State feedback control of continuous-time T–S fuzzy systems via switched fuzzy controllers

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Received 10 May 2007; received in revised form 31 October 2007; accepted 3 November 2007

Abstract

This paper studies the problem of state feedback control of continuous-time T–S fuzzy systems. Switched fuzzy controllers are exploited in the control design, which are switched based on the values of membership functions, and the control scheme is an extension of the parallel distributed compensation (PDC) scheme. Sufficient conditions for designing switched state feedback controllers are obtained with meeting an $H_\infty$ norm bound requirement and quadratic $\mathcal{D}$ stability constraints. It is shown that the new control design method provides less conservative results than the corresponding ones via the parallel distributed compensation (PDC) scheme. A numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: Takagi–Sugeno (T–S) fuzzy systems; Switched fuzzy controllers; State feedback; $H_\infty$ control; $\mathcal{D}$ stability; Linear matrix inequalities (LMIs)

1. Introduction

In the area of nonlinear control systems design, an important approach is to model the considered nonlinear systems as Takagi and Sugeno (T–S) fuzzy models by a set of If–Then rules, which are formed from linguistic variables and values [31] and by quantifying the meaning of the linguistic values using “membership functions”, the T–S fuzzy model can be obtained [21]. Compared with conventional modeling techniques which uses a single model to describe the global behavior of a nonlinear system, fuzzy modeling is essentially a multimodel approach in which simple sub-models (typically linear models) are fuzzily combined to describe the global behavior of a nonlinear system. As a result, the conventional linear system theory can be applied to...
the analysis and synthesis of the class of nonlinear control systems. Thus, numerous control problems have been studied based on this T–S fuzzy model. For example, state feedback control design problems are studied for discrete T–S fuzzy systems in \([4,9,27,30,28]\) and for continuous cases in \([14,23,25]\). Methods for designing static output feedback controllers and dynamic output feedback controllers are given in \([12,13,17]\), respectively. Moreover, robust control design methods are presented in \([24,29]\), respectively, for continuous and discrete-time-delay fuzzy systems. The advantage of these results is that the stability analysis and controller design can be converted into convex optimization problems in terms of linear matrix inequalities (LMIs) \([1]\), which can be solved efficiently \([7]\).

Recently, considerable efforts have been contributed to the design of an \(H_\infty\) fuzzy controller for a class of nonlinear systems, which can be represented by a Takagi–Sugeno (T–S) fuzzy model. In \([15]\), sufficient conditions for designing \(H_\infty\) controllers are given by considering the interactions among the fuzzy subsystems. Robust control methods are obtained for uncertain fuzzy systems with Frobenius norm-bounded parameter uncertainties in all system matrices in \([16]\). Controller design conditions based on piecewise Lyapunov function approaches are presented for continuous-time \([6,30]\) and discrete-time \([2,26]\) T–S fuzzy systems by switching constant controller gains. In \([19,20]\), a switching fuzzy controller design approach is presented for a class of continuous-time nonlinear systems by constructing a switching fuzzy model. Moreover, \(H_\infty\) static and dynamic output control synthesis problems are studied in \([11,17]\), respectively. However, the most above mentioned papers only deal with disturbance rejection aspects and provide little control over its transient behavior. On the other hand, for linear time invariant (LTI) systems, satisfactory time response and closed-loop damping can often be achieved by forcing the closed-loop poles into a suitable region \([3]\). However, the notion of pole does not exist in the context of nonlinear systems. In order to achieve good control effect for nonlinear systems, an LPV controller is designed for a 6-DOF vehicle through the use of pole clustering for each frozen LTI system in \([8]\). Moreover, generalizations of existing definitions of poles and zeros for linear time-varying systems are studied in \([18]\). In particular, by extending the notion of \(\mathcal{D}\) stability for linear systems to nonlinear systems, the desired transient specification behaviors can be guaranteed by imposing the requirement of \(\mathcal{D}\) stability for closed-loop systems, see \([17,10,5]\).

In this paper, the problem of designing switched state feedback \(H_\infty\) controllers for continuous-time T–S fuzzy systems with meeting the requirement of \(\mathcal{D}\) stability will be studied. A new type of state feedback controllers, namely, switched parallel distributed compensation (PDC) controllers, are proposed, which are switched based on the values of membership functions. The new control scheme is an extension of the PDC scheme \([22]\) and the switched controller gain scheme \([6]\). The purpose is to combine the merits of the two control schemes to enhance the performances of closed-loop systems. Quadratic Lyapunov functions are exploited to derive a new method for designing switched PDC controllers for guaranteeing the stability and \(H_\infty\) performances of closed-loop nonlinear systems with meeting the requirement of \(\mathcal{D}\) stability. The design conditions are given in terms of solvability of a set of linear matrix inequalities (LMIs). It is shown that the new method provides better or at least the same results of the corresponding design methods via the pure PDC scheme. The paper is organized as follows. Section 2 presents the T–S fuzzy model, the new type of state feedback control scheme and some preliminaries. Section 3 provides a technique for designing an \(H_\infty\) state feedback controller with meeting requirements of \(\mathcal{D}\) stability for continuous-time T–S fuzzy systems. An example is given to illustrate the effectiveness of the new proposed method in Section 4. Concluding remarks are given in Section 5.

**Notation:** For a square matrix \(E\), \(\text{He}(E)\) is defined as

\[
\text{He}(E) = E + E^T
\]

and

\[
E > 0 \iff \text{He}(E) > 0
\]

\[
[H_{ij}]_{r\times s} =:
\begin{bmatrix}
H_{11} & H_{12} & \cdots & H_{1g} \\
H_{21} & H_{22} & \cdots & H_{2g} \\
\vdots & \vdots & \ddots & \vdots \\
H_{r1} & H_{r2} & \cdots & H_{rg}
\end{bmatrix}
\]

\(\otimes\) denotes the Kronecker product of matrices, e.g.,
2. System description and problem statement

2.1. T–S Fuzzy Models

The fuzzy dynamic model, proposed by Takagi and Sugeno [21] is described by fuzzy IF–THEN rules, which locally represents linear input–output relations of a system. The continuous-time Takagi–Sugeno (T–S) fuzzy model is of the following form: Plant Rule 

$$
\text{IF } v_1(t) \text{ is } M_{i1} \text{ and } v_2(t) \text{ is } M_{i2}, \ldots, v_p(t) \text{ is } M_{ip} \text{ THEN } 
\dot{x}(t) = A_i x(t) + B_{1i} w(t) + B_{2i} u(t) 
$$

$$
z(t) = C_i x(t) + D_{1i} w(t) + D_{2i} u(t)
$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input vector, $z(t) \in \mathbb{R}^{n_z}$ is the controlled output vector, and $w(t) \in \mathbb{R}^{n_w}$ is the exterior disturbance. $r$ is the number of IF–THEN rules, $v_i(t)$ are the premise variables, and $M_{ij}$ are the fuzzy sets. By using the fuzzy inference method with a singleton fuzzifier, product inference and center average defuzzifiers, the T–S fuzzy model is obtained as

$$
\dot{x}(t) = \frac{\sum_{i=1}^{r} w_i(v(t)) (A_i x(t) + B_{1i} w(t) + B_{2i} u(t))}{\sum_{i=1}^{r} w_i(v(t))} 
$$

$$
z(t) = \frac{\sum_{i=1}^{r} w_i(v(t)) (C_i x(t) + D_{1i} w(t) + D_{2i} u(t))}{\sum_{i=1}^{r} w_i(v(t))}
$$

where

$$w_i(v(t)) = \prod_{j=1}^{p} M_{ij}(v_j(t))$$

$M_{ij}(v_j(t))$ is the grade of membership of $v_j(t)$ in $M_{ij}$, where it is assumed that

$$\sum_{i=1}^{r} w_i(v(t)) > 0, \quad w_i(v(t)) \geq 0, \quad i = 1, 2, \ldots, r$$

Denote

$$\alpha_i(v(t)) = \frac{w_i(v(t))}{\sum_{i=1}^{r} w_i(v(t))}$$

then

$$0 \leq \alpha_i(v(t)) \leq 1 \quad \text{and} \quad \sum_{i=1}^{r} \alpha_i(v(t)) = 1$$

$\alpha_i(v(t))$ are said to be normalized membership functions. Denote $\alpha(v(t)) = [\alpha_1(v(t)), \alpha_2(v(t)), \ldots, \alpha_r(v(t))]^{T}$

For the convenience of notations, $\alpha_i(v(t))$ is denoted as $\alpha_i$, the vector $\alpha(v(t))$ as $\alpha$ and denote

$$\sum_{i=1}^{r} \alpha_i A_i = A(\alpha) \sum_{i=1}^{r} \alpha_i B_{1i} = B_1(\alpha)$$

$$\sum_{i=1}^{r} \alpha_i B_{2i} = B_2(\alpha) \sum_{i=1}^{r} \alpha_i C_i = C(\alpha)$$

$$\sum_{i=1}^{r} \alpha_i D_{1i} = D_1(\alpha) \sum_{i=1}^{r} \alpha_i D_{2i} = D_2(\alpha)$$

where $T$ is a matrix.
Then (2) can be written as follows:

\[
\dot{x}(t) = A(x)x(t) + B_1(x)w(t) + B_2(x)u(t)
\]

\[
z(t) = C(x)x(t) + D_1(x)w(t) + D_2(x)u(t)
\]

(5)

2.2. Switched PDC scheme

Denote

\[
\Omega = \left\{ x : 0 \leq x_i \leq 1, 1 \leq i \leq r, \sum_{i=1}^{r} x_i = 1 \right\}
\]

\[
\Omega_l = \left\{ x : 0 \leq x_i \leq x_{l_i}, 1 \leq i \leq r, \sum_{i=1}^{r} x_i = 1 \right\}
\]

(6a)

(6b)

where \( 1 \leq l \leq r \).

Remark 1. \( \Omega \) is the set of all the vectors \( x = [x_1, x_2, \ldots, x_r]^T \), and \( x_i \) \( (i = 1, \ldots, r) \) take all possible values of membership functions \( x_i \). \( \Omega_l \) is the set of all the vectors \( x \) with \( x_i \) \( (i = 1, \ldots, r) \) satisfying \( 0 \leq x_i \leq x_{l_i} \), which describes the case where the \( l \)th rule plays more important or at least the same role than other rules. In fact, for a fuzzy system, at some time instant \( t_s \), there must exist a fuzzy subsystem playing more important or at least the same role than other subsystems, if the subsystem is \( l \)th subsystem, i.e., \( x_i \leq x_{l_i}, 1 \leq i \leq r \). Then at the time instant \( t_s \), the vector \( x(v(t_s)) = [x_1(v(t_s)), x_2(v(t_s)), \ldots, x_r(v(t_s))]^T \in \Omega_l \). All vectors \( x(v(t)) \) with \( x_i(v(t)) \leq x_i(v(t)) \) consist of the set \( \Omega_l \). Obviously, \( \Omega = \bigcup_{l=1}^{r} \Omega_l \).

In this paper, a new type of controllers, namely switched PDC controllers, are considered as follows:

\[
u(t) = K(x)z(t)
\]

(7)

where

\[
K(x) = \begin{cases} 
K_1(x) = \sum_{j=1}^{r} x_j K_{j1} & \text{for } x \in \Omega_1 \\
\vdots & \\
K_r(x) = \sum_{j=1}^{r} x_j K_{jr} & \text{for } x \in \Omega_r 
\end{cases}
\]

(8)

Moreover, if at some time invariant \( t, x(v(t)) \in \Omega_i \cap \Omega_j \), then the controller gains are chosen as follows:

\[
K(x) = \begin{cases} 
K_i(x), & x(v(t^-)) \in \Omega_i \\
K_j(x), & x(v(t^-)) \in \Omega_j 
\end{cases}
\]

Remark 2. From (7), it can be seen that when \( x \in \Omega_i \), it implies that the \( i \)th local model plays a more important (or at least the same) role than other local models in the overall system, therefore, a specific controller gain \( K_i(x) \) is used for the case in (7). The switched PDC scheme is an extension of the PDC scheme [22] and the switched controller gain scheme [6]. In fact, when \( K_1(x) = \cdots = K_r(x) \), the switched control scheme described by (7) is reduced to the pure PDC scheme [22]. When \( K_i(x)(i = 1, \ldots, r) \) are constant gain matrices, i.e., independent of \( x \), then the switched PDC control scheme are reduced to the switched linear control gains scheme given in [6].

Combining (5) and (7), then we can obtain the following closed-loop fuzzy system:

\[
\dot{x}(t) = (A(x) + B_2(x)K(x))x(t) + B_1(x)w(t)
\]

\[
z(t) = (C(x) + D_2(x)K(x))x(t) + D_1(x)w(t)
\]

(9)
Then the problem considered in this paper is formulated as follows:

Given a prescribed $H_\infty$ performance $\gamma > 0$ and any LMI stability region $\mathcal{D}$ with the characteristic function (12), design a fuzzy state feedback controller of the form (7) such that:

1. The closed-loop system (9) is asymptotically stable.
2. The closed-loop fuzzy system (9) is quadratically stable in the given LMI stability region $\mathcal{D}$ described by (11) and (12).
3. \[ \int_0^\infty z^T(t)z(t) \leq \gamma^2 \int_0^\infty w^T(t)w(t) \] (10)

2.3. Preliminaries

We recall the following definition.

**Definition 3** (LMI regions [3,17]). A subset $\mathcal{D}$ of the complex plane is called an LMI region if there exist a symmetric matrix $\Gamma = [\Gamma_{kl}]_{g \times g} \in \mathbb{R}^{g \times g}$ and a matrix $\Pi = [\Pi_{kl}]_{g \times g} \in \mathbb{R}^{g \times g}$ such that

\[ \mathcal{D} = \{ z = x + jy \in \mathcal{C} : f_\mathcal{D}(z) < 0 \} \] (11)

with the characteristic function

\[ f_\mathcal{D}(z) = \Gamma + \Pi z + \Pi^T z = [\Gamma_{kl} + \Pi_{kl} z + \Pi_{kl}^T z]_{g \times g} \] (12)

**Definition 4** [17] Quadratic $\mathcal{D}$-stability. Given an LMI stability region $\mathcal{D}$ defined by (12), the nonlinear system $\dot{x}(t) = f(x(t))x(t)$ is said to be quadratically $\mathcal{D}$-stable if there exists a positive definite symmetric matrix $X \in \mathbb{R}^{g \times g}$ such that

\[ \Gamma \otimes X + \Pi \otimes (Xf(x(t))) + \Pi^T \otimes (Xf(x(t)))^T < 0 \] (13)

where $\otimes$ denotes the Kronecker product of the matrices.

**Remark 5.** Let $V(x(t)) = x^T(t)Xx(t)$. Pre-and post-multiplying (13) by $I \otimes x^T(t)$ and its transpose, then it follows that for all $x(t) \neq 0$:

\[ \Gamma \otimes V(x(t)) + \Pi \otimes \frac{1}{2} \dot{V}(x(t)) + \Pi^T \otimes \frac{1}{2} \dot{V}(x(t)) < 0 \] (14)

Knowing $\frac{1}{2} \dot{V}(x(t)) = x^T(t)Xf(x(t))x(t) = x^T(t)f^T(x(t))Xx(t)$ and dividing (14) by $V(x(t))$, then we have

\[ \Gamma \otimes 1 + \Pi \otimes \frac{1}{2} \frac{\dot{V}(x(t))}{V(x(t))} + \Pi^T \otimes \frac{1}{2} \frac{\dot{V}(x(t))}{V(x(t))} < 0 \]

According to Definition 3, we can obtain that $\frac{1}{2} \frac{V(x(t))}{V(x(t))} \in \mathcal{D}$. This implies that if $\mathcal{D}$ is an LMI stability region, then the system’s decay rates lie within the given LMI stability region $\mathcal{D}$.

Below are a few examples of $\mathcal{D}$ regions.

- If $\mathcal{D}$ is the open left plane, this corresponds to $\Gamma = 0$ and $\Pi = 1$. Then (14) reduces to $\dot{V}(x(t)) < 0$, which coincides with the standard quadratic stability inequality.
- If $\mathcal{D}$ is an open left plane $\mathcal{D}(z) = -\theta$, this corresponds to $\Gamma = 0$ and $\Pi = 1$. Then (14) becomes $\dot{V}(x(t)) < -\theta V(x(t))$, which constraints the system to have at least $\theta$ decay rate.

In particular, for nonlinear systems, the desired transient specification can be characterised by the decay rate and $\zeta = \frac{\sqrt{\gamma^2 - 2}}{\gamma}$. The more concrete details can be found in [17].
3. Main results

In this section, a method of designing switched PDC controllers via state feedback for continuous-time fuzzy systems with a combination of $H_\infty$ performance and quadratic $\mathcal{D}$-stability requirements is presented. The details are given in the following theorem.

**Theorem 6.** Given a prescribed $H_\infty$ performance $\gamma$, any LMI stability region $\mathcal{D}$ with the characteristic function \((12)\), if there exist matrices \(Q = Q^T > 0 \in \mathbb{R}^{n_x \times n_x}, Y_{ij} \in \mathbb{R}^{n_u \times n_x}, J_{aijl} = J_{aijl}^T \in \mathbb{R}^{n_x \times n_x}, J_{bjjl} = J_{bjjl}^T \in \mathbb{R}^{n_x \times n_x}, R_{aijl} = R_{aijl}^T \geq 0 \in \mathbb{R}^{n_x \times n_x}, R_{bjjl} = R_{bjjl}^T \geq 0 \in \mathbb{R}^{n_x \times n_x}, 1 \leq i \leq j \leq r, 1 \leq l \leq r\) satisfy the following LMIs:

\[
\begin{align*}
\lambda_{iil} &< J_{aiil}, \quad 1 \leq i \leq r, \quad 1 \leq l \leq r \quad (15) \\
\lambda_{ijl} + \lambda_{jil} &< J_{aijl} + J_{ajil}^T, \quad 1 \leq i < j \leq r, \quad 1 \leq l \leq r \quad (16) \\
[J_{aijl}]_{r \times r} + \text{He}(E_{ai}[R_{aijl}]_r) + E_{aj}[R_{aijl}]_{(r-1) \times (r-1)}E_{al} &< 0, \quad 1 \leq l \leq r \quad (17) \\
\Gamma \otimes Q + \Pi \otimes \Phi_{il} + \Pi^T \otimes \Phi_{iil}^T &< J_{bil}, \quad 1 \leq i \leq r, \quad 1 \leq l \leq r \quad (18) \\
2\Gamma \otimes Q + \Pi \otimes \Phi_{iil} + \Pi^T \otimes \Phi_{iil}^T + \Pi \otimes \Phi_{jil} + \Pi^T \otimes \Phi_{jil}^T &< J_{bjil} + J_{jbil}^T, \quad 1 \leq i < j \leq r, \quad 1 \leq l \leq r \quad (19) \\
[J_{bjil}]_{r \times r} + \text{He}(E_{bi}[R_{bjil}]_r) + E_{bi}[R_{bjil}]_{(r-1) \times (r-1)}E_{bl} &< 0, \quad 1 \leq l \leq r \quad (20)
\end{align*}
\]

where

\[
\begin{align*}
\Phi_{ijl} &= A_iQ + B_{2i}Y_{ijl} \\
\lambda_{ijl} &= \begin{bmatrix}
A_iQ + QA_i^T + B_{2i}Y_{ijl} + Y_{ijl}^TB_{2i}^T & B_{1i} & (C_iQ + D_{2i}Y_{ijl})^T \\
B_{1i}^T & -\gamma^2 I & D_{1i}^T \\
C_iQ + D_{2i}Y_{ijl} & D_{1i} & -I
\end{bmatrix}
\end{align*}
\]

\[
E_{ai} = \begin{bmatrix}
-I_{e_a \times e_a} & 0 & \cdots & 0 & I_{e_a \times e_a} & 0 & \cdots & 0 \\
0 & -I_{e_a \times e_a} & \cdots & 0 & I_{e_a \times e_a} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -I_{e_a \times e_a} & I_{e_a \times e_a} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & I_{e_a \times e_a} & -I_{e_a \times e_a} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & I_{e_a \times e_a} & 0 & \cdots & -I_{e_a \times e_a}^{(r-1)\times r}
\end{bmatrix}
\]

\[e_a = n_e + n_z + n_w\]

\[
E_{bl} = \begin{bmatrix}
-I_{e_b \times e_b} & 0 & \cdots & 0 & I_{e_b \times e_b} & 0 & \cdots & 0 \\
0 & -I_{e_b \times e_b} & \cdots & 0 & I_{e_b \times e_b} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -I_{e_b \times e_b} & I_{e_b \times e_b} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & I_{e_b \times e_b} & -I_{e_b \times e_b} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & I_{e_b \times e_b} & 0 & \cdots & -I_{e_b \times e_b}^{(r-1)\times r}
\end{bmatrix}
\]

\[e_b = g \times n_z\]
Then the controller (7) with
\[ K_{il} = Y_{il}Q^{-1}, \quad 1 \leq i, \quad l \leq r \]
such that the closed-loop system (9) is asymptotically stable in the given LMI stability region \( \mathcal{D} \) described by (11) and (12) and satisfies an \( H_{\infty} \) performance bounded by \( \gamma \).

**Proof.** Considering
\[
\begin{bmatrix}
\alpha_1 I \\
\vdots \\
\alpha_s I
\end{bmatrix}^T \left( \text{He} \left( E_{al}^T [R_{aijl}]_{(r-1) \times r} \right) + E_{al}^T [\overline{R}_{aijl}]_{(r-1) \times (r-1)} E_{al} \right) \begin{bmatrix}
\alpha_1 I \\
\vdots \\
\alpha_s I
\end{bmatrix} = \text{He} 
\begin{bmatrix}
(\alpha_l - \alpha_1) I \\
\vdots \\
(\alpha_l - \alpha_{l-1}) I \\
(\alpha_l - \alpha_{l+1}) I \\
\vdots \\
(\alpha_l - \alpha_s) I
\end{bmatrix}^T 
\begin{bmatrix}
\alpha_1 I \\
\vdots \\
\alpha_s I
\end{bmatrix} 
\quad \text{for } \alpha = [\alpha_1 \alpha_2 \cdots \alpha_s]^T \in \Omega_i, \text{ then we have} 
\begin{align*}
\alpha_l - \alpha_i & \geq 0, \quad 1 \leq i \neq l \leq r \\
\text{Combining it with } (22) \text{ and } R_{aijl} \geq 0, \ \overline{R}_{aijl} \geq 0, \text{ then it follows that:}
\begin{bmatrix}
\alpha_1 I \\
\vdots \\
\alpha_s I
\end{bmatrix}^T \left( \text{He} \left( E_{al}^T [R_{aijl}]_{(r-1) \times r} \right) + E_{al}^T [\overline{R}_{aijl}]_{(r-1) \times (r-1)} E_{al} \right) \begin{bmatrix}
\alpha_1 I \\
\vdots \\
\alpha_s I
\end{bmatrix} \geq 0, \quad 1 \leq l \leq r, \ \alpha \in \Omega_i
\end{align*}
\]
Considering it and (23), then we can obtain
\[
\begin{bmatrix}
    x_i I \\ 
    \vdots \\
    x_r I
\end{bmatrix}^T
\begin{bmatrix}
    [J_{a lj}]_{r \times r} \\
    \vdots \\
    [J_{a ij}]_{r \times r}
\end{bmatrix}
\begin{bmatrix}
    x_i I \\
    \vdots \\
    x_r I
\end{bmatrix} < 0, \quad 1 \leq i \leq r, \quad \alpha \in \Omega_i
\]
i.e.,
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} x_i x_j J_{a lj} < 0, \quad 1 \leq l \leq r, \quad \alpha \in \Omega_i
\]
(25)

Multiplying (15) and (16), respectively, by $x_i^2$, $1 \leq i \leq r$ and $x_i x_j$, $1 \leq i < j \leq r$ and summing them, then we have
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} x_i x_j A_{ij} < \sum_{i=1}^{r} \sum_{j=1}^{r} x_i x_j J_{a ij}, \quad 1 \leq l \leq r, \quad \alpha \in \Omega_i
\]
Combining it and (25), then yields
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} x_i x_j A_{ij} < 0, \quad 1 \leq l \leq r, \quad \alpha \in \Omega_i
\]
(26)

which can be rewritten as follows:
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} x_i x_j \begin{bmatrix}
    \text{He}(A_i Q + B_2 Y_{lj}) & B_{li} & (C_i Q + D_2 Y_{lj})^T \\
    B_{li}^T & -\gamma^2 I & D_{li}^T \\
    C_i Q + D_2 Y_{lj} & D_{li} & -I
\end{bmatrix} < 0, \quad 1 \leq l \leq r, \quad \alpha \in \Omega_i
\]
(27)

From (21), we have $Y_{lj} = K_{lj} Q$, substituting $K_{lj} Q$ for $Y_{lj}$ in (27), then it follows that:
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} x_i x_j \begin{bmatrix}
    \text{He}(A_i Q + B_2 K_{lj} Q) & B_{li} & (C_i Q + D_2 K_{lj} Q)^T \\
    B_{li}^T & -\gamma^2 I & D_{li}^T \\
    C_i Q + D_2 K_{lj} Q & D_{li} & -I
\end{bmatrix} < 0, \quad 1 \leq l \leq r, \quad \alpha \in \Omega_i
\]
which can be rewritten as follows:
\[
\begin{bmatrix}
    \text{He}(A(z) Q + B_2(z) K_i(z) Q) & B_1(z) & (C(z) Q + D_2(z) K_i(z) Q)^T \\
    B_1^T(z) & -\gamma^2 I & D_1^T(z) \\
    C(z) Q + D_2(z) K_i(z) Q & D_1(z) & -I
\end{bmatrix} < 0, \quad 1 \leq l \leq r, \quad \alpha \in \Omega_i
\]
(28)

where $A(z), B_1(z), B_2(z), C(z), D_1(z), D_2(z), K_i(z)$, are same as in (4) and $K_i(z)$ is same as in (8).

Let $P = Q^{-1}$, since $Q > 0$, then $P > 0$ and $P$ is invertible. Pre-and post-multiplying (28) by
\[
\begin{bmatrix}
    P & 0 & 0 \\
    0 & I & 0 \\
    0 & 0 & I
\end{bmatrix}
\]
and its transpose, then we can obtain
\[
\begin{bmatrix}
    \text{He}(PA(z) + PB_2(z) K_i(z)) & PB_1(z) & (C(z) + D_2(z) K_i(z) Q)^T \\
    B_1^T(z) P & -\gamma^2 I & D_1^T(z) \\
    C(z) + D_2(z) K_i(z) & D_1(z) & -I
\end{bmatrix} < 0, \quad 1 \leq l \leq r, \quad \alpha \in \Omega_i
\]
Applying Schur complement to the above inequality, then we have
which can be rewritten as follows:

\[
\begin{bmatrix}
\text{He}(PA(x) + PB_2(x)K_l(x)) \\
PB_1(x) \\
B_1^T(x)P \\
-\gamma^2 I
\end{bmatrix}
+ \begin{bmatrix}
(C(x) + D_2(x)K_l(x))^T \\
D_1^T(x)
\end{bmatrix}
\begin{bmatrix}
C(x) + D_2(x)K_l(x)D_1(x)
\end{bmatrix}
\]

\[< 0, \quad 1 \leq l \leq r, \quad x \in \Omega_l \quad (29)\]

where

\[
\begin{align*}
T_{11l} &= \text{He}(PA(x) + PB_2(x)K_l(x)) + (C(x) + D_2(x)K_l(x))^T(C(x) + D_2(x)K_l(x)) \\
T_{12l} &= PB_1(x) + (C(x) + D_2(x)K_l(x))^TD_1(x) \\
T_{22l} &= -\gamma^2 I + D_1^T(x)D_1(x)
\end{align*}
\]

Pre-and post-multiplying (29) by \([x^T(t)w^T(t)] \neq 0\)

and its transpose, then yields

\[
x^T(t)T_{11l}x(t) + 2x^T(t)T_{12l}w(t) + w^T(t)T_{22l}w(t)
= x^T(t)\text{He}(PA(x) + PB_2(x)K_l(x))x(t)
+ x^T(t)(C(x) + D_2(x)K_l(x))^T(C(x) + D_2(x)K_l(x))x(t)
+ 2x^T(t)(PB_1(x) + (C(x) + D_2(x)K_l(x))^TD_1(x))w(t)
+ w^T(t)(-\gamma^2 I + D_1^T(x)D_1(x))w(t) < 0, 1 \leq l \leq r, x \in \Omega_l \quad (30)
\]

Choose Lyapunov function

\[V(t) = x^T(t)Pv(t)\]

and considering

\[
\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t)
= 2x^T(t)P((A(x) + B_2(x)K_l(x))x(t) + B_1(x)w(t))
+ ((C(x) + D_2(x)K_l(x))x(t) + D_1(x)w(t))g^T((C(x)
+ D_2(x)K_l(x))x(t) + D_1(x)w(t)) - \gamma^2 w^T(t)w(t) \quad (31)
\]

For any \(x = [x_1, x_2, \ldots, x_r]\), we have \(x \in \Omega\) and there exists some \(l \in \{1, \ldots, r\}\), such that \(x \in \Omega_l\), then from (7) and (31)

\[
\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t)
= 2x^T(t)P((A(x) + B_2(x)K_l(x))x(t) + B_1(x)w(t)) + ((C(x) + D_2(x)K_l(x))x(t) + D_1(x)w(t))^T((C(x)
+ D_2(x)K_l(x))x(t) + D_1(x)w(t)) - \gamma^2 w^T(t)w(t)
\]

Combining it with (30), we have

\[
\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0 \quad (32)
\]

Integrating both sides of this inequality yields

\[
\int_0^\infty \dot{V}(t)dt + \int_0^\infty z^T(t)z(t)dt - \gamma^2 \int_0^\infty w^T(t)w(t)dt
= V(\infty) - V(0) + \int_0^\infty z^T(t)z(t)dt - \gamma^2 \int_0^\infty w^T(t)w(t)dt < 0
\]
Using the fact that $\dot{x}(0) = 0$ and $V(\infty) \geq 0$, we obtain
\[
\int_0^\infty z^T(t)z(t)dt \leq \gamma^2 \int_0^\infty w^T(t)w(t)dt
\]

Hence, (10) holds and the $H_\infty$ performance is fulfilled.

If the disturbance $w(t) = 0$, then from (32), we have $\dot{V}(t) < 0$. Hence, the closed-loop system (9) is asymptotically stable.

Moreover, pre-and post-multiplying (20) by

\[
[\begin{array}{c}
\alpha_1I \\
\vdots \\
\alpha_rI
\end{array}]
\]

and its transpose, yields

\[
\begin{bmatrix}
\alpha_1I \\
\vdots \\
\alpha_rI
\end{bmatrix}^T
= \begin{bmatrix}
[\begin{array}{c}
J_{bji1}\end{array}]_{r \times r} \\
\vdots \\
\begin{array}{c}
J_{bri}I
\end{array}
\end{bmatrix} + \begin{bmatrix}
\alpha_1I \\
\vdots \\
\alpha_rI
\end{bmatrix}^T (\text{He}(E_{bi}^T[R_{bji}]_{(r-1) \times r} + E_{bi}^T[\overline{R}_{bji}]_{(r-1) \times (r-1)}E_{bi}) \\
\begin{bmatrix}
\alpha_1I \\
\vdots \\
\alpha_rI
\end{bmatrix} < 0, \ 1 \leq l \leq r
\]

Combining it and (24), then yields

\[
\sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j J_{bji} < 0, \ 1 \leq l \leq r, \ \alpha \in \Omega_l
\] (33)

Substituting $K_{ji}Q$ for $Y_{ji}$ in $\Phi_{ji}$, then

\[
\Phi_{ji} = A_iQ + B_2K_{ji}Q
\]

Multiplying (18) and (19) respectively by $\alpha_i^2$, $1 \leq i \leq r$ and $\alpha_i \alpha_j$, $1 \leq i < j \leq r$ and summing them, then we have

\[
\Gamma \otimes Q + \Pi \otimes \left( \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \Phi_{ji} \right) + \Pi^T \otimes \left( \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \Phi_{ji} \right)^T < 0, \ 1 \leq l \leq r, \ \alpha \in \Omega_l
\]

From (33) and the above inequality, then yields

\[
\Gamma \otimes Q + \Pi \otimes \left( \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \Phi_{ji} \right) + \Pi^T \otimes \left( \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \Phi_{ji} \right)^T
= \Gamma \otimes Q + \Pi \otimes (A(x)Q + B_2(x)K_{ji}(x)Q) + \Pi^T \otimes (A(x)Q + B_2(x)K_{ji}(x)Q)^T < 0, \ 1 \leq l \leq r, \ \alpha \in \Omega_l
\]

Since $Q = Q^T > 0$, then from Definition 4, it follows that quadratic $\mathcal{D}$-stability requirement with the characteristic function (12) is satisfied. Thus, the proof is complete. $\square$

**Remark 7.** Theorem 6 presents an LMI-based condition for designing state feedback $H_\infty$ switched PDC controllers with meeting quadratic $\mathcal{D}$-stability requirements, which can be solved efficiently via LMI Control Toolbox [7]. The new proposed method can provide less conservative results than the existing methods via the PDC scheme [15]. In fact, if choose $Y_{aji} = Y_{aji}$, $Y_{bji} = Y_{bji}$, $R_{aji} = 0$, $R_{bji} = 0$, $1 \leq i, j, l \leq r$ in the condition of Theorem 6, it reduces to the LMI-based controller design approach via the PDC scheme in [15]. Moreover, if choose $Y_{ji} = Y_j$, $1 \leq j, l \leq r$ in the condition of Theorem 6, then a switched controller design method via a switched constant controller gain scheme is obtained. An example in Section 4 is given to validate the fact.

4. Example

In this section, an example is given to illustrate the effectiveness of the proposed method for designing $H_\infty$ switched PDC controllers. We consider the following problem of balancing an inverted pendulum on a cart. Equations of motion for the pendulum [15] are
\[ \dot{x}_1 = x_2 + 0.1w \]
\[ \dot{x}_2 = \frac{g \sin(x_1) - \frac{am l^2 \sin(2x_1)}{2} - a \cos(x_1)u}{\frac{a}{l} - am l \cos^2(x_1)} + 0.1w \]
\[ z = x_1 + x_2 + 0.1u \]

where \( x_1 \) denotes the angle of the pendulum from the vertical, \( x_2 \) is the angular velocity, \( g = 9.8 \text{ m/s}^2 \) is the gravity constant, \( w \) is the external disturbance variable, \( m \) is the mass of the pendulum, \( M \) is the mass of the cart, \( 2l \) is the length of the pendulum, and \( u \) is the force applied to the cart. \( a = 1/(m + M) \). We choose \( m = 2.0 \text{ kg}, M = 8.0 \text{ kg}, 2l = 1.0 \text{ m} \) in the simulation. The fuzzy model (1) with two rules is used to describe the nonlinear system, where

\[
A_1 = \begin{bmatrix} 0 & 1.0000 \\ 17.2941 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1.0000 \\ 12.6305 & 0 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 \\ -0.0779 \end{bmatrix},
\]

\[
C_1 = [11], \quad C_2 = [11], \quad D_{11} = D_{12} = 0, \quad D_{21} = D_{22} = 0.1
\]

and the membership functions are

\[
a_1(x_1) = \left( \frac{1}{1 + \exp \left( -7(x_1 - 1) \right)} \right) \left( \frac{1}{1 + \exp \left( -7(x_1 + 1) \right)} \right)
\]
\[
a_2(x_1) = 1 - a_1(x_1)
\]

Let us consider a quadratic \( \mathcal{D} \)-stability requirement of the closed-loop system (9) within an LMI disk region with center \( q = -1.1 \) and radius \( r = 1 \); see Fig. 1. Notice that the LMI disk region has the following characteristic function:

\[
f_\mathcal{D}(z) = \begin{bmatrix} -r & q + z \\ q + z & -r \end{bmatrix} = \begin{bmatrix} -r & q \\ q & -r \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T z
\]

which implies that the values of the parameters in (12) are

\[
\Pi = \begin{bmatrix} -r & q \\ q & -r \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

Then, (18) is the following concrete form:

\[
\begin{bmatrix} -r & q \\ q & -r \end{bmatrix} \otimes Q + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \Phi_{\text{incl}} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \Phi_{\text{incl}}^T < J_{\text{incl}}
\]

Fig. 1. LMI disk region with \( q = -1.1 \) and \( r = 1 \).
For this example, the switched PDC controller is the following form:

$$
\begin{bmatrix}
-rQ & qQ + A_1Q + B_{1i}Y_{il} \\
* & -rQ
\end{bmatrix} < J_{biil}
$$

Similarly, (19) can be rewritten as follows:

$$
\begin{bmatrix}
-2rQ & 2qQ + (A_i + A_j)Q + B_{2i}Y_{ji} + B_{2j}Y_{jl} \\
* & -2rQ
\end{bmatrix} < J_{bijl} + J_{bijl}^T
$$

For this example, the switched PDC controller is the following form:

$$
u = \begin{cases}
K_1(x) = \sum_{i=1}^{2} \alpha_i K_{i1}x & \text{for } x \in \Omega_1 \\
K_2(x) = \sum_{i=1}^{2} \alpha_i K_{i2}x & \text{for } x \in \Omega_2
\end{cases}
$$

(34)

Applying the switched PDC controller design method given in Theorem 6, we can obtain

$$
Y_{i1} = [2.5304 -3.9807], \quad Y_{i2} = [3.3694 -5.1515], \quad Y_{i1} = [1.8456 -3.0461],
$$

$$
Y_{21} = [3.9348 -5.9165], \quad K_{11} = [113.3879 17.7581], \quad K_{21} = [164.6772 31.6881],
$$

$$
Y_{12} = [69.5737 5.2438], \quad K_{12} = [201.4682 42.3813], \quad Q = \begin{bmatrix}
0.0304 & -0.0518 \\
-0.0518 & 0.1068
\end{bmatrix},
$$

$$
\gamma_{opt} = 2.4752
$$

What it follows, the condition of Theorem 6 with $R_{bijl} = 0$, $R_{biil} = 0$, $Y_{ij} = Y_{jj}$, $1 \leq i, j, l \leq r$ is used to design PDC controllers (See Remark 7). The obtained results are given as follows:

$$
Q = \begin{bmatrix}
0.0311 & -0.0494 \\
-0.0494 & 0.1002
\end{bmatrix}, \quad Y_{11} = Y_{12} = [2.5109 -3.6771]
$$

$$
Y_{21} = Y_{22} = [3.9157 -5.4564], \quad K_1 = [103.9749 14.5747]
$$

$$
K_2 = [182.5540 35.5643], \quad \gamma_{opt} = 4.1909
$$

and the corresponding controller is

$$
u = \sum_{i=1}^{2} \alpha_i K_{i}x
$$

(35)

When the condition of Theorem 6 with $Y_{ij} = Y_{11}$, $1 \leq i, l \leq r$ (a controller design method via switched constant controller gain scheme, see Remark 7) is applied, there is no feasible solution. From the computational results, it can be seen that the obtained optimal $H_{\infty}$ performance via the switched PDC scheme is smaller than the PDC scheme, and the switched constant gain scheme is infeasible.

Now the controllers (34) and (35) are, respectively, used to simulate the responses of $x(t), z(t)$, under assuming that

$$w(t) = \begin{cases}
2, & 2 \leq t \leq 3 \\
2, & 5 \leq t \leq 6 \\
0, & \text{others}
\end{cases}
$$

and the initial condition to be $x(0) = [10]^T$. The trajectories of membership functions $\alpha_1(x_1(t))$ and $\alpha_2(x_1(t))$ are given in Fig. 2. The responses of $x(t), z(t), u(t), w(t)$ are given in Figs. 3–7 and the switches of the two gains of the controller (34) during the simulation are given in Fig. 8. From Figs. 3–5, it can be easily seen that the fuzzy controller (34) (switched PDC scheme) gives better $H_{\infty}$ performance than the controller (35) (PDC scheme).
In this paper, an LMI-based controller design method (Theorem 6) is given for continuous-time T–S fuzzy systems. On the other hand, some convex techniques in [22,25] are applicable for designing $H_\infty$ controllers.

Fig. 2. The trajectories of membership functions $\alpha_1(x_1(t))$ and $\alpha_2(x_1(t))$.

Fig. 3. State $x_1$ with initial condition $x(0) = [10]^T$.

Fig. 4. State $x_2$ with initial condition $x(0) = [10]^T$. 
Fig. 5. The controlled output $z$.

Fig. 6. The control input $u$.

Fig. 7. The disturbances $w$. 
with quadratic $\mathcal{D}$-stability constraints. What it follows, the comparisons between those techniques and Theorem 6 will be given in Table 1 by the obtained $H_\infty$ performance bounds. From Table 1, it can be seen that the new technique (Theorem 6) can give less conservative results than those techniques in [22,25]. Therefore, in contrast to the existing controller design methods, the new proposed method also is a good alternative.

5. Conclusion

In this paper, the problem of designing switched $H_\infty$ controllers with quadratic $\mathcal{D}$-stability constraints via state feedback for continuous-time T–S fuzzy systems has been studied. A new control scheme, namely, switched PDC control scheme, is proposed, which is an extension of the PDC scheme and the switched controller gain scheme. Quadratic Lyapunov functions are exploited to derive a new method for designing switched PDC controllers with guaranteeing the stability and $H_\infty$ performances of closed-loop nonlinear systems with quadratic $\mathcal{D}$-stability constraints. The design conditions are given in terms of solvability of a set of linear matrix inequalities (LMIs). It is shown that the new method is less conservative than the corresponding design methods via the pure PDC scheme or switched constant controller gain scheme. A numerical example is given to illustrate the effectiveness of the proposed method.

Acknowledgements

This work was supported in part by Program for New Century Excellent Talents in University (NCET-04-0283), the Funds for Creative Research Groups of China (No. 60521003), Program for Changjiang Scholars and Innovative Research Team in University (No. IRT0421), the State Key Program of National Natural Science of China (Grant No. 60534010), the Funds of National Science of China (Grant No. 60674021) and the Funds of PhD program of MOE, China (Grant No. 20060145019).

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