\ = \;& s(aA+bB+cC) - (a^2A + b^2B + c^2C) \quad \text{normalized.} \end{aligned}$$

This shows that X_9 is on the line joining the incentre I to the symmedian point $K = a^2 : b^2 : c^2$. Noting that

$$a^2 + b^2 + c^2 = 2[s^2 - (4R + r)r], \tag{4}$$

where R and r are respectively the circumradius and the inradius of the triangle, (see [8]), we have the symmedian point K dividing the segment IX_9 externally in the ratio $(4R + r)r : s^2$. While this certainly leads to a construction of X_9, it is much easier to locate X_9 by finding another collinearity relation. Since

$$a(s-a) + (s-b)(s-c) = (s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a)$$

is a symmetric function in a, b, c, say, $f(a, b, c)$, we also have

$$\begin{aligned} & a(s-a)A + b(s-b)B + c(s-c)C \quad \text{normalized} \\ = \; & f(a,b,c)G - [(s-b)(s-c)A + (s-c)(s-a)B + (s-a)(s-b)C] \\ & \quad \text{normalized} \end{aligned}$$

From this it follows that X_9 is collinear with the centroid G and the Gergonne point G_e. It can therefore be located as the intersection of the two lines IK and GG_e. The fact that X_9 lies on both of these lines is stated in [2], along with other lines containing the same point.

5.4. *Isogonal conjugates of the Nagel and Gergonne points.* The isogonal conjugate N_a^* of the Nagel point has homogeneous barycentric coordinates

$$\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c} = a^2(s-b)(s-c) : b^2(s-c)(s-a) : c^2(s-a)(s-b).$$

Making use of the formula

$$\sin^2\frac{\alpha}{2} = \frac{(s-b)(s-c)}{bc},$$

we write this in barycentric coordinates:

$$\begin{aligned} N_a^* &= \left(a\sin^2\frac{\alpha}{2}\right)A + \left(b\sin^2\frac{\beta}{2}\right)B + \left(c\sin^2\frac{\gamma}{2}\right)C \quad \text{normalized} \\ &= a(1-\cos\alpha)A + b(1-\cos\beta)B + c(1-\cos\gamma)C \quad \text{normalized} \\ &= 2s\cdot I - ((a\cos\alpha)A + (b\cos\beta)B + (c\cos\gamma)C) \quad \text{normalized} \\ &= RI - rO \quad \text{normalized.} \end{aligned}$$

Here, we have made use of

$$(a\cos\alpha)A + (b\cos\beta)B + (c\cos\gamma)C = \left(\frac{2rs}{R}\right)O.$$

A similar calcuation shows that the isogonal conjugate G_e^* of the Gergonne point, namely,

$$G_e^* = a^2(s-a) : b^2(s-b) : c^2(s-c)$$

has barycentric coordinate $\frac{1}{R+r}(RI+rO)$. This means that the isogonal conjugates of the Gergonne point and the Nagel point divide the segment OI harmonically in the ratio of the circumradius and the inradius. These are respectively the internal and external centres of similitude of the circumcircle and the incircle.

We close by mentioning an interesting geometric property of each of these centers of similitude of the circumcircle and the incircle.

(i) Through the internal center of similitude G_e^*, there are three congruent circles each tangent to two sides of the triangle. See ([9]).

(ii) The mixtilinear incircles of a triangle are the three circles each tangent to two sides and to the circumcircle internally. The three segments each joining a vertex of the triangle to the point of tangency of the circumcircle with the mixtilinear incircle in that angle intersect at the external center of similitude N_a^*. See ([10]).

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