Homogeneous orthocomplete effect algebras are covered by MV-algebras

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Outline

1. Introduction
2. Basic definitions
3. The condition \((W+)\)
4. Main theorem and its generalizations
Introduction

Joint work of J. Niederle and J. Paseka

Special types of effect algebras $E$ called homogeneous were introduced by G. Jenča.

The aim of our paper is to show that every block of an Archimedean homogeneous effect algebra satisfying the property $(W^+)$ is lattice ordered. Therefore, any Archimedean homogeneous effect algebra satisfying the property $(W^+)$ is covered by MV-algebras.

As a corollary, this yields that every block of a homogeneous orthocomplete effect algebra is a Heyting effect algebra.
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Basic definition – effect algebras

Definition (D. Foulis and M.K. Bennett, 1994)

A partial algebra $(E; \oplus, 0, 1)$ is called an **effect algebra** if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on $E$ which satisfy the following conditions for any $x, y, z \in E$:

(Ei) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

(Eii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$),

(Eiv) if $1 \oplus x$ is defined then $x = 0$.

Example

Let $E = [0, 1] \subseteq \mathbb{R}$. We put $x \oplus y = x + y$ iff $x + y \leq 1$. Hence $\frac{3}{4} \oplus \frac{4}{5}$ does not exist in $E$. 
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A subset $Q \subseteq E$ is called a \textit{sub-effect algebra} of $E$ if

(i) $1 \in Q$

(ii) if out of elements $x, y, z \in E$ with $x \oplus y = z$ two are in $Q$, then $x, y, z \in Q$.

An effect algebra $E$ is called an \textit{orthoalgebra} if $x \oplus x$ exists implies that $x = 0$.

On every effect algebra $E$ the partial order $\leq$ and a partial binary operation $\ominus$ can be introduced as follows:

$x \leq y$ and $y \ominus x = z$ iff $x \oplus z$ is defined and $x \oplus z = y$.

If $E$ with the defined partial order is a (complete) lattice then $(E; \oplus, 0, 1)$ is called a \textit{(complete) lattice effect algebra}.
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If $E$ with the defined partial order is a (complete) lattice then $(E; \oplus, 0, 1)$ is called a (complete) lattice effect algebra.
An effect algebra $E$ satisfies the *Riesz decomposition property* (or \(\text{RDP}\)) if, for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2$, there are $u_1, u_2$ such that $u_1 \leq v_1, u_2 \leq v_2$ and $u = u_1 \oplus u_2$.

(i) Every lattice effect algebra with \(\text{RDP}\) can be organized into an MV-algebra and conversely.

(ii) Every MV-algebra which is an orthoalgebra is a Boolean algebra.

An effect algebra $E$ is called *homogeneous* if, for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2 \leq u'$, there are $u_1, u_2$ such that $u_1 \leq v_1, u_2 \leq v_2$ and $u = u_1 \oplus u_2$.

A subset $B$ of $E$ is called a *block* of $E$ if $B$ is a maximal sub-effect algebra of $E$ with the Riesz decomposition property.
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A subset $B$ of $E$ is called a **block** of $E$ if $B$ is a maximal sub-effect algebra of $E$ with the Riesz decomposition property.
(i) Every orthoalgebra is homogeneous.
(ii) Every lattice effect algebra is homogeneous.
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An element $w$ of an effect algebra $E$ is called *sharp* if $w \wedge w' = 0$.

The well known fact is that in every lattice effect algebra $E$ the subset $S(E) = \{ w \in E \mid w \wedge w' = 0 \}$ is a sub-lattice effect algebra of $E$ being an orthomodular lattice.
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Blocks of $E$ and $S(E)$ - Example 1 - The diamond

$S(E) = \{0, 1\}$ is a Boolean algebra, but $E$ has two blocks, $\{0, a, 1\}$ and $\{0, b, 1\}$.

For any block $B$ of $E$, $S(E) \cap B = \{0, 1\}$ is a block of $S(E)$.

$a \oplus a = b \oplus b = 1$
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Blocks of $E$ and $S(E)$ - Example 2

$S(E) = \{0, a, b, 1\}$ is a Boolean algebra and $E$ has again two blocks. Namely, there are two blocks here, the Boolean algebra $S(E)$ and a 3-element chain $C_3 = \{0, c, 1\}$.

$S(E) \cap C_3 = \{0, 1\}$ is not a block of $S(E)$. 

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$a \oplus b = c \oplus c = 1$
Meager and hypermeager elements

In what follows set

\[ M(E) = \{ x \in E \mid \text{if } v \in S(E) \text{ satisfies } v \leq x \text{ then } v = 0 \}. \]

We also define

\[ HM(E) = \{ x \in E \mid \text{there is } y \in E \text{ such that } x \leq y \text{ and } x \leq y' \} \]

and

\[ UM(E) = \{ x \in E \mid \text{for every } y \in S(E) \text{ such that } x \leq y \text{ it holds } x \leq y \oplus x \}. \]

An element \( x \in M(E) \) is called \textit{meager}, an element \( x \in HM(E) \) is called \textit{hypermeager} and an element \( x \in UM(E) \) is called \textit{ultrameager}. 
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An element \( x \in M(E) \) is called meager, an element \( x \in \text{HM}(E) \) is called hypermeager and an element \( x \in \text{UM}(E) \) is called ultrameager.
Meager and hypermeager elements - Example 3

$1 = 4b = a \oplus b \oplus c$

$3b = a \oplus c$

$UM(E)$

$a \oplus b$

$b \oplus c$

$a$

$c$

$b$

$2b$

$M(E) = HM(E)$

$0$
Meager and hypermeager elements - Example 4

1 = 6b = a \oplus b \oplus c

\begin{align*}
a \oplus b & \quad 5b = a \oplus c & b \oplus c \\
a & \quad b & c
\end{align*}

\begin{align*}
4b & \\
3b & \\
2b & \\
0 & \quad \text{HM}(E) & \quad \text{M}(E)
\end{align*}

\text{UM}(E)
Meager and hypermeager elements

Lemma

Let $E$ be an effect algebra. Then $\text{UM}(E) \subseteq \text{HM}(E) \subseteq M(E)$. Moreover, for all $x \in E$, $x \in \text{HM}(E)$ iff $x \oplus x$ exists and, for all $y \in M(E)$, $y \neq 0$ there is $h \in \text{HM}(E)$, $h \neq 0$ such that $h \leq y$.

Lemma

In every homogeneous effect algebra $E$, $\text{UM}(E) = \text{HM}(E)$.

For an element $x$ of an effect algebra $E$ we write $\text{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ ($n$-times) exists for every positive integer $n$ and we write $\text{ord}(x) = n_x$ if $n_x$ is the greatest positive integer such that $n_x x$ exists in $E$. An effect algebra $E$ is Archimedean if $\text{ord}(x) < \infty$ for all $x \in E$, $x \neq 0$. 
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Orthogonal systems

We say that a finite system $F = (x_k)_{k=1}^n$ of not necessarily different elements of an effect algebra $E$ is \textit{orthogonal} if $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ (written $\bigoplus_{k=1}^n x_k$ or $\bigoplus F$) exists in $E$.

An arbitrary system $G = (x_\kappa)_{\kappa \in H}$ of not necessarily different elements of $E$ is called \textit{orthogonal} if $\bigoplus K$ exists for every finite $K \subseteq G$.

We say that for a orthogonal system $G = (x_\kappa)_{\kappa \in H}$ the element $\bigoplus G$ exists iff $\bigvee \{\bigoplus K \mid K \subseteq G \text{ is finite}\}$ exists in $E$ and then we put $\bigoplus G = \bigvee \{\bigoplus K \mid K \subseteq G \text{ is finite}\}$. We say that $\bigoplus G$ is the \textit{orthogonal sum} of $G$ and $G$ is \textit{orthosummable}. (Here we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (x_\kappa)_{\kappa \in H_1}$). We denote $G^\oplus := \{\bigoplus K \mid K \subseteq G \text{ is finite}\}$.

$E$ is called \textit{orthocomplete} if every orthogonal system is orthosummable.
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We say that a finite system \( F = (x_k)_{k=1}^n \) of not necessarily different elements of an effect algebra \( E \) is orthogonal if \( x_1 \oplus x_2 \oplus \cdots \oplus x_n \) (written \( \bigoplus_{k=1}^n x_k \) or \( \bigoplus F \)) exists in \( E \).

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Outline

1. Introduction
2. Basic definitions
3. The condition \((W+)\)
4. Main theorem and its generalizations
The condition \((W+)\)

An effect algebra \(E\) fulfills the condition \((W+)\) (introduced by Tkadlec) if for each orthogonal subset \(A \subseteq E\) and each two upper bounds \(u, v\) of \(A^{\oplus}\) there exists an upper bound \(w\) of \(A^{\oplus}\) below \(u, v\).

Statement (Tkadlec 2010)

Lattice effect algebras and orthocomplete effect algebras fulfill the condition \((W+)\).

Every orthocomplete effect algebra is Archimedean.

Proposition

Let \(E\) be an Archimedean effect algebra fulfilling the condition \((W+)\). Then every meager element of \(E\) is the orthosum of a system of hypermeager elements.
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Let $E$ be an Archimedean effect algebra fulfilling the condition (W+). Then every meager element of $E$ is the orthosum of a system of hypermeager elements.
Lemma (Shifting lemma)

Let $E$ be an Archimedean effect algebra fulfilling the condition $(W^+)$, let $u, v \in E$, and let $a_1, b_1$ be two maximal lower bounds of $u, v$. There exist elements $y, z$ and two maximal lower bounds $a, b$ of $y, z$ for which $y \leq u$, $z \leq v$, $a \leq a_1$, $b \leq b_1$, $a \land b = 0$, $a, b$ are maximal lower bounds of $y, z$ and $y, z$ are minimal upper bounds of $a, b$. Furthermore, $(y \lor a) \land (z \lor a) = 0$, $(y \lor b) \land (z \lor b) = 0$, $(y \lor a) \land (y \lor b) = 0$, $(z \lor a) \land (z \lor b) = 0$.

The Shifting lemma provides the following minimax structure.
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Let $E$ be an Archimedean homogeneous effect algebra fulfilling the condition (W+). Every two hypermeager elements $u, v$ possess $u \land v$.

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Corollary

Let $E$ be an Archimedean homogeneous effect algebra fulfilling the condition (W+). For every element $u$, $u \land u'$ and $u \lor u'$ exist and $[0, u \land u'] \subseteq B$ for every block $B$ containing $u$. 
Meets in Archimedean homogeneous effect algebra fulfilling the condition (W+)

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Corollary

Let \( E \) be an Archimedean homogeneous effect algebra fulfilling the condition (W+). For every element \( u \), \( u \wedge u' \) and \( u \vee u' \) exist and \([0, u \wedge u'] \subseteq B\) for every block \( B\) containing \( u \).
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Let \(E\) be an Archimedean homogeneous effect algebra fulfilling the condition \((W+)\). For every element \(u\), \(u \wedge u'\) and \(u \vee u'\) exist and \([0, u \wedge u'] \subseteq B\) for every block \(B\) containing \(u\).
Main theorem

Theorem

Let $E$ be an Archimedean homogeneous effect algebra fulfilling the condition $(W+)$. Then every block in $E$ is a lattice and $E$ can be covered by MV-algebras.

Corollary

Let $E$ be an orthocomplete homogeneous effect algebra. Then $E$ can be covered by Heyting MV-effect algebras.
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Generalization of the Main theorem

**Definition**

An effect algebra $E$ has the *maximality property* if \{u, v\} has a maximal lower bound $w$ for every $u, v \in E$.

It is easy to see that an effect algebra $E$ has the maximality property if and only if \{u, v\} has a maximal lower bound $w$, $w \geq t$ for every $u, v, t \in E$ such that $t$ is a lower bound of \{u, v\}. As noted by Tkadlec $E$ has the maximality property if and only if \{u, v\} has a minimal upper bound $w$ for every $u, v \in E$.

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Let $E$ be a homogeneous effect algebra having the maximality property. Then every block $B$ in $E$ is a lattice and $E$ can be covered by MV-algebras.

Corollary (Riečanová 2000)

Let $E$ be a lattice effect algebra. Then $E$ can be covered by MV-algebras which are blocks of $E$. 
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Example 5 - orthoalgebra $E$

$a$ and $b$ have two different minimal upper bounds, $a \oplus b = f'$ and $a \oplus c = e'$. 


References


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Thank you for your attention.