Abstract—In this paper, we consider a statistical multiplexer fed by heterogeneous ON and OFF sources. We provide closed-form results on the tail behavior of the queue-length distribution for statistical multiplexers, where the tail behavior can be expressed in terms of traffic parameters such as the first and second moments of ON and OFF periods for individual sources. Since our results are of closed-form, they can be used to provide a simple and efficient way to predict the tail behavior and the impact of traffic parameters on performance.

Index Terms—Asynchronous transfer mode, ON and OFF source, queueing analysis, statistical multiplexer, tail distribution.

I. INTRODUCTION

In future asynchronous transfer mode (ATM) networks, various communication services will be provided; and due to the uncertainty of traffic characteristics and the flexibility of ATM, new issues have appeared which need to be solved in order to implement efficient and robust ATM networks. One of the most important issues is to investigate the impact of traffic characteristics on the performance.

A statistical multiplexer is designed to accommodate various traffic generated by multiple sources and to obtain statistical gain by multiplexing those traffic. In heavy traffic conditions, it is well known that the queue-length distribution of the statistical multiplexer exhibits a geometric tail behavior [6], [7], [11]. Therefore, it is important to find a simple way to calculate the coefficient constant \( c \) and the geometric parameter \( z_0 \) [see (2)] in order to obtain a good and efficient approximation for the high percentile of the queue-length distribution. There have been many studies on this issue in literature. Daigle, Lee, and Magalhães [2] studied discrete time queues with phase-dependent arrivals by using inverse fast Fourier transform (IFFT). Sohraby [5] studied a statistical multiplexer with heterogeneous binary Markov sources. He obtained an approximate formula for tail behavior, which is expressed in terms of mean burst size of the source. By the same approach, he studied the tail behavior of a statistical multiplexer with general ON and OFF sources, and proposed a new call admission control policy based on his results [6]. Sohraby [7] studied the tail behavior of statistical multiplexers with Ternary Markov sources. An approximate analysis of a system with train arrivals was studied in Xiong and Bruneel [10], and multiplexers with three-state bursty sources were considered in Steyaert and Bruneel [8]. Xiong and Bruneel [11] gave a simple approach to obtain tight upper bounds for the asymptotic queueing behavior of statistical multiplexers with heterogeneous 2-state Markov modulated Bernoulli processes (MMBP’s). Ishizaki et al. [9] studied the loss probability approximation of a statistical multiplexer and they proposed a new call admission control based on their results.

In this paper, our aim is to derive simple and efficient expressions on the geometric parameter \( z_0 \) [see (2)] for the tail behavior of statistical multiplexers fed by heterogeneous ON and OFF sources. Starting from a well-known approximate expression [given in (5)] on the geometric parameter, we show that \( z_0 \) is of closed form, which is a function of the first and second moments of ON and OFF periods for individual sources. Since our results are simple, they can be used to provide an efficient way to predict the tail behavior of a statistical multiplexer and the impact of traffic parameters on performance.

The organization of this paper is as follows. In Section II, we describe a queueing model for a statistical multiplexer and give an approximate formula for the geometric tail behavior. In Section III, we consider various ON and OFF source models and derive our main results on the geometric tail behavior. Specifically, we consider a source with phase type ON and OFF periods in Section III-A, a source with general ON and OFF periods in Section III-B, and an ON and OFF source with batch arrivals during ON periods in Section III-C. We give some numerical results in Section IV and conclude in Section V.

II. MATHEMATICAL MODEL DESCRIPTION

In this paper, we consider a statistical multiplexer in ATM networks, which accommodates cells from \( N \) sources. All links are synchronized on a time slot basis, and one slot is needed to transmit a cell. The cell arrivals from each source is characterized by alternating independent ON and OFF periods. For source \( m \), let \( f_{n+1}(m) = j \) if the previous slot is in an OFF period, and \( f_{n+1}(m) = i \) if the source is in an ON period during slot \( n \). Let \( A_{k_{ij}}^{(m)} \) be the conditional probability that there are \( k \) arrivals from source \( m \) during a slot with state \( f_{n+1}^{(m)} = j \), given that the previous slot is in state \( f_n^{(m)} = i \). Let \( A_{k_{ij}}^{(m)} \) be a matrix whose \( (i, j) \)-th component is \( A_{k_{ij}}^{(m)}(z) \) and \( A^{(m)}(z) \) be defined by

\[
A^{(m)}(z) = \sum_{k=0}^{\infty} A_{k_{ij}}^{(m)} z^k.
\]
Let $B(z)$ be the PGM (probability generating matrix) of the superposed arrival process. We have

$$B(z) = A^{(1)}(z) \odot \cdots \odot A^{(N)}(z).$$

We assume that $B(1)$ is irreducible, so that the steady state probability vector $\pi$ of $B(1)$ exists. Then $\pi$ satisfies

$$\pi B(1) = \pi, \quad \pi e = 1$$

where $e$ is the column vector all of whose elements are equal to 1. Let $\rho$ be the traffic intensity of our system, defined by

$$\rho = \pi B'(1)e.$$

We assume that $\rho$ is less than 1 for stability.

Let $X_n$ be the number of cells in the system, and $J_n = (J^{(1)}_n, \ldots, J^{(N)}_n)$ denote the state of the superposed arrival process at slot $n$. Then $(X_n, J_n)$ is a Markov chain with a state transition probability matrix of M/G/1 type, given by

$$(B_0 \quad B_1 \quad B_2 \quad B_3 \quad \cdots)$$
$$(B_0 \quad B_1 \quad B_2 \quad B_3 \quad \cdots)$$
$$(0 \quad B_0 \quad B_1 \quad B_2 \quad \cdots)$$
$$(\vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots).$$

In many queueing systems of interest, it is well known that the tail behavior of the queue-length distribution in the steady state can be approximated by a geometric distribution as

$$P\{X = k\} \approx ce^{-\alpha_0}(z)$$  \quad (2)

where $c$ is a constant and $\alpha_0$ is the smallest root outside the unit disc of the equation

$$z = \chi(z).$$  \quad (3)

Here, $\chi(z)$ is the Perron–Frobenius (PF) eigenvalue of $B(z)$ satisfying $\chi(1) = 1$, $\chi'(1) = \rho$. Therefore, it is important to find out a simple way to calculate the constant $c$ and the root $\alpha_0$ in order to obtain a good and efficient approximation for the high percentile of the queue-length distribution. In this paper, we address a simple way to find out the root $\alpha_0$. For the constant $c$, interesting readers may refer to Ishizaki et al. [9] and Xiong and Bruneel [11].

By expanding (3) around $z = 1$, Sohraby [6] showed that $\alpha_0$ can be approximated by

$$\alpha_0 \approx 1 + \frac{2(1-\rho)}{\chi''(1)}$$ \quad (4)

and Neuts [4] showed $\chi''(1)$ satisfies

$$\chi''(1) = \pi B'(1)e - 2\rho^2 + 2\pi B'(1)[I - B(1) + ce\kappa]^{-1}B'(1)e.$$ \quad (5)

Although (5) looks simple, $[I - B(1) + ce\kappa]^{-1}$ in (5) have to be calculated, and in general this needs some computational efforts. So, in this paper we provide a simpler version of (5) which is expressed as a function of the first and second moments of ON and OFF periods for individual sources.

Recall that the PF eigenvalue $\chi(z)$ of the superposed arrival process is the product of PF eigenvalues of individual sources [2]. Let $\chi_i(z)$ be the PF eigenvalue of source $i$, $1 \leq m \leq N$. Then

$$\chi''(1) = \sum_{m=1}^{N} \chi''_{m}(1) + \rho^2 - \sum_{m=1}^{N} \rho_m^2$$

where $\rho_m$ is the traffic intensity offered by source $m$. This means that we have only to find out the second derivatives of PF eigenvalues of individual sources to derive the second derivative of the PF eigenvalue $\chi(z)$ of the superposed arrival process. This is why we pay attention to a single source in the next section.

III. MAIN RESULTS

In this section we first consider a single ON and OFF source which generates one cell per slot during ON periods and no cell during OFF periods. Since we limit our attention to a single source, we omit the superscript $(m)$ of $A^{(m)}(z)$ given in (1) for simplicity from now on. For a single ON and OFF source, we assume that $A(z)$ is divided into submatrices

$$\begin{pmatrix} A_{00} & A_{01} & z \\ A_{10} & A_{11} \end{pmatrix}$$

where $A_{00}$ is an $m_0 \times m_0$ matrix and $A_{11}$ is an $m_1 \times m_1$ matrix. Then $\chi''(1)$ in (5) is reduced to

$$\chi''(1) = -2\rho^2 + 2\pi A'(1)[I - A(1) + ce\kappa]^{-1}A'(1)e.$$ \quad (6)

Here we use $\pi$ to denote the steady-state probability vector of $A(1)$.

Let a column vector $\kappa$ be defined by

$$\kappa = [I - A(1) + ce\kappa]^{-1}A'(1)e$$ \quad (7)

and

$$\kappa = \begin{bmatrix} \kappa_0 \\ \kappa_1 \end{bmatrix}$$

where $\kappa_0$ and $\kappa_1$ are $m_0 \times 1$ and $m_1 \times 1$ column vectors, respectively. For later use, let $\pi$ be decomposed into two row vectors $\pi_0$ and $\pi_1$ such that

$$\pi = [\pi_0, \pi_1]$$

where $\pi_0$ and $\pi_1$ are $1 \times m_0$ and $1 \times m_1$ row vectors, respectively. Then (6) is reduced to

$$\chi''(1) = -2\rho^2 + 2\pi_1 \kappa_1.$$ \quad (8)

By multiplying $\pi$ on both sides of (7), we have

$$\pi \kappa = \rho.$$ \quad (9)

Multiplying $[I - A(1) + ce\kappa]$ on both sides of (7) yields

$$A'(1)e = [I - A(1)]\kappa + \rho e.$$ \quad (10)

Premultiplying

$$[I - A_{00} \quad 0 \\ 0 \quad I - A_{11}]^{-1}$$
on both sides of (10), we have
\[
\begin{bmatrix}
I & -A_0 \\
A_{10} & [I - A_{11}]^{-1}
\end{bmatrix}
\begin{bmatrix}
0 \\
A_{01}
\end{bmatrix}
= \begin{bmatrix}
I \\
-I - A_{11}^{-1}A_{01}
\end{bmatrix} \kappa
\]
\[= \begin{bmatrix}
-I - A_{11}^{-1}A_{01} \\
0
\end{bmatrix} \kappa
\]
\[+ \begin{bmatrix}
-I - A_{11}^{-1} \\
0
\end{bmatrix} [I - A_{11}]^{-1} \rho e.
\]
After some algebraic manipulations, we get
\[
\kappa_0 = \begin{bmatrix}
-I - A_{11}^{-1}A_{01} \\
0
\end{bmatrix} \kappa
\]
\[+ \begin{bmatrix}
-I - A_{11}^{-1} \\
0
\end{bmatrix} [I - A_{11}]^{-1} \rho e.
\] (11)

From (11), we have
\[
\kappa_0 = \begin{bmatrix}
-I - A_{11}^{-1}A_{01} \\
0
\end{bmatrix} \kappa
\]
\[-\rho I - A_{11}^{-1}e
\]
\[\kappa_1 = \begin{bmatrix}
-I - A_{11}^{-1}A_{01} \\
0
\end{bmatrix} \kappa
\]
\[-e + (1 - \rho)[I - A_{11}]^{-1}e.
\] (12)

Starting from (12) and (13), we will derive a simpler version of (5) in the following subsections.

A. Phase Type ON and OFF Periods

In this subsection we assume OFF and ON periods are of phase type with \((\alpha_0, H_0)\) and \((\alpha_1, H_1)\), respectively. Let \(H_0^0\) and \(H_1^1\) be column vectors such that
\[
H_0^0 e + H_0^0 = e, \quad H_1^0 e + H_1^0 = e.
\]
Then, we see that \(A_{00}, A_{01}, A_{02}, A_{10}, A_{11}\) are given by
\[
A_{00} = H_0^0, \quad A_{01} = H_0^0 \alpha_1, \quad A_{02} = H_0^0 \alpha_0, \quad A_{11} = H_1^0.
\]
Multiplying \(\pi_0\) on both sides of (12) gives
\[
\alpha_0 \kappa_0 = \alpha_1 \kappa_1 + 1 - p \rho [I - H_0]^{-1} e
\] (14)
By multiplying \(\pi_0\) on both sides of (12), we get
\[
\pi_0 \kappa_0 = \begin{bmatrix}
1 - \rho \alpha_1 \kappa_1 \\
0
\end{bmatrix} \pi_0 e + \begin{bmatrix}
1 - \rho \alpha_0 \kappa_0 \\
0
\end{bmatrix} [I - H_0]^{-1} e
\] (15)
where we need \(\pi_0 e = 1 - \rho\) and \(\pi_1 e = \rho\). Similarly, by multiplying \(\pi_1\) on both sides of (13), we get
\[
\pi_1 \kappa_1 = \rho \alpha_0 \kappa_0 - \rho + (1 - \rho) \pi_1 [I - H_1]^{-1} e
\] (16)
From (9), adding (15) and (16) yields
\[
\rho = (1 - \rho) \alpha_1 \kappa_1 + \rho \alpha_0 \kappa_0 + 1 - 2 \rho
\]
\[-\rho \pi_0 [I - H_0]^{-1} e + (1 - \rho) \pi_1 [I - H_1]^{-1} e.
\] (17)

By substituting (18) into (16), we have
\[
\pi_1 \kappa_1 = 2 \rho^2 - \rho - (1 - \rho)^2 \alpha_0 [I - H_0]^{-1} e
\]
\[+ \rho^2 \pi_0 [I - H_0]^{-1} e + (1 - \rho)^2 \pi_1 [I - H_1]^{-1} e.
\] (19)

Note that \(\alpha_0 [I - A_{11}]^{-1} e\) is the mean duration of the OFF period, \((\pi_0) / (\pi_0 e) [I - A_{11}]^{-1} e\) is the mean duration of the remaining sojourn time of the OFF period, and \((\pi_1) / (\pi_1 e) [I - A_{11}]^{-1} e\) is the mean duration of the remaining sojourn time of the ON period in the steady state. Let \(M_0 (M_1)\), respectively be the first moment of the OFF (ON) probability period, and \(S_0 (S_1)\), respectively, be the second moment of the OFF (ON) probability period. From renewal theory, we see
\[
\alpha_0 [I - H_0]^{-1} e = M_0
\]
\[\pi_0 [I - H_0]^{-1} e = (1 - \rho) \left( \frac{S_0}{M_0^2} + \frac{1}{2} \right)
\]
\[\pi_1 [I - H_1]^{-1} e = \rho \left( \frac{S_1}{M_1^2} + \frac{1}{2} \right).
\]
Substituting the above equations into (19) yields
\[
\pi_1 \kappa_1 = \frac{3}{2} \rho^2 - \frac{1}{2} \rho - (1 - \rho)^2 M_0
\]
\[+ \rho^2 (1 - \rho) \frac{S_0}{M_0^2} + (1 - \rho)^2 \rho \frac{S_1}{M_1^2}.
\] (20)

By substituting (20) into (8), we finally get a closed-form expression
\[
\chi''(1) = \rho^2 - 2(1 - \rho) \rho^2 M_0
\]
\[+ \rho^2 (1 - \rho) \frac{S_0}{M_0^2} + (1 - \rho)^2 \rho \frac{S_1}{M_1^2}.
\] (21)

Next, we consider some examples for a source with phase type ON and OFF periods.

1) Example 1: A 2-state ON and OFF source [1]
For a 2-state ON and OFF source, when the source is in OFF (ON) state, it remains in the OFF (ON) state with probability \(\delta (\gamma)\) per slot and changes to the ON (OFF) state with probability \(1 - \delta (1 - \gamma)\). So, \(A(z)\) is given by
\[
\begin{bmatrix}
\delta & (1 - \delta) z \\
1 - \gamma & \gamma z
\end{bmatrix}
\]
and a 2-state ON and OFF source is a special case of phase type ON and OFF source where the parameters for phase type distributions are given by \(\alpha_0 = 1, \alpha_1 = 1, H_0 = \delta, \) and \(H_1 = \gamma\). Further, we see that
\[
\rho = \frac{1 - \delta}{2 - \gamma - \delta}
\]
\[M_0 = \frac{1}{1 - \delta}, \quad M_1 = \frac{1}{1 - \gamma}
\]
\[S_0 = \frac{1 + \delta}{(1 - \delta)^2}, \quad S_1 = \frac{1 + \gamma}{(1 - \gamma)^2}.
\]
Using the equations above and (21), we derive \(\chi''(1)\) for a 2-state ON–OFF source.
2) Example 2: A 3-state ON and OFF source [8].

For a 3-state ON and OFF source, we have two types of ON periods, say states 1 and 2, and one type of OFF period. The time durations spent in the ON period \( i \) and in the OFF period are assumed to have geometrically distributed with parameter \( \gamma_i \) and \( \delta \), respectively \((i = 1, 2)\). At the end of each OFF period, the next active period is of type 1 with probability \( p \) and of type 2 with probability \( 1 - p \), and at the end of each ON period, the next period is OFF period. Then \( A(x) \) is given by

\[
\begin{bmatrix}
\delta & (1 - \delta)p & (1 - \delta)(1 - p) \\
1 - \gamma_1 & \gamma_1 & 0 \\
1 - \gamma_2 & 0 & \gamma_2 \\
\end{bmatrix}
\]

and the parameters for phase-type distributions are given by

\[
\alpha_0 = 1, \quad H_0 = \delta \\
\alpha_1 = (p, 1 - p), \quad H_1 = \begin{bmatrix}
\gamma_1 & 0 \\
0 & \gamma_2 \\
\end{bmatrix}.
\]

Using the above parameters, we have

\[
\rho = \frac{p}{1 - \gamma_1} + \frac{1 - p}{1 - \gamma_2} + \frac{1}{1 - \beta} \\
M_0 = \frac{1}{1 - \delta}, \quad S_0 = \frac{1 + \delta}{(1 - \delta)^2} \\
M_1 = p \frac{1}{1 - \gamma_1} + (1 - p) \frac{1}{1 - \gamma_2} \\
S_1 = p \frac{1 + \gamma_1}{(1 - \gamma_1)^2} + (1 - p) \frac{1 + \gamma_2}{(1 - \gamma_2)^2}.
\]

Using the above equations and (21), we derive \( \chi''(1) \) for a 3-state ON–OFF source.

B. General ON and OFF Periods

In this subsection we assume that ON and OFF periods are distributed with general distributions. We assume that ON and OFF periods are independent and bounded, so the first and second moments of each period exist. Note that our bounded assumption is meaningful in practical situations.

To describe the source model explicitly, we introduce two random variables, \( I_n \) and \( T_n \). \( I_n \) denotes the state of the source at slot \( n \), i.e.,

\[
I_n = \begin{cases}
0, & \text{if the source is in state "OFF"}, \\
1, & \text{if the source is in state "ON"},
\end{cases}
\]

\( T_n \) denotes the elapsed time of the OFF period (ON period, respectively) when \( I_n = 0 \) (\( I_n = 1 \), respectively) at slot \( n \). Then \( \{(I_n; T_n)\} \) is a Markov chain and \( A(z) \) defined in (1) has the same form as given in the first part of Section III.

Let’s start with (12) and (13). Recall that the term \( (I - A(0) - A(b(0))_{ij}) \) is the conditional probability that, given the OFF period starts in state \( (0, i) \), the first state entered when the source changes from OFF to ON is \( (1, j) \). Since \( T_n \) means the elapsed sojourn time of the period, we have

\[
[I - A_{00}]^{-1}A_{01} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{bmatrix}.
\]

Similarly, we have

\[
[I - A_{11}]^{-1}A_{31} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{bmatrix}.
\]

Let \( \kappa_{ij} \) be the first element of \( \kappa_i \). By using (22) and (23), (12) and (13) are reduced to

\[
\kappa_0 = \kappa_{11} e + e - \rho[I - A_{00}]^{-1}e \\
\kappa_1 = \kappa_{00} e - e + (1 - \rho)[I - A_{11}]^{-1}e.
\]

Then, starting from (24) and (25) and following the similar arguments as given in Section III-A, we can easily derive

\[
\chi''(1) = \rho^2 - \rho - 2(1 - \rho)\rho^2 M_0 \\
+ \rho^2(1 - \rho)\frac{S_0}{M_0} + (1 - \rho)^2 \rho^2 \frac{S_1}{M_1}.
\]

Remark: By introducing the coefficients of variation \( C^2_{ON} \) and \( C^2_{OFF} \) for ON and OFF period distributions, respectively, given by

\[
C^2_{ON} = \frac{S_1}{M_1} - 1, \quad C^2_{OFF} = \frac{S_0}{M_0} - 1
\]

(26) can be reduced to

\[
\chi''(1) = (C^2_{ON} + C^2_{OFF})\rho(1 - \rho)^2 M_1 - \rho(1 - \rho).
\]

Sohraby [6] derived the above formula in a different way.

C. A Batch Arrival Source

In the previous two sections we allow only one cell per slot during ON periods, but in this subsection we allow batch arrivals during ON periods. We assume ON and OFF periods are distributed with general distributions as given in Section III-B, and the batch sizes are independent and identically distributed with its PGF \( b(z) \). Using the same notations as in the first part of Section III-A, we see

\[
A(z) = \begin{bmatrix}
A_{00} & A_{01}b(z) \\
A_{10} & A_{11}b(z)
\end{bmatrix}
\]

and

\[
\pi_0[I - A_{00}]^{-1}e = M_0 \\
\pi_0[I - A_{00}]^{-1}e = \left(1 - \frac{\rho}{b'(1)}\right)\left(\frac{S_0}{2M_0} + \frac{1}{2}\right) \\
\pi_1[I - A_{00}]^{-1}e = \left(\frac{\rho}{b'(1)}\right)\left(\frac{S_1}{2M_1} + \frac{1}{2}\right).
\]
Following the same arguments as before, we get $\chi''(1)$ for the ON and OFF source with batch arrivals

$$
\chi''(1) = \frac{b''(1)}{b'(1)} \rho - \rho b'(1) \left( 1 - \frac{\rho}{b'(1)} \right) + \rho b'(1) M_1 \left( 1 - \frac{\rho}{b'(1)} \right)^2 \left( \frac{S_0}{M_0^2} + \frac{S_1}{M_1^2} - 2 \right).
$$

In the next section, we give some numerical results based on our formulas.

IV. NUMERICAL RESULTS

First, we consider a statistical multiplexer fed by $N$ homogeneous sources. Table I gives numerical results for the root $z_0$ obtained by using our formulas and the exact values. Here we use an ON and OFF source given in Example 1 and $N = 10$. Error is measured in percent.

In Table II we consider a superposition of two kinds of 2-state ON and OFF sources. Class 1 consists of homogeneous ON and OFF sources with $\delta = 0.95$ and $\gamma = 0.3$, and class 2 consists of homogeneous ON and OFF sources with $\delta = 0.90$ and $\gamma = 0.1$. In this case, we change the number of sources and the batch size distribution in each class. Let $N$ and $M$ be the numbers of sources in class 1 and class 2, respectively, and $b_k(x)$, $i = 1, 2$ denote the PGF for the batch size distribution of the sources in class $i$. Error is measured in percent.

From Tables I and II, we see that our approximation is accurate, especially in heavy traffic load.

V. CONCLUSIONS

In this paper, we derive closed-form expressions on the geometric tail behavior of statistical multiplexers, which are functions of the first and second moments of ON and OFF periods. Some applications of them in real situations are given in Sohraby [6] and Ishizaki et al. [9]. As mentioned in [6], our expressions can be used to calculate the loss probability approximated by $P\{X > i\} \approx (1/z_0)^i$ for large $i$, and the CAC (connection admission control) based on the approximation can be implemented in real time. However, it is known that this approximation may overestimate the loss probability by several orders of magnitude even under heavy traffic condition. So, a tighter bound should be derived, and many previous works showed that more accurate values for the coefficient $c$ of $1/z_0$ could give better approximation results [9], [11]. In conclusion, since our results are of simple closed form, they can be used to provide a simple and efficient way to predict the tail behavior and the impact of traffic parameters on performance.

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