Global stability problem for feedback control systems of impulsive fractional differential equations on networks

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Abstract

This paper studies a novel class of feedback control systems of impulsive fractional differential equations on networks (FCSIFDENs). By combining some graph theory and the Lyapunov method, we provide a systematic method for constructing a global Lyapunov function for FCSIFDENs. Consequently, a new global asymptotic stability principle and a new global Mittag-Leffler stability principle, which have a close relation to the topology property of the network, are given. Finally, numerical examples are given to demonstrate the effectiveness of the theoretical results.

1. Introduction

Complex networks widely exist in many different areas in real world including the Internet networks, biological neural networks, social connection networks, etc., and become an important part of our daily lives. In recent years, dynamical behaviors of the coupled systems of integer-order differential equations on networks (CSDENs) have attracted current research interests and some results have been reported, see [1–11] and references therein. As we know, stability analysis, such as Lyapunov stability, is one of central tasks in the study of the coupled systems of differential equations on networks (CSDENs). Therefore, how to construct systematically a global Lyapunov function for CSDENs is an interesting question in the research. Recently, graph theory was proposed to construct a global Lyapunov function for coupled systems of integer-order differential equations on networks, and the global stability was explored in [2,3]. Furthermore, their results were applied to several classes of coupled systems on networks in engineering, ecology and epidemiology, see [4–8] and references therein. In [8], Su et al. investigated the stability problem of feedback control systems on of integer-order differential equations on networks (FCSIDENs). Moreover, the graph theoretic technique was developed to the stochastic systems and discrete-time coupled systems on networks [9–11]. However, to the best of our knowledge, the graph theoretic technique has not extended to the coupled systems of fractional differential equations on networks.

Fractional differential equations are generalizations of integer-order differential equations. These generalizations are not mere mathematical curiosities but rather they have interesting applications in many areas of science and engineering such as electrochemistry, control, porous media, and electromagnetism, see [12–17]. Meanwhile, many processes are characterized by the fact that at certain moments of time they experience abrupt changes of state. Consequently, it is natural to assume that these perturbations act instantaneously, that is, act in the form of impulses. Due to the intensive development in the theory of impulsive differential equations and fractional calculus as well as their wide applications in diverse fields, the study of impulsive fractional differential equations has become a new hot topic, and many important results have been performed, see [18–27]. In [24], Stamova investigated the global stability of zero solution of impulsive fractional differential equations by using a new comparison principle.

Motivated by the above discussion, it is easy to see that the incorporation of impulsive fractional differential equations into complex networks is an extremely important improvement. Stability of coupled systems of impulsive fractional differential equations on networks is seldom studied except [28]. In [28], Stamova discussed the global Mittag-Leffler stability of impulsive fractional-order neural networks, nevertheless, author did not give the relation between stability and the topology property of the network. In this paper, we
investigate the global stability problem of FCSIFDENS. The main contribution of this paper lies in the following aspects. Firstly, we provide a systematic method for constructing a global Lyapunov function for FCSIFDENS by using graph theory and Lyapunov method. Secondly, a new global asymptotic stability principle and a new global Mittag–Leﬄer stability principle, which have a close relation to the topology property of the network, are given.

The organization of the paper is as follows. In Section 2, some notations, deﬁnitions, lemmas and problem formulation are stated. In Section 3, we investigate the global asymptotic stability and global Mittag–Leﬄer stability of FCSIFDENS, and some criteria are given. In Section 4, numerical examples are provided to illustrate the effectiveness of the theoretical results. In the last section, we give a brief discussion.

2. Preliminaries

In this section, we need to state some useful notations, lemmas and deﬁnitions. Moreover, the problem formulation is also presented.

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with norm $\| \cdot \|$ and let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{N} = \{1, 2, \ldots, n\}$. Let $\| \cdot \|$ denote the Euclidean norm for vectors. A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called as a $\kappa$-function if it is continuous, strictly increasing and $\psi(0) = 0$. Let $\kappa_n$ present the family of all convex functions $\psi \in \kappa$.

Since the coupled system considered in this paper is built on a directed graph, it is necessary to show the basic concepts and notations on graph theory [3]. A directed graph $G = (V, E)$ contains a set $V = \{1, 2, \ldots, n\}$ of vertices and a set $E$ of arcs $(i, j)$ leading from initial vertex $i$ to terminal vertex $j$. A digraph $\tilde{G}$ is weighted if each arc $(i, j)$ is assigned a positive weight $\alpha_{ij}$. Here $\alpha_{ij} > 0$ if and only if there exists an arc from vertex $j$ to vertex $i$ in $G$. The weight $W(\tilde{G})$ of a subgraph $\tilde{G}$ is the product of the weights on all its arcs. A directed path $P$ in $G$ is a subgraph with distinct vertices $\{i_1, i_2, \ldots, i_m\}$ such that its set of arcs is $\{(i_k, i_{k+1}) : k = 1, 2, \ldots, m-1\}$. If $i_m = i_1$, we call $P$ a directed cycle. A digraph $\tilde{G}$ is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. Given a weighted digraph $\tilde{G}$ with $n$ vertices, deﬁne the weight matrix $A = (a_{ij})_{n \times n}$, whose entry $a_{ij}$ equals the weight of arc $(i, j)$ if it exists, and 0 otherwise. Denote the directed graph with weight matrix $A$ as $(G, A)$. A weighed digraph $(G, A)$ is said to be balanced if $W(C) = W(C')$ for all directed cycle $C$. Here, $-C$ denotes the reverse of $C$ and is constructed by reversing the direction of all arcs in $C$. For a unicyclic graph $G$ with cycle $C_0$, let $\tilde{G}$ be the unicyclic graph obtained by replacing $C_0$ with $-C_0$. Suppose that $(G, A)$ is balanced, then $W(\tilde{G}) = W(G)$. The Laplacian matrix of $(G, A)$ is deﬁned as $L = (p_{ij})_{n \times n}$, where $p_{ij} = -a_{ij}$ for $i \neq j$ and $p_{ii} = \sum_{k \neq i} a_{ik}$ for $i = j$.

Lemma 2.1 (Li and Shuai [3]). Assume $n \geq 2$. Let $c_i$ denote the cofactor of the i-th diagonal element of $L$. Then the following identity holds:

$$\sum_{i=1}^{n} c_i a_{ij} F_\xi(x_i, x_j) = \sum_{\eta \in \mathcal{Q}} W(\eta) \sum_{(i, j) \in \mathcal{E}(\eta)} F_\eta(x_i, x_j).$$

(1)

Here $F_\xi(x_i, x_j)$, $i,j \in \mathbb{N}$, are arbitrary functions, $\mathcal{Q}$ is the set of all spanning unicyclic graphs of $(G, A)$, where $W(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $C_0$ denotes the directed cycle of $Q$. In particular, if $(G, A)$ is strongly connected, then $c_i > 0$ for $i \in \mathbb{N}$.

We begin with the construction of some FCSIFDENSs. Assume that the i-th node dynamic is described by a nonlinear control system of impulsive fractional differential equations as follows:

$$\left\{\begin{array}{l}
\dot{D^\alpha x_i}(t) = f_i(t, x_i(t), \xi_i(t)), \quad t \neq t_k, \\
\Delta x_i(t) = I_i(x_i(t)), \quad t = t_k, \quad k = 1, 2, \ldots, \\
x_i(0) = x_{i0},
\end{array}\right.$$

(2)

where $\dot{D^\alpha}$ is the Caputo’s fractional derivative of order $\alpha$, $0 < \alpha < 1$, $x_i \in \mathbb{R}^m$ is the state variable of the i-th dynamical node at time $t$, $\xi_i(t) \in \mathbb{R}^m$ is the control input, which is a measurable locally essentially bounded function, and $f_i : [t_0, \infty) \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies $f_i(t, 0, 0) = 0$. Assume that all the current values of the states are available, the state feedback control $\xi_i(t)$ can be represented by a feedback law function $\xi_i(t) = K_i(t, x_i(t))$. Consequently, $\psi_i = K_i : [t_0, \infty) \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ represents the influence of node $i$ on node $i$. If $\psi_i = 0$ if there exists no arc from node $j$ to node $i$ in $G$. Hence, by replacing $f_i$ by $f_i + \sum_{j \in N(i)} \psi_{ij}$, we have the following feedback control system of impulsive fractional differential equations on graph $G$:

$$\left\{\begin{array}{l}
\dot{D^\alpha x_i}(t) = f_i(t, x_i(t), K_i(t, x_i(t))) + \sum_{j \in N(i)} \psi_{ij}(x_j(t), x_i(t)), \quad t \neq t_k, \\
\Delta x_i(t) = I_i(x_i(t)), \quad t = t_k, \quad k = 1, 2, \ldots, \\
x_i(t_0) = x_{i0},
\end{array}\right.$$

(3)

where $i \in \mathbb{N}$, $\Delta x_i(t_k) = x_j(t_k^+) - x_j(t_k^-), \quad l_k : \mathbb{R}^m \to \mathbb{R}^m$ satisfies $l_k(0) = 0$, $k = 1, 2, \ldots, t_k < t_k^+ < \cdots < t_k < t_{k+1} < \cdots$, and $\lim_{t \to \infty} x_i(t) = x_{i0}$.

Let $\mathbb{N}_0 = \{1, 2, \ldots\}$ denote by $x(t) = (x(t_1), \ldots, x(t_n)) = (x_1(t, t_0), x_2(t, t_0), \ldots, x_n(t, t_0))$, the solution of system (3), satisfying the initial condition $x(t_0) = x_{t_0}$. We suppose that the functions $f_i$, $g_i$ and $l_k$ are smooth enough on $[t_0, \infty) \times \mathbb{R}^m \times \mathbb{R}^m$, $[t_0, \infty) \times \mathbb{R}^m \times \mathbb{R}^m$ and $\mathbb{R}^m$, respectively, to guarantee existence and uniqueness of the solution $x(t, t_0, x_{t_0})$ of the initial value problem (3) on the interval $[t_0, \infty)$ for each $x_{t_0}$. In general, the solution $x(t, t_0, x_{t_0})$ is piecewise continuous functions with points of discontinuity of first type at which they are left continuous, that is, at the moments $t_k$, $k = 1, 2, \ldots$, the following relations are satisfied: $x(t_k^+) = x(t_k^-)$ and $x(t_k^+) = x(t_k^-) + l_k(x(t_k))$. Moreover, we refer the reader to [24,25] for the more interpretations of impulsive fractional differential equations.

Definition 2.1 (Podlubny [12], Stanovka [24]). For any $t \geq t_0$, Caputo's fractional derivative of order $0 < \alpha < 1$ with the lower limit $t_0$ for a function $f \in C([t_0, b], \mathbb{R})$, $b > t_0$, is defined as

$$\dot{D^\alpha f}(t) = \frac{1}{\Gamma(1- \alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} f'(s) ds,$$

where $f(\cdot)$ is the Gamma function.

Lemma 2.2 (Anguraj and Maheswari [23]). For $0 < \alpha < 1$, and $l$ as a suitable function, we have

$$\int_{t_0}^{t} (t-s)^{\alpha-1} l(s) ds, \quad t > t_0,$$

where $l(\cdot)$ is the Gamma function.

Definition 2.2 (Li and Chen [29]). Let $x(t) = x(t, t_0, x_0)$ be a given solution of (3) existing for $t > t_0$, then the solution of system (3) is said to be globally Mittag–Leﬄer stable, if $|x(t, t_0, x_0)| \leq \psi_0 \leq (c(t-t_0))^{\alpha}, \quad t > t_0$, where $E_\alpha$ is the corresponding Mittag–Leﬄer function, $c > 0$, $d > 0$, $\gamma(0) = 0$, $\gamma(X) \geq 0$ and $\gamma(X)$ is Lipschitzian with respect to $x \in \mathbb{R}^m$.

Definition 2.3. A function $V : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}_+$ is said to belong to class $V_0$ if
(B1) \( V \) is continuous in \((t_{k-1}, t_k) \times \mathbb{R}^m \), and for each \( x \in \mathbb{R}^m \), there exists

\[
\lim_{(t,y) \to (t_k^{-}, x)} V(t, y) = V(t_k^{-}, x).
\]

(B2) \( V \) is locally Lipschitzian in \( x \).

**Lemma 2.3 (Stamova [24]).** Assume that the function \( V \in \mathcal{V}_0 \) is such that

\[
V(t^+, x + I_k(x)) \leq V(t, x), \quad x \in \mathbb{R}^m, \quad t = t_k, \quad k = 1, 2, \ldots,
\]

\[
\frac{d}{dt} V(t, x) \leq 0, \quad (t, x) \in (t_{k-1}, t_k) \times \mathbb{R}^m.
\]

Then,

\[
V(t, x(t_0, x_0)) \leq V(t_0^+, x_0).
\]

**Lemma 2.4 (Stamova [24]).** Assume that the function \( V \in \mathcal{V}_0 \) is such that

\[
V(t^+, x + I_k(x)) \leq V(t, x), \quad x \in \mathbb{R}^m, \quad t = t_k, \quad k = 1, 2, \ldots,
\]

\[
\frac{d}{dt} V(t, x) \leq MV(t, x), \quad (t, x) \in (t_{k-1}, t_k) \times \mathbb{R}^m.
\]

Then,

\[
V(t, x(t_0, x_0)) \leq V(t_0^+, x_0) E_M(M(t - t_0)^{q/2}),
\]

**Lemma 2.5 (Yu et al. [30]).** Suppose function \( g(t) \) is nondecreasing and differentiable on \( t \in [0, \infty) \), then for any constant \( h \) and \( t \in [0, \infty) \),

\[
\frac{d}{dt} (g(t) - h)^2 \leq 2(g(t) - h)\frac{d}{dt} g(t),
\]

where \( 0 < a < 1 \).

The upper right-hand derivative of \( V_i \) in Caputo’s sense of order \( q \), \( 0 < q < 1 \) with respect to \( \tau \) is defined by

\[
\frac{d^q}{dt^q} V(t, x) = \lim_{h \to 0^+} \frac{1}{h^q} \left[ V(t + h, x + \frac{h}{2} J_i(t, x, K_i)) - V(t, x) \right],
\]

where \( V_i : (t_{k-1}, t_k) \times \mathbb{R}^m \to \mathbb{R}^m \). Let \( \mathbb{R}^{m_1} = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_n} \), \( m = m_1 + m_2 + \cdots + m_n, \quad x = (x_1, x_2, \ldots, x_n) \). We denote function \( V_i : \mathbb{R}^n \to \mathbb{R}^m \), as follows:

\[
V(t, x) = \sum_{i=1}^n c_i V_i(t, x_i),
\]

where \( c_i \) is defined in Lemma 2.1.

**3. Main results**

The goal of this section is to establish some criteria of the global asymptotic stability and global Mittag-Leffler stability based on the methods of Lyapunov function and graph theory.

**Theorem 3.1.** Suppose that the following conditions hold:

(H1) For \( i \neq j \in I \), there exist constants \( a_{ij} \geq 0 \), \( \kappa \)-function \( \psi(\cdot) \), and functions \( F_i(t, x_i, x_j) \) such that

\[
\frac{d}{dt} V_i(t, x_i) \leq \sum_{j=1}^n a_{ij} F_i(t, x_i, x_j) + \psi_i(\|K_i(t, x_i)\|) \leq \phi_i(\|x_i\|),
\]

where \( t \in (t_{k-1}, t_k), \quad x_i \in \mathbb{R}^m, \quad k = 1, 2, \ldots \).

(H2) Let weighted digraph \((G, A)\) be strongly connected, in which matrix \( A = (a_{ij})_{n \times n} \) and along each directed cycle \( c \) of weighted digraph \((G, A)\), there is

\[
\sum_{(i,j) \in E(c)} F_i(t, x_i, x_j) \leq 0, \quad t \geq t_0, \quad x_i \in \mathbb{R}^m, \quad x_j \in \mathbb{R}^m.
\]

(H3) \( V(t^+, t, x + I_k(x)) \leq V(t, x), \quad t = t_k \).

(H4) For any \( i \in I \), there exist a \( \kappa \)-function \( a_i(\cdot) \) and a \( \kappa \)-function \( b_i(\cdot) \) such that \( a_i(\|x_i\|) \leq V(t, x_i) \leq b_i(\|x_i\|) \).

Then function \( V(t, x) = \sum_{i=1}^n c_i V_i(t, x_i) \) is a Lyapunov function for (3), in which \( c_i \) is defined in Lemma 2.1. Furthermore, the trivial solution of (3) is uniformly globally asymptotically stable.

**Proof.** Let \( V(t, x) = \sum_{i=1}^n c_i V_i(t, x_i) \), where \( c_i \) denotes the cofactor of the \( i \)-th diagonal element of \( L \). The property of strong connectedness of digraph \((G, A)\) implies that \( c_i > 0 \) for any \( i \in I \). According to condition (H1), when \( t \in (t_{k-1}, t_k) \), we have

\[
\frac{d}{dt} V(t, x) = \sum_{i=1}^n c_i \frac{d}{dt} V_i(t, x_i) \leq \sum_{i=1}^n c_i \phi_i(\|x_i\|),
\]

Define \( \sigma_i(\cdot) = \psi_i^{-1}(\frac{1}{2} \phi_i(\cdot)) \). Recalling the fact \( \psi_i(\cdot) \) is a \( \kappa \)-function and \( \phi_i(\cdot) \) is a \( \kappa \)-function, we have that \( \sigma_i(\cdot) \) is a \( \kappa \)-function. Choose any feedback law \( K_i(t, x_i) \), \( \leq \sigma_i(\|x_i\|) \), we have

\[
\frac{d}{dt} V(t, x) = \sum_{i=1}^n c_i \frac{d}{dt} V_i(t, x_i) \leq \sum_{i=1}^n c_i \phi_i(\|x_i\|) \leq \sum_{i=1}^n c_i \phi_i(\|x_i\|) \leq \phi_i(\|x_i\|).
\]

(4)

Let \( c = \min(c_1, c_2, \ldots, c_n), \quad C = \max(c_1, c_2, \ldots, c_n) \) and \( \phi(\|x\|) = \frac{1}{2} \min\{c_1 \phi(c/C) \|x\|, c_2 \phi(c/C) \|x\|, \ldots, c_n \phi(c/C) \|x\|\} \).

Since function \( \phi(\cdot) \in \kappa \), there exists function \( \phi(\cdot) \in \kappa \) such that \( \phi(\cdot) \leq \phi_i(\cdot) \) for arbitrary \( i \in I \), using the property of convex function \( \phi(\cdot) \), we have

\[
-\frac{1}{2} \sum_{i=1}^n c_i \phi_i(\|x_i\|) \leq \sum_{i=1}^n c_i \phi_i(\|x_i\|) \leq \sum_{i=1}^n c_i \phi_i(\|x_i\|) \leq -\phi(\|x\|).
\]

(5)

In view of Lemma 2.1, condition (H2) and fact \( W(\|x\|) > 0 \), we have

\[
\sum_{i=1}^n c_i \phi_i(\|x_i\|) \leq \sum_{i=1}^n W(\|x_i\|) \sum_{j=1}^n F_i(x_i, x_j) \leq 0.
\]

(6)

Substituting (5) and (6) into (4), we obtain that

\[
\frac{d}{dt} V(t, x) \leq -\phi(\|x\|), \quad t \in (t_{k-1}, t_k).
\]

(7)

When \( t = t_k \), from condition (H3), we have

\[
V(t^+, x(t^+)) = \sum_{i=1}^n c_i V_i(t, x_i) \leq \sum_{i=1}^n c_i \phi_i(\|x_i\|) \leq -\phi(\|x\|).
\]

(8)
Let $b(||x||) = n \max(c_i b_1(||x_1||), c_2 b_2(||x_2||), ..., c_n b_n(||x_n||))$. From condition (H3), we have
\[
V(t, x) = \sum_{i=1}^{n} c_i V_i(t, x_i) \\
\leq \sum_{i=1}^{n} c_i b_i(||x_i||) \\
\leq b(||x||).
\]
(9)

On the other hand, from function $a(\cdot) \in \kappa$, there exists function $a(i) \in \kappa_i$, such that $a(i) \leq a(i)$ for arbitrary $i \in \mathbb{L}$, using the property of convex function $a(\cdot)$, we have
\[
V(t, x) \geq \sum_{i=1}^{n} c_i a_i(||x_i||) \\
\geq \sum_{i=1}^{n} c_i \sum_{k=1}^{n} \left( \frac{c_i}{\sum_{k=1}^{n} c_k} a(\sum_{k=1}^{n} c_k ||x_i||) \right) \\
\geq \sum_{i=1}^{n} c_i a \left( \sum_{k=1}^{n} \frac{c_k}{\sum_{k=1}^{n} c_k} ||x_i|| \right) \\
\geq a(||x||),
\]
(10)

where $a(||x||) = n \min(c_i a((c/C) ||x_i||), c_j a((c/C) ||x_j||), ..., c_n a((c/C) ||x_n||))$.

Next, we will show that the zero solution of (3) is uniformly stable. Let $x(t_0, x_0)$ be the solution of the initial value problem (3) satisfying the initial condition $x(t_0^+, t_0, x_0) = x_0$, for any $\varepsilon > 0$, choose $\delta = b^{-1}(\varepsilon/a(\varepsilon)) > 0$, when $||x_0|| < \delta$, since all the conditions of Lemma 2.3 are met, then we have
\[
a(||x(t_0, x_0)||) \leq V(t, x(t_0, x_0)) \leq V(t_0^+, x_0), \quad t \geq t_0.
\]

From the above inequalities and (9), we get the following inequalities:
\[
a(||x(t_0, x_0)||) \leq V(t_0^+, x_0) \leq b(||x_0||) < b(\delta) = a(\varepsilon),
\]
from which it follows that $||x(t_0, x_0)|| < \varepsilon$ for $t \geq t_0$. This proves the uniform stability of zero solution of (3).

For any $M > 0$, there exists a $\alpha = a(M)$ such that $||x(t_0, x_0)|| < \alpha$ for $t \geq t_0$ and $x_0 \leq M$. In fact, for any $M > 0$, we choose $\alpha = a^{-1}(b(M))$. Since all conditions of Lemma 2.3 are met, from (9) and (10), we have
\[
a(||x(t_0, x_0)||) \leq V(t, x(t_0, x_0)) \leq V(t_0^+, x_0) \leq b(||x_0||) \leq b(M),
\]

hence $V(t, x(t_0, x_0)) < \alpha$ for $t \geq t_0$. This implies that the solutions of (3) are uniformly bounded. Now, we will prove the zero solution of (3) is uniformly globally attractive. For any $\varepsilon > 0$ small enough, we choose $\eta = \eta(\varepsilon) > 0$ such that $b(\eta) < a(\varepsilon)$ and let
\[
T > \left( \frac{b(\alpha) T(1 + \eta)}{\phi(\eta)} \right)^{1/\eta}.
\]

If we assume that the inequality $||x(t_0, x_0)|| \geq \eta$ is valid for each $t \in [t_0, t_0 + T]$, then from (7)–(9) and Lemma 2.2, we have
\[
V(t, x(t_0, x_0)) \leq V(t_0^+, x_0) - \frac{1}{T(1 + \eta)} \int_{t_0}^{t} \phi(||x(s, t_0, x_0)||)(s - T)^{\eta - 1} \, ds \\
\leq b(\alpha) - \frac{\phi(\eta) T^{\eta}}{T(1 + \eta)} \int_{t_0}^{t} \phi(||x(s, t_0, x_0)||)(s - T)^{\eta - 1} \, ds \\
< 0,
\]
which contradicts (10). The contradiction obtained shows that there exists $t^* \in [t_0, t_0 + T]$ such that $||x(t^*, t_0, x_0)|| < \eta$. Then from (7)–(10), it follows that for $t \geq t^*$ (hence for any $t \geq t_0 + T$) the following inequalities hold:
\[
a(||x(t_0, x_0)||) \leq V(t, x(t_0, x_0)) \leq V(t^*, x^*(t_0, x_0)) < b(\eta) < a(\varepsilon),
\]
thus,
\[
||x(t_0, x_0)|| < \varepsilon \quad \text{for} \quad t \geq t_0 + T.
\]

Therefore the zero solution $x \equiv 0$ is uniformly globally asymptotically stable. □

**Remark 3.1.** The definition of uniformly globally asymptotic stability can be seen in Definitions 2.2–2.4 of [24]. The definition of Lyapunov function can be seen in Theorem 1 of [4], Theorem 1 of [5] and Theorem 3.1 of [6].

**Theorem 3.2.** Let conditions (H2) and (H3) hold. Suppose that the following conditions hold:
\[
(H_2) \quad \varepsilon D^p V(t, x) = \sum_{i=1}^{n} a_i F_i(t, x_i, x_0) - \varepsilon V_i(t, x_i)\quad \text{if} \quad t \in [t_k, t_{k+1}], \quad \varepsilon > 0, \quad i, k \in \mathbb{L}.
\]
\[
(H_3) \quad \rho_1 ||x|| \leq V_i(t, x_i) \leq \rho_1 ||x||, \quad p \geq 2, \quad \rho_1 > 0, \quad \mu_1 > 0.
\]

Then FCSIFDENs (3) is globally Mittag–Leffler stable.

**Proof.** Since $(G, A)$ is strongly connected, then $c_i > 0$ for $i \in \mathbb{L}$, where $c_i$ denotes the cofactor of the $i$-th diagonal element of $L$. Let $V(t, x) = \sum_{i=1}^{n} c_i V_i(t, x_i)$, according to the conditions (H2) and (H3), Lemma 2.1 and the fact $W(\mathbb{Q}) > 0$, when $t \in [t_k, t_{k+1}]$, we have
\[
f^p V(t, x) = \sum_{i=1}^{n} c_i f^p V_i(t, x_i) \\
\leq \sum_{i=1}^{n} c_i \rho_1 \sum_{i=1}^{n} a_i F_i(t, x_i, x_0) - \varepsilon V_i(t, x_i) \\
= \sum_{i=1}^{n} c_i a_i \rho_1 F_i(t, x_i, x_0) - \varepsilon \sum_{i=1}^{n} c_i V_i(t, x_i) \\
\leq \sum_{Q \in \mathbb{Q}} W(Q) \sum_{i, j \in E(Q)} F_{ij}(x_i, x_j) - \varepsilon V(t, x) \\
\leq -\varepsilon V(t, x),
\]
(11)

where $\varepsilon = \min(\varepsilon_1, \varepsilon_2, ..., \varepsilon_q)$. When $t = t_k$, from condition (H3), we have
\[
V(t^{+}, x(t^{+})) \leq V(t, x(t)).
\]
(12)

Denote $\mu = \sum_{i=1}^{n} c_i \mu_i$ and
\[
\rho = \left( \frac{\sum_{i=1}^{n} c_i \mu_i}{\min_{i} c_i \mu_i} \right)^{p/2}.
\]

Then it is easy to see from condition (H3) that
\[
V(t, x) = \sum_{i=1}^{n} c_i V_i(t, x_i) \leq \sum_{i=1}^{n} c_i \mu_i ||x||^p \leq \sum_{i=1}^{n} c_i \mu_i ||x||^p = \mu ||x||^p,
\]
and
\[
V(t, x) \geq \sum_{i=1}^{n} c_i \rho_1 ||x||^p \\
= \sum_{j=1}^{n} c_{j} \rho_1 \sum_{i=1}^{n} \frac{c_i}{\sum_{k=1}^{n} c_k} ||x||^2 \rho^2/2 \\
\geq \sum_{j=1}^{n} c_{j} \rho_1 \sum_{i=1}^{n} \frac{c_i}{\sum_{k=1}^{n} c_k} ||x||^2 \rho^2/2 \\
\geq \sum_{i=1}^{n} c_i \rho_1 \left( 1 - \rho^{p/2} \right) \mu ||x||^p / \rho^p \\
= \rho ||x||^p,
\]
(13)

Hence
\[
\rho ||x||^p \leq V(t, x) \leq \mu ||x||^p.
\]
(14)

Let $x(t_0, x_0)$ be the solution of the initial value problem (3) satisfying the initial condition $x(t_0^+, t_0, x_0) = x_0$. From (11) and (12), by Lemma 2.4, the following inequality is valid:
\[
V(t, x(t_0, x_0)) \leq V(t_0^+, x_0) - \varepsilon (t - t_0)^{1/p}, \quad t \geq t_0.
\]

From the above inequality and (14), we obtain
\[
||x(t, x_0)|| \leq \left( \frac{V(t_0^+, x_0) - \varepsilon (t - t_0)^{1/p}}{\rho} \right)^{1/p} \leq \left[ \frac{\mu ||x||^p (t - t_0)^{1/p}}{\rho} \right]^{1/p}, \quad t \geq t_0.
\]
Let \( \gamma(x_0) = (\mu/\rho)\|x_0\|^p \), then we have
\[
\|x(t, t_0, x_0)\| \leq \|x_0\| \exp(-\psi(t-t_0)), \quad t \geq t_0,
\]
which means that the zero solution of (3) for \( p = q = 1 \) is globally exponentially stable.

Note that if \((\mathcal{G}, A)\) is balanced, then
\[
\sum_{i,j=1}^n c_{ij}F_g(t, x_i, x_j) = \frac{1}{\rho} \sum_{i,j=1}^n W(Q) \sum_{g \in E(Q)} \left[ F_g(t, x_i, x_j) + F_g(t, x_j, x_i) \right].
\]
In this case, if condition (H2) in Theorem 3.1 is replaced by
\[
\sum_{g \in E(Q)} \left[ F_g(t, x_i, x_j) + F_g(t, x_j, x_i) \right] \leq 0, \quad t \geq t_0, \quad x_i, x_j \in \mathbb{R}^m,
\]
we get the following corollary.

**Corollary 3.1.** Suppose that \((\mathcal{G}, A)\) is balanced. Then the respective conclusions of Theorems 3.1 and 3.2 hold if (H2) is replaced by (H2).

Conditions (H2) and (H2) can be readily verified if there exist functions \( G(t, x_i) \), \( i \in \mathbb{N} \) such that
\[
F_g(t, x_i, x_j) \leq G(t, x_i) - G(t, x_j), \quad i, j \in \mathbb{N},
\]
we have the following corollary.

**Corollary 3.2.** The respective conclusions of Theorems 3.1 and 3.2 hold if (H2) is replaced by (H2).

**Remark 3.3.** In the study of the stability of FCSIFDENS, how to construct an appropriate Lyapunov function is a formidable task. Theorems 3.1 and 3.2 offer a technique to systematically construct the Lyapunov function for (3) by using Lyapunov function \( V \) of each vertex system, that avoids the difficulty of finding directly Lyapunov function of (3).

**Remark 3.4.** Theorems 3.1 and 3.2 show uniformly globally asymptotic stability and global Mittag–Leﬄer stability of FCSIFDENS require the strong connectedness of the network, which is a topology property of network.

**4. Numerical examples**

In this subsection, we give two examples and their numerical simulations to illustrate the effectiveness of the theoretical results obtained in the above subsection.

**Example 4.1.** Consider the following feedback control systems of impulsive fractional differential equations on networks (FCSIFDENS)
\[
\begin{aligned}
&x_i(t^+) = x_i(t^-) + \sum_{j=1}^n \beta_{ij} x_j(t^-) + k_i x_i(t^-), \quad t \neq t_k, \\
&x_i(t_k^+) = \alpha_i x_i(t_k^-), \quad k = 1, 2, ..., \\
&x_i(t_k^-) = x_0.
\end{aligned}
\]
where \( x_i \in \mathbb{R}, \, i \in \mathbb{N}, \beta_{ij} \leq 0, \beta_{ji} = -\beta_{ij} \) and \( \beta_{ij} \neq 0 \). Let \( G \) be a digraph with \( n \) vertices, then (15) can be regarded as a FCSIFDENs on \( G \).

**Corollary 4.1.** Suppose that the following conditions hold:
- \((H_9)(G, A)\) is strongly connected and balanced.
- \((H_{10}) \quad d_i - k_i > 0, \quad i \in \mathbb{N}.
- \((H_{11}) \quad -1 \leq \alpha \leq 1.

Then the trivial solution of (15) is uniformly globally asymptotically stable.

**Proof.** We choose \( V_i(t, x_i(t)) = |x_i(t)|^4 \), where \( i \in \mathbb{N} \). It is easy to verify that condition (H4) holds for \( a_i(x_i(t)) = |x_i(t)|, \quad 0 \leq t \leq 1 \) and \( b_i(x_i(t)) = \varepsilon |x_i(t)|, \quad \varepsilon \geq 1 \).

For \( t \in (t_{k-1}, t_k), \quad k = 1, 2, ..., \) if \( x_i(t) = 0, \quad i \in \mathbb{N} \), then
\[
\frac{d}{dt} \left| x_i(t) \right|^4 = \frac{1}{t(1-q)} \int_0^t x_i(s) \left( \frac{t}{s} - s \right)^{q-1} \, ds = -D^q x_i(t). 
\]
If \( x_i < 0, \quad i \in \mathbb{N} \), then
\[
\frac{d}{dt} \left| x_i(t) \right|^4 = -\frac{1}{t(1-q)} \int_0^t x_i(s) \left( \frac{t}{s} - s \right)^{q-1} \, ds = -D^q x_i(t).
\]
Therefore, \( D^q |x_i(t)| = \text{sgn}(x_i(t)) D^q |x_i(t)| \).

Then, from (15), we have
\[
\frac{d}{dt} \left| x_i(t) \right|^4 = \text{sgn}(x_i(t)) D^q \left| x_i(t) \right| = \frac{1}{t(1-q)} \int_0^t x_i(s) \left( \frac{t}{s} - s \right)^{q-1} \, ds.
\]

Therefore, \( D^q |x_i(t)| = \text{sgn}(x_i(t)) D^q |x_i(t)| \).

Thus along each directed cycle \( C \) of the weighted digraph \((G, A), \)
\[
\sum_{i,j \in E(C)} |F_{ij}(x_i, x_j) + F_{ji}(x_j, x_i)| = 0.
\]

On the other hand,
\[
V_i(t_k^+, x_i(t_k^+)) = |x_i(t_k^+)| = |\alpha_i x_i(t_k^-)| \leq |x_i(t_k^-)| = V_i(t_k, x_i(t_k^-)).
\]

According to Corollary 3.1, we can conclude that system (15) is uniformly globally asymptotically stable.
In the simulation, for the sake of simplicity, we consider system (15) with $x = (x_1, x_2)^T$, $q = 0.9$, $d_1 = 2$, $k_1 = 0.8$, $d_2 = 2$, $k_2 = 1$, $p_{11} = p_{22} = 0$, $p_{12} = 2$, $p_{21} = -2$, $\alpha = 0.6$ and $r = t_k - t_{k-1} = 0.1$, $k = 1, 2, \ldots$. Since $a_{ij} = |\beta_{ij}|$, we have

$$A = (a_{ij})_{2 \times 2} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. $$

Obviously, (H$_3$)-(H$_4$) hold, according to Corollary 4.1, system (15) is uniformly globally asymptotically stable. Numerical simulations can be seen in Fig. 1, where we choose three group initial values $(x_{10}, x_{20})^T = (1, -1)^T$, $(2, -2)^T$, $(3, -3)^T$, respectively. From Fig. 1, we find that all of the solutions of system (15) which through these initial values will converge to the zero solution. □

Example 4.2. Consider the following feedback control systems of impulsive fractional differential equations on networks (FCSIDENs)

$$\frac{cD^\alpha x_i(t)}{dt} = -d_i x_i(t) - \sum_{j=1}^n a_{ij} x_j(t)|/x_j(t)| + k_i x_i(t), \quad t \neq t_k, $$

$$x_i(t_k^+) = \alpha x_i(t_k), \quad k = 1, 2, \ldots, $$

$$x_i(t_0^+) = x_{i0}, $$

where $x_i \in \mathbb{R}$, $i \in \mathbb{Z}$ and $A = (a_{ij})_{n \times n}$, constants $a_{ij} \geq 0$. Let $G$ be a digraph with $n$ vertices, then (16) can be regarded as a FCSIDENs on $G$. That is to say, each vertex $i$ is assigned the $i$-th vertex dynamics of (16).

Corollary 4.2. Suppose that the following conditions hold:

(H$_{12}$) $(G, A)$ is strongly connected.

(H$_{13}$) $d_i - k_i \leq 0$, $i \in \mathbb{Z}$.

(H$_{14}$) $-1 \leq \alpha \leq 0$.

Then the zero solution of (16) is globally Mittag–Leffler stable.

Proof. We choose $V_i(t, x_i(t)) = \frac{1}{2}\xi_i^T(t)\xi_i(t)$, where $i \in \mathbb{Z}$. It is easy to verify that condition (H$_2$) holds for $q_i x_i(t) = \frac{1}{2}\xi_i^2(t)$, $0 \leq r < 1$ and $b_i x_i(t) = (c/2)\xi_i^2(t)$, $c > 1$. For $t \in (t_k-1, t_k)$, $k = 1, 2, \ldots$. From Lemma 2.5 and (16), we have

$$\frac{cD^\alpha + V_i(t, x_i(t))}{dt} \leq \frac{1}{2}D^\alpha + \xi_i^2(t)$$

$$\leq \xi_i(t)|/d_i x_i(t)| - \sum_{j=1}^n a_{ij} x_j(t)|/x_j(t)| + k_i x_i(t)$$

where $F_i(t, x_i(t), x_j(t)) = -d_i x_i(t)|/x_i(t)|$. On the other hand, for $t = t_k$, we have

$$V_i(t_k^+, x_i(t_k^+)) = \xi_i^2(t_k^+) = \alpha \xi_i^2(t_k) \leq \xi_i^2(t_k) = V_i(t_k, x_i(t_k)).$$

Thus, all the conditions of Theorem 3.2 are satisfied, according to Theorem 3.2, we can conclude that system (16) is globally Mittag–Leffler stable.

In the simulation, for the sake of simplicity, we consider system (16) with $q = 0.95$, $x = (x_1, x_2, x_3, x_4)^T$, $d_1 = d_4 = 3$, $d_2 = d_3 = 2$, $k_1 = k_4 = 2$, $k_2 = k_3 = 1$, $\alpha = 0.6$, $r = t_k - t_{k-1} = 0.1$, $k = 1, 2, \ldots$, and

$$A = (a_{ij})_{4 \times 4} = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix}. $$

Obviously, (H$_{12}$)-(H$_{14}$) hold, according to Corollary 4.2, system (16) is globally Mittag–Leffler stable. Numerical simulations can be seen in Fig. 2, where we choose three group initial values $(x_{10}, x_{20}, x_{30}, x_{40})^T = (1, 3, -1, -3)^T$, $(15, 2.5, -1, -2.5)^T$, $(2, 0.5, -2, -0.5)^T$, respectively. From Fig. 2, we find that all of the solutions of system (16) which through these initial values will converge to the zero solution. □

5. Conclusion

In this paper, we have studied some stabilities of FCSIDENs. First, a model of FCSIDENs has been presented. Next, a systematic method for constructing a global Lyapunov function for FCSIDENs by using graph theory has been given. Furthermore, several criteria are established to guarantee the global asymptotic stability and global Mittag–Leffler stability. At last, numerical simulations have been provided to verify the proposed results. The results obtained are original. In future work, we will study the global asymptotic stability and global Mittag–Leffler stability of FCSIDENs with time-varying delays.

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