FALL COLORING OF GRAPHS I

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Abstract

A fall coloring of a graph $G$ is a proper coloring of the vertex set of $G$ such that every vertex of $G$ is a color dominating vertex in $G$ (that is, it has at least one neighbor in each of the other color classes). The fall coloring number $\chi_f(G)$ of $G$ is the minimum size of a fall color partition of $G$ (when it exists). Trivially, for any graph $G$, $\chi(G) \leq \chi_f(G)$.

In this paper, we show the existence of an infinite family of graphs $G$ with prescribed values for $\chi(G)$ and $\chi_f(G)$. We also obtain the smallest non-fall colorable graphs with a given minimum degree and determine their number. These answer two of the questions raised by Dunbar et al.

Keywords: fall coloring of graphs, non-fall colorable graphs.

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1. Introduction

Let $G = (V, E)$ be a simple connected undirected graph. A proper coloring of a graph $G$ is a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of the vertex set $V$ of $G$ into independent subsets of $V$. Each $V_i$ is called a color class of $\Pi$. A vertex $v \in V_i$ is a color dominating vertex (c.d.v.) with respect to $\Pi$, if it is adjacent to at least one vertex in each color class $V_j, j \neq i$. A $k$-coloring $\Pi = \{V_1, V_2, \ldots, V_k\}$ of $G$ is a fall coloring of $G$ if each vertex of

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$G$ is a c.d.v. with respect to $\Pi$. In this case, $\Pi$ is called a $k$-fall coloring of $G$. The least positive integer $k$ for which $G$ has a $k$-fall coloring is the fall chromatic number of $G$ and denoted by $\chi_f(G)$. A graph $G$ may or may not have a fall coloring. For example, the cycle $C_n$ has a fall coloring if and only if $n$ is multiple of 3 or even [3]. Trivially, $\chi_f(K_n) = n$ and hence all complete graphs are fall colorable. Clearly, if $G$ is fall colorable, $\chi(G) \leq \chi_f(G) \leq \delta(G) + 1$, where $\delta(G)$ is the minimum degree of $G$.

In Sections 2 and 3, we answer two of the questions raised by Dunbar et al. — one relating to the existence of graphs with prescribed chromatic and fall chromatic numbers and the other relating to the determination of all smallest non-fall colorable graphs with prescribed minimum degree. Notation and terminology not mentioned here can be found in [2].

2. Existence of Graphs $G$ with Prescribed Values for $\chi$ and $\chi_f$

In this section, we show that given any two positive integers $a$ and $b$ with $2 < a < b$, there exists an infinite sequence of graphs $\{H_i\}$ with $\chi(H_i) = a$ and $\chi_f(H_i) = b$. First we define a new graph $G^*$ from a given graph $G$.

Let $V(G) = \{x_1, x_2, \ldots, x_n\}$, and let $G^*$ be the graph with vertex set $V(G^*) = V(G) \cup V'(G)$, where $V'(G) = \{y_i : x_i \in V(G)\}$, $V(G) \cap V'(G) = \emptyset$, and edge set $E(G^*) = E(G) \cup \{x_iy_j : i \neq j\}$.

Lemma 2.1 brings out the relation between the chromatic numbers of $G^*$ and $G$. The proof is straightforward.

**Lemma 2.1.** If $G$ is not complete, then $\chi(G^*) = \chi(G) + 1$.

The following remarks will be used to determine, for any graph $G$, the fall chromatic number of $G^*$.

**Remark 2.2.** Let $G$ be a graph having a fall coloring. Then $G$ has a universal vertex if and only if any fall color partition of $G$ contains at least one singleton color class.

**Remark 2.3.** Consider the partition $\{x_i, y_i\}, i = 1, 2, \ldots, |V(G)|$ of $V(G^*)$. Clearly this partition is a fall color partition of $G^*$. Thus the graph $G^*$ is fall colorable irrespective of $G$ being fall colorable or not. Moreover, $\chi_f(G^*) \leq |V(G)|$. 
Remark 2.4. In any fall coloring of $G^*$, all the vertices of $V'(G)$ either receive the same color or else receive distinct colors. Also, $V'(G)$ is an independent subset of $G^*$.

Theorem 2.5. If $G$ has fall coloring and $G$ has no universal vertex, then $\chi_f(G^*) = \chi_f(G) + 1$.

Proof. As $G$ has no universal vertex, by Remark 2.2, in any fall color partition of $G$, each color class contains at least two vertices. Consequently, if $k = \chi_f(G)$, then $G$ has a $k$-fall color partition with each color class containing at least two vertices. Give a new color $k + 1$ to all vertices of $V'(G)$ which yields a $(k + 1)$-fall coloring of $G^*$. Thus $\chi_f(G^*) \leq \chi_f(G) + 1$.

Suppose $\chi_f(G^*) \leq \chi_f(G)$. If $l = \chi_f(G^*)$, then $l < n$, where $n = |V(G)|$. By Remark 2.4, all vertices of $V'(G)$ receive the same color, say, $l$. Then the remaining $(l - 1)$ colors must appear in $G$ and this coloring induces a $(l - 1)$-fall coloring of $G$ and hence $\chi_f(G) \leq l - 1$, contradiction to the assumption that $\chi_f(G^*) \leq \chi_f(G)$. Therefore $\chi_f(G^*) = \chi_f(G) + 1$.

Theorem 2.6. For any graph $G$, $\chi_f(G^*) = |V(G)|$ if and only if
(i) $G$ has no fall coloring or
(ii) $G$ has a fall coloring and contains a universal vertex.

Proof. Suppose $\chi_f(G^*) = |V(G)|$. If $G$ has no fall coloring, then we are done. If not, $G$ has a fall coloring. Suppose $G$ has no universal vertex, then by Theorem 2.5, $\chi_f(G^*) = \chi_f(G) + 1$ and by Remark 2.2, in any fall color partition of $G$, each color class contains at least two vertices. Thus $|V(G)| \geq 2\chi_f(G)$ and $|V(G)| \geq 4$. Therefore, $\chi_f(G^*) \leq \frac{|V(G)|}{2} + 1$, a contradiction to the fact that $\chi_f(G^*) = |V(G)|$.

Conversely, assume (i) so that $G$ has no fall coloring and $k = \chi_f(G^*) < |V(G)|$. Then by Remark 2.4, if $\Pi$ is a $k$-fall coloring of $G^*$, then $V'(G)$ will be a color class receiving the same color, say, $k$ of $\Pi$. Now it is clear that in a fall coloring of a graph $H$, the union $S$ of any subset of color classes will induce a fall coloring on the subgraph of $H$ induced by $S$. Therefore, $\Pi - V'(G)$ will be a fall coloring of $G$, a contradiction.

Now assume (ii) so that $G$ has a fall coloring and that $G$ has a universal vertex. By Remark 2.2, any fall color partition of $G$ contains at least one singleton color class. Suppose $k = \chi_f(G^*) < |V(G)|$. By Remark 2.4, in any $k$-fall color partition of $G^*$, all vertices of $V'(G)$ receive the same color, say, $k$, and the remaining $(k - 1)$-colors are present in $G$. These $(k - 1)$ colors
induce a \((k-1)\)-fall coloring of \(G\), say \(\Pi\). By our assumption, \(\Pi\) contains at least one singleton color class, say, \(V_i = \{x\}\), then its corresponding vertex \(y\) in \(V'(G)\) is not adjacent to the vertex \(x\) (the only vertex of color \(i\)), a contradiction.

Corollary 2.7. For any positive integers \(a, b\) with \(3 \leq a < b\), there is an infinite sequence of graphs \(\{H_i\}\) with \(\chi(H_i) = a\) and \(\chi_f(H_i) = b\).

**Proof.** Let \(G_{a,b}\) be a graph obtained by attaching \(b-a+1\) pendant edges at a vertex of \(K_{a-1}\). Then \(|V(G_{a,b})| = b\). If \(a = 3\), then \(G_{a,b}\) has a fall coloring and being a star it has a universal vertex. If \(a \geq 4\), then \(G_{a,b}\) has no fall coloring (as the condition \(\chi \leq \delta + 1\) is violated). Therefore by Theorem 2.6, \(\chi_f(G^*) = b\).

Since \(G_{a,b}\) is not complete and by Lemma 2.1, \(\chi(G_{a,b}^*) = a\) (as \(\chi(G_{a,b}) = a - 1\)).

This construction can be used to generate an infinite sequence \(\mathcal{H}_{a,b} = \{H_i\}\) of graphs with \(\chi = a\) and \(\chi_f = b\) as follows:

Start with \(G_{a,b}\) and get \(H_1 = G_{a,b}^*\). Form \(H_2\) by concatenating a copy of \(G_{a,b}^*\) at a vertex of \(H_1\), and in general, form \(H_i\) by concatenating a copy of \(G_{a,b}^*\) at a vertex of \(H_{i-1}\) (Recall that a concatenation of a graph \(G\) with a graph \(H\) is the graph got by linking \(G\) and \(H\) by the identification of a vertex of \(G\) with a vertex of \(H\)). Each graph in \(\mathcal{H}_{a,b} = \{H_i\}\) has \(\chi(H_i) = a\) and \(\chi_f(H_i) = b\).

3. Smallest Non-Fall Colorable Graphs with Given Minimum Degree

In this section, we determine the smallest (with respect to both order and size) non-fall colorable graphs with given minimum degree \(\delta\).

**Theorem 3.1.** The graph \(G = C_{p_1} \cup C_{p_2} \cup \cdots \cup C_{p_l}\), (where \(\cup\) stands for disjoint union), has no fall coloring if and only if for at least one \(i\), \(p_i\) is odd and \(p_i \geq 5\).

**Proof.** Assume that \(G\) has no fall coloring and that no \(p_i\) is odd and greater than or equal to 5 (that is, if \(p_i\) is odd, then \(p_i = 3\)). Without loss of generality, let \(p_1, \ldots, p_r\) be even and \(p_{r+1}, \ldots, p_l\) be odd. Then it is easy to give a fall color partition of \(G\) as follows: Just pair off the consecutive
vertices of $C_{p_i}$ for each $i$, $1 \leq i \leq r$, and treat each such part as a color class (for instance, for $C_{2k}$, color the vertices consecutively by $1, 1; 2, 2; \ldots; k, k$), and in the case when $j \geq r + 1$, we can treat each of $V(C_{p_j}) = V(C_3)$ as a color class. Thus, we get a contradiction.

Conversely, assume that for at least one $i$, $p_i \geq 5$ and odd. Then $G$ has no fall coloring, the reason being some vertex of $C_{p_i}$ cannot be a c.d.v. in $G$.

**Theorem 3.2.** Any graph $G$ with $|V(G)| \leq \delta(G) + 2$, where $\delta(G)$ is the minimum degree of $G$, has a fall coloring.

**Proof.** There are only two cases to consider.

(i) $|V(G)| = \delta(G) + 1$. In this case $G = K_{\delta(G)+1}$ and hence $G$ has a fall coloring.

(ii) $|V(G)| = \delta(G) + 2$. Let $S = \{x \in V(G) : d(x) = \delta(G)\}$ and $T = V(G) - S$. Then $\langle T \rangle$, the subgraph induced by $T$, is a clique in $G$ and for every $x \in S$, there exists a unique vertex $y (\neq x)$ in $S$ such that $xy \notin E(G)$. Thus $|S|$ must be even and there are exactly $\frac{|S|}{2}$ pairs of nonadjacent vertices in $G$. For $1 \leq i \leq r := \frac{|S|}{2}$, let $S_i$ be the pair $\{x_i, y_i\}$ of vertices in $S$ such that $x_iy_i \notin E(G)$. Let $T = \{u_1, u_2, \ldots, u_k\}$.

Define $c : V(G) \to \{1, 2, \ldots, r, r + 1, \ldots, r + k\}$ by

$$c(v) = \begin{cases} i & \text{if } v \in S_i, \\ r + j & \text{if } v = u_j \text{ for some } j, 1 \leq j \leq k. \end{cases}$$

Clearly $c$ is a proper coloring of $G$ and every vertex of $G$ is a c.d.v.. Thus $G$ has a fall coloring.

Hence a smallest non-fall colorable graph of minimum degree $\delta$ must be of order at least $\delta + 3$ and size at least $\frac{\delta(\delta+3)}{2}$.

Naturally, any such graph $G$ must be $\delta$-regular graph and order $\delta + 3$ and hence its complement must be a disjoint union of cycles.

We can take $G = C_{p_1} \cup C_{p_2} \cup \cdots \cup C_{p_r}$, where $\sum_{i=1}^{r} p_i = \delta(G) + 3$, all $p_i \geq 3$ and at least one $p_i$ is odd and $p_i \geq 5$. Then, clearly, $G$ is a $\delta(G)$-regular graph and by Theorem 3.1, $G$ has no fall coloring. This $G$ is our required graph. Clearly, $G$ is not unique if $\delta \geq 6$ and unique if $\delta = 5$.

The smallest non-fall colorable graphs with $\delta \leq 4$ have been determined earlier in [3]. The extremal graph, for $\delta = 2$, is $C_5 \cong C_5$, and for $\delta = 4$, it is $C_7$. These coincide with the extremal graphs given in [3]. For $\delta = 3$,...
there are two smallest non-fall colorable graphs, namely, $P_3 \cup K_3$ and the wheel on 6 vertices and these are given in [3]. In this case, as $\delta + 3 = 6$ does not have a partition in the way we required, we do not get the smallest non-fall colorable graphs by our result. However, if we treat $C_5 \cup C_1$ as a degenerate case, we get the wheel on 6 vertices. For $\delta \geq 4$, our result gives all the smallest non-fall colorable graphs. Their exact number (where $\delta \geq 4$) can be obtained as follows: Let $N(k)$ denote the number of partitions of $k$ in which each part is of size at least 3 and one part is odd and of size at least 5. Then $N(k)$ gives the number of smallest non-fall colorable graphs of order $k$ (with minimum degree $k - 3$).

Let $p(n)$ be the well-known partition function of $n$ [1]. Sort each partition from smallest part to largest part. Then, $p(n) - p(n - 1) - p(n - 2) + p(n - 3)$ gives the number of partitions of $n$ not beginning with a 1 or 2. Doubling each part of a partition of $\frac{n}{2}$ gives an even partition of $n$, and so the number of even partitions which do not begin with 2 is $p\left(\frac{n}{2}\right) - p\left(\frac{n}{2} - 1\right)$. The remaining partitions to be excluded are those with smallest part equal to 3, whose remaining parts are even. Removing the first $m$ copies of 3 (a fixed portion of the partition), the remaining even partitions can be given by $p\left(\frac{n-3m}{2}\right)$, and to ensure that the even portion does not begin with two, we subtract $p\left(\frac{n-3m}{2} + 1\right)$. Let $p(n) = 0$ if $n$ is not an integer, and we have the following expression for $N(k)$:

$$N(k) = (p(k) - p(k - 1) - p(k - 2) + p(k - 3))$$

$$- \sum_{m=0}^{\lfloor k/3 \rfloor} \left(p\left(\frac{k-3m}{2}\right) - p\left(\frac{k-3m}{2} + 1\right)\right).$$

For example, $N(8) = 1$ and $N(11) = 4$. $N(8)$ corresponds to the unique graph $C_3 \cup C_5$, while $N(11)$ corresponds to the four graphs $C_{11}, C_4 \cup C_7, C_5 \cup C_6$ and $C_3 \cup C_3 \cup C_5$.

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