

Subsets of Rectifiable curves in Hilbert Space-The Analyst's TSP

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Abstract

We study one dimensional sets (Hausdorff dimension) lying in a Hilbert space. The aim is to classify subsets of Hilbert spaces that are contained in a connected set of finite Hausdorff length. We do so by extending and improving results of Peter Jones and Kate Okikiolu for sets in \mathbb{R}^d . Their results formed the basis of quantitative rectifiability in \mathbb{R}^d . We prove a quantitative version of the following statement: a connected set of finite Hausdorff length (or a subset of one), is characterized by the fact that inside balls at most scales around most points of the set, the set lies close to a straight line segment (which depends on the ball). This is done via a quantity, similar to the one introduced in [Jon90], which is a geometric analog of the Square function. This allows us to conclude that for a given set K , the ℓ_2 norm of this quantity (which is a function of K) has size comparable to a shortest (Hausdorff length) connected set containing K . In particular, our results imply that, with a correct reformulation of the theorems, the estimates in [Jon90, Oki92] are **independent of the ambient dimension**.

1 Introduction

1.1 Basic Notation and Definitions

Cubes, Grids, Balls, and Nets. Multiresolution Families

A cube Q in \mathbb{R}^d is a set of the form $I_1 \times I_2 \times \dots \times I_d$, where I_1, I_2, \dots, I_d are intervals satisfying $|I_1| = |I_2| = \dots = |I_d| = l(Q)$. We call $l(Q)$ the side-length of Q . We denote by λQ the cube with the same center as Q , but with side-length $\lambda l(Q)$.

A dyadic cube is a cube of the form

$$Q = \left[\frac{i_1}{2^j}, \frac{i_1+1}{2^j} \right] \times \dots \times \left[\frac{i_d}{2^j}, \frac{i_d+1}{2^j} \right]$$

where i_1, \dots, i_d, j are integers. The standard dyadic grid on \mathbb{R}^d is

$$\mathcal{D} = \left\{ Q = \left[\frac{i_1}{2^j}, \frac{i_1+1}{2^j} \right] \times \dots \times \left[\frac{i_d}{2^j}, \frac{i_d+1}{2^j} \right] : i_1, \dots, i_d, j \text{ integers} \right\}.$$

A ball Q is a set

$$\text{Ball}(x, r) := \{y : \|y - x\| \leq r\}.$$

We denote by $l(Q)$ the diameter of the ball Q and by λQ the ball with the same center as Q , but with radius λr instead of r (we call this a dilation by λ of Q).

We say that X is an ε – net for K if

- (i) $X \subset K$
- (ii) $\|x_1 - x_2\| > \varepsilon, \forall x_1, x_2 \in X$
- (iii) $\forall y \in K, \exists x \in X$ such that $\|x - y\| \leq \varepsilon$

Hence $K \subset \bigcup_{x \in X} \text{Ball}(x, \varepsilon)$. Note that if $X' \subset K$ satisfies $\|x_1 - x_2\| > \varepsilon, \forall x_1, x_2 \in X'$ then X' can be extended to an ε – net X since a maximal subset of K satisfying (ii), will satisfy (iii).

Fix a set K . Denote by X_n^K a sequence of 2^{-n} – nets for K , such that $X_n^K \subset X_{n+1}^K$. Set

$$\hat{\mathcal{G}}^K = \{\text{Ball}(x, A2^{-n}) : x \in X_n^K, n \text{ an integer}, n \geq n_0\}$$

for a constant $A > 1$ and n_0 an arbitrary (possibly negative) integer. Existence of such a sequence of nets is assured since we may start by choosing a maximal subset of K satisfying (ii) for n_0 and then proceed inductively for $n > n_0$. (This is the only use of n_0 . All results will be independent of n_0 and hence we will suppress it in the notation.)

We call $\hat{\mathcal{G}}^K$ a *multiresolution family*. Note that $\hat{\mathcal{G}}^K$ depends on K . We also call the standard dyadic grid a *multiresolution family*.

For a multiresolution $\hat{\mathcal{G}}$, we denote by $\lambda \hat{\mathcal{G}}$ the multiresolution given by dilating each element in $\hat{\mathcal{G}}$ by λ .

Neighborhoods

We denote the ε neighborhood of a set E by $\mathcal{N}_\varepsilon(E)$.

Hausdorff Length and Arclength

For a set K we denote by $\mathcal{H}^1(K)$ the one dimensional Hausdorff measure, which we call *Hausdorff length*. See [Mat95] for definition and discussion. For a Lipschitz function (see below) $\tau : [a, b] \rightarrow H$ (a Hilbert space) we will denote by $\ell(\tau)$ the arclength of τ . We will also extend this definition to Borel sets of the domain of a given Lipschitz function and use it for the push-forward of this measure.

Hilbert Space

We shall concern ourselves with subsets of a Hilbert space which are subsets of finite length connected sets. All finite length sets are separable. Hence we shall only concern ourselves with separable subspaces of Hilbert spaces, which in turn, are separable Hilbert spaces. Those are all isometric to subspaces of ℓ_2 as vector spaces. Note that since the isometries in question are of vector spaces, straight lines go to straight lines. This will be crucial so that we do not loose generality. Hence we restrict our discussion to separable sets K and fix $H = \ell_2$ as our Hilbert space.

Lipschitz Functions, Rectifiable Sets, Rectifiable curves

A function $f : \mathbb{R}^k \rightarrow H$ is said to be *Lipschitz* if

$$\frac{\|f(x) - f(y)\|}{\|x - y\|} \leq C_f, \forall x, y \in \mathbb{R}^k.$$

A set is called *k-rectifiable* if it is contained in a countable union of images of Lipschitz functions $f_j : \mathbb{R}^k \rightarrow H$, except for a set of k -dimensional Hausdorff measure zero. For more details see [Mat95], where one can also find an excellent discussion of rectifiability in the setting of \mathbb{R}^d , part of which carries over to the setting of H .

A set is called a *rectifiable curve* if it is the image a Lipschitz function defined on \mathbb{R} .

The Jones β Numbers. The Jones Function

Assume we have a set K lying in \mathbb{R}^d or H . Consider Q a cube or ball. We define the Jones β number as

$$\begin{aligned} \beta_K(Q) &= \frac{2}{\text{diam}(Q)} \inf_{L \text{ line}} \sup_{x \in K \cap Q} \text{dist}(x, L) \\ &= \frac{\text{width of thinnest cylinder containing } K \cap Q}{\text{diam}(Q)}. \end{aligned}$$

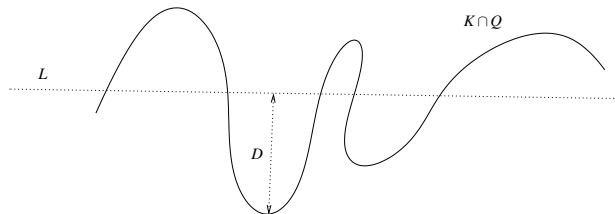


Figure 1: $D = \frac{1}{2}\beta_K(Q)\text{diam}(Q)$

Hence if $K' \supset K$ then $\beta_{K'}(Q) \geq \beta_K(Q)$. Note that we have defined a quantity which is scale independent. This quantity is usually referred to as the Jones β_∞ number in order to differentiate it from its L^p variants (extensively developed by David and Semmes in [DS93] and generalized in [Ler03]). We omit the ∞ subscript as we will always use β_∞ . We will occasionally use the notation $\beta_K(x, r) := \beta_K(\text{Ball}(x, r))$. We will often omit K from the notation when it is obvious what it is.

Fix a multiresolution family. We define the Jones function $J(x)$ as follows:

$$J(x) = \sum \beta^2(Q) \chi_Q(x)$$

where we sum over the multiresolution family of Q 's we have fixed, and χ_Q is the indicator function of Q . This should be thought of as an analog of (the square of) the Square function for certain categories of sets. In many cases one adds for each Q a weight in the above sum. See [BJ90, DS93, Jon90, Ler03, Oki92] for explicit and implicit appearances. One can view the remainder of this essay as an explanation of the right way of generalizing this notion to a larger category of sets. See subsection 1.3 for more details and more precise statements.

1.2 Subsets of Rectifiable Curves. The Analyst's Traveling Salesman Problem

Overview

Given a set $K \subset H$ one can ask under what conditions is K contained in Γ , the image of a single Lipschitz function $\gamma: [0, 1] \rightarrow H$. One can also ask for estimates on the minimal arc-length of such γ . (Recall that, up to multiplicative constants, the arclength of a Lipschitz curve γ is equivalent to the 1-dimensional Hausdorff length of the image of γ . See Lemma 3.7) In [Jon90] Peter Jones gave such an estimate for a *planar* set K in the form of an ℓ_2 sum. In fact, he gave a necessary and sufficient condition for a *planar* set K to be contained in the image of a Lipschitz function by requiring this ℓ_2 sum to be finite. He also gave a construction of a curve whose image Γ_0 contains K , such that $\mathcal{H}^1(\Gamma_0) \lesssim \mathcal{H}^1(\Gamma_{MST})$. Here \lesssim means 'less than a constant multiple of', $\mathcal{H}^1(\cdot)$ is the one dimensional Hausdorff measure and Γ_{MST} is a shortest connected set containing K , whose existence is assured using Arzela-Ascoli and Golab's Theorem. The curve he constructed enjoys some useful properties (for example, see [BJ90]). Philosophically, one should view this ℓ_2 sum as the expectation of how much K deviates from being flat (along a line segment) in a random window. The surprise was that this quantity ended up being equivalent to the length of the shortest curve containing K . In [Oki92] Okikiolu extended this result to sets $K \subset \mathbb{R}^d$, rather than \mathbb{R}^2 . These results formed the basis of a theory now called 'quantitative rectifiability' which was extended by many authors, of which we should give special mention to Guy David and Stephen Semmes whose work on Uniform Rectifiability inspired part of this essay. For example see [DS93, BJ90, Ler03]. Also see [Paj02] for a more complete survey and bibliography! As it turns out, many aspects of the quantitative form in which things will be presented are *parallel* to the theory of wavelets, and a dictionary (discovered by Peter Jones) can be written (see Appendix B in [Sch05]).

Unfortunately, the dependence of the constants in [Jon90, Oki92] on d (as in \mathbb{R}^d) is exponential. This gives motivation to have a Hilbert space version of this theory, which is equivalent to obtaining dimension free estimates. This is the goal of this essay. One should note other examples in harmonic analysis where one was able to obtain dimension free estimates such as the boundedness of the ball Maximal function and the norm of the size of the Riesz vector. See [Ste83, SS83].

A very natural question to ask is 'How does this relate to the Euclidean TSP or Euclidean MST?' (the classical TSP is finding a shortest Hamiltonian cycle on a finite graph; the classical MST problem is finding a minimal spanning tree in a finite graph; their Euclidean counterparts are when the graphs are embedded in Euclidean space). If one wants a polynomial time algorithm and is willing to accept an answer that is not 'the shortest', but 'the shortest up to a constant multiple' then there are readily available algorithms (see e.g. [JM02] for a description of several algorithms). They either do not come with a multiresolution analysis (such as a greedy algorithm which gives multiplicative constant 2), or have exponential (super-exponential!) dependence on the dimension

d (such as [Aro03] which gives multiplicative constant $1 + \varepsilon$). The results of [Jon90], [Oki92] can be used (and are used) to give such algorithms, but with exponentially bad dependence of the constant on the dimension of the ambient space. As an example of an application of our theorem we give a proof that a *local version* of the Farthest Insertion algorithm for the MST converges to a connected set no longer than a constant multiple of the length of the MST. Our proof gives constants independent of the ambient dimension d ! One should note that the constants given by our proof are not as good as the ones experimentally found, as discussed in [JM02].

The results of [Jon90, Oki92] can also be used to prove results regarding existence of Spanning Trees for rectifiable curves. In particular they can be used to give an alternative geometric construction to [KK92] that works in \mathbb{R}^d . We have not yet been able to extend this result to the setting of a Hilbert space.

More Details

We now discuss the results of [Jon90, Oki92] in a little more detail.

Jones ([Jon90]) proved that for any curve γ with image $\Gamma \subset \mathbb{R}^2$ (or equivalently, for any connected set $\Gamma \subset \mathbb{R}^2$)

$$\sum_{\mathcal{D}} \beta_{\Gamma}^2(3Q)l(Q) \lesssim \mathcal{H}^1(\Gamma). \quad (1.1)$$

where \mathcal{D} is the dyadic grid on \mathbb{R}^2 and $l(Q)$ is the side length of a cube $Q \in \mathcal{D}$.

Jones also gave a construction that, given a set $K \subset \mathbb{R}^d$ (and in particular \mathbb{R}^2), yields a connected set $\Gamma_0 \supset K$. The length of Γ_0 satisfies

$$\mathcal{H}^1(\Gamma_0) \lesssim \text{diam}(K) + \sum_{\mathcal{D}} \beta_K^2(3Q)l(Q)$$

where \mathcal{D} is a dyadic grid on \mathbb{R}^d . This construction is a multi-scale algorithm, starting from the "roughest" scale and then refining. This multi-scale method also allows one to form approximations of the final connected set by applying only a finite number of iterations. Combining this length estimate with (1.1) one gets

$$\mathcal{H}^1(\Gamma_0) \lesssim \mathcal{H}^1(\Gamma_{MST})$$

and

$$\text{diam}(K) + \sum_{\mathcal{D}} \beta_K^2(3Q)l(Q) \sim \mathcal{H}^1(\Gamma_{MST})$$

as $\beta_K \leq \beta_{\Gamma_{MST}}$ and $\text{diam}(K) \leq \mathcal{H}^1(\Gamma_{MST})$.

The proof given in [Jon90] for (1.1) relied (quite heavily) on complex analysis. In [Oki92] Okikiolu extended (1.1) to $\Gamma \subset \mathbb{R}^d$ replacing complex analysis with Euclidean geometry and some ℓ_2 type computations. The constants that hide behind the use of the symbol \lesssim are exponential in d . This arises from the fact that a multi-scale dyadic grid is used, from some accounting methods (which can in turn be related to the dyadic grid as well) and from ideas such as covering the unit sphere in \mathbb{R}^d with balls of radius δ .

Note that (1.1) can be reformulated without defining \mathcal{D} by

$$\int_0^\infty \int_\Gamma \frac{\beta_\Gamma^2(\text{Ball}(x, At))}{\mathcal{H}^1(\Gamma \cap \text{Ball}(x, t))} dx dt \lesssim \mathcal{H}^1(\Gamma) \quad (1.2)$$

where A is a constant. See Lemma 3.2 for some further details (but not all of them as we consider a different multiresolution in that lemma).

1.3 New Results

We prove a Hilbert space version of the above results.

Let a set $K \subset H$ be given. For $n > n_0$ define $X_n^K \subset K$ to be a 2^{-n} net such that $X_n^K \subset X_{n+1}^K$. Define a replacement for \mathcal{D} :

$$\hat{\mathcal{G}}^K = \{Q = \text{Ball}(x, A2^{-n}) : x \in X_n^K, n \text{ an integer}, n \geq n_0\}.$$

where $A > 1$ is a constant and n_0 is a (possibly negative) integer.

We show (in Section 3)

Theorem 1.1.

$$\sum_{Q \in \hat{\mathcal{G}}^K} \beta_\Gamma^2(Q) \text{diam}(Q) \lesssim \mathcal{H}^1(\Gamma)$$

for any connected set Γ containing K . The constant behind the symbol \lesssim depends only on the choice of A (which can be given any value greater than 1). In particular, the constant is independent of our choice of $\{X_n^K\}_{n \geq n_0}$ and the choice of n_0 .

Equivalently (see Corollary 3.3),

Theorem 1.2.

$$\int_0^\infty \int_\Gamma \frac{\beta_\Gamma^2(\text{Ball}(x, At))}{\mathcal{H}^1(\Gamma \cap \text{Ball}(x, t))} dx dt \lesssim \mathcal{H}^1(\Gamma)$$

for any connected set Γ . The constant behind the symbol \lesssim depends only on the choice of A (which can be given any value greater than 1).

Remark 1.3. Our proof actually gives more information. Consider a set E which is a countable union of connected sets Γ_i

$$E = \cup \Gamma_i.$$

Let $K \subset E$, and

$$\mathcal{G}^{E,K} = \{Q \in \hat{\mathcal{G}}^K : \forall i, \Gamma_i \cap (H \setminus 4Q) \neq \emptyset\}.$$

Then inspection of the proof we give shows

$$\sum_{Q \in \mathcal{G}^{E,K}} \beta_E^2(Q) \text{diam}(Q) \lesssim \sum \mathcal{H}^1(\Gamma_i)$$

Remark 1.4. For the statement and proof of theorem 1.1 and for the previous remark we do not actually need the condition $X_{n+1} \supset X_n$ in the definition of $\hat{\mathcal{G}}^K$. This condition is however used for the statement and proof of the following results.

In Section 4 we modify Jones' construction to give:

Theorem 1.5. *There is a constant A_0 , such that for all $A > A_0$, for any set $K \subset H$ there exists a connected set $\Gamma_0 \supset K$ satisfying*

$$\mathcal{H}^1(\Gamma_0) \lesssim \text{diam}(K) + \sum_{Q \in \hat{\mathcal{G}}^K} \beta_K^2(Q) \text{diam}(Q). \quad (1.3)$$

The constant behind the symbol \lesssim depends only on the choice of A . In particular, the constant is independent of our choice of $\{X_n^K\}_{n \geq n_0}$. We require $2^{-n_0} \geq \text{diam}(K)$.

We would like to note recent independent work done by Immo Hahlmaa (see [Hah05]) containing a generalization of Theorem 1.5 to the setting of Metric Spaces! (Where one needs to use Menger curvature to define β .) See also [Hah] for other analogues. There is also work by Ferrari, Franchi and Pajot for a version of this theorem in certain geodesic spaces (such as the Heisenberg group).

Analogs of Theorem 1.1 (and Theorem 1.2) can be obtained for metric spaces satisfying certain axioms by inspecting the proof appearing in this essay. See also [Sch] for other analogues.

As immediate corollaries (by combining the above theorems), we get the following results, which in \mathbb{R}^2 were the motivation for [Jon90].

Corollary 1.6. *For $A > A_0$ and $n_0 \leq -\log(\text{diam}(K))$*

$$\mathcal{H}^1(\Gamma_0) \lesssim \mathcal{H}^1(\Gamma_{MST}).$$

$$\text{diam}(K) + \sum_{Q \in \hat{\mathcal{G}}^K} \beta_K^2(Q) \text{diam}(Q) \sim \mathcal{H}^1(\Gamma_{MST})$$

for any set K , where Γ_{MST} is a shortest connected set containing K .

We show the existence of Γ_{MST} in Appendix 5.2.

Corollary 1.7. *For $A > A_0$*

$$\text{diam}(\Gamma) + \int_0^\infty \int_\Gamma \frac{\beta_\Gamma^2(\text{Ball}(x, At))}{\mathcal{H}^1(\Gamma \cap \text{Ball}(x, t))} dx dt \sim \mathcal{H}^1(\Gamma).$$

$$\text{diam}(\Gamma) + \sum_{Q \in \hat{\mathcal{G}}^\Gamma} \beta_\Gamma^2(Q) \text{diam}(Q) \sim \mathcal{H}^1(\Gamma)$$

for any connected set Γ .

This automatically gives that the relevant constants in [Jon90, Oki92] need not be exponential in d , if \mathcal{D} is replaced by $\hat{\mathcal{G}}$ (super-indexed correctly). The construction of Γ_0 remains the same as Jones', except for one part (specifically, the case when $\beta > \varepsilon$ becomes slightly more complicated). The proof of Theorem 1.1 is done by a modification (as described in the following paragraphs) of Okikiolu's method which results in dimension free estimates.

One should note that in the case of one dimensional Uniformly Rectifiable sets (see [DS93] for definition of Uniformly Rectifiable) these results are obtained with much less difficulty by combining [Dav91, DS93, Jon88, Jon90, Oki92]. The key idea is that using [Jon88] one gets that Ahlfors regular curves contain what is called 'big pieces of chord-arc curves' (see [DS93] for a definition). For chord-arc curves we have (using a modification of [Oki92]) desired estimates, which can be used with machinery from [DS93] to extend to Ahlfors regular curves. All this requires an inspection of some proofs given in the above references, which results in the observation that, even though they are not stated as such, they are dimension independent for the relevant cases (or can be made so with very minor modifications; for example [Oki92] can be made dimension independent in the case of chord-arc curves). Inspecting the results in [DS93] was suggested to the author by Guy David.

Outline

We prove Theorems 1.1 and 1.2 in Section 3. We do this by considering the geometry of the set Γ inside the different balls. Let us give a vague intuitive description.

We call an arc τ (delimited by a given ball Q and contained in $\Gamma \cap Q$) an 'almost flat arc' if $\beta(\tau)$ is small in comparison with $\beta(Q)$ (i.e. τ is close to a straight line segment). For a given Q the collection of these arcs is called S_Q . We also designate an arc going through the center of Q (existing by the definition of $\hat{\mathcal{G}}$) by γ_Q .

In subsection 3.2 we discuss balls Q for which either γ_Q is not an 'almost flat arc' ('non-flat arc') or $\beta(Q)$ is not controlled by $\beta_{S_Q}(Q)$ ($= \beta$ restricted to the 'almost flat arcs'). The latter balls also contain an arc τ which is 'very non-flat' in the sense that it is 'non-flat' enough so that $\beta(\tau)$ controls $\beta(Q)$ (see Figure 2 (a) and (c)). We make use of this by employing ideas of Okikiolu, as well as ideas similar to ones of G. David and M. Christ (see e.g. [Chr90], and [Dav91] page 93 for a simple version). Note however that one main difference with the case of Christ and David is the lack of homogeneity assumption on Γ ! This makes Okikiolu's ideas harder to use. Subsection 3.2 corresponds to the first half of [Oki92].

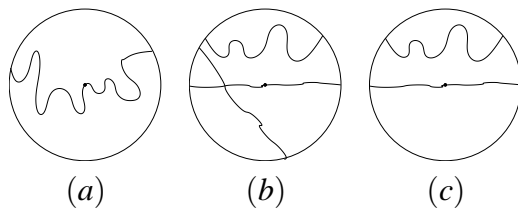


Figure 2: Three examples of balls

In subsection 3.3 we discuss the rest of the balls. These balls Q satisfy $\beta(Q) \lesssim \beta_{S_Q}(Q)$ (see

Figure 2 (b)). The key idea here is to use the curve itself as the notebook for the bookkeeping (as in [Oki92, Ler03, BJ90]). This is done explicitly in this subsection (whereas in this preceding subsections this idea is used implicitly). This subsection corresponds to the second part of [Oki92], where Okikiolu allots segments whose length controls $\beta(Q)\text{diam}(Q)$. We substitute allotting segments by allotting densities. We do so by constructing for each ball $Q \in \mathcal{G}$, a weight w_Q supported in Q , satisfying $\int_Q w_Q d\ell \geq \beta(Q)\text{diam}(Q)$ and $\sum_Q w_Q(x) \leq C$ for almost every $x \in \Gamma$. An important point is that the construction of each w_Q is done in a multiscale fashion (as a martingale), which allows the assurance of the above properties. This assurance is not so straight forward and most of subsection 3.3 is devoted to it. This gives us that for these balls $\sum \beta_\Gamma(Q)\text{diam}(Q) \lesssim \mathcal{H}^1(\Gamma)$ (note that we do not square the terms!).

We prove Theorem 1.5 in Section 4. We do so by a construction which is a modification of the *farthest insertion* algorithm. This is not far from what is written in [Jon90].

Constants and a computational note

We will fix certain constants in the following proofs. Some of them will depend on each other. In particular, in subsection 4 we will obtain a value A_0 so that we will require $A > A_0$. $A_0 = 200$ will suffice. Then we will use a (any) choice of A and derive from it a choice for the constant that we name ε_2 , which first appears in subsection 3.1. In subsection 3.3 we have a constant C whose choice depends on A . The constant J appears several times when we wish to skip (Jump over) scales. Only in subsection 3.3 is it required that it depend on A .

All other named constants are independent of A .

The constant n_0 introduced in this section is not used in any other constant! It is used solely as a starting point for an inductive argument.

By following the proofs one gets that the dependence of the constant hiding behind the symbol \sim in Corollary 1.7 on A is $A^{\frac{9}{2}} \log A$. We have made no effort to get this dependence to be as minimal as possible while proving the theorems. The reason this is of interest is that one may find this useful when trying to use these theorems in an \mathbb{R}^d numerical setting, and hence go back to considering dyadic cubes (which renders the choice of $A \sim \sqrt{d}$ reasonable).

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3 Proof of Theorems 1.1 and 1.2

3.1 Preliminaries, Notation and Definitions

Remark 3.1. *We would like to show*

$$\sum_{Q \in \hat{\mathcal{G}}} \beta(Q)^2 \text{diam}(Q) \leq C \mathcal{H}^1(\Gamma). \quad (3.1)$$

We have

$$\lim_{n \rightarrow \infty} \sum_{\substack{Q \in \hat{\mathcal{G}} \\ \text{center}(Q) \in X_n}} \beta_{X_n}(Q)^2 \text{diam}(Q) = \sum_{Q \in \hat{\mathcal{G}}} \beta(Q)^2 \text{diam}(Q).$$

If we choose $d = \#(X_n)$ then we can project the problem to \mathbb{R}^d , and so if we prove (3.1) for $\Gamma \subset \mathbb{R}^d$ with C independent of d we are done. Hence one should note that we may just as well in subsections 3.1 - 3.3 assume that we are working in \mathbb{R}^d , not in H . This will however, be of no consequence to us, as our proof works in H .

Assume $K \subset \Gamma \subset H$ as in the statement of the theorems. A is fixed to a constant larger or equal to the constant A_0 that we get from Chapter 4. We will omit the superscripts/subscripts K, Γ whenever possible to simplify notation.

We start with a discretization lemma.

Lemma 3.2. *Assume that $K \subset \Gamma$ and $\mathcal{H}^1(B(x, t) \cap \Gamma) > 0, \forall x \in \Gamma, t > 0$. Let $8A' \leq A \leq \frac{1}{8}A''$. Then*

$$(i) \int_0^\infty \int_\Gamma \frac{\beta_\Gamma^2(A' \text{Ball}(x, t))}{\mathcal{H}^1(\Gamma \cap \text{Ball}(x, t))} dx dt \lesssim \sum_{Q \in \hat{\mathcal{G}}^\Gamma} \beta_\Gamma(Q)^2 \text{diam}(Q)$$

$$(ii) \int_0^\infty \int_\Gamma \frac{\beta_\Gamma^2(A'' \text{Ball}(x, t))}{\mathcal{H}^1(\Gamma \cap \text{Ball}(x, t))} dx dt \gtrsim \sum_{Q \in \hat{\mathcal{G}}^K} \beta_\Gamma(Q)^2 \text{diam}(Q).$$

Proof. Notice that for a ball B we have $\beta(B) \leq C\beta(CB), \forall C \geq 1$.

$$\begin{aligned} \int_0^\infty \int_\Gamma \frac{\beta^2(A' \text{Ball}(x, t))}{\mathcal{H}^1(\Gamma \cap \text{Ball}(x, t))} dx dt &\lesssim \int_0^\infty \int_\Gamma \frac{\beta^2(A' \text{Ball}(x, 2t))}{\mathcal{H}^1(\Gamma \cap \text{Ball}(x, 2t))} dx dt \\ &\lesssim \sum_{n \in \mathbb{Z}} 2^{-n} \int_\Gamma \frac{\beta^2(4A' \text{Ball}(x, 2^{-n}))}{\mathcal{H}^1(\Gamma \cap 2\text{Ball}(x, 2^{-n}))} dx \\ &\lesssim \sum_{n \in \mathbb{Z}} 2^{-n} \sum_{x \in X_n^\Gamma} \beta^2(8A' \text{Ball}(x, 2^{-n})) \frac{\mathcal{H}^1(\Gamma \cap \text{Ball}(x, 2^{-n}))}{\mathcal{H}^1(\Gamma \cap \text{Ball}(x, 2^{-n}))} \\ &\lesssim \sum_{Q \in \hat{\mathcal{G}}^\Gamma} \beta(Q)^2 \text{diam}(Q) \end{aligned}$$

as long as $A \geq 8A'$. The change of variable $t \rightarrow \frac{1}{2}t$ was used for the first inequality. Conversely,

$$\begin{aligned}
\sum_{Q \in \hat{\mathcal{G}}^K} \beta(Q)^2 \text{diam}(Q) &= \sum_{n \in \mathbb{Z}} 2A2^{-n} \sum_{x \in X_n^K} \beta^2(\text{Ball}(x, A2^{-n})) \\
&\lesssim \sum_{n \in \mathbb{Z}} 2^{-n} \int_{x \in \Gamma} \beta^2(2\text{Ball}(x, A2^{-n})) \frac{1}{\mathcal{H}^1(\Gamma \cap \frac{1}{2}\text{Ball}(x, 2^{-n}))} dx \\
&\lesssim \int_0^\infty \int_{x \in \Gamma} \beta^2(4\text{Ball}(x, At)) \frac{1}{\mathcal{H}^1(\Gamma \cap \frac{1}{2}\text{Ball}(x, t))} dx dt \\
&\lesssim \int_0^\infty \int_{x \in \Gamma} \frac{\beta^2(8A\text{Ball}(x, t))}{\mathcal{H}^1(\Gamma \cap \text{Ball}(x, t))} dx dt
\end{aligned}$$

so take $A'' \geq 8A$. The change of variable $t \rightarrow 2t$ was used for the last inequality. \square

This immediately gives us

Corollary 3.3. *Theorem 1.2 and Theorem 1.1 are equivalent.*

For the remainder of this section we will concern ourselves only with Theorem 1.1. We state some point-set topology lemmas. For completeness we give proofs for these lemmas in the appendix.

Lemma 3.4. *Assume Γ is connected. Then $\mathcal{H}^1(\Gamma) = \mathcal{H}^1(\Gamma^{\text{closure}})$.*

Lemma 3.5. *Assume $\Gamma \subset H$ is a closed connected set with $\mathcal{H}^1(\Gamma) < \infty$. Then Γ is compact.*

Lemma 3.6. *Let $C_1, C_2 > 0$ be given. Given a compact connected set $\Gamma \subset H$ the set $E := \{x \in H : x = tx_1 + (1-t)x_2, x_i \in \Gamma, -C_1 \leq t \leq C_2\}$ is compact.*

Lemma 3.7. *Let $\Gamma \subset H$ be a compact connected set of finite length. Then we have a Lipschitz function $\gamma : [0, 1] \rightarrow H$ such that $\text{Image}(\gamma) = \Gamma$ and $\|\gamma\|_{\text{Lip}} \leq 32\mathcal{H}^1(\Gamma)$*

Corollary 3.8. *Let $\Gamma \subset H$ be a compact connected set of finite length. Then we have a Lipschitz function $\gamma : \mathbb{T} \rightarrow H$ such that $\text{Image}(\gamma) = \Gamma$ and $\|\gamma\|_{\text{Lip}} \leq 32\mathcal{H}^1(\Gamma)$*

Proofs for the above lemmas can be found in the appendix.

Since Theorem 1.1 is trivially satisfied for Γ satisfying $\mathcal{H}^1(\Gamma) = \infty$ we will assume $\mathcal{H}^1(\Gamma) < \infty$. We may replace Γ by its closure without loss of generality for the purpose of proving this theorem. This will not affect the Jones- β numbers, the connectedness of Γ , or the length of Γ . Hence we will assume Γ is compact from now on. By re-scaling we may also assume $\text{diam}(\Gamma) \leq 1$ and $n_0 = 0$.

Using Corollary 3.8, we fix a parameterization γ for Γ , $\gamma : \mathbb{T} \rightarrow H$, such that we have $\ell(\gamma) \leq 32\mathcal{H}^1(\Gamma)$. **From here on we also use $\ell(\cdot)$ as the push-forward by γ of the arc-length measure.**

We will show (for a given A) that Theorem 1.1 holds, i.e.

$$\sum_{Q \in \mathcal{G}} \beta(Q)^2 \text{diam}(Q) \leq C\mathcal{H}^1(\Gamma).$$

We recall that we have (after re-scaling)

$$\hat{\mathcal{G}} = \{Q = \text{Ball}(x, A2^{-n}), x \in X_n; n \geq 0\}.$$

We define

$$\mathcal{G} = \{Q \in \hat{\mathcal{G}} : \Gamma \cap (H \setminus 4Q) \neq \emptyset\}.$$

(\mathcal{G} is the collection of balls Q that are small enough so that Γ must exit $4Q$.)

Consider $\hat{\mathcal{G}} \setminus \mathcal{G}$.

Lemma 3.9. $\sum_{Q \in \hat{\mathcal{G}} \setminus \mathcal{G}} \beta(Q)^2 \text{diam}(Q) \leq C \mathcal{H}^1(\Gamma).$

Proof. Set $L = \ell(\gamma)$ (the arc-length of a parameterization γ of Γ assured by Corollary 3.8) and $D = \text{diam}(\Gamma)$.

We have at most:

$$\begin{array}{ll} \frac{8 \cdot 2AL}{D} & \text{balls of diameter } \frac{1}{8}D \text{ in } \hat{\mathcal{G}} \setminus \mathcal{G}. \text{ They are of } \beta \leq 1. \\ \frac{4 \cdot 2AL}{D} & \text{balls of diameter } \frac{1}{4}D \text{ in } \hat{\mathcal{G}} \setminus \mathcal{G}. \text{ They are of } \beta \leq 1. \\ \frac{2 \cdot 2AL}{D} & \text{balls of diameter } \frac{1}{2}D \text{ in } \hat{\mathcal{G}} \setminus \mathcal{G}. \text{ They are of } \beta \leq 1. \\ \frac{2AL}{D} & \text{balls of diameter } D \text{ in } \hat{\mathcal{G}} \setminus \mathcal{G}. \text{ They are of } \beta \leq 1. \\ \frac{2AL}{2D} & \text{balls of diameter } 2D \text{ in } \hat{\mathcal{G}} \setminus \mathcal{G}. \text{ They are of } \beta \leq \frac{1}{2}. \\ \frac{2AL}{4D} & \text{balls of diameter } 4D \text{ in } \hat{\mathcal{G}} \setminus \mathcal{G}. \text{ They are of } \beta \leq \frac{1}{4}. \\ \vdots & \\ \frac{2AL}{AL} & \text{balls of diameter } AL \text{ in } \hat{\mathcal{G}} \setminus \mathcal{G}. \text{ They are of } \beta \leq \frac{D}{AL} \\ 1 & \text{ball of diameter } 2AL \text{ in } \hat{\mathcal{G}} \setminus \mathcal{G}. \text{ It is of } \beta \leq \frac{D}{2AL} \\ 1 & \text{ball of diameter } 4AL \text{ in } \hat{\mathcal{G}} \setminus \mathcal{G}. \text{ It is of } \beta \leq \frac{D}{4AL} \\ \vdots & \end{array}$$

(Since the centers must be along γ). Hence

$$\sum_{Q \in \hat{\mathcal{G}} \setminus \mathcal{G}} \beta(Q)^2 \text{diam}(Q) \leq \sum_{n=-3}^{\log(\frac{AL}{D})} AL2^{-2n} + \sum_{\log(\frac{AL}{D})}^{\infty} AL2^{-n} \leq 32AL + D \leq (32A + 1)L.$$

□

We need some more notation:

$$\Lambda(Q) := \{ \tau = \gamma|_{[a,b]} : [a,b] \subset \mathbb{T}; [a,b] \text{ a connected component of } \gamma^{-1}(\Gamma \cap Q) \}.$$

We will freely use $\tau \in \Lambda(Q)$ as both a parameterization of an arc (given by restriction of γ), and its image. We will denote by $\text{diam}(\tau)$ the diameter of the image of τ .

We define for an arc $\tau : [a,b] \rightarrow H$

$$\tilde{\beta}(\tau) := \sup_{t \in [a,b]} \frac{\text{dist}(\tau(t), [\tau(a), \tau(b)])}{\text{diam}(\tau)},$$

where $[x,y]$ is the straight line segment connecting x and y . (This is how we define the Jones β number of an arc).

Consider $\tau \in \Lambda(Q)$. We call τ *almost flat* iff

$$\tilde{\beta}(\tau) \leq \varepsilon_2 \beta(Q).$$

Set:

$$S_Q := \{ \tau \in \Lambda(Q) : \tilde{\beta}(\tau) \leq \varepsilon_2 \beta(Q) \}.$$

This is the collection of *almost flat* arcs in $\Lambda(Q)$. See Figure 3.

Consider all balls $Q \in \mathcal{G} \cup 2\mathcal{G} \cup 4\mathcal{G}$. For each of them, we fix $\gamma_Q \in \Lambda_Q$ an arc containing the center of Q . If there is more than one option, choose so that if $Q_1 = \frac{1}{2}Q_2$ then $\gamma_{Q_1} \subset \gamma_{Q_2}$. This can be easily done by working top-down (as opposed to $\hat{\mathcal{G}}$, \mathcal{G} has a coarsest scale).

Set for $j \in \{0, 1, 2\}$

$$\begin{aligned} \mathcal{G}_1^j &= \{ Q \in \mathcal{G} : \tilde{\beta}(\gamma_{2^j Q}) > \varepsilon_2 \beta(2^j Q) \} \\ \mathcal{G}_2^j &= \{ Q \in \mathcal{G} : \tilde{\beta}(\gamma_{2^j Q}) \leq \varepsilon_2 \beta(2^j Q); \beta_{S_{2^j Q}}(2^j Q) > \varepsilon_1 \beta(Q) \} \\ \mathcal{G}_3^j &= \{ Q \in \mathcal{G} : \tilde{\beta}(\gamma_{2^j Q}) \leq \varepsilon_2 \beta(2^j Q); \beta_{S_{2^j Q}}(2^j Q) \leq \varepsilon_1 \beta(Q) \} \end{aligned}$$

where by $\beta_{S_{2^j Q}}(\cdot)$ we mean $\beta_{\cup\{\tau: \tau \in S_{2^j Q}\}}(\cdot)$. This is an abuse of notation we will keep on using throughout this essay. Clearly for every j , $\mathcal{G} = \mathcal{G}_1^j \cup \mathcal{G}_2^j \cup \mathcal{G}_3^j$ and $\gamma_{2^j Q} \in S_{2^j Q}$, for all $Q \in \mathcal{G}_2^j \cup \mathcal{G}_3^j$. We will fix ε_1 independent of A to be sufficiently small. We will fix ε_2 in subsection 3.3 such that $A\varepsilon_2$ is sufficiently small. See Figure 4 for examples of the above sets. For most of this essay the reader can be content with simply fixing $j = 0$ (the only exception will be \mathcal{G}_3^j).

We will show

$$\sum_{Q \in \mathcal{G}_1^j} \beta(Q)^2 \text{diam}(Q) \leq C \mathcal{H}^1(\Gamma) \quad (3.2)$$

for any $j \in \{0, 1, 2\}$ in subsection 3.2. We will also show

$$\sum_{Q \in \mathcal{G}_3^1 \cap \mathcal{G}_3^2 \cap \mathcal{G}_3^3} \beta(Q)^2 \text{diam}(Q) \leq C \mathcal{H}^1(\Gamma) \quad (3.3)$$

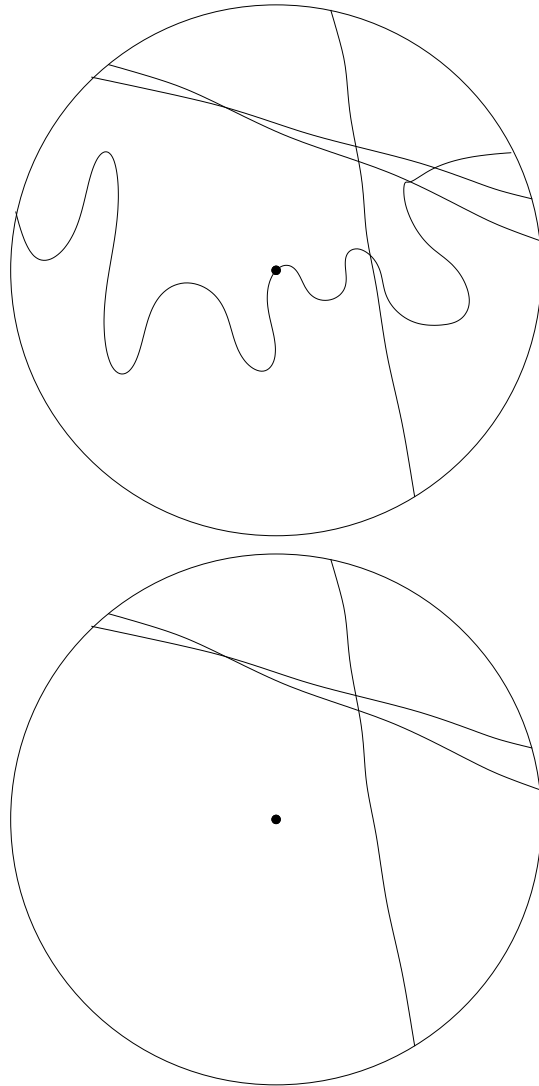
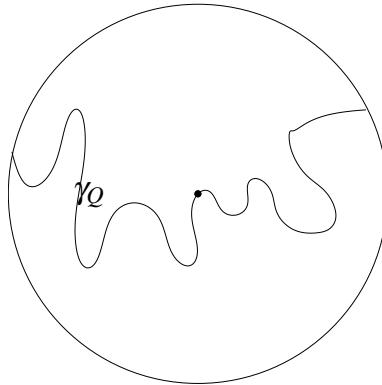
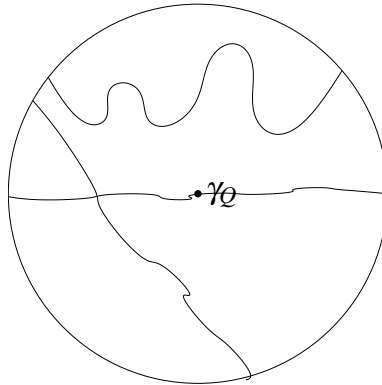


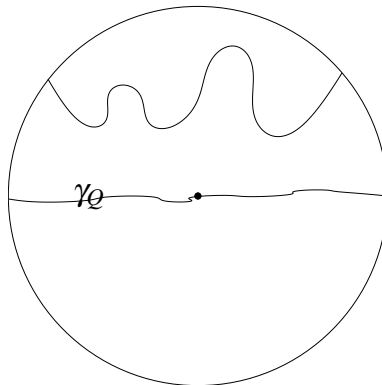
Figure 3: Example of Q (top) and S_Q (bottom)



An example of a \mathcal{G}_1 ball



An example of a \mathcal{G}_2 ball



An example of a \mathcal{G}_3 ball

Figure 4: Examples of the three different types of balls

in subsection 3.2. We will show

$$\sum_{Q \in \mathcal{G}_2^j} \beta(Q)^2 \text{diam}(Q) \leq C \mathcal{H}^1(\Gamma) \quad (3.4)$$

for any $j \in \{0, 1, 2\}$ in subsection 3.3.

This will give us.

$$\sum_{Q \in \mathcal{G}} \beta(Q)^2 \text{diam}(Q) \leq C \mathcal{H}^1(\Gamma)$$

3.2 Non-Flat Arcs

In this subsection we prove (3.2) and (3.3). The tools developed in this subsection will also be used in subsection 3.3.

Remark 3.10. *In the following we will be discussing various sub-arcs. They are parameterized by the global parameterization of Γ . It is important that when we want to say something about the intersection or union of two sub-arcs, that we talk about their **domain** and not their **range**! In contrast, when we discuss the diameter of an arc, we will be discussing the diameter of its **image**.*

Even though the setup is slightly different, the proof for the following lemma is copied almost word for word from [Oki92].

Lemma 3.11. *Suppose we are given a family of sub-arcs $\mathcal{F} = \bigcup_{i=0}^{\infty} \mathcal{F}_i$ with the following properties:*

- (1) $\tau' \in \mathcal{F}_{n+1} \Rightarrow \exists! \tau \in \mathcal{F}_n$ such that $\tau' \subset \tau$
- (2) $\tau \in \mathcal{F}_n \Rightarrow 2^{-nJ} \leq \text{diam}(\tau) \leq A2^{-nJ+2}$
- (3) $\tau, \tau' \in \mathcal{F}_n \Rightarrow \#(\tau \cap \tau') \in \{0, 1, 2\}$ (the intersection is an empty set, a single point, or two points)
- (4) $\bigcup_{\mathcal{F}_0} \tau = \bigcup_{\mathcal{F}_n} \tau, \forall n$

(we will call such a family a **filtration**). Then we have:

$$\sum_{\tau \in \mathcal{F}} \tilde{\beta}(\tau)^2 \text{diam}(\tau) \lesssim \ell\left(\bigcup_{\mathcal{F}_0} \tau\right).$$

Proof. Set for $\tau \in \mathcal{F}_n$,

$$\mathcal{F}_{\tau,k} = \{\tau' \subset \tau : \tau' \in \mathcal{F}_{n+k}\}.$$

Set for $\tau \in \mathcal{F}_n$ such that $\tau : [\text{initial}, \text{final}] \rightarrow H$,

$$I_{\tau} = [\tau(\text{initial}), \tau(\text{final})]$$

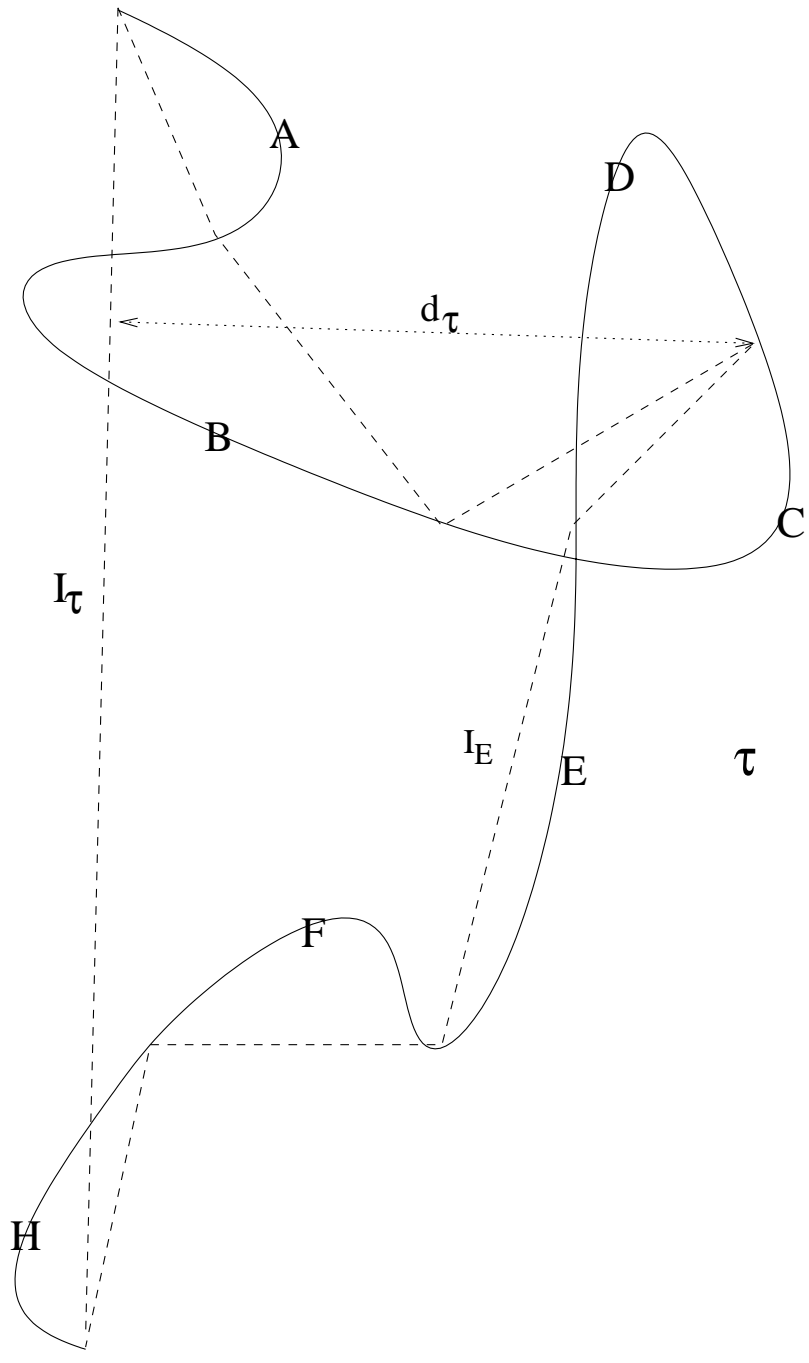


Figure 5: (Here $A, B, C, D, E, F, G, H \in F_{\tau,1}$ and $A \cup B \cup C \cup D \cup F \cup G \cup H = \tau$)

and

$$d_\tau = \sup_{\substack{\tau' \in \mathcal{F}_{\tau,1} \\ x \in I_{\tau'}}} \text{dist}(x, I_\tau).$$

This is in fact a maximum and not a supremum: $\bigcup_{\tau' \in \mathcal{F}_{\tau,1}} I_{\tau'}$ is compact by the compactness of τ . For each τ and k , let $\tau_k \in \mathcal{F}_{\tau,k}$ be chosen such that d_{τ_k} is maximal among all arcs in $\mathcal{F}_{\tau,k}$. Again, this is a maximum and not a supremum.

We have for $\tau \in \mathcal{F}_n$:

$$\tilde{\beta}(\tau) \text{diam}(\tau) \leq \sum_{k=0}^{\infty} d_{\tau_k} \quad (3.5)$$

by the following:

Consider a sequence: $\tau^0 = \tau$, $\tau^{k+1} \in \mathcal{F}_{\tau^k,1}$. We have

$$\tilde{\beta}(\tau) \text{diam}(\tau) = \sum_0^{\infty} (\tilde{\beta}(\tau^k) \text{diam}(\tau^k) - \tilde{\beta}(\tau^{k+1}) \text{diam}(\tau^{k+1}))$$

by the fact that we have a telescoping series with the summand going to 0 by (2). If we choose $\tau^{k+1} \in \mathcal{F}_{\tau^k,1}$ such that $\tilde{\beta}(\tau^{k+1}) \text{diam}(\tau^{k+1})$ is maximal among all arcs in $\mathcal{F}_{\tau^k,1}$, then

$$d_{\tau_k} \geq (\tilde{\beta}(\tau^k) \text{diam}(\tau^k) - \tilde{\beta}(\tau^{k+1}) \text{diam}(\tau^{k+1}))$$

by the triangle inequality and our choices of τ_k and τ^{k+1} . Hence we get (3.5)

We also have

$$\frac{d_\tau^2}{\text{diam}(\tau)} \lesssim \left(\sum_{\tau' \in \mathcal{F}_{\tau,1}} \mathcal{H}^1(I_{\tau'}) \right) - \mathcal{H}^1(I_\tau). \quad (3.6)$$

We see this as follows:

By compactness of τ we have a point $P_1 \in \bigcup_{\tau' \in \mathcal{F}_{\tau,1}} I_{\tau'}$ such that

$$d_\tau = \text{dist}(P_1, I_\tau).$$

From the fact that the union is a union of line segments and I_τ is a line segment we have $P_1 \in \tau$. Let $P_2 \in I_\tau$ satisfy $\text{dist}(P_1, I_\tau) = \text{dist}(P_1, P_2)$. Set

$$\begin{aligned} c_1 &= \text{dist}(P_1, \tau(\text{initial})) \\ c_2 &= \text{dist}(P_1, \tau(\text{final})) \\ a_1 &= \text{dist}(P_2, \tau(\text{initial})) \\ a_2 &= \text{dist}(P_2, \tau(\text{final})). \end{aligned}$$

Hence (by say, the Cosine Theorem)

$$\begin{aligned} d_\tau^2 &\leq c_1^2 - a_1^2 \\ d_\tau^2 &\leq c_2^2 - a_2^2 \end{aligned}$$

and so

$$2d_\tau^2 \leq c_1^2 - a_1^2 + c_2^2 - a_2^2 \quad (3.7)$$

$$= (c_1 - a_1)(c_1 + a_1) + (c_2 - a_2)(c_2 + a_2) \quad (3.8)$$

$$\leq (c_1 - a_1)2\text{diam}(\tau) + (c_2 - a_2)2\text{diam}(\tau) \quad (3.9)$$

$$= 2\text{diam}(\tau)(c_1 + c_2 - (a_1 + a_2)). \quad (3.10)$$

Finally, by the triangle inequality,

$$c_1 + c_2 \leq \sum_{\tau' \in \mathcal{F}_{\tau,1}} \mathcal{H}^1(I_{\tau'})$$

and we also have

$$a_1 + a_2 = \mathcal{H}^1(I_\tau)$$

which gives

$$d_\tau^2 \leq \text{diam}(\tau) \left(\sum_{\tau' \in \mathcal{F}_{\tau,1}} \mathcal{H}^1(I_{\tau'}) - \mathcal{H}^1(I_\tau) \right)$$

which gives (3.6) as desired.

This means, by summing (3.6) over \mathcal{F}_n :

$$\sum_{\tau \in \mathcal{F}_n} \frac{d_\tau^2}{\text{diam}(\tau)} \lesssim \sum_{\tau \in \mathcal{F}_{n+1}} \mathcal{H}^1(I_\tau) - \sum_{\tau \in \mathcal{F}_n} \mathcal{H}^1(I_\tau).$$

Summing over all n we get:

$$\sum_{\tau \in \mathcal{F}} \frac{d_\tau^2}{\text{diam}(\tau)} \lesssim \sup_n \left(\sum_{\tau \in \mathcal{F}_n} \mathcal{H}^1(I_\tau) \right) \leq \ell \left(\bigcup_{\mathcal{F}_0} \tau \right).$$

We can now compute in an ℓ_2 fashion:

$$\begin{aligned}
\left(\sum_{\tau \in \mathcal{F}} \tilde{\beta}(\tau)^2 \text{diam}(\tau)\right)^{\frac{1}{2}} &\leq \left(\sum_{\tau \in \mathcal{F}} \frac{(\sum_{k=0}^{\infty} d_{\tau_k})^2}{\text{diam}(\tau)}\right)^{\frac{1}{2}} \\
&\leq \sum_{k=0}^{\infty} \left(\sum_{\tau \in \mathcal{F}} \frac{d_{\tau_k}^2}{\text{diam}(\tau)}\right)^{\frac{1}{2}} \\
&\leq \sum_{k=0}^{\infty} 2^{-J\frac{k}{2}} \left(\sum_{\tau \in \mathcal{F}} \frac{d_{\tau_k}^2}{2^{-Jk} \text{diam}(\tau)}\right)^{\frac{1}{2}} \\
&\lesssim \sum_{k=0}^{\infty} 2^{-J\frac{k}{2}} \left(\sum_{\tau \in \mathcal{F}} \frac{d_{\tau_k}^2}{\text{diam}(\tau_k)}\right)^{\frac{1}{2}} \\
&\lesssim \sum_{k=0}^{\infty} 2^{-J\frac{k}{2}} \left(\ell\left(\bigcup_{\mathcal{F}_0} \tau\right)\right)^{\frac{1}{2}} \\
&\lesssim \ell\left(\bigcup_{\mathcal{F}_0} \tau\right)^{\frac{1}{2}}
\end{aligned}$$

where the penultimate inequality follows from the fact that $\tau \neq \tau' \Rightarrow \tau_k \neq \tau'_k$ unless τ is one of the $\log(4A)$ immediate consecutive forefathers of τ' or vice-versa. (More careful notation would eliminate this need for a factor of $\log(4A)$.) \square

Remark 3.12. *If one follows the computation one gets*

$$\sum_{\tau \in \mathcal{F}} \tilde{\beta}(\tau)^2 \text{diam}(\tau) \leq CA^{\frac{1}{2}} \log(A) \ell\left(\bigcup_{\mathcal{F}_0} \tau\right),$$

where C is a universal constant, independent of A .

Remark 3.13. *Since there was nothing special about $\bigcup_{\mathcal{F}_0} \tau$, we get*

$$\sum_{\{\tau' \in \mathcal{F}; \tau' \subset \tau\}} \tilde{\beta}(\tau')^2 \text{diam}(\tau') \lesssim \ell(\tau)$$

for all $\tau \in \mathcal{F}$. Hence we can set

$$w_{\tau}(x) = \frac{\tilde{\beta}(\tau)^2 \text{diam}(\tau)}{\ell(\tau)}, \forall x \in \tau$$

and get

- (1) $\int_{\tau} w_{\tau} dl = \tilde{\beta}(\tau)^2 \text{diam}(\tau)$
- (2) $w(x) := \sum_{\tau \in \mathcal{F}} w_{\tau}(x) \in BMO_{\mathcal{F}}$
- (3) $\text{supp}(w_{\tau}) \subset \tau$

where

$$f \in BMO_{\mathcal{F}} \iff \sup_{\tau \in \mathcal{F}} \frac{1}{\ell(\tau)} \int_{\tau} |f - \frac{1}{\ell(\tau)} \int_{\tau} f dl| dl < \infty.$$

We now turn to the construction of filtrations. As before, when we discuss the intersection or union of arcs, we do this in the parameter space (\mathbb{T}). When we discuss the diameter of an arc, we do so in the image space Γ .

The idea for the proof of the lemma comes from now classical constructions of Dyadic Cubes on Homogeneous Spaces (see e.g. [Chr90], and [Dav91] page 93 for a simple version). One should note that condition (2) replaces a doubling condition.

Lemma 3.14. *There is a universal constant $J > 0$ ($J = 10$ suffices) such that, given a collection of arcs $\mathcal{F}^0 = \bigcup_{i=0}^{\infty} \mathcal{F}_i^0$ with the following properties:*

- (1) $\tau \in \mathcal{F}_n^0 \Rightarrow \#\{\tau' \in \mathcal{F}_n^0 : \tau' \cap \tau \neq \emptyset\} \leq C$
- (2) $\tau \in \mathcal{F}_n^0 \Rightarrow 2^{-n} \leq \text{diam}(\tau) \leq A2^{-n+1}$.

Then we have $2CJ$ families of arcs, each of which will be a filtration (see requirements of previous lemma). Furthermore, we will have: $\forall \tau' \in \mathcal{F}_{nJ}^0 \exists \tau \in \mathcal{F}_n$ for one of the filtrations. This τ will satisfy: $\tau' \subset \tau$ and $\text{diam}(\tau) < 2\text{diam}(\tau')$ (and hence $\tilde{\beta}(\tau) \geq c_0\tilde{\beta}(\tau')$). This mapping can be made to be injective.

Proof. We will now construct $\leq (2CJ)$ filtrations: $\{\mathcal{F}^j\}_{j=1}^k, k \leq (2CJ)$. A single filtration will be denoted by $\mathcal{F} = \{\mathcal{F}_i\}_0^{\infty}$ (omitting the superscript) and will have the properties required for the previous lemma.

We will use an order on the arcs given by the flow along a universally chosen parameterization of Γ .

First, divide each \mathcal{F}_n^0 into C collections such that at every level n each collection is composed of disjoint arcs. Then divide each of these into 2 collections, such that within each collection any two arcs at the same level n will be separated by an arc of diameter at least 2^{-n} .

Select a single collection from each level n and call it \mathcal{F}_n^1 . Do NOT confuse this superscript with the enumeration of the different final filtrations; this is merely a step in the construction of a single filtration, \mathcal{F} .

Now, ‘dilute’ each $\{\mathcal{F}_n^1\}_{n=0}^{\infty}$ by skipping J generations at a time, multiplying the number of collections by J . Call a single collection $\mathcal{F}^2 = \{\mathcal{F}_n^2\}_{n=0}^{\infty}$ (renumbering $Jn \rightarrow n$).

We now want to turn $\{\mathcal{F}_n^2\}_{n=0}^{\infty}$ to a nested family. Consider $\tau^2 \in \mathcal{F}_n^2$. Set

$$\begin{aligned} \tau_0 &= \tau^2 \\ \tau_{k+1} &= \tau_k \cup \left(\bigcup_{\substack{\tau' \in \mathcal{F}_{n+k+1}^2 \\ \tau_k \cap \tau' \neq \emptyset}} \tau' \right) \\ \tau^3 &= \lim_k \tau_k. \end{aligned}$$

Denote by \mathcal{F}^3 the family given by $\tau^2 \rightarrow \tau^3$.

Note that $\tilde{\beta}$ decreases by only a small factor (dependent on J) by $\tau^2 \rightarrow \tau^3$, as the diameter increases by at most a factor of $1 + 2^{-J}$. We have that \mathcal{F}^3 is almost the family which we desire (a filtration) - requirement (4) and the existence part of (1) (see previous lemma) are not yet satisfied.

Suppose $n = 0$. Consider

$$R = \Gamma \setminus \bigcup_{\tau \in \mathcal{F}_n^3} \tau = \cup R_j$$

where $\{R_j\}$ are connected components, ordered by γ . Consider an R_j . Note that $\text{diam}(R_j) > 2^{-n}$. If

$\text{diam}(R_j) > A2^{-n}$ then chop it up into a finite number of connected parts with $\text{diam} \leq A2^{-n}$. Rename them to be $\{R_j\}$. Go over them in order. If an element R_j has $\text{diam}(R_j) < 2^{-n}$ then join it to the following element in $\mathcal{F}_n^3 \cup \{R_j\}$. Now perturb each new ∂R_j so that elements of \mathcal{F}_{n+1}^3 have a father (unique). We call the collection of these new sets \mathcal{F}_n^4 . (We remind the reader this was the all for $n = 0$.) This gives the requirements for the previous lemma for $n = 0$.

Suppose we have the requirements for n and we want to get them for $n + 1$. Consider $R = \Gamma \setminus (\bigcup_{\tau \in \mathcal{F}_{n+1}^3} \tau \cup \bigcup_{\tau \in \mathcal{F}_n^4} \partial \tau)$. As before, we may write $R = \cup R_j$, where the R_j are connected components.

Also, as before, we may subdivide R_j arcs to get arcs of diameter at most $A2^{-n-1}$. We then rejoin them if necessary to adjoining R_j arcs to make sure they are of diameter at least 2^{-n-1} . (Note that an R_j arc must have been of diameter at least 2^{-n-1} before being subdivided.) By perturbing each new ∂R_j we make sure \mathcal{F}_{n+2}^3 have a father (unique). We call the collection of these new sets \mathcal{F}_{n+1}^4 .

We get that \mathcal{F}^4 is the desired filtration \mathcal{F} . Clearly by making different initial choices we get a total of at most $2CJ$ filtrations. \square

Lemma 3.15. *We have (3.2) for $j \in \{0, 1, 2\}$.*

Proof. Fix $j \in \{0, 1, 2\}$. We can fix $\varepsilon_0 > 0$ below, independent of all other constants.

Notice that $\tilde{\beta}(\tau)$ is continuous in the endpoints of τ . Also notice that if $n = -\log(\frac{\text{diam}(2^j Q)}{A})$ and $\tilde{\beta}(\tau) \leq \varepsilon_0$ for $\tau \in \Lambda(Q)$ then $\sharp(\tau \cap X_n) < 2^{j+1}A + 1$.

If $\sharp(\gamma_{2^j Q} \cap X_n) \leq 2^{j+1}A + 1$ set $\tau_Q = \gamma_{2^j Q}$. Otherwise, set τ_Q to be a sub-arc of $\gamma_{2^j Q}$ such that it contains *center*(Q) and $\sharp(\tau_Q \cap X_n) = 2^{j+1}A + 1$.

Using the definition of \mathcal{G}_1 , we get that in both of the above cases

$$\begin{aligned} \tilde{\beta}(\tau_Q) &\gtrsim \varepsilon_2 \beta(Q), \\ \sharp(\tau_Q \cap X_n) &\leq 2^{j+1}A + 1, \\ \text{diam}(\tau_Q) &\sim \text{diam}(Q) \text{ (with constant } A), \text{ and} \\ Q \rightarrow \tau_Q &\text{ is at most } (2^{j+1}A + 1) : 1. \end{aligned}$$

Furthermore,

$$\forall Q \in \mathcal{G}_1 : \sharp\{Q' \in \mathcal{G}_1 : \text{diam}(Q) = \text{diam}(Q'); \tau_{Q'} \cap \tau_Q \neq \emptyset\} \leq (3 \cdot 2^{j+1}A + 3). \quad (3.11)$$

To see this, we use the order on $\{Q' : \text{diam}(Q') = \text{diam}(Q)\}$, given by the parameterization to the centers. Let $\tau_{Q'_1}$ be the largest, and $\tau_{Q'_2}$ the smallest such elements. Then every other such $\tau_{Q'}$ must have the center of Q' contained in the union $\tau_{Q'_1} \cup \tau_Q \cup \tau_{Q'_2}$ which gives (3.11).

For all of that we need ε_0 fixed sufficiently small. We now use the previous lemmas: first we use Lemma 3.14 and then Lemma 3.11.

We have

$$\sum_{Q \in \mathcal{G}_1^j} \beta(Q)^2 \text{diam}(Q) \lesssim \sum_{Q \in \mathcal{G}_1^j} \tilde{\beta}(\tau_Q)^2 \text{diam}(Q) \lesssim \sum_{i=1}^C \sum_{\tau \in \mathcal{F}^i} \tilde{\beta}(\tau)^2 \text{diam}(\tau) \lesssim \ell(\gamma) \lesssim \mathcal{H}^1(\Gamma).$$

□

Remark 3.16. *Note that if one follows the computation one gets that (3.2) is satisfied with constant $\sim \frac{1}{\varepsilon_2} A^{\frac{5}{2}} \log(A)$. We will have $\varepsilon_2 \sim \frac{1}{A}$ and so we get $\sim A^{\frac{9}{2}} \log(A)$.*

Lemma 3.17. *We have (3.3).*

Proof. Consider $Q \in \mathcal{G}_3^0 \cap \mathcal{G}_3^1 \cap \mathcal{G}_3^2$. From the definition of \mathcal{G}_3^0 we have the existence of $\xi_Q^0 \in \Lambda_Q \setminus S_Q$. We consider two cases. If $\xi_Q^0 \in \mathcal{N}_{\frac{1}{10}\text{diam}(Q)}(\gamma_Q)$, set $\tau_Q = \xi_Q^0$. If $\xi_Q^0 \notin \mathcal{N}_{\frac{1}{10}\text{diam}(Q)}(\gamma_Q)$ then denote by ξ_Q^1 the extension of ξ_Q^0 to an element of Λ_{2Q} . Set $\tau_Q = \xi_Q^1$. In both cases we get

$$\text{diam}(\tau_Q) \geq \frac{1}{2} \text{diam}(Q),$$

and by the definition of \mathcal{G}_3^0 and \mathcal{G}_3^1 (using the inequality $\beta(2Q) \geq \frac{1}{2}\beta(Q)$) we have

$$\tilde{\beta}(\tau_Q) \gtrsim \varepsilon_2 \beta(Q).$$

(This follows from the fact we may reduce ε_1 in the definition of \mathcal{G}_i^j .) We also have

$$\#\{Q' \in \mathcal{G}_3^0 \cap \mathcal{G}_3^1 \cap \mathcal{G}_3^2 : \text{diam}(Q) = \text{diam}(Q'); \tau_Q \cap \tau_{Q'} \neq \emptyset\} \leq 8A$$

which follows from $\gamma_{4Q} \supset \gamma_Q$ and $\beta_{S_{4Q}}(Q)$ being small. Now we use Lemma 3.14 and then Lemma 3.11 as we did in the proof of the previous lemma.

□

Remark 3.18. *Note that if one follows the computation one gets that (3.3) is satisfied with constant controlled by that of equation (3.2).*

3.3 Almost Flat Arcs

In this subsection we prove (3.4). Throughout this subsection, the reader is urged to consider the example of Γ being a finite union of straight line segments, ignoring any problems that may arise at the end-points of these segments. The proofs simplify somewhat if they are reduced to just this example, however almost all of the ideas will remain!

In [Oki92] Okikiolu proved a corresponding result by allotting for each cube Q a segment seg_Q whose length controlled $\beta(Q)\text{diam}(Q)$ (Okikiolu used the dyadic grid as her multiresolution family). We follow in the same spirit by allotting a density (we use the word 'weight') for every ball. Let us give a vague idea of our plan:

For each Q we will define a density (weight) function w_Q . We will have several families of balls (the number of such families is bounded by some universal constant). Every ball Q will have a core U_Q . Within each family, these cores will have nice nesting properties between different balls.

We will get a constant $q < 1$ such that if $U_{Q_1} \supset U_{Q_2}$ then $w_{Q_1}(x) \leq q \cdot w_{Q_2}(x)$. Hence, within each family, we will get that the sum of the densities at a given point is a geometric sum and hence bounded by a constant. To be slightly more accurate (in this vague setting), the above only happens for almost every point ($d\ell$), which is enough. Furthermore, $\int_Q w_Q d\ell$ is enough to control the Jones beta number of the ball Q (scaled correctly).

We start with a preliminary lemma. We build for each ball Q a *core* $U_Q \subset Q$ such that these *cores* are divided into J families. Within each family they will have a nice 'nesting' structure (see property (4)).

To see the origins of the idea for the statement and proof of Lemma 3.19, see [Dav91] page 93.

Lemma 3.19. *Given $c \leq \frac{1}{4A}$ and $J \geq 10$, there exist J families of connected sets in H such that (denoting a single family by $\{U_n^{c,k}\}_{n=0, k=0}^{n=\infty, k=k_n}$):*

- (1) for every $x \in X_n$ there exists a unique k such that $cQ \subset U_n^{c,k}$ for some family, where $\text{radius}(Q) = A2^{-n}$
- (2) $cA2^{-n} \leq \text{diam}(U_n^{c,k}) \leq (1 + 2^{-J+1})cA2^{-n}$
- (3) if $k \neq k'$ then $U_n^{c,k} \cap U_n^{c,k'} = \emptyset$ as long as they are in the same family. Furthermore, $\text{dist}(U_n^{c,k}, U_n^{c,k'}) \geq 2^{-n-1}$
- (4) if $U_m^{c,k} \cap U_n^{c,k'} \neq \emptyset$, they are in the same family and $m > n$ then $U_m^{c,k} \subset U_n^{c,k'}$.

Proof. Let $Q_n^k = \text{Ball}(x^k, A2^{-n})$ where $x^k \in X_n$ for the proof of this lemma. Then if $k \neq k'$ then $\text{dist}(cQ_n^k, cQ_n^{k'}) > 2^{-n}$. Set

$$\begin{aligned} U_{m,0}^{c,k} &:= cQ_m^k \\ U_{m,i+1}^{c,k} &:= U_{m,i}^{c,k} \cup \bigcup_{cQ_{m+iJ+J}^{k'} \cap U_{m,i}^{c,k} \neq \emptyset} cQ_{m+iJ+J}^{k'} \\ U_m^{c,k} &:= \lim U_{m,i}^{c,k} \end{aligned}$$

The j^{th} family is

$$\{U_m^{c,k} : m \in j + J\mathbb{N}\}.$$

□

Remark 3.20. *When changing c we change only the parental relationship in each family, and when we change J we mix between families.*

Set

$$\begin{aligned} c_0 &:= \frac{1}{64A} \\ U_Q &:= U_n^{c_0,k} \\ U_Q^x &:= U_n^{8c_0,k} \\ U_Q^{xx} &:= U_n^{16c_0,k} \end{aligned}$$

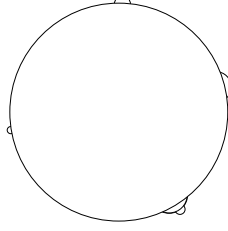


Figure 6: Example of U_Q

where $Q = \text{Ball}(x^k, A2^{-n})$ and $x^k \in X_n$.

The purpose of the following is to prove (3.4) for $j \in \{0, 1, 2\}$. We will have

$$\sum_{Q \in \mathcal{G}_2^j} \beta(Q)^2 \text{diam}(Q) \lesssim \sum_{Q \in \mathcal{G}_2^j} \beta_{S_{2^j Q}} (2^j Q)^2 \text{diam}(2^j Q) \leq C\ell(\gamma). \quad (3.12)$$

All but the last inequality are obvious.

We will show

Proposition 3.21.

$$\sum_{Q \in \mathcal{G}_2} \beta_{S_Q}(Q) \text{diam}(Q) \leq C\mathcal{H}^1(\Gamma). \quad (3.13)$$

Since A is arbitrary, this will give us the last inequality in (3.12) and hence (3.4) for $j \in \{0, 1, 2\}$.

Remark 3.22. *One should note the lack of the power 2.*

Set

$$\begin{aligned} \Delta_1 &= \{Q \in \mathcal{G}_2 : C\beta_{S_Q}(U_Q^x) > \beta_{S_Q}(Q)\} \\ \Delta_2 &= \mathcal{G}_2 \setminus \Delta_1 \end{aligned}$$

Remark 3.23. *We have (taking C large enough) for all $Q \in \Delta_2$*

- (a.) $\beta_{S_Q}(4c_0 Q) < \varepsilon_0$. (If we want ε_0 small independent of A then we need $C \sim A$.)
- (b.) *the existence of $\tau_Q \in S_Q$ such that $\tau_Q \cap U_Q = \emptyset$.*
- (c.) *(b) implies that we have that $\beta_{S_Q}(Q) \gtrsim 1$ for $Q \in \Delta_2$.*

Set

$$\begin{aligned} \Delta_{2,2} &= \{Q \in \Delta_2 : (\Lambda(Q) \setminus S_Q) \cap U_Q^x \neq \emptyset\} \\ \Delta_{2,1} &= \Delta_2 \setminus \Delta_{2,2} \end{aligned}$$

We first show control over

$$\sum_{Q \in \Delta_{2,2}} \beta_{S_Q}(Q) \text{diam}(Q).$$

This is a slightly stronger version of what is done in section subsection 3.2 for the proof of (3.2).

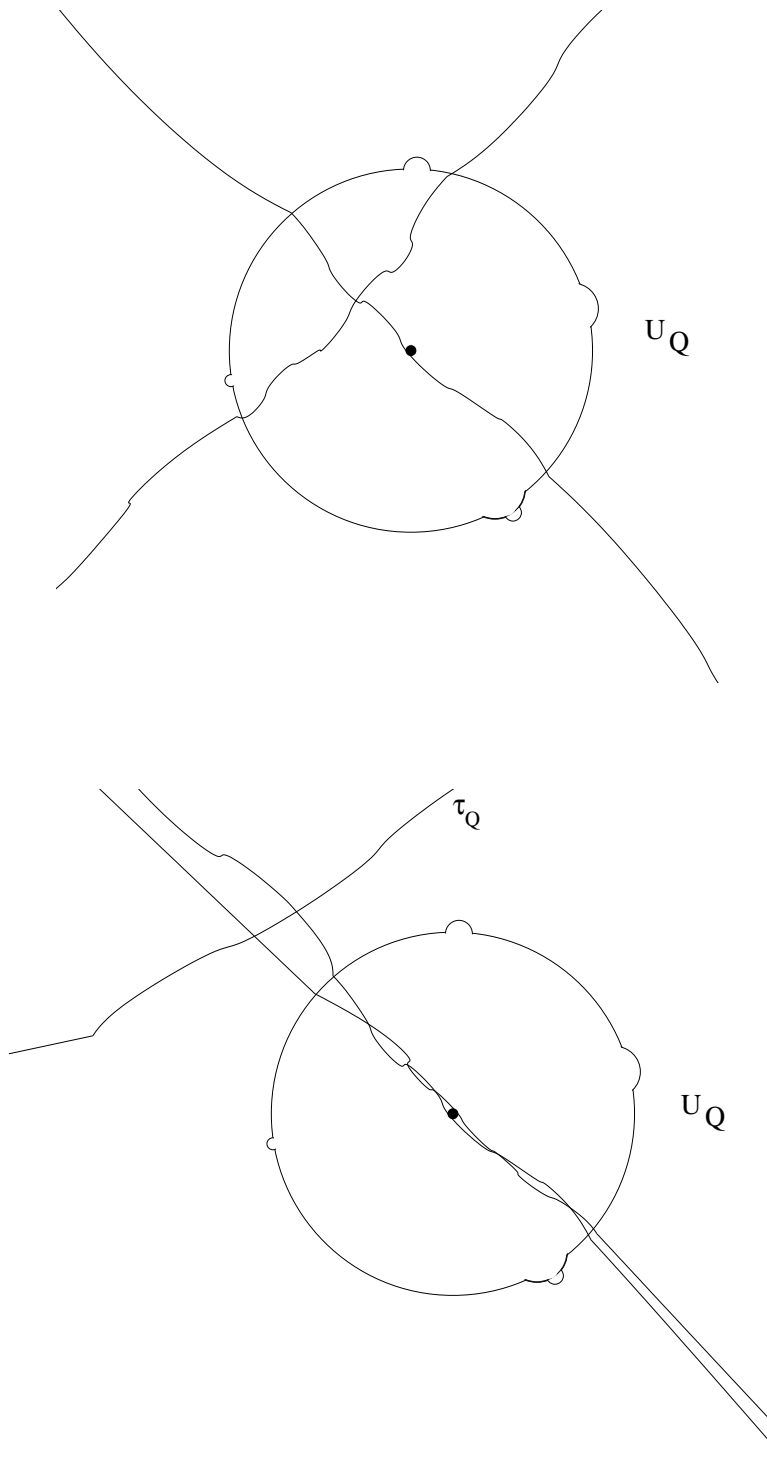


Figure 7: Example of a Δ_1 element (top) and a Δ_2 element (bottom)

Lemma 3.24.

$$\sum_{Q \in \Delta_{2,2}} \beta_{S_Q}(Q) \text{diam}(Q) \lesssim \ell(\gamma).$$

Proof. Let $\xi_Q^0 \in \Lambda(Q) \setminus S_Q$ such that

$$\xi_Q^0 \cap U_Q^x \neq \emptyset.$$

This implies

$$\text{diam}(\xi_Q^0 \cap U_Q^{xx}) > 8c_0 \text{diam}(Q).$$

Suppose

$$\xi_Q^0 \cap U_{Q_i}^x \neq \emptyset$$

for different balls $Q_i \in \Delta_{2,2}$ of the same diameter as $\text{diam}(Q)$. We have

$$\text{diam}(\xi_Q^0 \cap U_{Q_i}^{xx}) > 8c_0 \text{diam}(Q)$$

for all but maybe 2 (on $\partial \xi_Q^0$). If we have an arc of diameter 1 decomposed into n disjoint sub-arcs, such that all but maybe two are of diameter $8c_0$ then $n > 9A$ implies the arc has $\beta \geq \varepsilon$.

Hence we can do as follows: If ξ_Q^0 intersects $\leq 9A$ other $U_{Q_i}^x$'s of the same scale, then set $\xi_Q = \xi_Q^0$. Otherwise take a sub-arc which intersects exactly $9A$ other $U_{Q_i}^x$'s of the same scale (and also intersects our U_Q^x) to be ξ_Q . We have

$$\tilde{\beta}(\xi_Q) \gtrsim \varepsilon_2 \beta(Q),$$

$$\text{diam}(\xi_Q) \sim \text{diam}(Q) \text{ (with constant } A).$$

Furthermore,

$$\#\{Q' \in \Delta_{2,2} : \text{diam}(Q) = \text{diam}(Q'); \xi_{Q'} \cap \xi_Q \neq \emptyset\} \leq 9A.$$

We are now in position to use the lemma's of subsection 3.2. First we use Lemma 3.14 and then Lemma 3.11. This gives us

$$\sum_{Q \in \Delta_{2,2}} \beta_{S_Q}(Q)^2 \text{diam}(Q) \lesssim \ell(\gamma).$$

By Remark 3.23 (c) this gives us the lemma. □

We now turn to deal with $\Delta_{2,1}$ and Δ_1 .

Δ_1

Fix $M \in \mathbb{N}$. Take K such that $1 \leq K < MJ$. Let $\Delta \subset \Delta_1$, such that

$$\Delta := \{Q \in \Delta_1 : U_Q^{xx}, \text{radius}(Q) \in A2^{K+MJ\mathbb{N}}\}.$$

In other words, Δ is obtained by taking a single family (as constructed in Lemma 3.19) and *thinning* it by taking every M^{th} element (starting at some offset).

Suppose $\Delta' \subset \Delta$ is a family of balls, such that

$$\forall Q \in \Delta' : 2^{-M} \leq \beta(U_Q^x) < 2^{-M+1}.$$

Consider $Q \in \Delta'$. Write

$$U_Q^{xx} \cap \Gamma = \left(\bigcup_i U_{Q^i}^{xx} \cap \Gamma \right) \cup R_Q \tag{3.14}$$

where $U_{Q^i}^{xx}$ is maximal in U_Q^{xx} , such that $Q^i \in \Delta'$ and $R_Q = \Gamma \cap U_Q^{xx} \setminus \bigcup_i U_{Q^i}^{xx}$. (By $U_{Q^i}^{xx}$ ‘maximal’ in U_Q^{xx} we mean that there does not exist $Q' \in \Delta'$ such that $U_{Q'}^{xx} \subset U_{Q^i}^{xx} \subset U_Q^{xx}$.)

Lemma 3.25. *Suppose Δ' is as above. Then*

$$\sum_{Q \in \Delta'} \beta(Q) \text{diam}(Q) \leq C_4 2^{-M} \ell(\gamma).$$

Proof. We will construct weights that satisfy (i), (ii) and (iii):

- (i) $\int_Q w_Q d\ell > C_1 \text{diam}(Q)$.
- (ii) for almost every $x_0 \in \Gamma$, $\sum_{Q \in \Delta'} w_Q(x) < C_2$.
- (iii) $\text{supp}(w_Q) = U_Q^{xx} \cap \Gamma$.

Consider $Q \in \Delta'$. We construct w_Q as a martingale. We will write $w_Q(V)$ for $\int_V w_Q d\ell$ for any (measurable) set V .

Set

$$w_Q(U_Q^{xx}) := \text{diam}(U_Q^{xx}).$$

Given $w_Q(U_{Q^i}^{xx})$, where

$$U_{Q^i}^{xx} \cap \Gamma = \left(\bigcup U_{Q^j}^{xx} \cap \Gamma \right) \cup R_{Q^i}$$

as in equation (3.14), then

$$w_Q(R_{Q^i}) := \frac{w_Q(U_{Q^i}^{xx})}{s'} \ell(R_{Q^i})$$

and

$$w_Q(U_{Q^i}^{xx}) := \frac{w_Q(U_{Q^i}^{xx})}{s'} \text{diam}(U_{Q^i}^{xx})$$

where

$$s' := \ell(R_{Q'}) + \sum_j \text{diam}(U_{Q^j}^{xx}).$$

Note that if we have an arc $\xi \subset \hat{\xi} \cap U_{Q'}^{xx}$ where $\hat{\xi} \in \Lambda(Q')$ then

$$\text{diam}(\xi) \leq \ell(R_{Q'} \cap \xi) + \sum_{U_{Q^j}^{xx} \cap \xi \neq \emptyset} \text{diam}(U_{Q^j}^{xx}).$$

Also note that

$$s' \leq (1 + 2^{-J+1})\ell(\Gamma \cap U_{Q'}^{xx}) < \infty$$

by considering $\gamma_{Q^j} \cap U_{Q^j}^{xx}$. Now,

Step 1:

There exists a universal constant $q < 1$ such that

$$\frac{\text{diam}(U_{Q'}^{xx})}{s'} \leq q.$$

To see this, notice we have an arc $\xi^0 \subset S_{Q'}$ such that

$$\beta_{\xi^0 \cup \gamma_{Q'}}(U_{Q'}^x) > \frac{1}{2}\beta(U_{Q'}^x)$$

Let $\eta_{Q'}$ be a largest connected component of $\gamma_{Q'} \cap U_{Q'}^{xx}$. Let ξ be the largest connected component of $\xi^0 \cap U_{Q'}^{xx}$. By considering both ξ and $\eta_{Q'}$ we get

$$\begin{aligned} \frac{\text{diam}(U_{Q'}^{xx})}{s'} &= \frac{\text{diam}(U_{Q'}^{xx})}{\ell(R_{Q'}) + \sum_j \text{diam}(U_{Q^j}^{xx})} \\ &\leq \frac{\text{diam}(U_{Q'}^{xx})}{\ell(R_{Q'}) + \sum_{U_{Q^j}^{xx} \cap \eta_{Q'} \neq \emptyset} \text{diam}(U_{Q^j}^{xx}) + \sum_{\substack{U_{Q^j}^{xx} \cap \xi \neq \emptyset \\ U_{Q^j}^{xx} \cap \eta_{Q'} = \emptyset}} \text{diam}(U_{Q^j}^{xx})} \\ &\leq \frac{(1 + 2^{-J+1})16c_0 \text{diam}(Q')}{\text{diam}(\eta_{Q'}) + \ell(R_{Q'} \setminus \eta_{Q'}) + \sum_{\substack{U_{Q^j}^{xx} \cap \xi \neq \emptyset \\ U_{Q^j}^{xx} \cap \eta_{Q'} = \emptyset}} \text{diam}(U_{Q^j}^{xx})} \\ &\leq \frac{(1 + 2^{-J+1})16c_0 \text{diam}(Q')}{16c_0 \text{diam}(Q') + \frac{1}{10}c_0 \text{diam}(Q')} \\ &= \frac{(1 + 2^{-J+1})}{1 + \frac{1}{160}} \\ &\leq q. \end{aligned}$$

where $q < 1$ is a universal constant. (As we may impose $J > 10$.)

Step 2:

We now have

$$\begin{aligned} \frac{w_Q(U_{Q^{j^*}}^{xx})}{\text{diam}(U_{Q^{j^*}}^{xx})} &= \frac{w_Q(U_{Q'}^{xx})}{s'} \\ &= \frac{w_Q(U_{Q'}^{xx})}{\text{diam}(U_{Q'}^{xx})} \frac{\text{diam}(U_{Q'}^{xx})}{s'} \\ &\leq q \frac{w_Q(U_{Q'}^{xx})}{\text{diam}(U_{Q'}^{xx})} \end{aligned}$$

where $q < 1$ is a universal constant.

Step 3:

We observe that *Step 2* gave us more. Suppose now that $x \in U_{Q_N}^{xx} \subset \dots \subset U_{Q_1}^{xx}$. Using *step 2* with

$$Q = Q_1; \quad Q' = Q_n; \quad Q^{j^*} = Q_{n+1}$$

we get:

$$\begin{aligned} \frac{w_{Q_1}(U_{Q_N}^{xx})}{\text{diam}(U_{Q_N}^{xx})} &\leq q \frac{w_{Q_1}(U_{Q_{N-1}}^{xx})}{\text{diam}(U_{Q_{N-1}}^{xx})} \\ &\leq q^2 \frac{w_{Q_1}(U_{Q_{N-2}}^{xx})}{\text{diam}(U_{Q_{N-2}}^{xx})} \leq \dots \leq q^{N-1} \frac{w_{Q_1}(U_{Q_1}^{xx})}{\text{diam}(U_{Q_1}^{xx})} = q^{N-1}. \end{aligned}$$

Hence, we have $w_{Q_1}(x) \leq 2q^{N-1}$, and so we have (ii) as a sum of a geometric series. \square

Corollary 3.26.

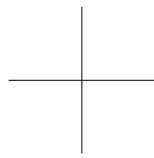
$$\begin{aligned} \sum_{Q \in \Delta_1} \beta_{S_Q}(Q) \text{diam}(Q) &\leq \sum_{M \geq 0} \sum_{\substack{Q \in \Delta_1 \\ 2^{-M-1} < \beta_{S_Q}(Q) \leq 2^{-M}}} \beta_{S_Q}(Q) \text{diam}(Q) \\ &\leq CJ \sum_{M \geq 0} M 2^{-M} \ell(\gamma) \\ &\lesssim \ell(\gamma) \end{aligned}$$

$\Delta_{2,1}$

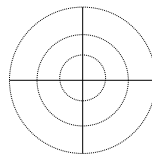
Consider a set $\Delta' \subset \Delta_{2,1}$, such that Δ' contains only balls from a single family as constructed in Lemma 3.19. Consider $Q \in \Delta'$. Write

$$U_Q \cap \Gamma = \left(\bigcup_i U_{Q_i} \cap \Gamma \right) \cup R_Q \tag{3.15}$$

where U_{Q_i} is maximal in U_Q , such that $Q_i \in \Delta'$ and $R_Q = \Gamma \cap U_Q \setminus \bigcup_i U_{Q_i}$. For a given j , We will denote the continuations of τ_{Q_j} and γ_{Q_j} to arcs in $\Lambda(Q)$ by $\widehat{\tau}_{Q_j}$ and $\widehat{\gamma}_{Q_j}^i$ respectively.



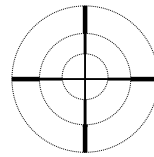
2 crossing segments with



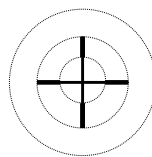
$U_{Q_1} \supset U_{Q_2} \supset U_{Q_3}$.

We have $M = 1$.

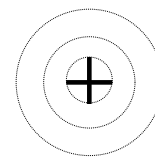
We assume these are the only U_Q 's.



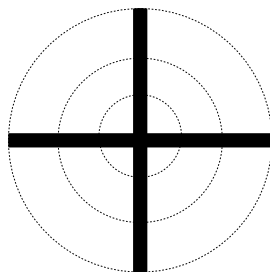
w_{Q_1}



w_{Q_2}

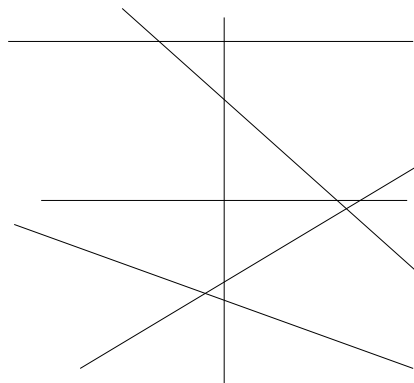


w_{Q_3}

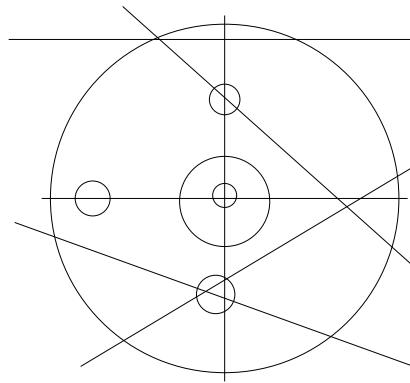


The total weight = $w_{Q_1} + w_{Q_2} + w_{Q_3}$.

Figure 8: An example of weight distribution arising from the martingale. First example



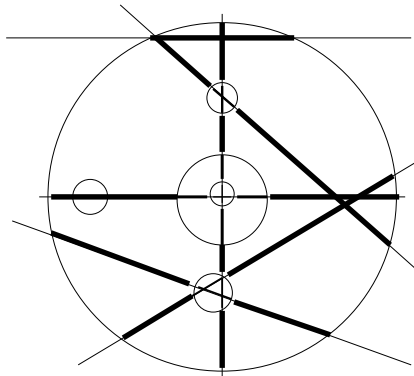
Several crossing segments.



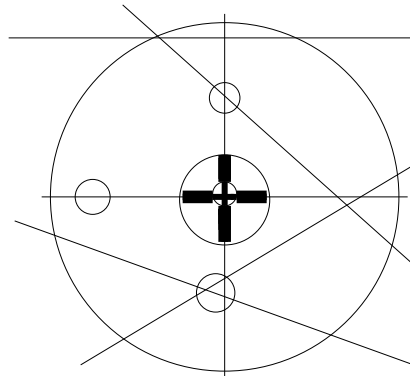
With U_Q 's.

We have $M = 1$.

We assume these are the only U_Q 's.



U_{Q_1}



U_{Q_2}

Figure 9: An example of weight distribution arising from the martingale. Second example

Remark 3.27. A key observation we will use is that if J (from Lemma 3.19) is large enough ($J \sim \log(A)$ suffices) we have

$$\widehat{\tau}_{Q^j} \in S_Q,$$

since otherwise we would have had $Q \in \Delta_{2,2}$. We also have $\widehat{\tau}_{Q^j} \cap 3c_0Q \neq \emptyset$. Combining the two we get $\widehat{\tau}_{Q^j} \cap c_0Q \subset \mathcal{N}_{\varepsilon_0 4c_0 \text{diam}(Q)}(\gamma_Q)$ and $\text{diam}(\widehat{\tau}_{Q^j} \cap U_Q) > (1 - 8\varepsilon_0)\text{diam}(U_Q)$. Similarly, $\widehat{\gamma}_{Q^j}$ has the same properties.

Lemma 3.28. Suppose Δ' is as above. Then

$$\sum_{Q \in \Delta'} \beta(Q) \text{diam}(Q) \leq C_4 \ell(\gamma).$$

Proof. We will construct weights that satisfy (i), (ii) and (iii):

- (i) $\int_Q w_Q d\ell > C_1 \beta(Q) \text{diam}(Q)$
- (ii) for almost every $x \in \Gamma$, $\sum_{Q \in \Delta'} w_Q(x) < C_2$
- (iii) $\text{supp}(w_Q) = U_Q \cap \Gamma$.

Consider $Q \in \Delta'$. We construct w_Q as a martingale. We do so in a similar manner to what is done in Lemma 3.25. We must be more careful here.

Set

$$w_Q(U_Q) := \text{diam}(U_Q).$$

Given $w_Q(U_{Q'})$, where

$$U_{Q'} \cap \Gamma = (\bigcup U_{Q^j} \cap \Gamma) \cup R_{Q'}$$

as in equation (3.15), then

$$w_Q(R_{Q'}) := \frac{w_Q(U_{Q'})}{s'} \ell(R_{Q'}) \cdot 2$$

and

$$w_Q(U_{Q^j}) := \frac{w_Q(U_{Q'})}{s'} \text{diam}(U_{Q^j})$$

where

$$s' := \ell(R_{Q'}) \cdot 2 + \sum_j \text{diam}(U_{Q^j}).$$

Note that

$$s' \leq 2(1 + 2^{-J+1}) \ell(\Gamma \cap U_Q) < \infty$$

by considering $\gamma_{Q^j} \cap U_{Q^j}$. Now,

Step 1:

There exists a universal constant $q < 1$ such that

$$\frac{\text{diam}(U_{Q'})}{s'} \leq q.$$

Proof: Let $\eta = \eta_{Q'}$ be a largest connected component of $\gamma_{Q'} \cap U_{Q'}$. We know

$$(1 + 2^{-J+1})\text{diam}(\eta_{Q'}) \geq \text{diam}(U_{Q'}) \geq \text{diam}(\eta_{Q'}) \geq (1 - 8\varepsilon_0)c_0\text{diam}(Q).$$

Consider $I_\eta := [\eta(\text{initial}), \eta(\text{final})]$. Let $\pi : \Gamma \cap (1 - 4\varepsilon_0)c_0Q' \rightarrow I_\eta$ be the radial (orthogonal) projection. If

$$U_{Q^j} \subset (1 - 2^{-J+1} - 4\varepsilon_0)c_0Q'$$

then

$$\pi(U_{Q^j}) \cap I_1 = \emptyset$$

where

$$I_1 := \{x \in I_\eta : \#\pi^{-1}(x) = 1\}.$$

Hence, if

$$U_{Q^j} \cap (1 - 2^{-J+2} - 4\varepsilon_0)c_0Q' \neq \emptyset,$$

then

$$\pi(U_{Q^j}) \cap I_1 = \emptyset.$$

Now,

$$\begin{aligned} & \frac{\text{diam}(U_{Q'})}{(1 + 2^{-J+1})(1 - 2^{-J+2} - 4\varepsilon_0)^{-1}(1 - 8\varepsilon_0)^{-1}} \\ & \leq \frac{\text{diam}(\eta_{Q'})}{(1 - 2^{-J+2} - 4\varepsilon_0)^{-1}} \\ & \leq \int_{I_\eta \cap (1 - 2^{-J+2} - 4\varepsilon_0)c_0Q'} 1 \\ & \leq \left(\int_{(1 - 2^{-J+2} - 4\varepsilon_0)c_0Q' \cap I_1} 1 + \int_{(1 - 2^{-J+2} - 4\varepsilon_0)c_0Q' \cap (I_\eta \setminus I_1)} 1 \right) \\ & \leq \frac{1}{2} \left(\int_{(1 - 2^{-J+2} - 4\varepsilon_0)c_0Q' \cap I_1} 2 + \int_{(1 - 2^{-J+2} - 4\varepsilon_0)c_0Q' \cap (I_\eta \setminus I_1)} 2 \right) \\ & \leq \frac{1}{2}s'. \end{aligned}$$

Take $q < 1$ such that

$$(1 + 2^{-J+1})(1 - 2^{-J+2} - 4\varepsilon_0)^{-1}(1 - 8\varepsilon_0)^{-1} \frac{1}{2} \leq q$$

by enlarging J and reducing ε_0 if need be.

Step 2 and **Step 3** are as in Lemma 3.25 replacing U^{xx} with U . □

Corollary 3.29.

$$\sum_{Q \in \Delta_{2,1}} \beta_{S_Q}(Q) \text{diam}(Q) \lesssim \ell(\gamma)$$

Putting it all together

We now have

$$\begin{aligned} \sum_{Q \in \mathcal{G}_2} \beta_{S_Q}(Q) \text{diam}(Q) &\leq \sum_{Q \in \Delta_2} \beta_{S_Q}(Q) \text{diam}(Q) + \sum_{Q \in \Delta_1} \beta_{S_Q}(Q) \text{diam}(Q) \\ &\leq \sum_{Q \in \Delta_{2,1}} \beta_{S_Q}(Q) \text{diam}(Q) + \sum_{Q \in \Delta_{2,2}} \beta_{S_Q}(Q) \text{diam}(Q) + \sum_{Q \in \Delta_1} \beta_{S_Q}(Q) \text{diam}(Q) \\ &\lesssim \ell(\gamma) \\ &\lesssim \mathcal{H}^1(\Gamma) \end{aligned}$$

We are done since we have shown equation (3.13) □

Remark 3.30. *If one follows the computations one gets that the total constant here is dominated by $\sim A^{\frac{7}{2}} \log^2 A$ which comes from $\Delta_{2,2}$.*

4 Proof of Theorem 1.5

Farthest Insertion - A Local Version

The following is a variation of what appears in [Jon90]. The theorem can also be deduced as a special case from the work in [Hah05]. We include a proof for completeness.

Let $K \subset H$ and $X^K = \cup X_n^K$ be given. We construct a connected set Γ_0 containing K such that

$$\mathcal{H}^1(\Gamma_0) \lesssim \text{diam}(K) + \sum_{Q \in \hat{\mathcal{G}}^K} \beta_K^2(AQ) \text{diam}(Q). \quad (4.1)$$

By rearranging the constants ($A \rightarrow A^2$) we then get Theorem 1.5. We do this via minor variations of the construction in [Jon90], which in turn is based on ‘Farthest Insertion’ (see [JM02]) applied to the MST (Minimal Spanning Tree) problem. The only innovation here beyond [Jon90] is the treatment of the case $\beta \geq \varepsilon_0$. All other cases are treated in the same spirit.

If the left hand side of the inequality (4.1) is infinite, then any connected set containing K will suffice, and so we assume it is finite. We may assume K is closed without loss of generality. We then get that K is compact. To see this, consider the contrary. We then have an infinite δ -net, $\{a_i\}$. Since $\text{diam}(K) < \infty$, we may assume WLOG that there is no infinite 2δ -net. By perturbing, we may assume WLOG that $X_{\log(\delta)+3}^K \supset \{a_i\}$ and there is no infinite 3δ -net. Hence there is an $x \in X^K$ such that $\text{Ball}(x, 3\delta) \cap X_{\log(\delta)+3}^K$ is infinite. If A is large enough then this is a contradiction to $\sum_{Q \in \mathcal{G}} \beta(AQ)^2 \text{diam}(Q) < \infty$. Hence we will assume K is compact.

We assume an order on X^K , such that all points in X_n^K come before (are smaller than) points in $X_{n+1}^K \setminus X_n^K$. We write $p_1 < p_2 < \dots$ where $X = \{p_1, p_2, \dots\}$. We define

$$O(p_i) := \{p_j : j < i\}$$

and

$$d_i := \text{dist}(p_i, O(p_i)).$$

The construction below uses the following scheme. We inductively construct a sequence of graphs G_i with vertices in X^K . We freely confuse the graph G_i with the set underlying the edges+vertices. We will also have a ‘virtual graph’ H_i which will simply be an addition to G_i which will be used as an accounting tool. Start with the segment $[p_1, p_2]$. Call it G_2 . Now, inductively, obtain G_{i+1} from G_i by connecting p_i . This may involve connecting it to G_{i-1} by modifying an edge that has both endpoints inside $\text{Ball}(p_i, A \cdot d_i)$ or by adding a new edge (the ‘cheaper’ of the two options in a sense which will be clear later). Since changes are done only in $\text{Ball}(p_i, A \cdot d_i)$ we refer to this as a *local version*. In some cases we will perform some preemptive constructions.

We prove that the resulting set has length comparable to the MST (up to a universal constant, which we made no effort to minimize). By the results of the previous sections this will be done if the length is controlled by

$$\text{diam}(K) + \sum_i \beta^2(p_i, A^2 2^{k(i)}) A^2 2^{k(i)}$$

where $p_i \in X_{k(i)} \setminus X_{k(i)-1}$. We will get estimates for $\mathcal{H}^1(G_n \cup H_n)$. We set $H_2 = [p_1, p_1 + A(p_1 - p_2)] \cup [p_2, p_2 + A(p_2 - p_1)]$.

We introduce some more notation:

At the induction stage we will add the point $x_0 = p_{n_1} \in X_k^K \setminus X_{k-1}^K$. We will call $Q = Q_{x_0} = B(x_0, A2^k)$. If $x_0 \in G_{n_1-1}$ (underlying set) then we do nothing. Otherwise, for convenience of notation, denote the nearest point to it in $O(x_0)$ by 0 (the origin). Denote by \mathbb{R} the line containing x_0 and passing through the origin, such that $x_0 > 0$. We will call $\pi_{\mathbb{R}} : H \rightarrow \mathbb{R}$ the orthogonal projection onto \mathbb{R} . We will also write $\Re(z) = \pi_{\mathbb{R}}(z)$, borrowing notation from complex variable. Let

$$W = W_{\mathbb{R}} = \{z \in \frac{1}{2}Q : \frac{-\Re(z)}{\text{dist}(z, \mathbb{R})} \leq \frac{1}{\sqrt{3}}\} \setminus \{0\}.$$

$$W^* = W_{\mathbb{R}}^* = \{z \in \frac{1}{2}Q : \frac{\Re(z)}{\text{dist}(z, \mathbb{R})} \leq \frac{1}{\sqrt{3}}\} \setminus \{0\}.$$

We assume that the following properties hold (we say that they hold *at 0*):

- (P1) If $x_0 \notin G_{n_1-1}$ and $\beta(Q) \leq \varepsilon_0$, and $O(x_0) \cap W \neq \emptyset$, then let $y_1 \in W$ minimize $\|z\|$ on $O(x_0) \cap W$. We have $G_{n_1-1} \cup H_{n_1-1} \supset [0, y_1]$
- (P2) If $x_0 \notin G_{n_1-1}$, $\beta(Q) \leq \varepsilon_0$, and if $O(x_0) \cap W = \emptyset$ then there exists an interval $I \subset (G_{n_1-1} \cup H_{n_1-1}) \cap W$ such that $I = [0, z], z \in \partial(\frac{1}{2}Q)$.
- (P3) If $x_0 \notin G_{n_1-1}$, $\beta(Q) \leq \varepsilon_0$, and if $O(x_0) \cap W^* = \emptyset$ then there exists an interval $I \subset (G_{n_1-1} \cup H_{n_1-1}) \cap W^*$ such that $I = [0, z], z \in \partial(\frac{1}{2}Q)$.

The way in which we will assure the assumptions (P2) and (P3) will be by having segments starting from 0 which have an angle between them which is large enough.

Case 1: $\beta(Q) \geq \varepsilon_0$.

Add segments in order to most efficiently connect x_0 to G_{n_1-1} by connecting to vertices in $O(x_0) \cap Q$. This can involve adding a segment connecting x_0 to a vertex in $O(x_0) \cap Q$, or modifying a segment in G_{n_1-1} . Further, we make sure (P2) and (P3) are preserved at x_0 by adding to H_{n_1} segments (if necessary) at x_0 . The costs of the addition to $G_{n_1-1} \cup H_{n_1-1}$ is a total of $\lesssim \beta(Q)^2 \text{diam}(Q)$. To

make sure (P1) is preserved we need only consider points in $X_{k+\log(A)}^K$. For each of those we make sure (P1) is preserved by adding the required segment. We may do so for **ALL** points of X_k^K at a total cost of $\lesssim \sum_{\substack{Q=B(x,A2^{-k}) \\ x \in X_{k+\log(A)}^K}} \beta(AQ)^2 \text{diam}(Q)$. This is the key difference with [Jon90], where Jones

used local compactness of \mathbb{R}^d to give bounds on the length added, and was hence able to be more ‘wasteful’ in adding segments.

Case 2: $\beta(Q) < \varepsilon_0$; $O(x_0) \cap W \neq \emptyset$; $O(x_0) \cap W^* \neq \emptyset$.

Let $y_1 \in W$ minimize $\|z\|$ on $O(x_0) \cap W$. By P1 we have $[0, y_1] \subset G_{n_1-1} \cup H_{n_1-1}$. If $[0, y_1] \subset G_{n_1-1}$ we replace it with $[0, x_0], [x_0, y_1]$ in G_{n_1} .

If $[0, y_1] \subset H_{n_1-1}$ then we replace it with $[0, x_0], [x_0, y_1]$, placing the shorter of the above intervals in G_{n_1} and the longer one in H_{n_1} .

We get that (P1), (P2) and (P3) are maintained. By the Pythagorean theorem (and a first order approximation) we get that the length added to $G_{n_1-1} \cup H_{n_1-1}$ is bounded by $C\beta(Q)^2 \text{diam}(Q)$.

Case 3: $\beta(Q) < \varepsilon_0$; $O(x_0) \cap W \neq \emptyset$; $O(x_0) \cap W^* = \emptyset$.

This corresponds to Cases 3 and 4 in [Jon90]. Let $y_1 \in W$ minimize $\|z\|$ on $O(x_0) \cap W$. If $[0, y_1] \subset G_{n_1-1}$ then replace it with $[0, x_0], [x_0, y_1]$ in G_{n_1} .

If $[0, y_1] \subset H_{n_1-1}$ then replace it with $[0, x_0], [x_0, y_1]$, placing the shorter of the above intervals in G_{n_1} and the longer one in H_{n_1} .

(P1), (P2) and (P3) are maintained at x_0 since $\beta < \varepsilon_0$.

We now make sure they are maintained at 0 and y_1 . Denote by z_1 the point maximizing $\|z\|$ in $(\{0\} \cup W^*) \cap X_{k+\log(A)}^K$. Since $W^* \cap O(x_0) = \emptyset$ we have $\|z_1\| \leq 2^{-k+1}$. Let $z_1, \dots, z_N = y_1$ be the points in $\text{Ball}(0, \|y_1\|) \cap X_{k+\log(A)}^K$, ordered by increasing ‘Real’ value $\Re(\cdot)$. Add on to G_n the segments $[z_1, z_2], \dots, [z_{N-1}, z_N]$.

If $\|z_1\| \neq 0$ add on to H_n the segment $[z_1, 2^k \frac{z_1}{\|z_1\|}]$. Otherwise, add on to H_n the segment $[0, -2^{-k}]$.

If y_1 also maximizes $\|z\|$ on $O(x_0) \cap W$ then do a similar (symmetric) construction near y_1 . Otherwise do nothing there.

The point is that the segments we have added will give us (P1), (P2), and (P3) $\log(A)$ scales into the future, and until then we have done all constructions needed.

We now need to account for the length we have added. (P2) assures us a line segment in the form of $[0, z] \subset G_{n_1-1} \cup H_{n_1-1}$ where $\|z\| = A2^{-k-1}, z \in Q$. Set $I_Q = [A2^{-k-4}, A2^{-k-3}]$. I_Q will not be altered (or moved) at any future stage (as it is far from K and deep inside W^*). Hence $I_Q \subset \bigcap_{k=0}^{\infty} (G_{n_1-1+k} \cup H_{n_1-1+k})$. The length that we added is bounded by

$$C\beta^2(Q) \text{diam}(Q) + 2 \cdot 2^{-k} + 3 \cdot 2^{-k+1} \left(\frac{1}{A} + 2\varepsilon_0 A \right) A \leq \frac{100}{A} \mathcal{H}^1(I_Q)$$

by reducing ε_0 . We also note that $x \in I_Q$ will be used as such at most once.

Case 4: $\beta(Q) < \varepsilon_0$; $O(x_0) \cap W = \emptyset$; $O(x_0) \cap W^* \neq \emptyset$

This corresponds to Case 5 in [Jon90], and is similar in accounting to Case 3 above. Let $y_{-1} \in W$ minimize $\|z\|$ on $O(x_0) \cap W^*$. We have $\|x_0\| \leq \|y_{-1}\|$. Let z_1, \dots, z_N be the points in $(\{0\} \cup W) \cap X_{k+\log(A)}^K$, ordered by increasing ‘Real’ value $\Re(\cdot)$. Add on to G_n the segments $[z_1, z_2], \dots, [z_{N-1}, z_N]$. Add on to H_n the segment $[z_N, 2z_N]$.

Case 5: $\beta(Q) < \varepsilon_0$; $O(x_0) \cap W = \emptyset$; $O(x_0) \cap W^* = \emptyset$

This corresponds to Case 6 in [Jon90], and is similar in accounting to Case 3 and 4 above. We have $\|x_0\| \leq 2^{-k+1}$. Let z_1, \dots, z_N be the points in $\text{Ball}(0, 2^{-k+1}) \cap X_{k+\log(A)}^K$, ordered by increasing ‘Real’ value $\Re(\cdot)$. Add on to G_n the segments $[z_N, z_{N-1}], \dots, [z_2, z_1]$. Add on to H_n the segment $[z_N, 2z_N]$. If $\|z_1\| \neq 0$ add on to H_n the segment $[z_1, 2^k \frac{z_1}{\|z_1\|}]$. Otherwise, add on to H_n the segment $[0, -2^{-k}]$.

This concludes all the cases.

Inductively we get:

$$\begin{aligned}
& \mathcal{H}^1(G_n \cup H_n) - \mathcal{H}^1(G_2 \cup H_2) \\
& \leq \sum_{x \in O(p_n)} [C\beta(Q_x)^2 \text{diam}(Q_x) + \frac{100}{A} \mathcal{H}^1(I_{Q_x})] + C \log(A) \sum_{Q \in \mathcal{G}} \beta(AQ)^2 \text{diam}(Q) \\
& \leq C \sum_{Q \in \mathcal{G}} \beta(Q)^2 \text{diam}(Q) + C \sum_{Q \in \mathcal{G}} \beta(AQ)^2 \text{diam}(Q) + \sum_{Q=Q_x: x \in O(p_n)} \frac{100}{A} \mathcal{H}^1(I_Q) \\
& \leq C \sum_{Q \in \mathcal{G}} \beta(Q)^2 \text{diam}(Q) + C \sum_{Q \in \mathcal{G}} \beta(AQ)^2 \text{diam}(Q) + \frac{100}{A} \mathcal{H}^1(G_n \cup H_n) \\
& \leq C \sum_{Q \in \mathcal{G}} \beta(AQ)^2 \text{diam}(Q) + \frac{100}{A} \mathcal{H}^1(G_n \cup H_n).
\end{aligned}$$

Hence

$$\mathcal{H}^1(G_n) \leq \mathcal{H}^1(G_n \cup H_n) \lesssim \sum_{Q \in \mathcal{G}} \beta(AQ)^2 \text{diam}(Q) + \text{diam}(K)$$

by choosing A large enough. Denote such a choice by A_0 .

Set

$$E := \{x \in H : x = tx_1 + (1-t)x_2, x_i \in K', -A \leq t \leq A\}.$$

We get E is compact by Lemma 3.6. We also have $G_n \subset E$. We use Lemma 3.7 to get $\gamma_n : [0, 1] \rightarrow G_n$ Lipschitz with uniformly bounded Lipschitz norm. Since E is compact we can use Arzela-Ascoli to obtain a limit γ . We have

$$K^{\text{closure}} \subset \text{Image}(\gamma) =: \Gamma_0$$

and we get the desired estimate on $\mathcal{H}^1(\Gamma_0)$.

□

Note that the exact same computation works for $G_n \cup H_n$.

Remark 4.1. One may take $X_1^K := \{p_1, p_2\}$ with $\text{dist}(p_1, p_2) = \text{diam}(K)$ and inductively or $i > 2$, p_i maximizing d_i , and define $X_n^K := \{p_i : d_i < 2^{-n}\}$ and take the order given by the induction above

in each X_n^K . Then aside from the preemptive constructions appearing in Cases 3-5, our algorithm gives a local version of the **Farthest Insertion** algorithm. Note however, that these preemptive constructions are exactly the constructions that would have appeared under a local version of the Farthest Insertion algorithm, by which we mean connecting p_n in the most efficient way to vertices in $O(p_n) \cap Q_{p_n}$. See [JM02] for the standard version.

5 Appendix

5.1 Proofs of Point-Set Topology Lemmas

Lemma 3.4 Assume Γ is connected. Then $\mathcal{H}^1(\Gamma) = \mathcal{H}^1(\Gamma^{closure})$.

Proof. First note that since $\mathcal{H}^1(\Gamma) \leq \mathcal{H}^1(\Gamma^{closure})$ we need only concern ourselves with the case $\mathcal{H}^1(\Gamma) < \infty$.

Let $\varepsilon > 0$ and $\delta > 0$ be given. By the definition of Hausdorff measure, there exists sets $\{E_i\}_0^\infty$ such that $\sum \text{diam}(E_i) \leq \mathcal{H}^1(\Gamma)(1 + \varepsilon)$ and $\text{diam}(E_i) \leq \delta$. Without loss of generality we may assume the sets E_i are convex and hence connected. We may also assume without loss of generality that they are open (by first taking sets that have diameters summing up to $\mathcal{H}^1(\Gamma)(1 + \frac{1}{2}\varepsilon)$ and then taking small neighborhoods of them). One may construct from them families $\{G_j\}_0^\infty$ and $\{B_j\}_0^\infty$ such that

- (i) $\delta \leq \text{diam}(G_j) < 2\delta$ and $\text{diam}(B_j) < \delta$
- (ii) both G_j and B_j are connected
- (iii) both G_j and B_j are unions of some sets E_i
- (iv) $\bigcup G_j \cup \bigcup B_j = \bigcup E_i$
- (v) $B_j \cap B_k = \emptyset$ for all $j \neq k$
- (vi) if $B_j \cap G_k \neq \emptyset$ then $\text{diam}(B_j \cup G_k) \geq 2\delta$.

This can be done by inductively going over the sets E_i and joining them whenever possible. Since Γ is connected, every set B_j intersects some set G_i (possibly more than one). Denote a choice of such a set by $G_{i(j)}$. Set $F_i = G_i \cup \bigcup_{\{j:i(j)=i\}} B_j$. We get by the triangle inequality

- (i) $\delta \leq \text{diam}(F_i) < 4\delta$
- (ii) $\sum \text{diam} F_i \leq \sum \text{diam}(E_i) \leq \mathcal{H}^1(\Gamma)(1 + \varepsilon)$

We conclude that there is a finite number of sets F_i and we may consider a $\frac{1}{2}\varepsilon\delta$ neighborhood of them, F'_i . We have

$$\bigcup F'_i \supset \Gamma^{closure}$$

We also have

$$\text{diam}(F'_i) \leq \text{diam}(F_i) + \varepsilon\delta \leq \text{diam}(F_i)(1 + \varepsilon)$$

and so

$$\sum \text{diam}(F'_i) \leq \mathcal{H}^1(\Gamma)(1 + \varepsilon)^2$$

We conclude that $\mathcal{H}^1(\Gamma^{closure}) \leq \mathcal{H}^1(\Gamma)$ and so we have equality as desired. \square

Lemma 3.5 Assume $\Gamma \subset H$ is a closed connected set with $\mathcal{H}^1(\Gamma) < \infty$. Then Γ is compact.

Proof. Assume Γ is not compact. Hence for arbitrarily small δ we can obtain an infinite δ -net for Γ : $\{a_n^\delta\}$. By connectedness we have $\mathcal{H}^1(\Gamma \cap \text{Ball}(a_n^\delta, \frac{1}{3}\delta)) \geq \frac{1}{3}\delta$. We also have $\text{Ball}(a_n^\delta, \frac{1}{3}\delta)$ are disjoint, which contradicts $\mathcal{H}^1(\Gamma) < \infty$. \square

Lemma 3.6 Let $C_1, C_2 > 0$ be given. Given a compact connected set $\Gamma \subset H$ the set $E := \{x \in H : x = tx_1 + (1-t)x_2, x_i \in \Gamma, -C_1 \leq t \leq C_2\}$ is compact.

Proof. Suppose $\{x^i\} \subset E$ is a sequence. We can write $x^i = t^i x_1^i + (1-t^i)x_2^i$ as in the definition of E . By the compactness of Γ we have i_k such that

$x_1^{i_k} \rightarrow x_1$. By compactness of Γ again, $x_2^{i_k} \rightarrow x_2$. By compactness of $[-C_1, C_2]$ we have $t^{i_k} \rightarrow t$. $x_1, x_2 \in \Gamma, t \in [-C_1, C_2]$. Hence $x^{i_k} \rightarrow tx_1 + (1-t)x_2 \in E$. \square

One can find an \mathbb{R}^d version of the following lemma in [DS93] with the Lipschitz norm depending on d . The following proof is a modification of the proof given there, which gives a result independent of d .

Lemma 3.7 Let $\Gamma \subset H$ be a compact connected set of finite length. Then we have a Lipschitz function $\gamma : [0, 1] \rightarrow H$ such that $\text{Image}(\gamma) = \Gamma$ and $\|\gamma\|_{Lip} \leq 32\mathcal{H}^1(\Gamma)$

Proof. We use a well known result from graph theory:

If G is a connected graph with finitely many edges, then there is a path that traverses each edge of G exactly twice (once in each direction). This result is easily seen by induction on the number of edges.

For $n \geq 0$, let $X_n = X_n^\Gamma$ (i.e. take $X_n \subset \Gamma$ a 2^{-n} -net such that $X_n \subset X_{n+1}$).

We want to get a connected set E_n . We do this by adding line segments inductively. Set $E_n^0 = X_n$. We get E_n^{i+1} from E_n^i by adding a line segment between points $x_1, x_2 \in X_n$ such that $\text{dist}(x_1, x_2) < 2^{-n+3}$ and they are not yet in the same connected component of E_n^i . If there are no two such points we stop and call the resulting set E_n . Let G_n be the obvious abstract graph associated to E_n . If G_n is not connected then $\text{Vertex}(G_n) = A \cup B$ with $\text{dist}(A, B) \geq 2^{-n+2}$ and A separated from B . By the construction of E_n and X_n we have that $\text{dist}(\mathcal{N}_{2^{-n}}(A), \mathcal{N}_{2^{-n}}(B)) \geq 2^{-n}$ and $\Gamma \subset \mathcal{N}_{2^{-n}}(A) \cup \mathcal{N}_{2^{-n}}(B)$. This is a contradiction to Γ being connected. Hence G_n is connected.

Note that $\mathcal{H}^1(E_n) \leq \#(X_n)2^{-n+3} \leq 16\mathcal{H}^1(\Gamma)$, where the final inequality follows from the fact that the balls $\{B(x, 2^{-n-1}) : x \in X_n\}$ are disjoint.

We can thus parameterize E_n by a Lipschitz curve of $\gamma_n : [0, 1] \rightarrow H$. The image of this parameterization is in E as defined in the previous lemma. By Arzela-Ascoli we have a subsequence converging to γ . We have that $\text{Image}(\gamma) = \Gamma$ by say

$$\sup_{x \in E_n} \text{dist}(x, \Gamma) + \sup_{y \in \Gamma} \text{dist}(E_n, y) \leq 3 \cdot 2^{-n} + 2^{-n} = 4 \cdot 2^{-n}$$

and a triangle inequality. \square

Hence, we also have

Corollary 3.8 Let $\Gamma \subset H$ be a compact connected set of finite length. Then we have a Lipschitz function $\gamma : \mathbb{T} \rightarrow H$ such that $\text{Image}(\gamma) = \Gamma$ and $\|\gamma\|_{Lip} \leq 32\mathcal{H}^1(\Gamma)$

5.2 Existence of the MST

Lemma 5.1. *Assume $K \subset \Gamma_0$, where Γ_0 is the image of an arc-length parameterization γ_0 and $\mathcal{H}^1(\Gamma_0) < \infty$. Then there exists a curve with image $\Gamma_{MST} \supset K$ such that $\mathcal{H}^1(\Gamma) \geq \mathcal{H}^1(\Gamma_{MST}), \forall \Gamma \supset K$*

Proof. We have $E := \{x \in H : x = tx_1 + (1-t)x_2, x_i \in \Gamma_0, 0 \leq t \leq 1\}$ is compact by Lemma 3.6. Consider also $F = K^{\text{closure}}$ which is also compact as a closed subset of a compact set.

Assume that γ_n are such that $\mathcal{H}^1(\Gamma_n)$ decreases to $L = \inf_{\Gamma \supset K} \mathcal{H}^1(\Gamma)$, where Γ_n is the image of γ_n . WLOG we may assume that for all n , γ_n is defined on \mathbb{T} and is Lipschitz with constant $C \leq 32(L+1)$. We will define γ_n^* as follows:

Assume $[a_i, b_i] \subset \mathbb{T}$ is a maximal interval with an interior whose image under γ_n is disjoint from F . Set $\gamma_n^*|_{[a_i, b_i]} = \pi_{\langle \gamma_n(a_i), \gamma_n(b_i) \rangle} \gamma_n$, where $\pi_{\langle \gamma_n(a_i), \gamma_n(b_i) \rangle}$ is projection onto the line going through $\gamma_n(a_i)$ and $\gamma_n(b_i)$. If $\gamma_n(a_i) = \gamma_n(b_i)$ define $\pi_{\langle \gamma_n(a_i), \gamma_n(b_i) \rangle} = \gamma_n(a_i)$.

We get that γ_n^* is continuous and has a derivative a.e. bounded by C (as the set of exit/entry points from E is countable). We also have that the image of every γ_n^* is in E . Now use Arzela-Ascoli to get a limit path γ with image Γ_{MST} (whose name is yet to be justified). WLOG assume that the original sequence γ_n converges to γ . Using Golab's Theorem (for subsets of \mathbb{R}^d) we have

$$L \leq \mathcal{H}^1(\Gamma_{MST}) = \lim_d (\mathcal{H}^1(\pi_d \Gamma_{MST})) \leq \lim_d (\liminf \mathcal{H}^1(\pi_d \Gamma_n)) \leq \lim_d (\liminf \mathcal{H}^1(\Gamma_n)) = L$$

and so $L = \mathcal{H}^1(\Gamma_{MST})$ (π_d is projection onto the first d coordinates). (The first equality above follows for instance by an idea similar to the one of the proof of Lemma 3.4.) \square

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